Local Conservation Equations

Material (Lagrangian)	**Spatial** (Eulerian)
Conservation of Mass	
$\varrho_0 = \varrho \det \mathbf{F}$	$\dfrac{\partial \varrho}{\partial t} + \text{div}(\varrho \mathbf{v}) = 0$
Conservation (Balance) of Linear Momentum	
$\text{Div}\mathbf{FS} + \mathbf{f}_0 = \varrho_0 \dfrac{\partial^2 \mathbf{u}}{\partial t^2}$	$\text{div}\mathbf{T} + \mathbf{f} = \varrho \left(\dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \text{ grad } \mathbf{v} \right)$
Conservation (Balance) of Angular Momentum	
$\mathbf{S} = \mathbf{S}^T$	$\mathbf{T} = \mathbf{T}^T$
Conservation of Energy	
$\varrho_0 \dot{e}_0 = \mathbf{S} : \dot{\mathbf{E}} - \text{Div}\mathbf{q}_0 + r_0$	$\varrho \dfrac{\partial e}{\partial t} + \varrho \mathbf{v} \cdot \text{ grad } e = \mathbf{T} : \mathbf{D} - \text{div}\mathbf{q} + r$
Second Law of Thermodynamics	
$\varrho \dot{\eta}_0 + \text{Div}\dfrac{\mathbf{q}_0}{\theta} - \dfrac{r_0}{\theta} \geq 0$	$\varrho \dfrac{\partial \eta}{\partial t} + \varrho \mathbf{v} \cdot \text{ grad } \eta + \text{div}\dfrac{\mathbf{q}}{\theta} - \dfrac{r}{\theta} \geq 0$

Maxwell's Equations

Integral Form	**Differential Form**
Gauss's Law	
$\epsilon_0 \displaystyle\int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} \, dA = q_\Omega = \int_\Omega \rho \, dx$	$\epsilon_0 \nabla \cdot \mathbf{E} = \rho$
Faraday's Law	
$\displaystyle\int \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_A \mathbf{B} \cdot \mathbf{n} \, dA$	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
The Ampere-Maxwell Law	
$\displaystyle\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_A \mathbf{j} \cdot \mathbf{n} \, dA + \mu_0 \epsilon_0 \int_A \mathbf{n} \cdot \frac{\partial \mathbf{E}}{\partial t} \, dA$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$
The Absence of Magnetic Monopoles	
$\displaystyle\int_{\partial\Omega} \mathbf{B} \cdot \mathbf{n} \, dA = 0$	$\nabla \cdot \mathbf{B} = 0$

Kinematics of Deformable Bodies

1. $u = \varphi(X) - X$
2. $F(X) = \nabla\varphi(X) = I + \nabla u(X)$
3. $C = F^T F$
4. $E = \frac{1}{2}(C - I)$
5. $v = \dot{x}(\varphi^{-1}(x,t),t)$
6. $\frac{D\psi}{Dt} = \frac{\partial\psi}{\partial t} + v\cdot \text{ grad } \psi$
7. $L = \text{ grad } v$
8. $L = D + W$
9. $D = \frac{1}{2}(L + L^T)$
10. $W = \frac{1}{2}(L - L^T)$
11. $L_m = \dot{F}F^{-1}$
12. $\dot{F} = \text{ grad } vF = L_m F$
13. $\dot{\det} F = \det F \text{div} v$
14. *Piola Transformation*
 $T_0(X) = \det F(X) T(x) F(X)^{-T}$
15. *Polar Decomposition*
 $F = RU = VR$
 $\to R$ orthogonal, U, V symmetric, PD.
 $\to C = U^T U$
16. *Reynolds Transport Theorem*
 $\frac{d}{dt}\int_\omega \Psi dx = \int_\omega \frac{\partial\Psi}{\partial t}dx + \int_{\partial\omega}\Psi v\cdot n\, dA$

Mass and Momentum

1. $M(B) = \int_{\Omega_t}\varrho dx$
2. $\int_{\Omega_0}\varrho_0(X)dX = \int_{\Omega_t}\varrho(x)dx$
3. *Material Cons. Mass*
 $\varrho_0(X) = \varrho(x)\det F(X)$
4. *Spatial Cons. Mass*
 $\frac{\partial\varrho}{\partial t} + \text{div}(\varrho v) = 0$
5. $\frac{dI(B,t)}{dt} = \int_{\Omega_t}\varrho\frac{dv}{dt}dx$

