Math Modeling Exam 1

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1 Dimensional Analysis

1.1 Math models

Math theories are formulated based on 1.) empirical observations, 2.) invariance principles and dependency assumptions, 3.) established theories and principles, and 4.) insightful guessing.

The invariance principles can constrain the closure models: 2 broadly useful invariance constraints are dimensional invariance and coordinate invariance.

The validity of a model relationship can't depend on the arbitrary references (like units and coord. systems) used to define the descriptors- because they're arbitrary. Measuring units (meters, etc) being arbitrary lead to the **principle of dimensional homogeneity**. The principle in which a physical system is invariant to the coordinate system in which we describe it leads to **tensor analysis**.

For the Buckingham Pi Theorem, the null space is $n - m \Rightarrow$ can form n-m dim-less quantities.

1.2 Tensor analysis

Tensors come in different ranks: 0.) scalars, 1.) vectors, 2.) homogeneous linear maps from vectors to vectors (ex: $Ax = b$), n >2.) homogeneous linear maps from m-rank tensors ($0 < m < n$) to $n - m$ rank tensors.

Too much work to carry around the summations and $\vec{e}_i$, so use Einstein summation convention.

1.) Tensors represented as indexed objects, e.g. x_i, y_{ij}, z_{ijk}
2.) In any term, an index can appear at most 2x
3.) Repeated index $\Rightarrow$ sum over that index
4.) Non-repeated index $i \Rightarrow$ mult with $\vec{e}_i$

Ex: $Y_{ij}x_jz_j$ not valid because 2x repeated indices usually occur because (e_j, e_k) = 1 so the orthogonality is pairwise and can only happen once.

Invariants of vectors/tensors are scalar fns of those vectors/tensors that are invariant to the coordinate system in which the tensors are expressed.

Two important results for rank-2 tensors in 3D are:

1.) **Cayley-Hamilton theorem:** a matrix is a soln to its own char. eqn. i.e. A^n for $n > 2$ is a lin. comb. of I, A, A^2. Why is this interesting? (look in SAQ)

2.) There are only 3 independent invariants of a 3D rank-2 tensor. e.g. Any scalar fn of a tensor A can be expressed as a fn of its e-values. A "standard" set of invariants are defined as the coefficients of the characteristic eqn. i.e. the e-values λ of A are solutions of:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

where

$$\begin{aligned} I_1 &= A_{ii} \\ I_2 &= \tfrac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji}) \\ I_3 &= det(A) = \tfrac{1}{6}\epsilon_{ijk}\epsilon_{pqr}A_{ip}A_{jq}A_{kr} \end{aligned}$$

Another commonly used set of invariants is:

$$\begin{aligned} I_A &= A_{ii} \\ II_A &= -\tfrac{1}{2}A_{ij}A_{ji} \\ III_A &= \tfrac{1}{3}A_{ij}A_{jk}A_{ji} \end{aligned}$$

Now, we can represent tensor fns, which we might use to formulate a model of a tensor quantity. The Cayley-Hamilton thm and the result on the number of independent scalar invariants allows us to write any analytic tensor fn $B = F(A)$ (complicated) into 3 simpler things:

$$\boxed{F(A) = \gamma_1(I_A)I + \gamma_2(I_A)A + \gamma_3(I_A)A^2}$$

where $\gamma(I_1, I_2, I_3)$ are fns of the invariants of A.

Suppose a 3D rank-2 tensor A is a fn of a vector r, that is $A = F(r)$. What is the most general form of this fn? Consider 2 arbitrary vectors a, b. We can form a scalar fn

$$a_iA_{ij}b_j = f(a, b, r)$$

but the scalar fn f must be expressed in terms of the invariants that can be formed from a, b, r. These are:

$$|a|, |b|, |r|, a \cdot b, a \cdot r, b \cdot r$$

but f must be bilinear in a, b, so the only possibilities are a_ib_i and $a_ir_ib_ir_i$, which can be multiplied by an arbitrary fn of $r \cdot r$. Thus,

$$\begin{aligned} a_iA_{ij}b_j &= g(|r|)(a \cdot b) + h(|r|)(r \cdot a)(r \cdot b) \\ &= a_i(g(|r|)\delta_{ij} + h(|r|)r_ir_j)b_j \\ \Rightarrow A_{ij} &= g(|r|)\delta_{ij} + h(|r|)r_ir_j \end{aligned}$$

1.3 Cross products and tensor consistency

With cross products in tensor notation, we introduce the "alternating tensor" or "Levi-Civita Symbol".

$$\epsilon_{ijk} \begin{cases} 1 & ijk = 123, 231, 312 \\ -1 & ijk = 132, 213, 321 \\ 0 & \text{otherwise} \end{cases}$$

An issue with this tensor is that if we swap any 2 of our basis vectors, we change the sign of the tensor $\Rightarrow$ this tensor is not coordinate system invariant. It is, however, almost coordinate system invariant, up to a sign, so we call it a **pseudo-tensor** (or **pseudo-vector** for vectors). Ex: ϵ_{ijk} is a 3rd rank pseudo-tensor.

Then the cross product is

$$\boxed{(a \times b)_i = \epsilon_{ijk} a_j b_k.}$$

Note: if we apply the tensor again (introduce another cross product in our term), the result would again be a vector ($(-1)^2 = 1$). Coordinate system invariance imposes the restriction that models for pseudo-vectors must be expressed in terms of pseudo-vectors.

Ex: $a \times b = c \times d$ and $a = b \times c \times d$ are permissible/valid models, but $a = b \times d$ is not.

Bottom line: tensor consistency and dimensional consistency impose significant constraints on model forms.

Tensor algebra is in the appendix.

2 Kinematics of Deformable Bodies

2.1 Motivation

The analog of $F = ma$ is

$$\frac{\partial}{\partial t} \int_\Omega \rho \vec{v} dx = \int_\Omega \vec{f_b} dx + \int_{\partial\Omega} \vec{f_s} dA$$

where ρ is the mass density, $\vec{f_b}$ is the body force per unit volume, and $\vec{f_s}$ is the surface force per unit area.

Models for f_s, f_b are needed to complete the description of the motion of the continuum. We neglect f_b and only focus on f_s for now. We hypothesize that for solids, internal surface forces f_s will depend on deformation (or strain) and on the rate for fluids.

2.2 Solids

The material points within a deformable body is labeled by a vector X, representing the position w.r.t. the origin in some "ref config".

The **motion** and deformation can be expressed by a fn $x = \varphi(X, t)$ (which is the spatial position of the material pt X at time t). We require:

1.) φ is differentiable
2.) φ is bijective - there is a 1-1 correspondence b/w pts in Ω_0 and Ω_t (except possibly at $\partial\Omega_0$)
3.) φ is *orientation preserving* (the material can't turn inside out), so $\det\nabla\varphi(X, t) > 0$

The **displacement** is

$$\boxed{u = \varphi(X) - X}$$

which is a vector field. The **deformation gradient** is

$$\boxed{F(X) = \nabla\varphi(X) = I + \nabla u(X)}$$

which is a rank-2 tensor $F_{ij} = \frac{\partial \varphi_i}{\partial X_j} = (\nabla\varphi)_{ij}$ because φ is a vector field. The motion is **rigid** if

$$\varphi(X) = \vec{a} + Qx$$

where $a \in \mathbb{R}^3$, is a translation and Q (the rigid body rotation) is a unitary tensor. This means the body is moving but does not deform.

2.3 Deformation Tensor

Consider the deformation of differential line segments. Let dX be a diff. segment in Ω_0. It will be mapped to $dx = FdX$. (For intuition, go to intuition pt 1.)

Non-rigid deformations will change the lengths of the line segments. Considering the length of these segments,

$$dS_0^2 = |dX|^2 = dX \cdot dX = dX_i dX_i$$
$$dS^2 = |dx|^2 = (Fdx) \cdot FdX = dX^{\mathrm{T}} F^{\mathrm{T}} F dX$$
$$\Rightarrow dS^2 = dX^{\mathrm{T}} C dX$$

Thus, we define the (right) **Cauchy-Green deformation tensor**, a rank-2 tensor C that quantifies "stretching" (the distortion) of continuum in different directions as

$$\boxed{C = F^{\mathrm{T}} F}$$

C is s.p.d. (For left Cauchy-Green deformation tensor, go to pg 8 of textbook). The change in line segment length is given by

$$dS^2 - dS_0^2 = dX^{\mathrm{T}} C dX - dX^{\mathrm{T}} dX$$

$$= dX^{\mathrm{T}}(2E)dX$$

where

$$\boxed{E = \frac{1}{2}(C - I)}$$

is the **Green-St. Venant Strain Tensor**, which denotes the change in length of line segments. (For intuition, go to intuition part 2.) **Note:** if φ is a rigid-body motion, then $E = 0$.

Since $F = I + \nabla u$, and $C = F^{\mathrm{T}}F$, then

$$\boxed{E = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}} + \nabla u^{\mathrm{T}}\nabla u).}$$

Since E is symmetric, it has real e-values and orthogonal e-vectors. These e-vectors are the **principal directions** of strain E. **Note:** the images of principal directions are also mutually orthogonal. (For more information, go to page 18 of the packet or pages 9, 20-23 of textbook.) **Note:** C and E have the same e-vecs.

