Impacts of Numerical Discretization on Large Eddy Simulation

Gopal Yalla (gyalla@utexas.edu)

Robert Moser, Todd Oliver

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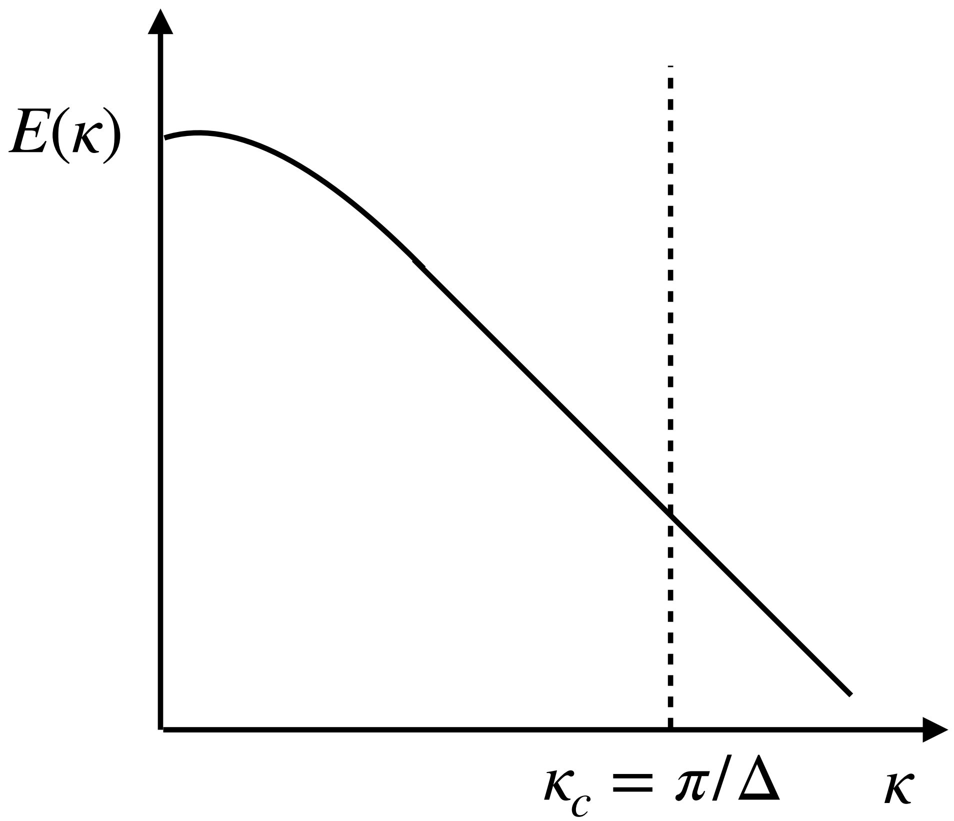
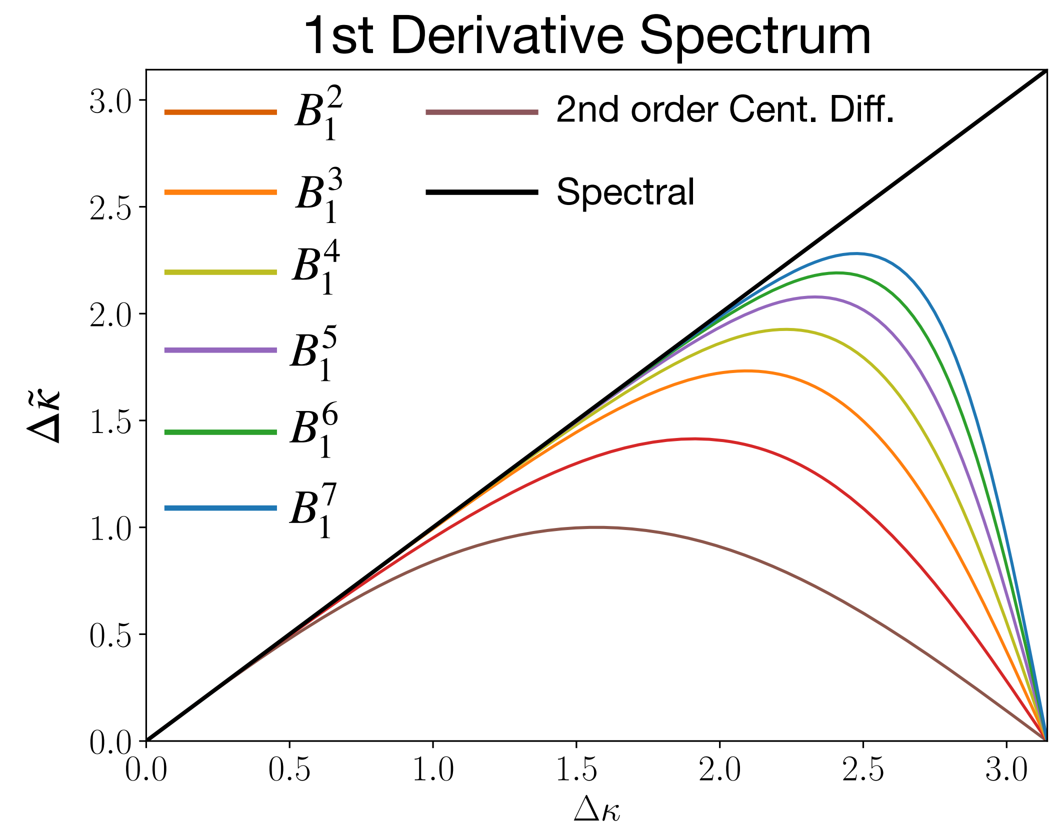
The University of Texas at Austin
Oden Institute for Computational Engineering and Sciences





