CSE389C: Practice Final

1. Provide short answers and *brief* explanations where needed for the following questions. No analysis is required. If you write more than a sentence or two, you are writing too much.

   (a) Why is it important for mathematical models to satisfy the following constraints?

      i. Dimensional Consistency

      ii. Tensorial Consistency

      iii. Material Frame Indifference

   (b) What are the principal directions of a second-rank tensor? Do all second rank tensors have principal directions?

   (c) $\mathbf{T}$ is the Cauchy stress tensor and $\mathbf{n}$ is a unit vector. What are $\mathbf{T}^T$ and $\mathbf{Tn}$?

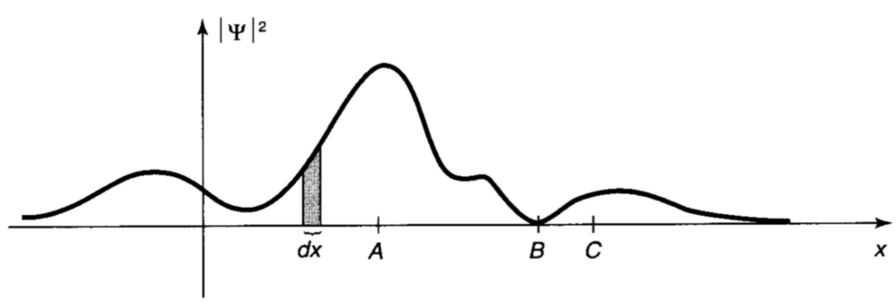
   (d) What are the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ and how are they defined?

   (e) What physical phenomenon is expressed in the following equation:

   $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

   (f) What is Born's probability rule?

   (g) Suppose a particle is in a quantum state $\psi(x)$, so that $|\psi|^2$ is given by:

   A measurement is then made about the position of the particle and it is found to be at a point C. *Shortly* after this measurement, a second measurement is made about the position of the particle. What does the measurement reveal? Why? Sketch the wave function after the measurement.

   (h) A particle is in a quantum state $\psi(x)$. Write an expression for the expected value of the momentum of the particle.

   (i) Describe the photoelectric effect.

   (j) Describe an experiment that indicated a wave-particle duality of electrons.

   (k) What does the Heisenberg Uncertainty Principle say about the position x and momentum p of a particle?

(l) Suppose the operators corresponding to two observables commute. What does this guarantee in a quantum system?

(m) What is represented by the following quantity:

$$\int_a^b |\psi(x,t)|^2 dx \ ?$$

(n) Just to make sure you *understand* quantum mechanics, what is one thing that confuses you about it?

2. We've just finished a semester-long course on modeling. Answer the following questions:

   (a) What is a mathematical model, and why are they useful?

   (b) Pick a mathematical model we've discussed and explain it. Some things to touch upon might be:

   - What physical phenomenon is being described.
   - What does each term in the model represent.
   - What are some of the assumptions that went into building this model.
   - What are the different parts of the model, such as which parts came from conservation laws and which parts are constitutive models.
   - Where is the model valid?

   (c) Describe a real world application of the model discussed in part (b).

3. **Sequential Measurements**. An operator $\tilde{A}$, representing observable A has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2, respectively. Operator $\tilde{B}$, representing observable B, has two normalized eigenstates ϕ_1 and ϕ_2, with eigenvalues b_1 and b_2. The eigenstates are related by:

$$\psi_1 = (3\phi_1 + 4\phi_2)/5 \qquad \psi_2 = (4\phi_1 - 3\phi_2)/5$$

   (a) Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?

   (b) If B is now measured, what are the possible results, and what are their probabilities?

   (c) Right after measurement of B, A is measured again. What is the probability of getting a_1?

   (d) Bonus: Why is this mysterious, and how do we explain what's going on in terms of the wave function?

4. Consider a particle of mass m constrained to move only along the x axis in a potential given by:

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}.$$

(a) Write the Schrodinger equation (time independent) describing the motion of the particle, stating any boundary conditions the wave function must satisfy?

(b) Determine the energy levels and associated wave functions.

In the same system, for the following three questions, suppose that the quantum state of the particle is given by:

$$\psi = Ax(a-x)\ . \tag{1}$$

(c) What is the value of A?