Now consider what happens to differential segments aligned with a more general set of orthogonal bases. The images of $dS_0\vec{e}_i{}'$ that aren't e-vecs of C and E aren't mutually orthogonal since the projn. along direction $\vec{e}_i$ is scaled according to λ_i which generally are not equal.

The apparent rotation of the non-principal basis vectors is encoded in the off-diagonal elements of E when expressed in these coordinates. They are referred to as **shear strains** γ_{ij} and can be viewed as a consequence of anisotropic stretching. The shear strain in the $X_1 - X_2$ plane is defined by the angle change $\gamma_{ij} \stackrel{\text{def}}{=} \frac{\pi}{2} - \theta$. Thus

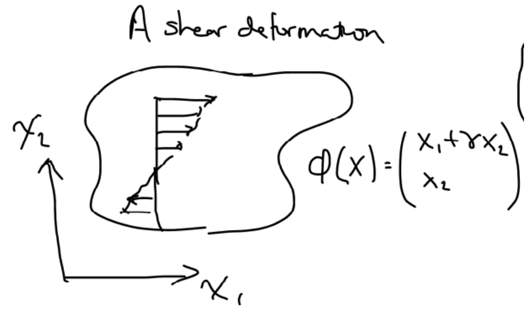
$$\boxed{\sin\gamma_{ij} = \frac{2E_{ij}}{\sqrt{1+2E_{ii}}\sqrt{1+2E_{jj}}}.}$$

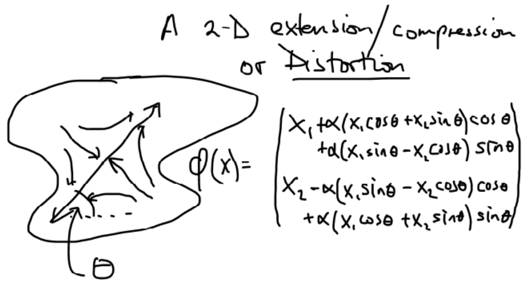
2.4 Relating shear and principal strain-the Polar Decomposition Theorem

Consider the two following deformations:
For a given γ, the two can be related by a scalar α and rotation angle θ. In particular, the 2 deformations will have the same C and E. To understand why, let's look at the **Polar Decomposition theorem**, which says that for any invertible F, $\exists$ unique R, U, V s.t.

1.) R is unitary (& orthonormal) [i.e. a rotation]
2.) U, V are s.p.d.
3.) $F = RU = VR$

Proof is on pg 19 of the textbook. **Note:** U and V are the right and left stretch tensors, respectively.





When F is the deformation gradient:

1.) $F = RU = VR$
2.) $C = F^{\mathrm{T}}F = U^{\mathrm{T}}R^{\mathrm{T}}RU = U^2$
3.) $B = FF^{\mathrm{T}} = VRR^{\mathrm{T}}V^{\mathrm{T}} = V^2$

$\Rightarrow$ C and B don't depend on R $\Rightarrow$ rotation doesn't impact C. This explains our example: in the shear case, $R \neq I$ and in the 2D distortion case, $R = I$. Thus, a pure shear in 2D (first image) is the composition of a 2D distortion (second image) and a rotation.

In general, the Polar Decomposition thm implies that any local deformation as characterized by the deformation gradient is composed of a distortion and a rotation **so the deformation tensors and strain tensors are independent of rotation.**

2.5 Fluids

We're looking at the rate of deformation now. Consider the motion as a fn of time:

1.) Lagrangian (material description) frame describes motion of a pt that originated at X throughout time

$$\dot{x} = v(x) = \frac{\partial\varphi(X,t)}{\partial t}$$
$$\ddot{x} = a(x) = \frac{\partial^2\varphi(X,t)}{\partial t^2}$$

2.) Eulerian (spatial description) frame describes motion of particles passing thru pt x throughout time

$$v(x) = \dot{x}(\varphi^{-1}(x,t), t)$$
$$a(x) = \ddot{x}(\varphi^{-1}(x,t), t)$$

Now consider the material time derivative of some field quantity ψ:

1.) Lagrangian

$$\boxed{\frac{d\psi_m(X,t)}{dt} = \frac{\partial \psi}{\partial t}\Big|_X}$$

2.) Eulerian (total derivative)

$$\boxed{\frac{D\psi}{Dt} := \frac{d\psi(x,t)}{dt}\Big|_X = \frac{\partial \psi}{\partial t} + v \cdot \operatorname{grad} \psi}$$

In the Eulerian description, we define the **velocity gradient tensor L** as

$$\boxed{L := \operatorname{grad} v(x,t).}$$

We can consider the rate of change of the (Lagrangian) deformation gradient F now as

$$\dot{F} = \frac{\partial}{\partial t}\nabla\varphi(X,t) = \nabla\frac{\partial \varphi}{\partial t} = \nabla v$$
$$\Rightarrow F_{ij} = \frac{\partial v_i}{\partial X_j} = \frac{\partial v_i}{\partial x_j}\frac{\partial x_i}{\partial X_j}$$

$$\boxed{\dot{F} = \operatorname{grad} v F = L_m F}$$

where L_m is L expressed in Lagrangian form. **Note:**

$$\boxed{L_m = \dot{F}F^{-1}}$$

can be decomposed into symmetric part D (**strain-rate tensor** or **deformation rate tensor**) and anti-symmetric part W (**rotation-rate tensor**) i.e.

$$\boxed{\begin{aligned} L &= D + W \\ D &= \frac{1}{2}(L + L^{\mathrm{T}}) \\ W &= \frac{1}{2}(L - L^{\mathrm{T}}) \end{aligned}} \tag{1}$$

Under the assumption that Ω_t is the ref. config (usually a good assumption for fluids as the choice of Ω_0 is somewhat arbitrary), $D = \dot{E}$. Under these assumptions, D is also rotation independent, which makes D a good candidate for modeling surface forces. (**Note:** for more, go to intuition part 3.)

In addition to quantifying the strain and rotation rates, we can quantify the rate of volume change as

$$\boxed{\dot{\overline{\det F}} = \det F \operatorname{div} v}$$

where $\det F$ is the volume element in Ω_t, and $\operatorname{div} v$ is the rate of volume change.

2.6 Reynold's Transport Theorem

Note: the divergence thm allows us to transition b/w volume and flux integrals. For some vector field Ψ,

$$\boxed{\int_\Omega \operatorname{div}\Psi dx = \int_{\partial\Omega} \Psi \cdot \hat{n} dA}$$

In writing conservation laws in an Eulerian description, it will be convenient to determine the material derivative of an intensive quantity integrated over some volume. We want to know $\frac{\partial}{\partial t}\int_{\omega_t} \Psi dx$ in a subdomain ω_t.

The material occupying ω_t was in some region $\omega_0 \in \Omega_0$, and the region occupied by this material evolves continually with time, which makes the integral of interest hard to evaluate.

This leads us to the **Reynold's Transport Theorem**:

$$\boxed{\frac{d}{dt}\int_{\omega_t} \Psi dx = \int_{\omega_t} \frac{\partial \Psi}{\partial t} dx + \int_{\partial\omega_t} \Psi v \cdot n dA}$$

where $\int_{\partial\omega_t} \Psi v \cdot n dA = \int_{\omega_t} \operatorname{div}(\Psi v) dx$. The surface integral can be interpreted as the net flux of the Ψ quantity carried by the material across the boundary of ω. (For the derivation, go to intuition part 4.)

2.7 Piola Transformation

Let $T = T(x) = T(\varphi(X))$ be a tensor field defined on $G \in \Omega_t$ and $T(x)\hat{n}(x)$ the flux of T across ∂G, $\hat{n}(x)$ being the unit normal to ∂G. We seek a relationship b/w $T_0(X)$ and $T(x)$ that will result in the same total flux through the surfaces ∂G_0 and ∂G, so that

$$\boxed{\int_{\partial G_0} T_0(X)\hat{n}_0(X) dA_0 = \int_{\partial G} T(x)\hat{n}(x) dA}$$

This relationship between T_0 and T is called the **Piola Transformation**. The above correspondence holds if

$$\boxed{T_0(X) = \det F(X) T(x) F(X)^{-\mathrm{T}} = T(x)\operatorname{Cof} F(X)}$$

This result can be used to establish the following correspondence b/w surface normals at diff times:

$$\boxed{\hat{n} = \frac{\operatorname{Cof} F \hat{n}_0}{||\operatorname{Cof} F \hat{n}_0||}.}$$

Note: more info on pgs 15-18 of textbook.

3 Eulerian Conservation (Fluids)

In this chapter, will present the conservation laws in the Eulerian reference frame.