$B_n^k =$ nth derivative operator, kth order bspline collocation

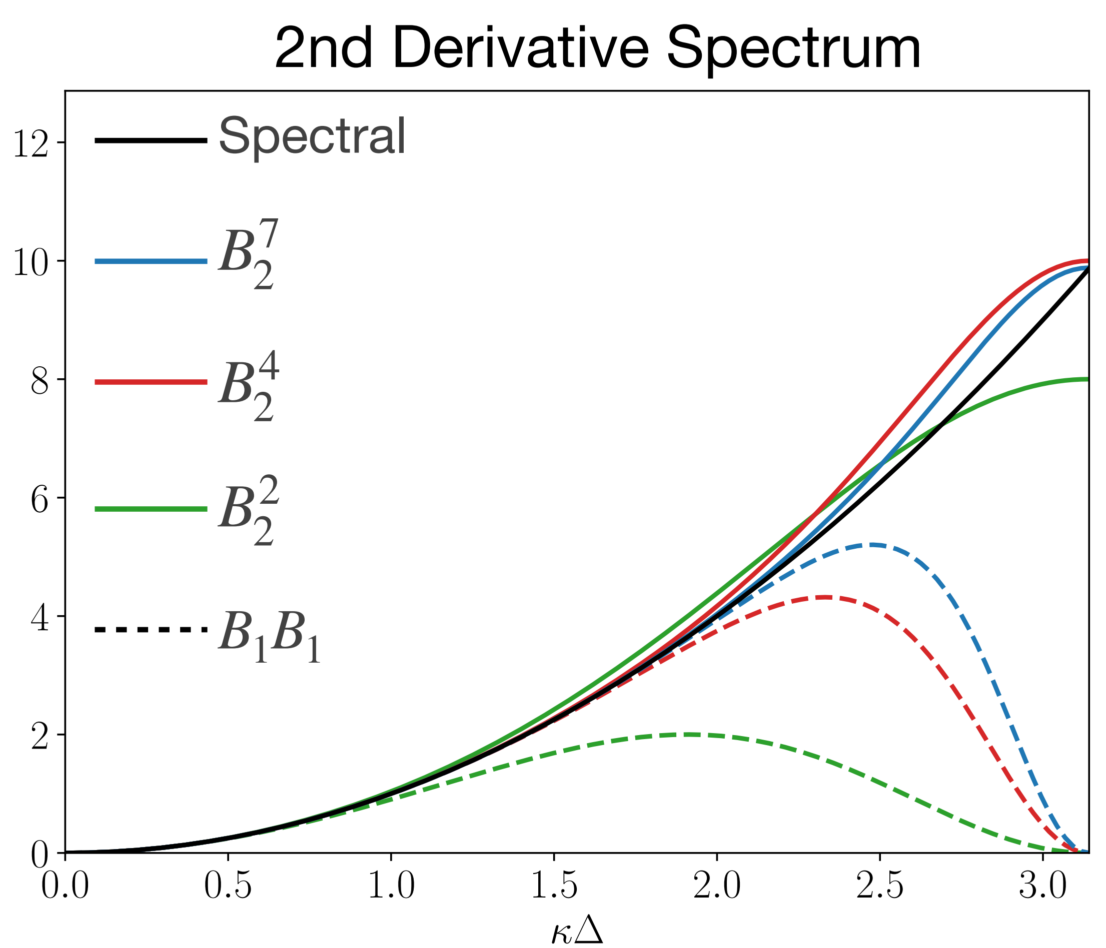
- Implicitly filtered LES is inherently under resolved.

