CSE389C: Practice Exam

1. (30 points) Provide short answers and *brief* explanations where needed for the following questions. No analysis is required. If you write more than a sentence or two, you are writing too much.

   (a) Why is dimensional analysis an important constraint on mathematical models of physical systems?

   (b) What processes are being represented by constitutive equations?

   (c) What is material frame indifference and how is this different than tensorial consistency?

   (d) What is the difference between the first and second Piola-Kirchhoff stress tensor?

   (e) What is the physical principle expressed in the following equation, and which quantities require constitutive modeling?

   $$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathrm{Div}\mathbf{F}\mathbf{S} + \mathbf{f}_0$$

   (f) What is the Eulerian form of conservation of energy? What do each of the terms represent in the equation?

   (g) At what length scales do our continuum equations/models hold? How do we treat processes that occur on scales smaller than this length scale?

2. (35 points)

   (a) State the Cauchy Stress Theorem.

   (b) The principle of balance of *angular* momentum states that:

   $$\int_{\Omega_t} \mathbf{x} \times \varrho \frac{d\mathbf{v}}{dt} dx = \int_{\Omega_t} \mathbf{x} \times \mathbf{f} dx + \int_{\partial\Omega t} \mathbf{x} \times \sigma(\mathbf{n}) dA \ . \qquad (1)$$

   Show that this implies $\mathbf{T} = \mathbf{T}^T$.

   (c) What does this imply about the second Piola-Kirchhoff stress tensor S? Show your justification.

3. (35 points) A cube of incompressible material of dimension a is subjected to a uniaxial load in the x_1 direction, resulting in a deformation that reduces the dimension in the x_1 direction to $\alpha^2 a$, with $0 < \alpha < 1$. Further, assume the material is homogeneous and isotropic, has uniform temperature and no heat flux. Then the Helmholtz free energy is given by:

   $$\psi = W(\mathbf{E})$$

   where W is the *strain energy density* function, and $\mathbf{E}$ is the Green-St. Venant strain tensor.

(a) Determine the deformation gradient tensor ($\mathbf{F}$), the right Cauchy-Green deformation tensor ($\mathbf{C}$), and the Green-St. Venant strain tensor ($\mathbf{E}$) in the material.

(b) What does tensorial consistency imply about the form of W?

(c) Assume the material has an incompressible neo-Hookean strain energy density function so that W only depends *linearly* on $\mathbf{E}$. What is the most general form of the corresponding model for W?

(d) Because the material is incompressible, there is a degeneracy in the stress relation, so that the second Piola-Kirchhoff stress is given by:

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} + p\mathbf{F}^{-1}\mathbf{F}^{-T}$$

where p can be determine from boundary conditions, in this case on the unloaded sides. Determine $\mathbf{S}$.