3.1 Momentum

Conservation of momentum is the fundamental relationship that describes motion of a body under the action of force. Given a motion φ of a body $\mathcal{B}$ of mass density ϱ, the **linear momentum** $I(\mathcal{B}, t)$ of $\mathcal{B}$ at time t is

$$I(\mathcal{B}, t) = \int_{\Omega_t} \varrho v dx.$$

From Newton's law,

$$\frac{dI(\mathcal{B},t)}{dt} = \frac{d}{dt}\int_{\Omega_t} \varrho v dx = \int_{\Omega_t} \varrho \frac{dv}{dt} dx = F_{\text{net}},$$

where F_{net} are the net forces on $\mathcal{B}$. These forces are **body forces** (F_b)–exerted volumetrically–and **surface forces** (F_s). We now have

$$\boxed{\frac{\partial}{\partial t}\int_{\Omega_t} \varrho \vec{v} dx = \int_{\Omega_t} \vec{f}_b dx + \int_{\partial\Omega_t} \vec{f}_s dA.}$$

Let $f(x,t)$ be the force per unit volume. Then body force F_b is $\vec{F}_b = \int_{\Omega_t} \vec{f}_b(x,t)dx$ (**Ex:** for gravity, $f = \varrho g$).

3.2 Cauchy Stress Theorem

Forces exerted on external surfaces may have different origins, but they enter the formulation in the same way. Let σ be the force per unit area acting on the body occupying Ω_t. In addition to external forces, there are **internal forces** (which depend on not just the pt that Γ passes thru but also the orientation of Γ) that can be imposed on the many surfaces passing thru any material pt.

To identify a model (dependencies) for surface stresses σ, we introduce the Cauchy Hypothesis and Cauchy Stress Theorem (proof on pg. 37 of txt and 9/16/19).

The **Cauchy Hypothesis** states $\exists$ a vector field $\sigma(\vec{x}, t, \hat{n})$ that defines force/area (stress) at pt x in Ω_t which depends on the normal (outward pointing) $\hat{n}$ to the surface $\partial\Omega$. Newton's laws imply $\sigma = \sigma(\vec{x}, t, \hat{n}) = -\sigma(\vec{x}, t, \hat{n})$.

The Cauchy Stress theorem says that if

1.) body forces f_b are continuous on Ω

2.) $\sigma(\vec{x}, t, \hat{n})$ is continuously differentiable w.r.t. $\hat{n}$ at constant x

3.) $\sigma(\vec{x}, t, \hat{n})$ is cont. diff. w.r.t. x at constant $\hat{n}$

then $\exists$ a tensor field $T(x,t)$ s.t. $\sigma(\vec{x}, t, \hat{n}) = T(x,t)n$ and $T(x,t) = T(x,t)^T$. $T(x,t)$ is a rank-2 tensor that maps the normal vector to force per unit area on $\partial\Omega$.

Now we can write the surfaces forces F_s as

$$\boxed{\vec{F}_s = \int_{\partial\Omega_t} T(x,t)\cdot \hat{n} dA_t.}$$

3.3 Eulerian Conservation of Momentum

Substituting the Cauchy Stress Tensor T for $\vec{f}_s$ and applying the Divergence Thm, we get

$$\boxed{\frac{d}{dt}\int_{\Omega} \varrho \vec{v} dx = \int_{\Omega} \vec{f}_b dx + \int_{\Omega} \operatorname{div} T dx}$$

Applying Reynold's transport thm to the term on the LHS and combining integrands, we get

$$\boxed{\int_{\Omega}\left[\frac{\partial(\varrho\vec{v})}{\partial t} + \operatorname{div}(\varrho\vec{v}\otimes\vec{v}) - \vec{f}_b - \operatorname{div} T\right] dx = 0.}$$

Since Ω is arbitrary, the integrand must be zero for the integral to be zero. This leads us to the PDE

$$\boxed{\frac{\partial(\varrho\vec{v})}{\partial t} + \operatorname{div}(\varrho\vec{v}\otimes\vec{v}) - \vec{f}_b - \operatorname{div} T = 0.}$$

The tensor $\operatorname{div}(\varrho\vec{v}\otimes\vec{v})$ is not symmetric, so there's some ambiguity on how the divergence operator should be applied. Writing in Cartesian tensor notation removes the ambiguity:

$$\boxed{\frac{\partial\varrho\vec{v}_i}{\partial t} + \frac{\partial\varrho\vec{v}_i\vec{v}_j}{\partial x_j} - \vec{f}_{b_i} - \frac{\partial T_{ij}}{\partial x_j} = 0.}$$

3.4 Modeling the Cauchy Stress Tensor T

Since the strain-rate tensor D is independent of rotations of the underlying coordinate system, it is a good candidate to model surface stresses σ for fluids. Thus, we assume the dependence of T on D s.t.

$$T = T(D).$$

Applying the Cayley-Hamilton thm, we get

$$T = aI + bD + cD^2$$

where a, b, c are scalar fns of the 3 scalar invariants of D. It's difficult to quantify all these fns experimentally or otherwise, so we assume *T is a linear fn of D*.

This assumption leads to the defn of **Newtonian Fluids**. This is a fair assumption for a wide class of fluids (like water, oil, and others) but is a poor one for others, including important fluids like blood. However, models for these non-Newtonian fluids often rely on similar assumptions.

Linearity implies $c = 0$ and b =constant (since scalar invariants of D depend on D). The only scalar invariant of D linear in D is tr$[D]$, so a must be a linear fn. of tr$[D]$. (i.e. $a = \alpha + \beta\text{tr}[D]$). Now we have

$$T = \alpha I + \beta\text{tr}[D]I + \gamma D$$

where α, β, γ are constants (in D). We can rewrite as

$$\boxed{T = pI + \kappa\text{tr}[D]I + 2\mu\left(D - \frac{\text{tr}[D]}{3}I\right)}$$

where p is hydrodynamic pressure, κ is bulk viscosity, and μ is shear viscosity. **Note:** tr$[D]$ = div$v$. (For more info, go to intuition pt 5.)

3.5 Eulerian Conservation of Mass

In the ref. config., the mass is given by

$$M_0(\mathcal{B}) = \int_{\Omega_0} \varrho_0 dX.$$

At time t, the mass is

$$M(\mathcal{B}) = \int_{\Omega_t} \varrho(x,t)dx$$

where ϱ is the mass density field. We have that

$$\int_{\Omega_0} \varrho_0(X)dX = \int_{\Omega_t} \varrho(x)dx.$$

The Eulerian conservation of mass can be obtained by observing that the above eqn implies

$$\frac{d}{dt}\int_{\Omega_t} \varrho(x)dx = 0.$$

Applying Reynold's transport theorem to $\varrho(x)$, we get

$$\int_{\Omega_t}\left[\frac{\partial\varrho(x)}{\partial t} + \text{div}(\varrho(x)v)\right]dx = 0.$$

And since Ω_t is arbitrary, we get

$$\boxed{\frac{\partial\varrho(x)}{\partial t} + \text{div}(\varrho(x)v) = 0.}$$

3.6 Eulerian Conservation of Energy

Let ε be the total energy per unit volume. The energy eqn will follow the general outline:

$$\frac{d}{dt}\int_{\Omega_t} \varepsilon dx = \text{Sources}.$$

Then we have

$$\frac{d}{dt}\int_{\Omega_t}(\kappa + U)dx = \mathcal{P} + \dot{Q}$$

where

$$\kappa = \frac{1}{2}\int_{\Omega_0} \varrho_0 \dot{u}\cdot\dot{u}dX = \frac{1}{2}\int_{\Omega_t} \varrho v\cdot v dx$$

and

$$U = \int_{\Omega_0} \varrho_0 e_0(X,t)dX = \int_{\Omega_t} \varrho e(x,t)dx.$$

Note: e is the internal energy density per unit mass.

The internal energy equation is

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \varrho e dx = \int_{\Omega_t}(T : D + r - \text{div}\,\vec{q})dx}$$

which enforces the first law of thermodynamics. Applying Reynold's transport thm on the LHS, we get the differential form of (Eulerian) internal energy conservation

$$\boxed{\frac{\partial\varrho e}{\partial t} + \text{div}(\varrho e\vec{v}) = T : D + r - \text{div}\,\vec{q}.}$$

where r is the volumetric heating rate, $\vec{q}$ is the heat flux, and $T : D$ denotes the contraction of T and D (or the trace of their product). (Derivation is on pg 34 of notes and 46 of textbook.)

3.7 Thermodynamics and Entropy

The second law of thermodynamics (the **entropy** S of a closed system cannot decrease) is not a conservation eqn. Instead, it is a constraint on the internal energy eqn– we are interested in entropy constraining the direction of heat flow.