(d) What are the expected values of the momentum and energy of the particle?

(e) A measurement of the energy of the particle will be performed. What is the probability that the result of the measurement will be the ground state energy from (b) above?

Now, in the same quantum system, suppose that there are three identical, non-interacting particles.

(f) Suppose all the particles are electrons (fermions with "spin" 1/2). Our analysis of the periodic table revealed that no more than 2 electrons can be in the first energy level (this is known as Pauli's exclusion principle). Given this information, what is the ground state energy of the system?

(g) Suppose all the particles are photons (bosons with "spin" 1). Pauli's exclusion principle does not apply to Bosons. Given this information, what is the ground state of the energy of the system?

Bonus: Suppose we are considering one particle in the infinite well that is in the ground state, i.e.,

$$\psi = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)\ , \tag{2}$$

and that the well suddenly expands to twice its original size! The right wall moving from a to $2a$ — leaving the wave function (momentarily) undisturbed. The energy of the particle is now measured.

(h) The most probable result for the energy is E_2, i.e., the second state when $n = 2$. What is the probability of getting that result?

(i) How would you determine the probability of getting the *next* most probable result for the energy?

Note: The integral for part (i) may be too involved for an actual exam, but make sure you know how to set up the problem and, since this isn't a really exam, try to figure out the answer and what the next most probable result for the energy is.

5. **This problem appeared on the 2016 Area C prelim. It is similar to one of your homework problems, so try to do it without looking back unless you**

get stuck. In a linearly elastic solid undergoing small (infinitesimal) deformations, the second Piola-Kirchoff stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = \lambda \mathbf{I} \nabla \cdot \mathbf{u} + \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right)$$

where $\mathbf{u}$ is the small displacement, $\mathbf{I}$ is the identity tensor, and λ and μ are constants.

(a) Express $\mathbf{S}$ in index notation.

(b) Assuming no body forces and starting with conservation of momentum, derive an equation for the evolution of $\mathbf{u}$. Again, proceed using index notation.

(c) Show that $\mathbf{u} = f(\mathbf{x} \cdot \mathbf{e} - \alpha t)\mathbf{e}$ for any sufficiently smooth f and any unit vector $\mathbf{e}$ is a solution. What is the value of α?

(d) Show that $\mathbf{u} = f(\mathbf{x} \cdot \mathbf{e} - \beta t)\mathbf{g}$ with $\mathbf{g}$ a unit vector such that $\mathbf{g} \cdot \mathbf{e} = 0$, is a solution. What is the value of β?

6. **Bonus: The problem below is from your midterm. Try to do this without notes. Moreover, if you previously did it using the Lagrangian approach, do it using the Eulerian approach, and vice-versa! Make sure you understand the difference between both, but can still arrive at the same answer**

A liquid of constant density ρ is flowing in a straight horizontal rectangular channel in a gravitational field with acceleration of gravity g. Let x be the coordinate along the length of the channel, let $h(x, t)$ be the height of the liquid about the flow of the channel at the point x, which does not vary across the width of the channel, and let $V(x, t)$ be the velocity in the x-direction, which does not vary over the width or over the depth of the liquid. Because nothing varies over the width, the equations to be derived below can be written in terms of the mass and momentum per unit width, in which case the width of the channel never enters.

(a) What is the total mass of liquid per unit width in the interval $x \in [a, b]$?

(b) Derive an integral relation expressing conservation of mass (per unit width) in the interval $x \in [a, b]$.

(c) Use the fact that $[a, b]$ is arbitrary to write a PDE in x and t expressing conservation of mass.

(d) What is the total x-momentum of the liquid in the interval $x \in [a, b]$?

(e) Derive an integral relation expressing conservation of x-momentum (per unit width) in the interval $x \in [a, b]$. What quantity needs to be modeled in this relation?

(f) Use the fact that $[a, b]$ is arbitrary to write a PDE in x and t expressing conservation of x-momentum.

(g) In a momentum conservation equation, we expect an internal force to appear. In this case, it would be the x-direction force per unit width f. Determine a model for f, assuming that f does not depend on V. Are there any undetermined constants? Does this model allow you to close the momentum equations derived above?