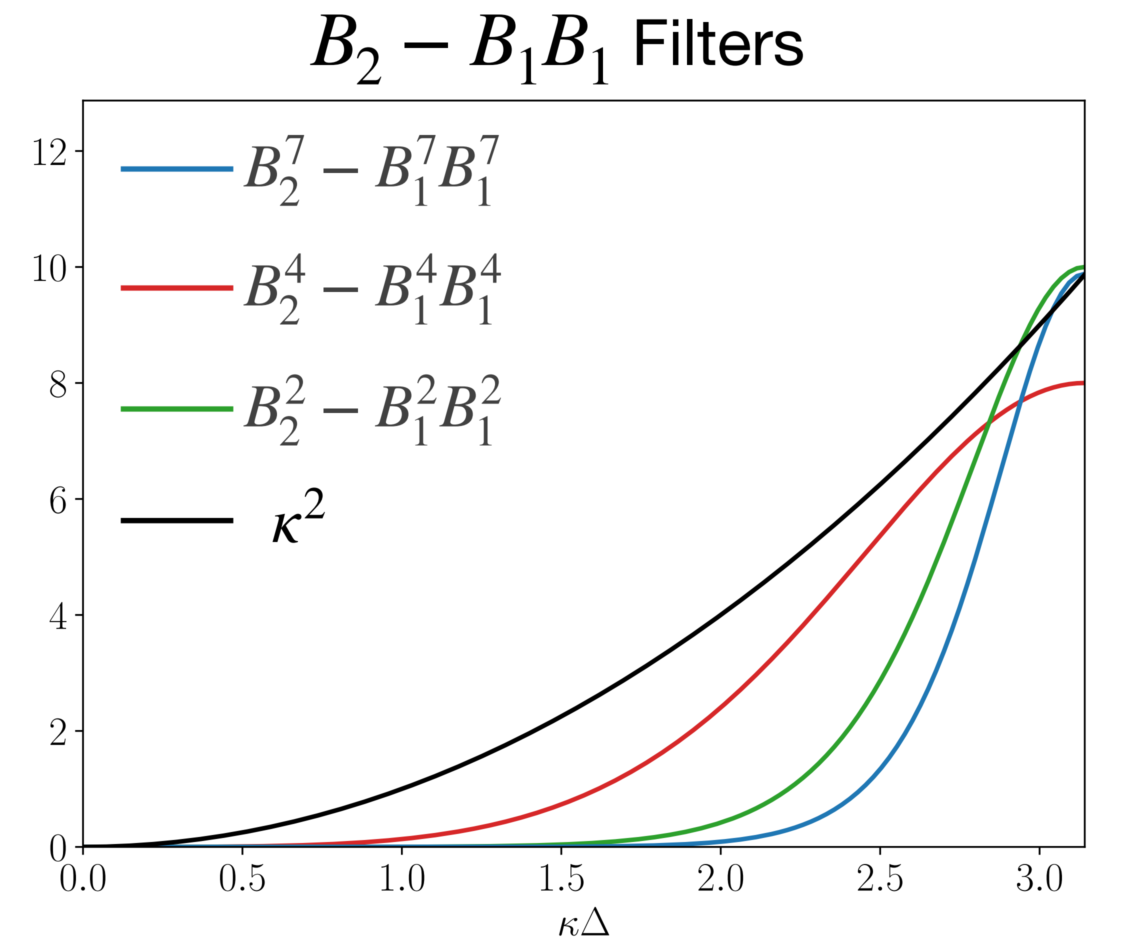
- The Fourier cutoff filter, combined with the discrete derivative operators, define the resolved scales whose dynamics are accurately represented.

- An a priori statistical requirement should be imposed on the flow so that the energy spectrum is sufficiently small for these poorly represented scales.

- Here, we investigate the statistical effects of these poorly resolved scales on the energy balance in homogeneous, isotropic turbulence in the context of inhomogeneous resolution and dispersion error.