Let Q be the heat added to a system at constant temperature θ between times $t_2 > t_1$. Then the second law of thermodynamics implies

$$S(t_2) - S(t_1) - \frac{\dot{Q}}{\theta} \geq 0.$$

When equality holds, a process is **reversible**. In classical thermodynamics, the change in entropy b/w two states of a system measures the quantity of heat received per unit temperature. For the time evolution, we have

$$\frac{dS}{dt} - \frac{\dot{Q}}{\theta} \geq 0.$$

For a continuum, we define η as the entropy density per unit mass, so

$$S = \int_{\Omega_t} \varrho\eta dx$$

and

$$\frac{d}{dt}\int_{\Omega_t} \varrho\eta dx + \int_{\partial\Omega_t} \frac{\vec{q}\cdot\hat{n}}{\theta}dA - \int_{\Omega_t} \frac{r}{\theta}dx \geq 0.$$

Note: for where $\dot{Q}$, look at the conservation of energy equation (on pg 34 of the notes). Applying the divergence thm to the surface integral and applying Reynold's transport thm, we get the Eulerian Entropy eqn

$$\boxed{\int_{\Omega_t}\left[\frac{\partial\varrho\eta}{\partial t} + \text{div}(\varrho\eta\vec{v}) + \text{div}\frac{\vec{q}}{\theta} - \frac{r}{\theta}\right]dx \geq 0.}$$

Since Ω_t is arbitrary, we can get the **Clausius Duhem Inequality**

$$\boxed{\frac{\partial \varrho\eta}{\partial t} + \text{div}(\varrho\eta\vec{v}) + \text{div}\frac{\vec{q}}{\theta} - \frac{r}{\theta} \geq 0.}$$

4 Lagrangian Conservation (Solids)

Now, we define similar conservation laws for a Lagrangian frame that are more convenient in dealing wit solids.

4.1 Mass

We start with

$$\int_{\Omega_0} \varrho_0(X)dX = \int_{\Omega_t} \varrho(x)dx.$$

The integral in x can be transformed into an integral in X since $x = \varphi(X)$ and $dx = \text{det}FdX$ where $\text{det}fdX$ is the ratio of differential volume dx to dX. We get

$$\int_{\Omega_0} (\varrho_0(X) - \varrho(\varphi(X))\text{det}\,FdX = 0.$$

Since Ω_t is arbitrary, we get

$$\boxed{\varrho_0(X) = \varrho(\varphi(X))\text{det}\,F(X).}$$

4.2 Momentum

For reference, here is the Eulerian conservation eqn:

$$\int_\Omega \left[\frac{\partial(\varrho\vec{v})}{\partial t} + \text{div}(\varrho\vec{v}\otimes\vec{v}) - \vec{f}_b - \text{div}\,T\right] dx = 0.$$

To get a Lagrangian representation, we can "transform" from x to X like we did for the Lagrangian conservation of mass eqn. This works for the first three terms, but for the last term, we need a way to relate the changing orientation and size (stretching) of differential areas. We can use the Piola Transform to find a tensor field that can achieve this. We get

$$\boxed{\text{Div}\,T_0 = \text{div}\,T\text{det}\,F.}$$

Observe

$$\int_{\Omega_0} \text{Div}\,T_0 dX = \int_{\Omega_0} \text{div}\,T\text{det}\,FdX = \int_\Omega \text{div}\,Tdx.$$

From the divergence thm, we get

$$\int_{\partial\Omega_0} T_0\hat{n}_0 dA_0 = \int_{\partial\Omega} T\hat{n}dA.$$

4.3 Piola-Kirchhoff Stress Tensors

T is a linear transformation from $\hat{n}$ (CC) to a CC force/CC area. T_0 is a linear transformation of a normal $\hat{n}_0$ (RF) to a CC force/RF area. (**Note:** more info on Piola Transformation in section 2.7.)

T_0 is given the symbol $P(X)$, the **first Piola-Kirchhoff Stress Tensor** s.t.

$$\boxed{P = \text{det}\,FTF^{-\text{T}} = T\text{Cof}\,F.}$$

Now we can write

$$\varrho_0\frac{\partial^2 u}{\partial t^2} = \text{Div}\,P + f_0$$

where

$$f_0 = f(\varphi(X))\text{det}\,F.$$

Note: P is not symmetric (since F and therefore its cofactor are not symmetric), but since $P = \text{det}\,FTF^{-\text{T}}$, then $PF^{\text{T}} = \text{det}\,FT$ is symmetric. A symmetric tensor is also recovered if we map the CC force/RF area to a RC force/RF area. Then we get the **second Piola-Kirchhoff Stress Tensor**

$$\boxed{S = F^{-1}P = \text{det}\,FF^{-1}TF^{-\text{T}}.}$$

We can write

$$\varrho_0\frac{\partial^2 u}{\partial t^2} = \text{Div}\,FS + f_0.$$

T: maps from CC $\hat{n}$ to CC force/CC area.

P: maps from RC $\hat{n}_0$ to CC force/RC area.

S: maps from RC $\hat{n}_0$ to RC force/RC area.

4.4 Energy

The derivation is on pg 42 of the notes. The Lagrangian energy conservation eqn is

$$\boxed{\varrho_0\dot{e}_0 = S : \dot{E} - \text{Div}\,\vec{q}_0 + r_0}$$

where

$$e_0 = e(\varphi(X))$$
$$r = r_0\text{det}\,F$$
$$\text{div}\,\vec{q} = \text{Div}\,\vec{q}_0$$
$$\vec{q}_0 = (\text{Cof}\,F)^{\text{T}}\vec{q}.$$

Note: e_0 is not scaled by the volume ratio det$F$ as it is measured per unit mass, not per unit volume.

The entropy eqn can be transformed by combining trivial analogues of the preceding transformations to get

$$\boxed{\varrho_0\dot{\eta}_0 + \text{Div}\frac{\vec{q}_0}{\theta} - \frac{r_0}{\theta} \geq 0}$$

which is the **Lagrangian Clausius Duhem Inequality**.

5 Constitutive Equations (Closure Models)

Now we want to use the eqns we developed to determine the behavior of a body. However, since the laws affect different materials in different ways, we have too many unknown variables and too few eqns. To complete the problem (to "**close**" the system of eqns), we must supplement the basic eqns with **constitutive equations**. These constitutive eqns impose constraints on the possible responses of the material body. We need constitutive models for $T, \vec{q}, e,$ and η.

We require our constitutive models follow these principles:

1.) **Determinism:** no dependence on the future state, only on the history

2.) **Material Frame Indifference (MFI):** invariance to changes in the ref. frame, i.e. no dependence on rigid motions (uniform velocity or rotation)

3.) **Physical Consistency:** cannot violate conservation laws or Clausius Duhem inequality (2^{nd} law)

4.) **Material Consistency:** if the material is invariant to a group of "unimodular" transformations (e.g. rotations; reflections; continuous rotation group or isotropy), then constitutive model must also be invariant to these transformations

5.) **Local Action:** constitutive models depend only on "local" state; essentially depends on the state and finite number of spatial derivatives at a pt

6.) **Dimensional Consistency:** dimensional and coordinate invariance

7.) **Other considerations:** well-posedness (existence or there exists solns to the problems resulting from use of the eqns), equipresence

5.1 Application of MFI to T for Solids

Two observers can be moving or rotating differently. The motion observed by one observer, $x(t)$, is related to the motion observed by the second observer, $x^*(t)$ by

$$x^*(t) = Q(t)x(t) + c(t)$$

where Q and c do not depend on x, and Q is unitary. A constraint is

$$\boxed{T^* = QTQ^{\mathrm{T}}.}$$

This implies that the relationship $T = \mathcal{T}(F)$ must satisfy

$$\boxed{\mathcal{T}(QF) = Q\mathcal{T}(F)Q^{\mathrm{T}}}$$

for all $Q(T)$. Any constitutive relationship for T depending on F must take the form

$$\boxed{T = \mathcal{T}(F) = FU^{-1}\mathcal{T}(U)U^{-1}F^{\mathrm{T}}}.$$

Note: derivation is on page 46 of notes.

5.2 Restricted Classes of Constitutive Relations

1.) **Thermo-elastic:** for this class of constitutive relations, we assume that quantities $T, \vec{q}, e, \eta$ at a pt depend only on present values of the following state variables at the pt

Solids: $F, \theta, \nabla\theta$

Fluids: $D, \theta, \nabla\theta, \rho$

2.) **Homogeneity:** for this class of constitutive eqns, we assume all the material pts of the body are composed of the same material (ref. config. uniform) so no explicit dependence on X.

5.3 Application of MFI to Fluids

An **ideal fluid** (said to be incompressible) is one in which the motions are isochoric (volume preserving), density $\varrho =$ constant, and T is isotropic, i.e.

$$\boxed{T = -\mathrm{p}(x,t)I}$$

which satisfies MFI since I commutes with tensor operations (note dependence of ϱ, θ suppressed here). In a viscous fluid, $T = -\mathrm{P}I + \mathcal{F}(L)$ where $L = \mathrm{grad}v$ and dependence on ϱ, θ has been repressed.

MFI for a fluid requires

$$\boxed{\mathcal{F}(QLQ^{\mathrm{T}} + \Omega) = Q\mathcal{F}(L)Q^{\mathrm{T}}}$$

where $L^* = QLQ^{\mathrm{T}} + \Omega$ or

$$\boxed{\mathcal{F}(QDQ^{\mathrm{T}}) = Q\mathcal{F}(D)Q^{\mathrm{T}}}$$

for an isotropic fluid where $D^* = QDQ^{\mathrm{T}}$. This is a Stokes fluid.