Force and Stress

1. *Total Force*
 $F = \int_{\Omega_t} f dx + \int_{\partial\Omega_t}\sigma(n)\, dA$

Balance Lin. & Ang Momentum

1. *Principle of Balance Lin. Momentum*
 $\frac{dI}{dt} = F\ (F = ma).$
2. $\int_{\Omega_t}\varrho\frac{dv}{dt}dx = \int_{\Omega_t} f dx + \int_{\partial\Omega_t}\sigma(n)\, dA.$
3. $\int_{\Omega_t} x\times\varrho\frac{dv}{dt}dx = \int_{\Omega_t} x\times f dx + \int_{\partial\Omega_t} x\times\sigma(n)\, dA$
4. *Cauchy's Theorem*
 $\sigma(n,x,t) = T(x,t)n \qquad T = T^T$
5. $\text{div}T + f = \varrho\frac{dv}{dt} = \varrho(\frac{\partial v}{\partial t} + v\cdot \text{ grad } v)$
6. *Cauchy Stress*
 $T = (\det F)^{-1}PF^T = (\det F)^{-1}FSF^T$
7. *First Piola-Kirchoff Stress*
 $P = (\det F)TF^{-T} = FS$
8. *Second Piola-Kirchoff Stress*
 $S = (\det F)F^{-1}TF^{-T} = F^{-1}P$
9. *Power*
 $P = \int_{\Omega_t} f\cdot v dx + \int_{\partial\Omega_t}\sigma(n)\cdot v\, dA$
 $P = \frac{dk}{dt} + \int_{\Omega_t} T : D dx$
10. $k = \frac{1}{2}\int_{\Omega_t}\varrho v\cdot v dx.$

Conservation Energy

1. Total Energy $= k + U$
 $\to k =$ kinetic energy, $U =$ internal energy.
2. *Principle Consv. Energy*
 $\frac{d}{dt}(k + U) = P + \dot{Q}.$
 $\to Q =$ internal heating.
3. $\varrho\frac{de}{dt} = T : D - \text{div}q + r$
4. $\varrho_0\dot{e_0} = S : \dot{E} - \text{Div}q_0 + r_0$
 $\to r =$ heat per unit volume.

2nd Law Thermodynamics

1. *Clausius-Duhem Inequality*
   $\varrho\frac{d\eta}{dt} + \text{div}\frac{q}{\theta} - \frac{r_0}{\theta}$
   $\varrho_0\dot{\eta}_0 + \text{Div}\frac{q_0}{\theta} - \frac{r_0}{\theta} > 0$
   $\rightarrow \theta =$temp, $\eta =$ entropy density

Constitutive Equations

1. *Material Frame Indifference*
   $x^* = Qx + c \implies T^* = QTQ^T$
2. *MFI Solids*
   $F^* = QF \qquad \det F^* = \det F$
3. *MFI Fluids*
   $T = -pI + 2\mu D$
4. *Coleman-Noll (Dissipative)*
   $S = \varrho_0\frac{\partial\Psi}{\partial E}, \eta_0 = -\frac{\partial\Psi}{\partial\theta}$
   $\frac{\partial\Psi}{\partial\nabla\theta} = 0, -\frac{1}{\theta}q_0 \cdot \nabla\theta \geq 0$
5. *Coleman-Noll* $S = F(E) + I(\dot{E})$
   $F(E) = \varrho_0\frac{\partial\Psi}{\partial E} \;, I(\dot{E}) : \dot{E} - \frac{1}{\theta}q_0 \cdot \nabla\theta \geq 0$