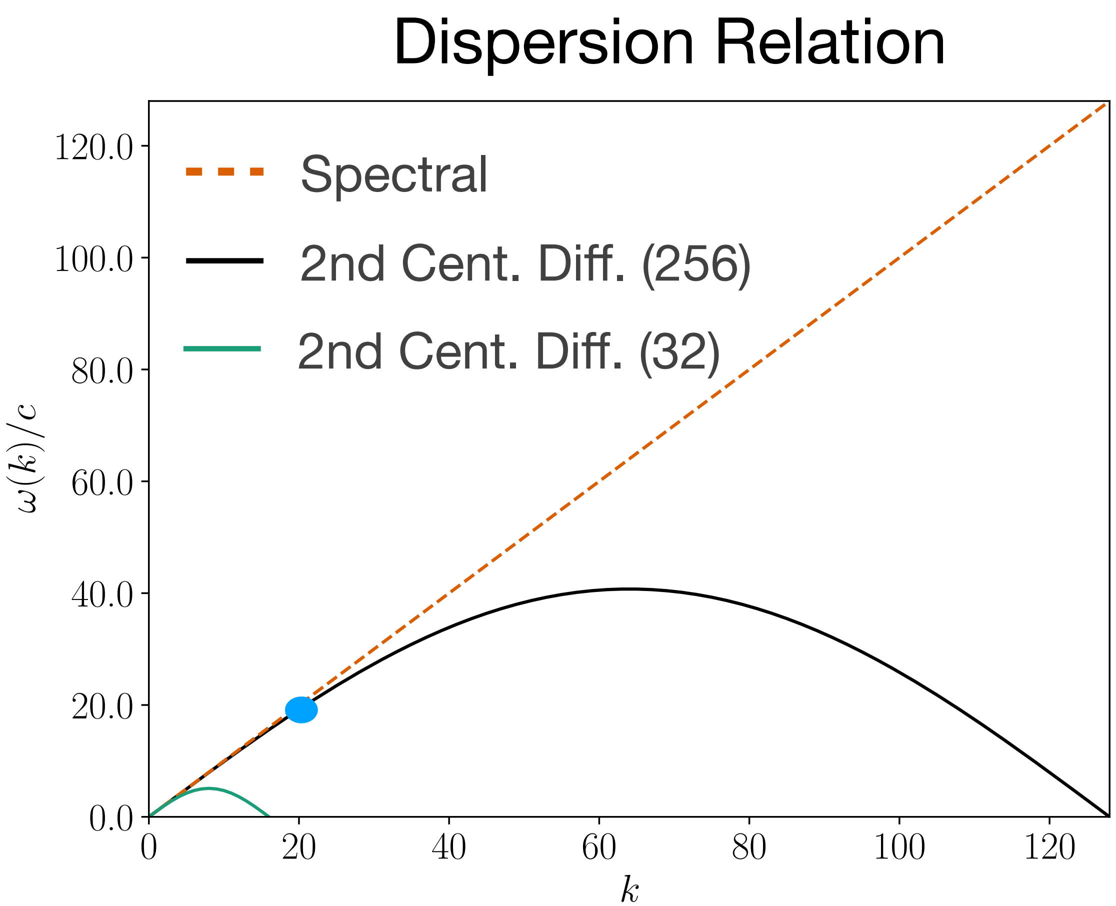
$B_2 - B_1B_1$ Numerical Filters





- Acts as a filter for scales whose dynamics are poorly represented.
- Operates as a hyper-viscosity, e.g., $B_2^7 - B_1^7B_1^7 \sim \Delta^8 \dfrac{d^{10}}{dx^{10}}$, without needing to resort to higher order numerics.
- Readily available as the B_2 and B_1 operators are already required by the governing equations.
- The $B_2 - B_1B_1$ operators scales to the resolvability of each mode.

1D Advection Example

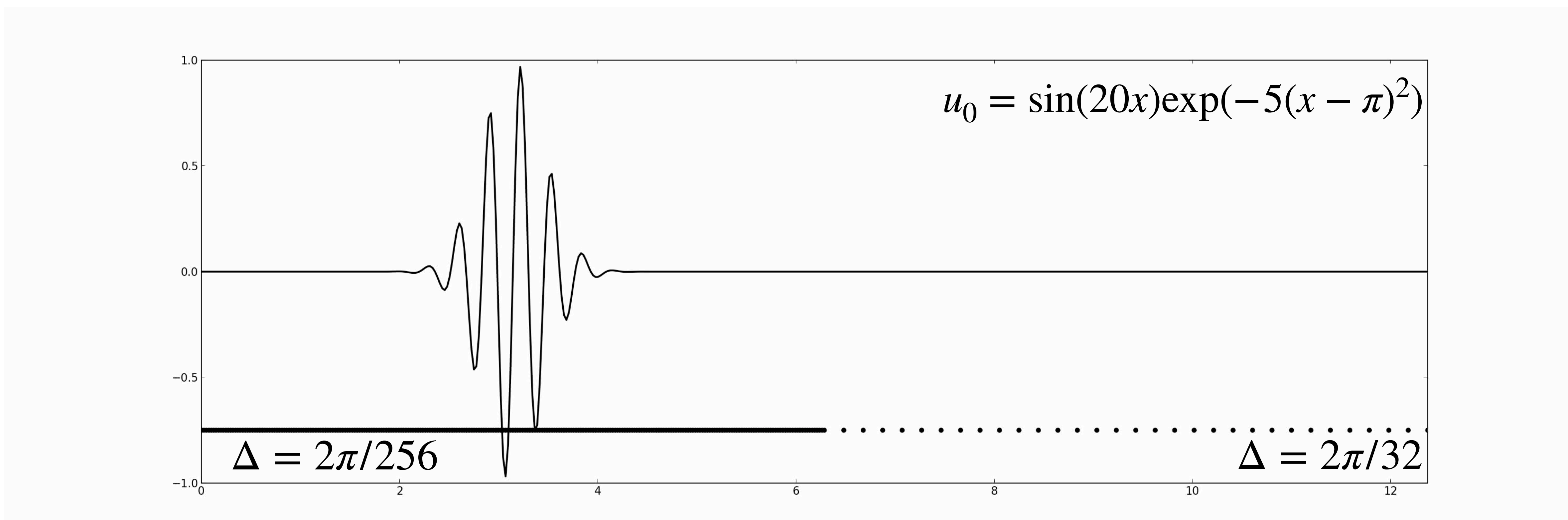


- Inhomogeneous resolution introduces non-local wavenumber interactions.