If we assume $\mathcal{F}$ is linear in D (because the particles move fast and aren't very far apart), we obtain the general form for a Newtonian Fluid

$$\boxed{T = -\mathrm{p}I + 2\mu\left(D - \frac{1}{3}\mathrm{tr}[D]I\right) + \kappa\mathrm{tr}[D]I}$$

where p is the (hydrostatic) pressure, μ is the (shear) viscosity and κ is the bulk viscosity (commonly assumed to be 0). P, μ, κ depend on ϱ, θ. The bulk viscosity represents the irreversibility of volume change and is typically small for fluids at reasonable conditions.

In the special case of **incompressible flow** (which is valid in the limit as $\frac{||\vec{v}||}{c} \to 0$ with c is the speed of sound. In this limit,

$$\text{tr}[L] = \text{div}\,\vec{v} = 0 \Rightarrow \text{tr}[D] = 0$$
$$\varrho = \text{constant}$$
$$T = -\text{p}_{\text{in}} + T_{\text{visc}} \text{ where } \text{tr}[T_{\text{visc}}] = 0$$

where a constitutive relation is needed for T_{visc}, and P_{in} is determined by $\text{div}v = 0$. For a Newtonian fluid, $T_{\text{visc}} = 2\mu D$.

5.4 2nd Law Consistency For Solids

We must adhere to the 2nd law of thermodynamics (satisfy Clausius Duhem inequality in section 4.4). We use the Coleman-Noll method which places restrictions on the nature of constitutive eqns imposed by the 2nd law.

Start with the **Helmholtz free energy** $\psi = e - \theta\eta$ where e is internal energy density, η the entropy density, θ the absolute temp. **Note:** this eqn is useful to work with because it puts into one variable three things we don't know (η is in the 2nd law of thermodynamics, so it makes sense to include it; we include e because it exposes the variables we want to model; we include θ for dimensions).

Subtracting the cons. eqn for e from θ times the Clausius-Duhem inequality for η, we get

$$\boxed{-\varrho\frac{d\psi}{dt} - \varrho\eta\frac{d\theta}{dt} + T : D - \frac{\vec{q}}{\theta}\cdot \text{grad}\,\theta \geq 0.}$$

In the ref. config. this takes on the form

$$\boxed{-\varrho_0\dot{\psi}_0 - \varrho_0\eta_0\dot{\theta} + S : \dot{E} - \frac{\vec{q}_0}{\theta}\cdot\nabla\theta \geq 0.}$$

Suppose the Helmholtz free energy eqn takes the form

$$\psi_0 = \Psi(E, \theta, \nabla\theta).$$

Then

$$\dot{\psi}_0 = \tfrac{\partial\Psi}{\partial E} : \dot{E} + \tfrac{\partial\Psi}{\partial\theta}\dot{\theta} + \tfrac{\partial\Psi}{\partial\nabla\theta}\cdot\nabla\dot{\theta}.$$

Substituting this into the previous inequality, we get,

$$(S - \varrho_0\tfrac{\partial\Psi}{\partial E}) : \dot{E} - \varrho_0(\tfrac{\partial\Psi}{\partial\theta} + \eta_0)\dot{\theta} - \varrho_0\tfrac{\partial\Psi}{\partial\nabla\theta}\cdot\nabla\dot{\theta} - \tfrac{\vec{q}_0}{\theta}\cdot\nabla\theta \geq 0$$

For this inequality to hold for rates $(\dot{E}, \dot{\theta}, \nabla\dot{\theta})$ of arbitrary sign, it is sufficient (and convenient) that the coefficients of the rates vanish, so

$$S = \varrho_0\tfrac{\partial\Psi}{\partial E}, \qquad \eta_0 = -\tfrac{\partial\Psi}{\partial\theta}, \qquad \tfrac{\partial\Psi}{\partial\nabla\theta} = 0$$

and

$$-\tfrac{\vec{q}_0}{\theta}\cdot\nabla\theta \geq 0.$$

In the current config. we have $-\vec{q}\cdot\text{grad}\,\theta \geq 0$. Now we have constitutive eqns for S and η_0. The free energy cannot depend on the gradient of the temperature, and heat must flow from hot to cold.

In this case where $S = \varrho_0\frac{\partial\Psi}{\partial E}$, the strain is non-dissipative (i.e. work done on the body to affect deformation can be recovered). More generally, S may depend on $\dot{E}$, which will lead to dissipation. Then

$$S = \mathcal{F}(E) + \mathcal{I}(\dot{E})$$

where

$$\mathcal{F}(E) = \varrho_0 = \tfrac{\partial\Psi}{\partial E}$$

and

$$\mathcal{I}(\dot{E}) : \dot{E} - \tfrac{\vec{q}_0}{\theta}\cdot\nabla\theta \geq 0.$$

Note: the assumption that $S = \varrho_0\frac{\partial\Psi}{\partial E}$ implies that instead of idenifying a tensor fn to model S, we may be able to identify a scalar fn. In particular, the previous equality assumes that material deformations are reversible.

5.5 2nd Law Consistency for Fluids

For a viscous fluid in the Eulerian representation, we assume Ψ depends on θ, $\text{grad}\,\theta$ only, and $\mathcal{T} = \mathcal{I}(D)$ where

$$\mathcal{I}(D) : D - \tfrac{q}{\theta}\cdot\text{grad}\,\theta \geq 0.$$

Then

$$\eta = -\tfrac{\Psi}{\theta}, \qquad \tfrac{\partial\Psi}{\partial\text{grad}\,\theta} = 0$$

leads to $\mu, \kappa > 0$.

Returning to the governing eqn for a Newtonian fluid, the constitutive law for T is

$$T = -\text{p}I + 2\mu\tilde{D} + \kappa\text{tr}[D]$$

where $\tilde{D} = D - \frac{1}{3}\text{tr}[D]I$ is the deviatoric part of D (constructed s.t. $\text{tr}[\tilde{D} = 0)$. Most often we have the following dependencies,

$$\mu = \mu(\theta), \qquad \kappa = 0, \qquad \text{p} = \text{p}(\varrho, \theta)$$

5.6 Navier Stokes Equations

The compressible eqns are

$$\boxed{\begin{aligned} &\frac{\partial\varrho}{\partial t} + \text{div}(\varrho\vec{v}) = 0 \\ &\varrho\frac{\partial\vec{v}}{\partial t} + \varrho\vec{v}\cdot\text{grad}\,\vec{v} = -\text{grad}\,\text{p} + \text{div}(2\mu\tilde{D}) \\ &\qquad\qquad\qquad\qquad + \text{grad}(\kappa\text{div}\,\vec{v}) \\ &\varrho C_\nu\frac{\partial\theta}{\partial t} + \varrho C_\nu\vec{v}\cdot\text{grad}\,\theta = -\text{p}\text{div}\,\vec{v} + 2\mu\tilde{D} : D \\ &\qquad\qquad\qquad\qquad + \kappa(\text{div}\,\vec{v})^2 + \text{div}(k\text{grad}\,\theta) \end{aligned}} \tag{2}$$

where $\frac{\partial e}{\partial \theta} = C_\nu$ is the specific heat at a constant volume. **Note:** the derivation is on pg 52 of the notes, and IC and BC's and more info on pg 53-54. This system is now closed if we have constitutive relations for parameters μ, κ, k and an eqn of state (e.g. ideal gas law) for p. The following dependencies are commonly used: $p(\theta, \varrho), \mu(\theta), k(\theta), \kappa = 0$.

For incompressible Navier Stokes, we have

$$\boxed{\begin{aligned} &\varrho_0 \frac{\partial \vec{v}}{\partial t} + \varrho_0 \vec{v} \cdot \operatorname{grad} \vec{v} = -\operatorname{grad} p' + \mu_0 \Delta \vec{v} + \vec{f} \\ &\operatorname{div} \vec{v} = 0 \end{aligned}} \tag{3}$$

5.7 Heat Equation

Take what we did with energy and heat flux, i.e. $\frac{de}{d\theta} = C_\nu$ and $\vec{q} = -k \operatorname{grad} \theta$. Then if there is no deformation, we get

$$\boxed{\varrho C_\nu \frac{\partial \theta}{\partial t} = \operatorname{div}(\kappa \operatorname{grad} \theta) + r}$$

where $k > 0$ required by the 2nd law.

5.8 Elasticity

For a deformable body (solid) with uniform θ, no heat flux ($\vec{q} = 0$), homogeneous, and isotropic: the free energy constitutive dependence simplifies to $\psi = \Psi(E)$. We then call ψ the stored energy fn (or **strain energy fn**). This is a hyperelastic constitutive relation.