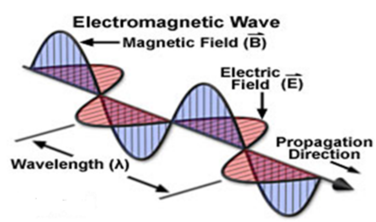
Electromagnetic Waves

1. *Coulomb's Law*
   $F = k\frac{|q_1||q_2|}{r^2}, \qquad k = \frac{1}{4\pi\epsilon_0}$
2. *Gauss's Law*
   $q_\Omega = \epsilon_0 \oint_{\partial\Omega} E \cdot n \, dA = \int_\Omega \rho \, dx$
   $\epsilon_0\nabla \cdot E = \rho$
   $\rightarrow \rho =$ charge density
3. *Ampere's Law*
   $\oint B \cdot ds = \mu_0 i_{\text{enclosed}}, \, i = \text{current.}$
4. *Ampere + Maxwell Law*
   $\oint B \cdot ds = \mu_0 i + \mu_0\epsilon_0\frac{d}{dt}\Phi_E$
   $\oint B \cdot ds = \mu_0 \int_A j \cdot n \, dA + \mu_0\epsilon_0\frac{d}{dt}\int_A E \cdot n \, dA$
   $\nabla \times B = \mu_0 j + \mu_0\epsilon_0\frac{\partial E}{\partial t}$
5. *Faraday's Law*
   $\oint E \cdot ds = -\frac{d}{dt}\int_A B \cdot n \, dA$
   $\nabla \times E = -\frac{\partial B}{\partial t}$
6. *No Magnetic Monopoles*
   $\int_{\partial\Omega} B \cdot n \, dA = 0$
   $\nabla \cdot B = 0$

Waves

1. $u(x, y) = \mu_0 e^{i(k\cdot x - \omega t)}$
   $\rightarrow \mu_0 =$ amplitude of wave
   $\rightarrow k =$ wave number
   $\rightarrow \omega =$ angular frequency
   $\rightarrow \lambda = 2\pi/k =$wave length
   $\rightarrow T = 2\pi/\omega =$period of oscillation
   $\rightarrow v = \omega/k =$ wave speed
2. *General Wave Equation*
   $\frac{\partial^2 u}{\partial t^2} = \left(\frac{\omega^2}{k^2}\right)\frac{\partial^2 u}{\partial x^2}$
3. *Electromagnetic Waves*
   $E = E_0 e^{i(k\cdot x - \omega t)}, \;\; B = B_0 e^{i(k\cdot x - \omega t)}$
   $\rightarrow \omega/|k| =$propagation speed
   $\rightarrow \hat{k} = k/|k| =$ direction on propagation
   $\rightarrow c = 1/\sqrt{\epsilon_0\mu_0} =$ speed of light
4. *Electromagnetic Wave Equation*
   $\frac{\partial^2 E}{\partial t^2} = \frac{1}{\mu_0\epsilon_0}\Delta E$
5. *E-M Waves & Maxwell's Equations*
   $k \cdot E = 0 \implies k \cdot E_0 = 0$
   $k \cdot B = 0 \implies k \cdot B_0 = 0$
   $k \times E = \omega B \implies \hat{k} \times E = cB$
   $k \times B = -\frac{1}{c^2}\omega E \implies \hat{k} \times B = -\frac{1}{c}E$

   $E \cdot B = 0$
   $|E_0| = c|B_0|$
   $E \times B = \frac{k}{\omega}|E|^2 = \frac{1}{c}|E|^2\hat{k}$



Quantum Mechanics

1. $E = \hbar w = h\nu$
2. $\lambda = \dfrac{h}{p}$
3. *Wave Equation*
   $\Psi(x,t) = \psi_0 e^{i(px-Et)/\hbar}$
   $\rightarrow \Psi(x,t) = \psi_0 e^{i(kx-wt)}$
   $\rightarrow k = 2\pi/\lambda = p/\hbar$
   $\rightarrow w = 2\pi\nu = E/\hbar$
4. $E\Psi = \left(-\dfrac{\hbar}{i}\dfrac{\partial}{\partial t}\right)\Psi$
5. $p\Psi = \left(\dfrac{\hbar}{i}\dfrac{\partial}{\partial x}\right)\Psi$
6. *Schrodinger's equation* (free particle)
   $i\hbar\dfrac{\partial \Psi}{\partial t} + \dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi}{\partial x} = 0$
   $\rightarrow E = p^2/2m$
7. *Hamiltonian Operator* $H(q,p) = E$
   $H(q,p) = \dfrac{p^2}{2m} + V(q) = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2} + V(x)$
8. *Schrodinger's equation* (time independent)
   $H\psi = E\psi$
   $\rightarrow E$ constant, i.e., eigenvalue
9. *Schrodinger's equation* (general)
   $i\hbar\dfrac{\partial \Psi}{\partial t} + \dfrac{\hbar^2}{2m}\dfrac{\partial^2\Psi}{\partial x^2} - V\Psi = 0$
10. $\Psi^*\Psi = |\Psi(x,t)|^2 = \rho(x,t)$
    $\rightarrow \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$
    $\rightarrow \dfrac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 0$