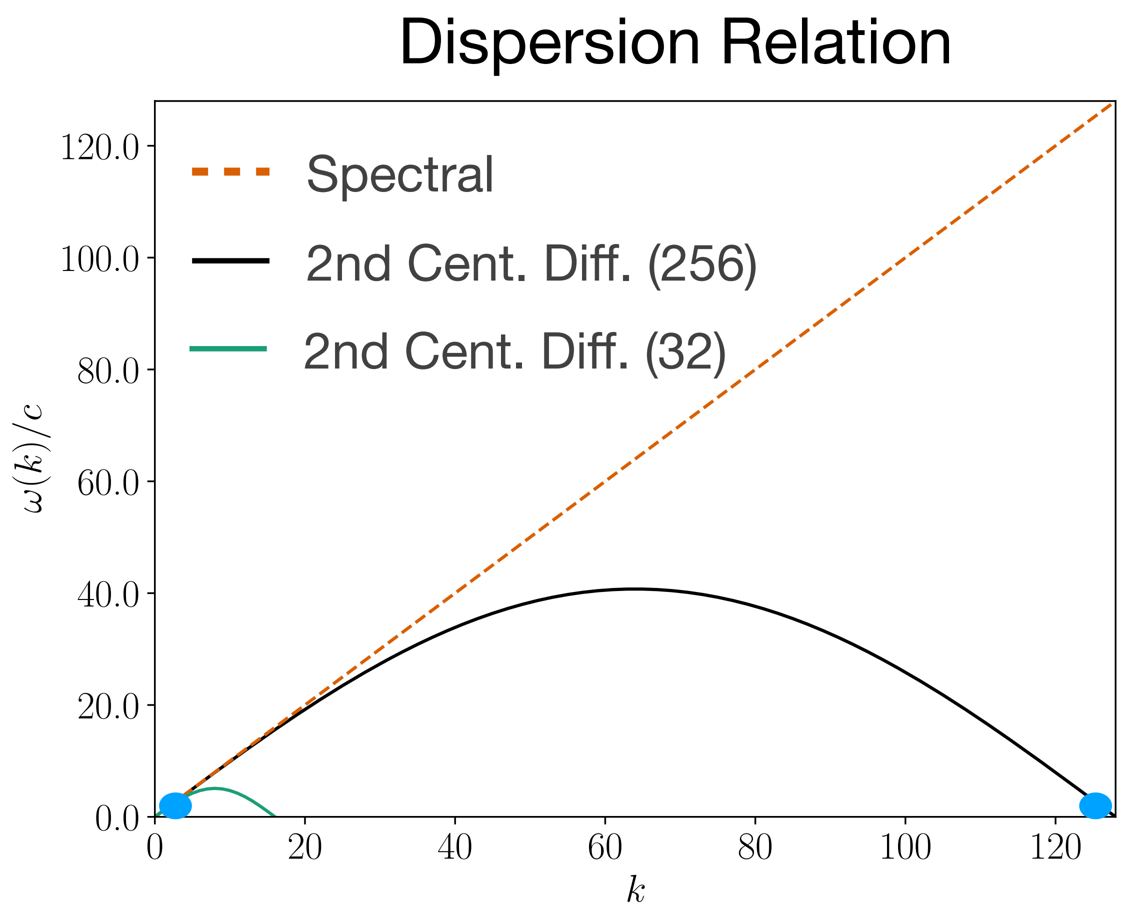
- Particularly damaging to the energy balance as,

  "It is generally accepted that the energy cascade in the internal range in dominated by interactions local in wave number, so that most of the energy is transferred between similar scales through triad interactions amongst wave vectors of about the same length"

  - Waleffe (1991) "The nature of triad interactions in homogeneous turbulence



1D Advection Example



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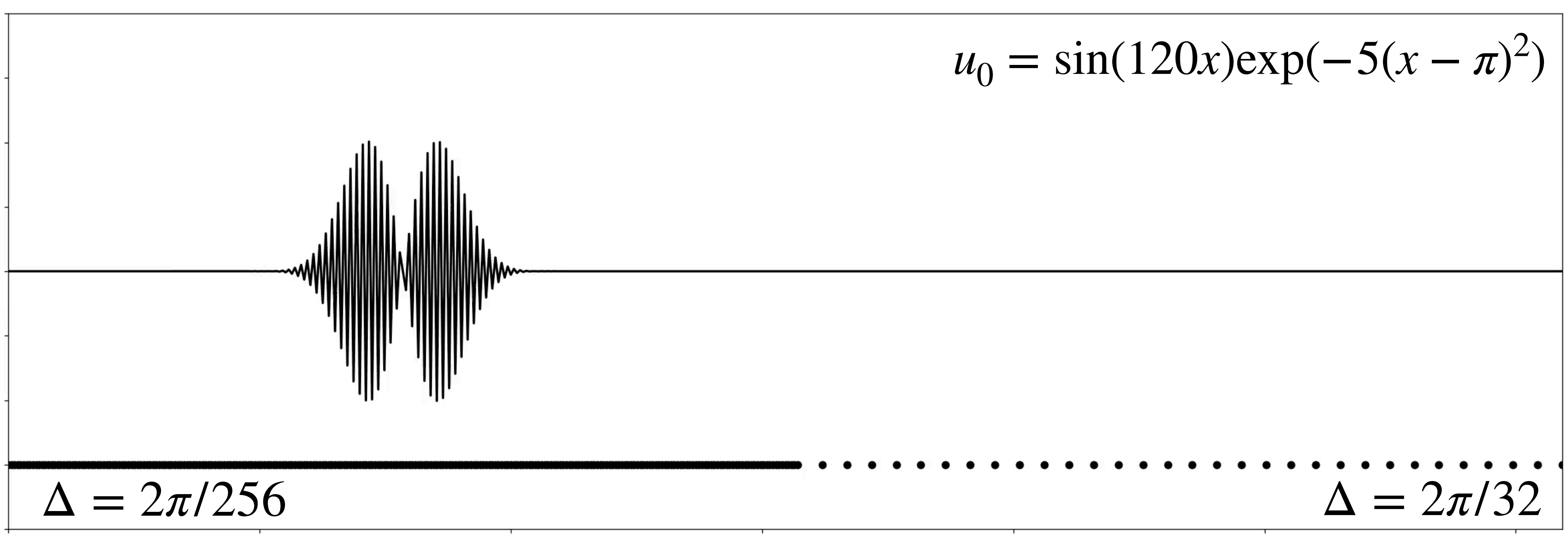