Since Ψ is an isotropic scalar fn of a tensor, it must take the following form

$$\Psi = W(I_E, II_E, III_E).$$

Then

$$S = \frac{\partial \Psi}{\partial E} = \frac{\partial W}{\partial I_E}\frac{\partial I_E}{\partial E} + \frac{\partial W}{\partial II_E}\frac{\partial II_E}{\partial E} + \frac{\partial W}{\partial III_E}\frac{\partial III_E}{\partial E}.$$

For example, $\frac{\partial E_{ii}}{\partial E_{jk}} = \delta_{jk}$ (because $E_{ii} = E_{11} + E_{22} + E_{33}$) implies that $\frac{\partial I_E}{\partial E} = I$. Similarly, we should get

$$\frac{\partial II_E}{\partial E} = \operatorname{tr}[E]I - E$$

$$\frac{\partial III_E}{\partial E} = \frac{1}{2}(\operatorname{tr}[E]^2 - \operatorname{tr}[E^2])I + (E^2 - \operatorname{tr}[E]E).$$

The eqns become

$$\boxed{\begin{aligned} &\varrho_0 \frac{\partial^2 u}{\partial t^2} = \operatorname{Div}\left((I + \nabla u)\frac{\partial W}{\partial E}\right) + f_0 \\ &E = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}} + \nabla u^{\mathrm{T}} \nabla u). \end{aligned}} \tag{4}$$

5.9 Linear Elasticity

Assume all displacements are small. Then non-linear terms become negligible, so

$$E \approx e = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}).$$

The Lamé eqns are

$$\boxed{(\lambda + \mu)\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 u_i}{\partial x_i \partial x_k} + \vec{f}_{0i} = \varrho_0 \frac{\partial^2 u_i}{\partial t^2}.}$$

The linearized eqns are

$$\boxed{\varrho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial X_j}\left(E_{ijk\ell}\frac{\partial u_k}{\partial X_\ell} + \vec{f}_{0,i}\right)}$$

.

For an isotropic material, the most general form is

$$\boxed{E_{ijk\ell} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})}$$

and μ, λ are the Lamé constants

$$\boxed{\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.}$$

E is Young's Modulus and ν is Poisson's ratio. Then

$$S = \lambda \operatorname{tr}[E]I + 2\mu E = \lambda \operatorname{div} u + u(\nabla u + \nabla u^{\mathrm{T}}).$$

Finally, notice that as $\nu \to \frac{1}{2}$, then $\lambda \to \infty$, so for S to remain finite, $\operatorname{tr}[E] \to 0$ and $\operatorname{div} u \to 0$.

6 Terms

Body forces: acting on material points of the body by its environment. **Ex:** weight-per-unit volume exerted by the body by gravity or forces per unit volume exerted by am external magnetic field

Buckingham Pi theorem: if there is a physically meaningful equation involving a certain number n of physical variables, then the original equation can be rewritten in terms of a set of $p = n - k$ dimensionless parameters $\pi_1, \pi_2, ..., \pi_p$ constructed from the original variables

Cauchy-Green deformation tensor: gives us the square of local change in distances due to deformation, i.e. $d\mathbf{x}^2 = d\mathbf{X} \cdot \mathbf{C} \cdot d\mathbf{X}$; the right and left tensors capture only the stretching part of a deformation, not the rotation

Cauchy Hypothesis: there exists a vector-valued surface force density $\sigma(\hat{n}, x, t)$ (called the stress vector field) giving the force per unit area on an oriented surface Γ through x with unit normal $\hat{n}$ at time t; the stress vector does not depend on the curvature of the boundary (assume the curvature is so big that the small intermolecular forces look flat)

Clausius Duhem Inequality: a way of expressing the 2nd law of thermodynamics; is particularly useful in determining whether the constitutive relation of a material is thermodynamically allowable; is a statement concerning the irreversibility of natural processes, especially when energy dissipation is involved

Closure model: a model made to close a set of equations; often representing effects and phenomena that are not accessible by theory (a set of eqns is closed when the number of variables is reduced to the number of governing eqns)

Cofactor: for any matrix **A**= $[A_{ij}]$ of order n and for each row i and col j, let $\mathbf{A}'_{ij}$ be the matrix of order $n-1$ obtained by deleting the ith row and jth col of **A**. let $d_{ij} = (-1)^{i+j}\det \mathbf{A}'_{ij}$. then the matrix

$$\text{Cof } \mathbf{A} = [d_{ij}]$$

is the cofactor matrix of A and d_{ij} is the (i, j) cofactor of A. **Note:** Cof **F** = (det **F**) $\mathbf{F}^{-T}$ (pg 7 of txtbook)

Constitutive models: are "made up" to be consistent with what is known and available data; a relation between two physical quantities that is specific to a material approximate the response of material to external stimuli; mathematical description of how materials respond to various loadings

D (strain-rate tensor): describes how fluid is deforming/stretching (how it's strained); a physical quantity that describes the rate of change of the deformation of a material

Deviatoric: the stress tensor σ_{ij} can be expressed as the sum of two other stress tensors–(1) a mean hydrostatic stress tensor or volumetric stress tensor or mean normal stress tensor, $\pi\delta_{ij}$, which tends to change the volume of the stressed body; and (2) a deviatoric component called the stress deviator tensor, s_{ij}, which tends to distort it.

Dimensional homogeneity: a principle; quality of eqns having quantities of same units on both sides

Dissipation: taking the kinetic energy and turning it into heat, which raises the entropy of the system; the result of an irreversible process that takes place in homogeneous thermodynamic systems. **Ex:** heat transfer is dissipative because it is a transfer of internal energy from a hotter body to a colder one

Energy: the quality of a physical system (e.g. a deformable body) measuring its ability to do work; a change in energy in causes work to be done by the forces acting on the system and a change in energy in time produces a rate of work (i.e. power)

Entropy: a thermodynamic quantity representing the unavailability of a system's thermal energy for conversion into mechanical work, often interpreted as the degree of disorder or randomness in the system

Eulerian (description of) velocity: the velocity at a fixed place $x \in \mathbb{R}^3$ is the speed and direction (at time t) of particles flowing thru the place x

Flux: rate of flow through a surface or substance; for transport phenomena, flux is a vector quantity, describing the magnitude and direction of the flow of a substance or property

Green-St. Venant Strain Tensor: denotes the change in length of line segment; gives a measure of the deformation of a material (**Note:** for more, go to intuition pt 2)

Helmholtz free energy: a thermodynamic potential that measures the useful work obtainable from a closed thermodynamic system at a constant temperature and volume (isothermal, isochoric)

Homogeneous deformation: a deformation is homogeneous if $F = C$ =constant; the deformation gradient is (uniform) independent of the coordinates; all straight lines in the solid remain straight under the deformation; planes deform to planes, cubes (no matter how large) deform to parallelepipeds, and spheres deform to ellipsoids Thus, every point in the solid experiences the same shape change; **Ex:** volume preserving uniaxial extension, simple shear, rigid rotation thru angle about axis

Hyperelastic: a type of constitutive model for ideally elastic material for which the stress–strain relationship derives from a strain energy density function; the elastic deformation can be extremely large

Incompressible flow (isochoric flow): a flow in which the material density is constant within a fluid parcel—an infinitesimal volume that moves with the flow velocity; the divergence of the flow velocity is zero. **Note:** incompressible flow does not imply that the fluid itself is incompressible

Inviscid fluid (ideal fluid): a nonviscous fluid (a fluid for which all surface forces exerted on the boundaries of each small element of the fluid act normal to these boundaries); the stress tensor reduces to the pressure; in the dynamics of an inviscid fluid, 1) no restraints are placed on the tangential component of the flow at a solid bounding surface, and 2) there is no dissipation of kinetic into thermal energy within the fluid; where an inviscid fluid flows along a surface, that surface is said to be a free slip surface.

Isotropic: invariant to rotation; having a physical property which has the same value when measured in different directions

Lagrangian (description of) velocity: the velocity of a material pt is the time rate of change of the position of the pt as it moves along its path

Mean free path: average distance an atom/molecule travels before colliding with another one; this is a characteristic time scale

Momentum: the quantity of motion of a moving body, measured as a product of its mass and velocity

Newtonion Fluids: fluids where internal stresses depend linearly on strain-rate; a fluid whose viscosity does not change with rate of flow

Principal direction and values: eigenvectors and eigenvalues; for the Cauchy Green deformation tensor, the principal directions indicate the direction of the stretching and the values are how much the material is stretched in the principal direction

Principal invariant: principal invariants of the second rank tensor $\mathbf{A}$ are the coefficients of the characteristic polynomial; do not change with rotations of the coordinate system (they are objective, or in more modern terminology, satisfy the principle of material frame-indifference)

Reynold's number: the ratio between the viscous forces in a fluid and the inertial forces; used to help predict flow patterns in different fluid flow-situations; at low Reynolds numbers, flows tend to be dominated by laminar (sheet-like) flow, while at high Reynolds numbers turbulence results from differences in the fluid's speed and direction, which may sometimes intersect or even move counter to the overall direction of the flow (eddy currents)

Scaling variables: the variables $x_1, \ldots, x_m$ used to non-dimensionalize the remaining variables

Shear strain: the ratio of the change in deformation to its original length perpendicular to the axes of the member due to shear stress

Shear stress: the component of stress coplanar with a material cross section; arises from the force vector component parallel to the cross section of the material; normal stress, on the other hand, arises from the force vector component perpendicular to the material cross section on which it acts

Simple shear: a deformation in which parallel planes in a material remain parallel and maintain a constant distance, while translating relative to each other

Stokes fluid: has all the same constraints as a Newtonian fluid but does not have to be linear in D

Strain: a description of deformation in terms of relative displacement of particles in the body that excludes rigid-body motions

Stress: a physical quantity that expresses the internal forces that neighbouring particles of a continuous material exert on each other

Surface forces: the contact of the boundary surfaces of the body with the exterior universe or contact of internal parts of the body on surfaces that separate them (ex. on pg 31 of textbook)

Vorticity: a pseudovector field that describes the local spinning motion of a continuum near some point (the tendency of something to rotate), as would be seen by an observer located at that point and traveling along with the flow; can be expressed as $\vec{w} = \nabla \times \vec{u}$ (here it is defined as the curl of the flow velocity $\vec{u}$); more on pg 22 of notes

W (rotation rate tensor): describes how the fluid is rotating

7 Short Answer Questions (SAQ)

7.1 Questions from notes

What is a math model? How are they formulated? Why are they useful? A math model is a relationship between quantities (or among mathematical descriptors of that system). They can be formulated by empirical observations- from these, we derive math "theories" to describe the data. They can help understand and predict phenomena.