Dynamic Variables & Observables

1. *Dynamic Variable*
   $Q = Q(q_1, q_2, \cdots, q_N; p_1, p_2, \cdots p_N)$
   $\tilde{Q} = \tilde{Q}(q_1, \cdots, q_N; -i\hbar\dfrac{\partial}{\partial q_1},, \cdots, -i\hbar\dfrac{\partial}{\partial q_N})$
2. $\langle Q\rangle = \langle \Psi, \tilde{Q}\Psi\rangle = \int \Psi^*\tilde{Q}\Psi dq$
3. *Hermitian*
   $\langle \psi, A\phi\rangle = \langle A\psi, \phi\rangle \quad \forall \phi, \psi \in L^2$
4. $\sigma_Q^2 = \langle Q^2\rangle - \langle Q\rangle^2$
5. *Uncertainty Principle*
   $\sigma_Q^2\sigma_M^2 \geq \left(\dfrac{1}{2i}\left\langle[\tilde{Q}, \tilde{M}]\right\rangle\right)^2$

Hydrogen Atom

1. *The complete hydrogen wave functions*
   $\psi_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta, \phi)$
   $n = 1, 2, \dots$ (describes energy level)
   $\ell = 0, 1, \dots, n-1$ (describes shape)
   $\rightarrow \ell =$ s,p,d,f
   $m = 0, \pm 1, \pm 2, \dots, \pm \ell$ (describes orientation)

Spin & Angular Momentum

1. $L = q \times p, \quad L_j = \epsilon_{rsj} q_r \dfrac{\hbar}{i}\dfrac{\partial}{\partial q_s}, \quad L^2 = L_1^2 + L_2^2 + L_3^2$
2. $[L_1, L_2] = i\hbar L_3, \quad [L_2, L_3] = i\hbar L_1, \quad [L_3, L_1] = i\hbar L_2$
3. $[L^2, L_i] = 0$
4. $L_\pm = L_1 \pm iL_2, \qquad L_3(L_\pm\phi) = (\mu \pm \hbar)L_\pm\phi$
5. *Assume for spin operator S*
   $[S_1, S_2] = i\hbar S_3, \quad [S_2, S_3] = i\hbar S_1, \quad [S_3, S_1] = i\hbar S_2$
6. $S^2 q_{sm} = \hbar^2 s(1+s) q_{sm}, \quad S_3 q_{sm} = \hbar m q_{sm}$
7. $S_1 = \dfrac{\hbar}{2}\sigma_1, \quad S_2 = \dfrac{\hbar}{2}\sigma_2, \quad S_3 = \dfrac{\hbar}{2}\sigma_3$
   $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
8. *Multielectron Systems*
   $\psi_\pm(r_1, r_2) = C\left(\psi_1(r_1)\psi_2(r_2) \pm \psi_2(r_1)\psi_1(r_2)\right)$
   $\rightarrow \psi_+(r_1, r_2) = +\psi_+(r_2, r_1)$ (Boson: $\mathbb{Z}$-spin)
   $\rightarrow \psi_-(r_1, r_2) = -\psi_-(r_2, r_1)$ (Fermions:$\frac{1}{2}\mathbb{Z}$-spin)
   $\implies \psi_1 \neq \psi_2$(Fermions)
9. *Slater determinants* $\rightarrow$ Simply a way to satisfy antisymmetry of wave functions.