Statistical Description of Commutation Error

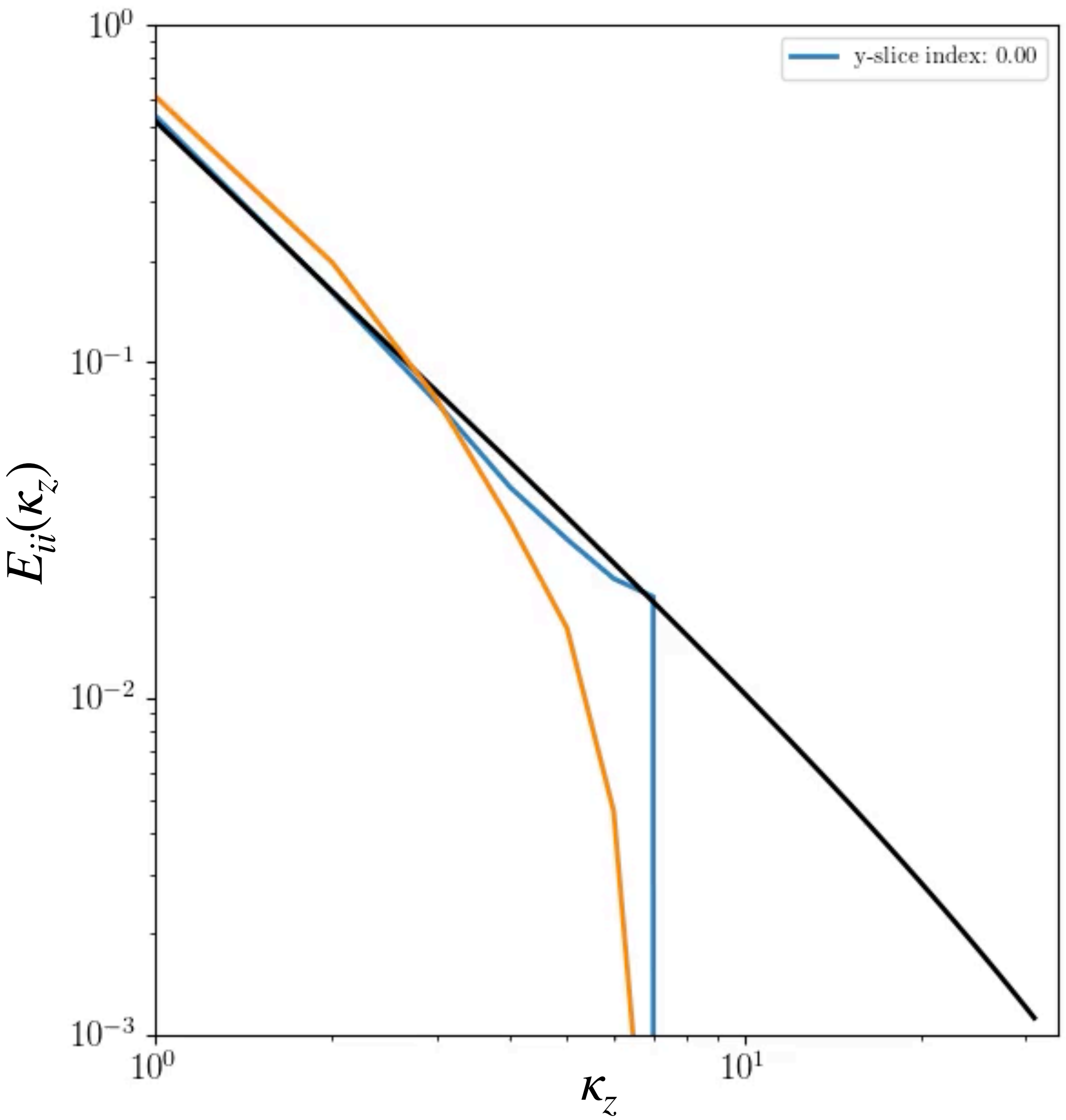
Consider LES of isotropic turbulence convecting at constant velocity U through a grid with vary resolution. If the resolution is varying over length scales that are long compared to the resolution scale, the flow can be considered in a frame moving at velocity U, with varying Fourier cutoff filter $\kappa_c(x)$. The evolution of the resolved turbulence kinetic energy $k^>$ is:

$$\frac{Dk^>}{Dt} = \underbrace{E(\kappa_c(x), t)U_j\frac{\partial \kappa_c}{\partial x_j}}_{\text{Dissipation from commutator}} + \int_0^{\kappa_c(x)} \underbrace{\frac{\partial E(\kappa, t)}{\partial t}}_{\text{Dissipation from SGS}} d\kappa$$

- A commutation model should have information about the mean velocity and the gradient of grid change.