Consider an object as a physical system. What descriptors might be useful? Dimensions, measures on properties

In many (most) situations, we know (or think we know) the fundamental laws that govern the physical systems we want to model. In principle, this should be all we need, but practically, why is it not? When modeling complex systems, we commonly have a set of applicable reliable theories, which are not "closed". We then need "closure models", often representing effects and phenomena that are not accessible by theory.

Why is the Cayley-Hamilton thm interesting? Suppose we want to model $B = F(A)$ where we postulate that a tensor is a fn of another tensor. Well, any analytic tensor fn of a tensor $F(A)$ is a linear combination of I, A, A^2 with coefficients that are scalar fns of the invariants of A.

In continuum mechanics, apply Newton's law $F = ma$ to a continuum, rather than discrete masses. Why might this be useful? What continua might be of interest? it is locally homogeneous in other words if you subdivide it sufficiently many times, all sub-divisions have identical properties (eg mass density)

7.2 Kinematics of Deformable Bodies

What is the momentum of the material in Ω? What forces are acting on it? The momentum is the sum (integral) of momentum of constituent particles. Two types of forces are considered: body and surface forces.

What physical phenomena might $\vec{f_b}$ and $\vec{f_s}$ represent? $\vec{f_b}$: gravity, $\vec{f_s}$: stress or pressure

Consider 2 subdomains of a continuum material, Ω_1, Ω_2. Looking at the following eqns, what should $\vec{f}_{s1}$ and $\vec{f}_{s2}$ depend on? How would this be different

for a fluid or a solid? Should $\vec{f}_{s1}$, $\vec{f}_{s2}$ be the same at points where $\partial\Omega_1$, $\partial\Omega_2$ intersect?

$$\frac{\partial}{\partial t}\int_{\Omega_1}\rho\vec{v}dx = \int_{\partial\Omega_1}\vec{f}_{s1}dA_1$$
$$\frac{\partial}{\partial t}\int_{\Omega_2}\rho\vec{v}dx = \int_{\partial\Omega_2}\vec{f}_{s2}dA_2$$

The key point here is that f_s will depend on the orientation of the curve (surface) $\partial\Omega$ (i.e. the surface normal of the boundary). For solids, we care about deformation, while for fluids, we care about rate of deformation. The surface forces will not be the same where domains intersect since they will depend on the orientation of the boundary.

7.3 Solids

Should the internal surface forces be affected by rigid body motion? No, they shouldn't since the continuum isn't compressed or altered, just shifted and rotated.

Why are the images of vectors $\vec{e}_i$ also mutually orthogonal? By the properties of the SVD of C, see conditions for orthogonality of right singular vectors V.

Recall (in section 2.4)-we started that discussion expecting internal forces to depend on the deformation. Is the deformation tensor/strain tensor a good candidate for this dependence? Yes (for a solid).

7.4 Fluids

Internal surface forces that depend on deformation make sense for solids. Do they make sense for fluids? If not, what makes sense? No they don't. Fluids may undergo deformation without carrying residual stresses into a new ref config. (e.g. shaken water bottle), so we consider the dependence of internal surface forces on the rate of deformation instead.

When is the incompressible approximation good? When the flow velocity is small compared to the speed of sound.

7.5 Momentum

What is momentum? Recalling our formulation of Newton's law in a continuum however we can write $\frac{dm\vec{v}}{dt} = \vec{F}$, thus it can also be seen as the integral of force. In particular, when no force is applied, momentum remains constant-Newton's first law. It is this relation that we seek to conserve when we speak of momentum conservation.

7.6 Mass

What is mass? How is it measured? It's a property of a body that is a measure of the amount of material it contains and causes it to have weight in a gravitational field (or a measure of how much stuff is in an area). It can be measured as density times volume.

How are M_0 and M_t related? Why? They are the same since the mass of particles does not change and Ω_t tracks particles thru space.

7.7 Energy

What constitutes the energy of a continuum? What are the possible sources? The energy consists of kinetic energy κ (the energy due to motion) and internal energy U (energy due to "everything else", like deformation, temperature, etc). Possible sources include power, or rate of work ($\mathcal{P}$) and heating rate $\dot{Q}$. **Note:** the internal energy depends on the deformation, temperature gradient, and other physical entities. This precise form of this dependency varies from material to material and depends on the physical "constitution" of the body.

What does $T : D$ mean? Looking at the boundary forces, part of the surface work went into kinetic energy. $T : D$ is the part that did not go into kinetic energy–it went into the internal energy. It may be dissipation or elastic energy storage.

7.8 Linear Elasticity

What do Young's Modulus (E) and Poisson's ratio (ν) mean? If there is uniaxial loading, Young's Modulus is the ratio between stress in the loading direction to the deformation. Poisson's ratio is the ratio of strain and direction orthogonal to the strain (ratio of transverse contraction strain to longitudinal extension strain in the direction of stretching force). We're looking at a situation where when a material is stretched in one direction it tends to get thinner in the other two directions. These are used instead of and because these are quantities that can be measured through experiments.

7.9 Gopal's Questions

Why is dimensional consistency important in the process of model development? Allows us to directly compare quantities to other physical quantities of the same kind, and we don't want the validity of our model to depend on the arbitrariness of units.

Why is tensorial consistency important in the process of model development? The tensorial invariance of our quantities are preserved.

How should you determine which scaling variables to choose in the process of dimensional analysis? Why is it useful to choose scaling variables this way? There is no exact set of rules, but typically, you want to choose variables you believe will have the strongest or most direct impact on the phenomenon you are trying to model (or accounts most for the variability of the system). This is especially important when later trying to simplify your model. If you chose an insignificant parameter, it is then difficult if not impossible to remove it without going back to square one and re-picking scaling variables.

Is the number of variables important? The most important part is having the *right* variables. If you have too many, you unnecessarily complicate your model. If you miss relevant variables, you'll get an inconsistent model/won't capture the true dynamic of the system. **Side note:** we can simplify models by eliminating variables that are dependent on other things or placing constraints on the model. Ex: if we assume helicopters are all proportional (radius, thickness of paper, tail length), we can remove a lot of the geometric variables for the helicopters.

How does dimensional analysis help? It reduces the number of parameters to model on.

What is a second rank tensor? Can you give an example? A second rank tensor is a homogeneous linear map from a vector to a vector. An example is a matrix or the stress tensor T which maps a normal n to a force per unit area.

How many invariants can a 2nd rank tensor have? The dimension of the space the tensor is in, so we usually use/have 3.

The models we'll formulate are rightly called mathematical models-but people commonly refer to computational models. What is the difference? A computational model is a mathematical model in computational science that requires extensive computational resources to study the behavior of a complex system by computer simulation. A mathematical model is a description of a system using mathematical concepts and language (can be solved analytically).

What is the Cauchy-Green deformation tensor? What does it express? It is a rank-2 deformation tensor C that quantifies "stretching" (the distortion) of continuum in different directions. It is $C = F^{\mathrm{T}}F$.

How is the Green-St Venant Tensor related to the Cauchy Green Tensor? The Green-St Venant Tensor gives the change in line segments, and it is half of the distance between C and I.

What do the principal values and directions of C represent? The principal directions indicate the direction of the stretching, and the principal values are how much the material is stretched in the principal direction.

Why are constitutive equations required/important in developing models for the physical world? What processes are being represented by the constitutive equations? They are made to be consistent with what is known (and available data), and are complementary eqns to the balance and kinematic eqns. They're important because they are mathematical simplifications of a quite complex physical behavior (and we can't get an "exact" model). The processes we're representing with constitutive eqns. are $T, \vec{q}, e$, and η.

What is a Newtonian Fluid? It is a fluid where its internal stresses depend linearly on strain-rate, and its viscosity does not change with the flow rate.

What is a fluid? How is our description of the stress tensor different for fluids versus solids? Why? A fluid is a substance that has no fixed shape and yields easily to external pressure; a gas or (especially) a liquid. Our stress tensor for fluids depends on the strain-rate tensor while the stress tensor for solids depends on the strain (amount of deformation). T depends on the strain rate because fluids may undergo deformation without carrying residual stresses into a new ref config. (e.g. shaken water bottle).