Ab Initio Methods

1. *Many Atom & Electron Systems*
   $H = T_e(r^N) + T_M(R^M) + V_{eM}(r^N, R^M)$
   $+ V_{MM}(R^M) + V_{ee}(r^N)$
2. *Born-Oppenheimer Approximation*
   $\psi(r^N, R^M) = \psi_e(r^N, R^M)\chi(R^M)$
   $\bullet H_{elec}\psi_e(r^n, R^M) = E_{elec}(R^M)\psi_e(r^N, R^M)$
   $\rightarrow R^M$ treated as parameter
   $\bullet E_\chi = \left(T_M(R^M) + V_{MM}(R^M) + E_{elec}(R^M)\right)\chi$

3. *Hartree Method*
   $h(r_i)\phi_i(r_i) = e_i\phi_i(r_i)$
   $\psi_H(r_1, \ldots, r_N) = \psi_1(r_1)\psi_2(r_2)\cdots\psi_N(r_N)$
   $\rightarrow$ Solve ψ_i one at a time
   $\rightarrow$ ignore elec-elec interaction

Density Functional Theory

1. $n(r) = N \int |\psi(r, r_1, r_2, \ldots, r_{N-1})|^2 dr$
   $\rightarrow$ pdf's are indistinguishable.

2. $\int n(r)dr = N$

3. $\left\langle \psi, \sum_{i=1}^{N} v(r_i)\psi \right\rangle = \int v(r)n(r)dr$
   $\rightarrow$ expected value of potential in field of nuclei.

4. $\left\langle \psi, \sum_{i=1}^{N} \sum_{j>i} U(r_i, r_j)\psi \right\rangle = \frac{c}{2}\left(1 - \frac{1}{N}\right) \int \int \frac{n(r)n(r')}{|r - r'|} dr dr'$

- **Divergence Theorem**: $\int_{\Omega} \nabla \cdot \Phi \, dx = \int_{\partial\Omega} \Phi \cdot \mathbf{n} \, ds$

- **Stokes Theorem**: $\int_{\Omega} \nabla \times \Phi \cdot \; dx = \oint_{\partial\Omega} \Phi \cdot ds$

Properties

- **Distributive Properties**
$\nabla(\psi + \phi) = \nabla\psi + \nabla\phi$
$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

- **Product Rule for gradient**
$\nabla(\psi\,\phi) = \phi\,\nabla\psi + \psi\,\nabla\phi$

- **Product of a scalar and a vector**
$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi(\nabla \cdot \mathbf{A})$
$\nabla \times (\psi\mathbf{A}) = \psi(\nabla \times \mathbf{A}) + (\nabla\psi) \times \mathbf{A}$

- **Quotient rule**
$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$
$\nabla \cdot \left(\frac{\mathbf{A}}{g}\right) = \frac{(\nabla \cdot \mathbf{A})g - \mathbf{A} \cdot \nabla g}{g^2}$
$\nabla \times \left(\frac{\mathbf{A}}{g}\right) = \frac{(\nabla \times \mathbf{A})g + \mathbf{A} \times \nabla g}{g^2}$

- **Chain rule**
$\nabla(f \circ g) = (f' \circ g)\nabla g$
$\nabla(f \circ \mathbf{A}) = (\nabla f \circ \mathbf{A})\nabla\mathbf{A}$
$\nabla \cdot (\mathbf{A} \circ f) = (\mathbf{A}' \circ f) \cdot \nabla f$
$\nabla \times (\mathbf{A} \circ f) = -(\mathbf{A}' \circ f) \times \nabla f$

- **Vector dot product**
$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$

- **Vector cross product**
$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\ &= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla \cdot (\mathbf{A}\mathbf{B}^{\mathrm{T}}) \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})\end{aligned}$$

Second Derivatives

- **Curl of the gradient**
$\nabla \times (\nabla\phi) = \mathbf{0}$

- **Divergence of the curl**
$\nabla \cdot (\nabla \times \mathbf{A}) = 0$

- **Divergence of the gradient**
  $\nabla^2 \psi = \nabla \cdot (\nabla \psi)$

- **Curl of the curl**
  $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

Important Identities

- **Addition and multiplication**
  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
  $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$
  $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$
  $(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}$
  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$
  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\,\mathbf{C}$
  $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\,\mathbf{A}$
  $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})\,(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})\,(\mathbf{A} \cdot \mathbf{D})$
  $(\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))\,\mathbf{D} = (\mathbf{A} \cdot \mathbf{D})\,(\mathbf{B} \times \mathbf{C}) + (\mathbf{B} \cdot \mathbf{D})\,(\mathbf{C} \times \mathbf{A}) + (\mathbf{C} \cdot \mathbf{D})\,(\mathbf{A} \times \mathbf{B})$
  $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{D}))\,\mathbf{C} - (\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}))\,\mathbf{D}$