- Ideally, the commutation model should only affect scales near the cutoff wavenumber.

Homogeneous, Isotropic Turbulence Example



$$\frac{Du_i}{Dt} = -\partial_i p + \nu \partial_j \partial_j u_i + \partial_j (C_s \Delta^2 (2 S_{lm} S_{lm})^{1/2} S_{ij})$$

$$\frac{Du_i}{Dt} = -\partial_i p + \nu \partial_j \partial_j u_i + C_1 \Delta^{4/3} \varepsilon^{1/3} \partial_j \partial_j u_i + C_2 \Delta^2 (2 S_{lm} S_{lm})^{1/2} (B_2 - B_1 B_1) u_i$$

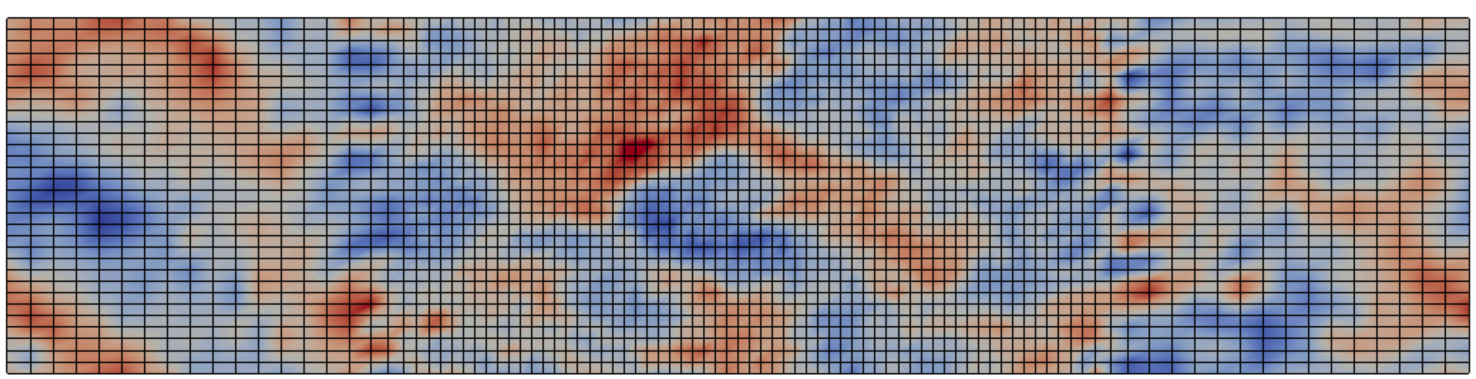
- 1D energy spectra in z for each x-location

$$E(\kappa_z) = \frac{1}{2} \int \phi_{ii}(x, \kappa_y, \kappa_z) d\kappa_z$$

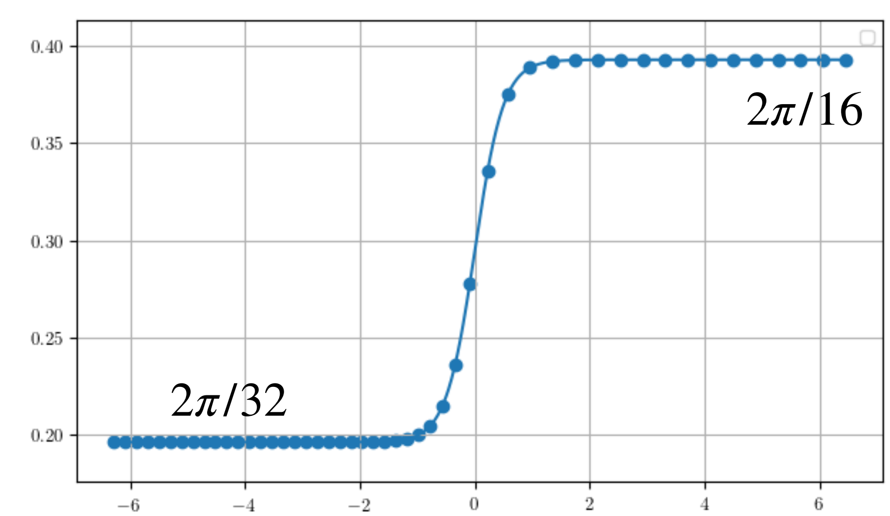
- 7th order Bspline used in each direction