What do the stress tensor T, the first Piola-Kirchhoff Stress Tensor, and the second Piola-Kirchhoff Stress Tensor represent? T maps from CC $\hat{n}$ to CC force/CC area. P maps from RC $\hat{n}_0$ to CC force/RC area. S maps from RC $\hat{n}_0$ to RC force/RC area.

8 Intuition

8.1 Deformation Tensor

Let's consider the deformation of differential line segments. Let dX be a differential segment in Ω_0. It will be mapped to $dx = FdX$.

Note: The distance between two close points in Ω_0 is $dX = X_2 - X_1 = X + dX - X$, and the distance for Ω_t is $dx = \varphi(X_2) - \varphi(X_1)$. If $dx > dX$, there is a stretch, and if $dx < dX$, there is compression.

We write $dx = FdX$ because the length of the line segment in Ω_t is the length of the line in Ω_0 transformed, with the transformation described by F (I think).

8.2 Green-St. Venant Strain Tensor

Note that tensor C can be viewed as the dilation of a line segment, while E denotes (half) the actual change in length of the segment.

Ex: a segment [of length 1] stretched to a ratio of 1.1 times its original size only changed 0.1 in length. 1.1 would be C in this direction, and 0.1 is twice the value of E in this direction.

8.3 Strain Rate Tensor

Consider a rate of change of a differential line segment dS, but first note that

$$\begin{aligned}\frac{\partial}{\partial t}(dX \cdot CdX) &= dX \cdot \frac{\partial C}{\partial t} dX \\ &= dX \cdot \frac{\partial}{\partial t}(F^T F) dX \\ &= dX \cdot (\dot{F}^T F + F^T \dot{F}) dX \\ &= dX \cdot ((LF)^T F + F^T (LF)) dX \\ &= dX \cdot (F^T (L + L^T) F) dX\end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial dS^2}{\partial t} &= dX \cdot [(F^T (L + L^T) F) dX] \\ &= (FdX) \cdot [(L + L^T)(FdX)] \\ &= (dx) \cdot [(L + L^T)(dx)] \\ &= 2(dx) \cdot [Ddx]\end{aligned}$$

Note: dX is in Ω_0 so it doesn't depend on time, and FdX is a line segment in Ω_t. So the rate of stretch can be written in terms of line segments in Ω_t. Ω_0 becomes irrelevant, which is convenient because everything gets tangled for a fluid.

8.4 Reynold's Transport Theorem

The time derivative we want is of the material occupying ω_t at time t, but following the material. So we can write

$$\frac{d}{dt}\int_{\omega_t} \Psi dx = \frac{d}{dt}\int_{\omega_0} \Psi_m \det F dX$$

$= \int_{\omega_0} \frac{d}{dt}(\Psi_m \det F)dX$
$= \int_{\omega_0} (\frac{\partial \Psi_m}{\partial t} + v \cdot \text{grad}\, \Psi_m) \det F dX + \int_{\omega_0} \Psi_m \dot{\overline{\det F}} dX$

The term inside the first integral is obtained for the time derivative of Ψ_m using the multivariate chain rule (**), as Ψ_m depends on variables $(x(t), t)$. Switching back to the Eulerian integral,
$= \int_{\omega_t} (\frac{\partial \Psi}{\partial t} + v \cdot \text{grad}\, \Psi)dx + \int_{\omega_t} \Psi \text{div} v dx$
$= \int_{\omega_t} (\frac{\partial \Psi}{\partial t} + \text{div}(\Psi v))dx$

** The following shows the multivariate chain rule applied to Ψ_m :

$$\frac{d\Psi_m(x(t),t)}{dt} = \frac{\partial \Psi_m}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \Psi_m}{\partial t}\frac{\partial t}{\partial t} = \frac{\partial \Psi_m}{\partial t} + v \cdot \text{grad}\, \Psi_m$$

8.5 Newtonian Fluids

For Newtonian Fluids, we have

$$\boxed{T = p + \kappa \text{tr}[D]I + 2\mu \left(D - \frac{\text{tr}[D]}{3}I\right)}$$

where p is hydrodynamic pressure, κ is bulk viscosity, and μ is shear viscosity.

κ is commonly assumed to be zero. This is the standard model for internal stresses of Newtonian fluids; substituting T into the the differential momentum conservation equation

$$\boxed{\frac{\partial \varrho \vec{v}_i}{\partial t} + \frac{\partial \varrho \vec{v}_i \vec{v}_j}{\partial x_j} - \vec{f}_{b_i} - \frac{\partial T_{ij}}{\partial x_j} = 0.}$$

gives the Navier-Stokes eqns. To modify this model for non-Newtonian fluids, a common practice is to make μ a fn of an invariant of D. The class of fluids that this extends to are known as shear-thinning or shear-thickening fluids, since the shear-viscosity is a fn of shear-rate.

Note that substituting T into the above equation results in 3 eqns, one for each component of velocity (plus 3 eqns. imposing the symmetry of T)-but we have a number of unknowns including: $v_i, \varrho, p, \kappa, \mu$. It turns out that μ will depend on additional variables like temperature θ, but this will be dealt with later when we consider energy conservation. Basically: **our system is severely underdetermined**.

9 Appendix

9.1 Invariants

The scalar-, vector-, and tensor-valued functions $\phi, \vec{a}, \text{T}$ of the scalar variable ϕ, vector variable $\vec{v}$, and second-order tensor variable B are isotropic fns if

$\phi(\text{Q}\vec{v}) = \phi(\vec{v})$	$\phi(\text{QBQ}^\text{T}) = \phi(\text{B})$
$\vec{a}(\phi) = \text{Q}\vec{a}(\phi)$	$\vec{a}(\text{Q}\vec{v}) = \text{Q}\vec{a}(\vec{v})$
$\vec{a}(\text{QBQ}^\text{T}) = \text{Q}\vec{a}(\text{B})$	
$\text{T}(\phi) = \text{QT}(\phi)\text{Q}^\text{T}$	$\text{T}(\text{Q}\vec{v}) = \text{QT}(\vec{v})\text{Q}^\text{T}$
$\text{T}(\text{QBQ}^\text{T}) = \text{QT}(\text{B})\text{Q}^\text{T}$	

for all orthogonal tensors Q. For more information and the integrity bases, go to packet.

9.2 Scientific Units

acceleration (a)	LT^{-2}	velocity per time-squared (m/s^2)
momentum (p)	MLT^{-1}	mass · velocity
mass density (ρ)	ML^{-3}	mass per volume
force (F)	MLT^{-2}	mass · acceleration (N)
energy (E), work (W)	ML^2T^{-2}	force · distance (Joules)
power (P)	ML^2T^{-3}	energy per time (Watts)
pressure (P), stress (σ)	$\text{ML}^{-1}\text{T}^{-2}$	force per area (Pascals)
frequency (ω)	T^{-1}	per time (Hertz)
internal energy (U)	ML^2T^{-2}	energy
specific heat (c)	$\text{L}^2\text{T}^{-2}\Theta^{-1}$	energy per mass per degree
heat flux (ϕ)	MT^{-3}	energy per time per area
thermal conductivity (K)	$\text{MLT}^{-3}\Theta^{-1}$	flux per length per degree
diffusivity (k)	L^2T^{-1}	$K/\rho c$
viscosity (μ)	$\text{ML}^{-1}\text{T}^{-1}$	mass per length per time
heat loss coefficient (h)	$\text{MT}^3\Theta^{-1}$	energy per volume · deg · time

9.3 Dimensional Analysis

These are the steps to using the Buckingham Pi Theorem.

1.) Count number of variables and state the dimensions of each of the variables
2.) Determine the fundamental dimensions
3.) Number of pi groups is $p = n - m$ where n is number of variables and m is number of dimensions
4.) Choose m scaling variables
5.) Scale the other variables with the variables from (4) (can do this by putting the dimensions for each variable into a matrix and solving for the null set)

Other considerations

1.) The final eqn is obtained in the form of $\pi_1 = f(\pi_2, ..., \pi_{n-m})$
2.) The π groups must be independent of each other and no group should be formed by multiplying together powers of other groups.

9.4 Tensor Algebra

Vector calculus:

1.) Gradient: $(\nabla\phi)_i = \frac{\partial}{\partial x_i}$ increases tensor rank by 1
2.) Divergence: $\nabla \cdot v = \frac{\partial v_i}{\partial x_i}$ decreases tensor rank by 1
3.) Curl: $\nabla \times v = \epsilon_{ijk}\frac{\partial v_j}{\partial x_i}$

Kronecker Delta and Levi-Civita identities:

1.) $\delta_{ii} = 3$
2.) $\delta_{ij}\delta_{jk} = \delta_{ik}$
3.) $\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}$
4.) $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$
5.) $\epsilon_{ijk}\epsilon_{ijk} = 6$

Tensor identities:

1.) $(a \otimes b)c = (b \cdot c)a$
2.) $(a \otimes b)^{\mathrm{T}} = b \otimes a$
3.) $(a \otimes b)(c \otimes d) = (b \cdot c)(a \otimes d)$
4.) $(u \otimes v) : (q \otimes p) = (u \cdot q)(v \cdot p)$
5.) $Av \cdot u = v \cdot A^{\mathrm{T}}u$