- **Differentiation**

* **Gradient**
  $\nabla(\psi + \phi) = \nabla\psi + \nabla\phi$
  $\nabla(\psi\,\phi) = \phi\,\nabla\psi + \psi\,\nabla\phi$
  $\nabla\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\,\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$

* **Divergence**
  $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
  $\nabla \cdot (\psi\mathbf{A}) = \psi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\psi$
  $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

* **Curl**
  $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
  $\nabla \times (\psi\mathbf{A}) = \psi\nabla \times \mathbf{A} + \nabla\psi \times \mathbf{A}$
  $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B}$

* **Second derivatives**
  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
  $\nabla \times (\nabla\psi) = \mathbf{0}$
  $\nabla \cdot (\nabla\psi) = \nabla^2\psi$
  $\nabla\,(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2\mathbf{A}$
  $\nabla \cdot (\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi$
  $\psi\nabla^2\phi - \phi\nabla^2\psi = \nabla \cdot (\psi\nabla\phi - \phi\nabla\psi)$
  $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi$
  $\nabla^2(\psi\mathbf{A}) = \mathbf{A}\nabla^2\psi + 2(\nabla\psi \cdot \nabla)\mathbf{A} + \psi\nabla^2\mathbf{A}$
  $\nabla^2(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla^2\mathbf{B} - \mathbf{B} \cdot \nabla^2\mathbf{A} + 2\nabla \cdot ((\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{B} \times \nabla \times \mathbf{A})$

* **Third derivatives**
  $\nabla^2(\nabla\psi) = \nabla(\nabla \cdot (\nabla\psi)) = \nabla(\nabla^2\psi)$
  $\nabla^2(\nabla \cdot \mathbf{A}) = \nabla \cdot (\nabla(\nabla \cdot \mathbf{A})) = \nabla \cdot (\nabla^2\mathbf{A})$
  $\nabla^2(\nabla \times \mathbf{A}) = -\nabla \times (\nabla \times (\nabla \times \mathbf{A})) = \nabla \times (\nabla^2\mathbf{A})$

- **Integration**

* **Surface-volume integrals**

$${}_{\partial V}\mathbf{A}\cdot d\mathbf{S} = \iiint_V (\nabla\cdot\mathbf{A})\, dV$$

$${}_{\partial V}\psi d\mathbf{S} = \iiint_V \nabla\psi\, dV$$

$${}_{\partial V}\left(\hat{\mathbf{n}}\times\mathbf{A}\right) dS = \iiint_V (\nabla\times\mathbf{A})\, dV$$

$${}_{\partial V}\psi\left(\nabla\varphi\cdot\hat{\mathbf{n}}\right) dS = \iiint_V \left(\psi\nabla^2\varphi + \nabla\varphi\cdot\nabla\psi\right) dV$$

$${}_{\partial V}\left[(\psi\nabla\varphi - \varphi\nabla\psi)\cdot\hat{\mathbf{n}}\right] dS = {}_{\partial V}\left[\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right] dS = \iiint_V \left(\psi\nabla^2\varphi - \varphi\nabla^2\psi\right) dV$$

* **Curve-surface integrals**

$$\oint_{\partial S}\mathbf{A}\cdot d\boldsymbol{\ell} = \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{s}$$

$$\oint_{\partial S}\psi d\boldsymbol{\ell} = \iint_S (\hat{\mathbf{n}}\times\nabla\psi)\, dS$$

$${}_{\partial S}\mathbf{A}\cdot \mathrm{d}\boldsymbol{\ell} = -{}_{\partial S}\mathbf{A}\cdot \mathrm{d}\boldsymbol{\ell}.$$