- Inhomogeneous, isotropic resolution changing in the x-direction

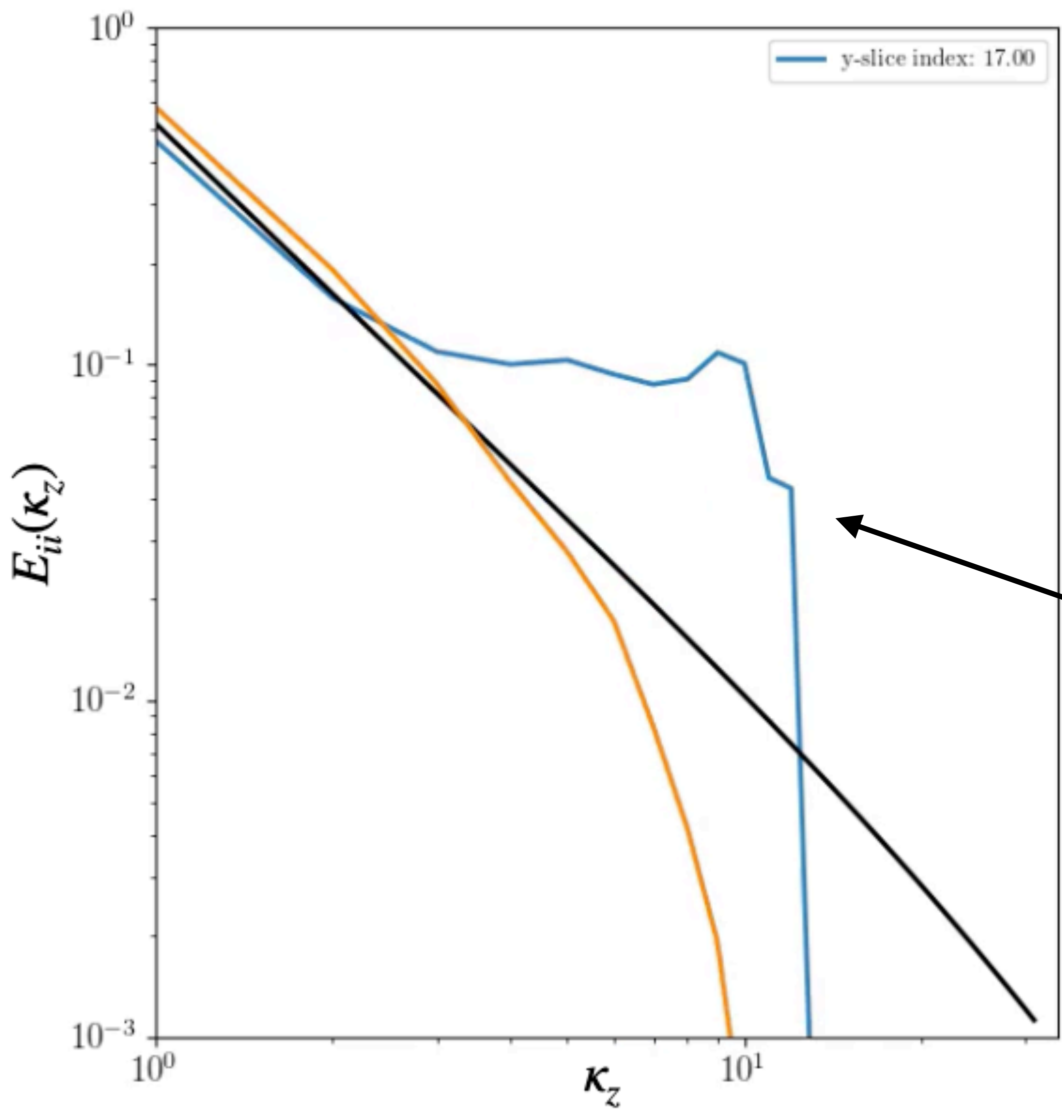
$\Delta(x)$



$\varepsilon = 1 \;,\; \nu = 0 \;,\; \langle u \rangle = 0$



Homogeneous, Isotropic Turbulence Example



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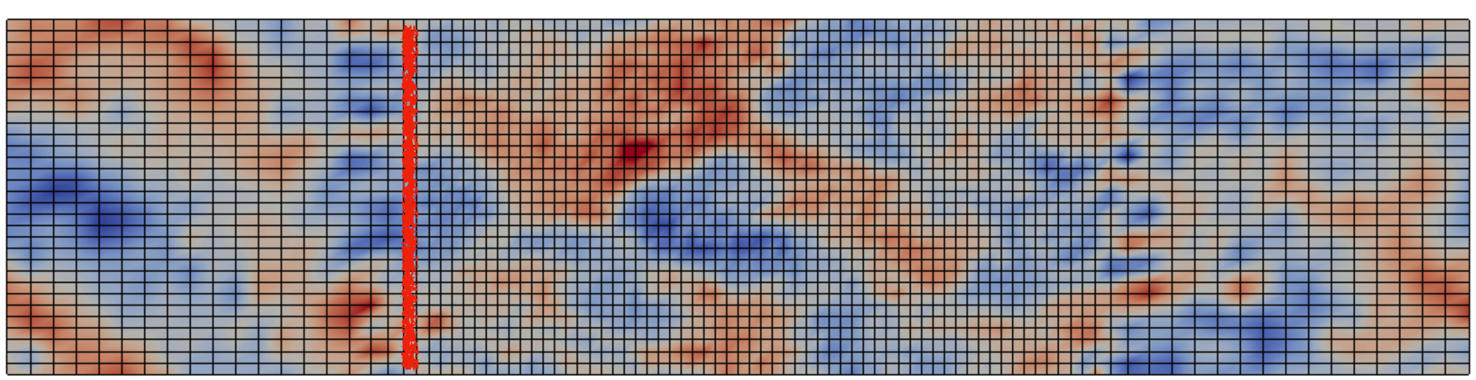
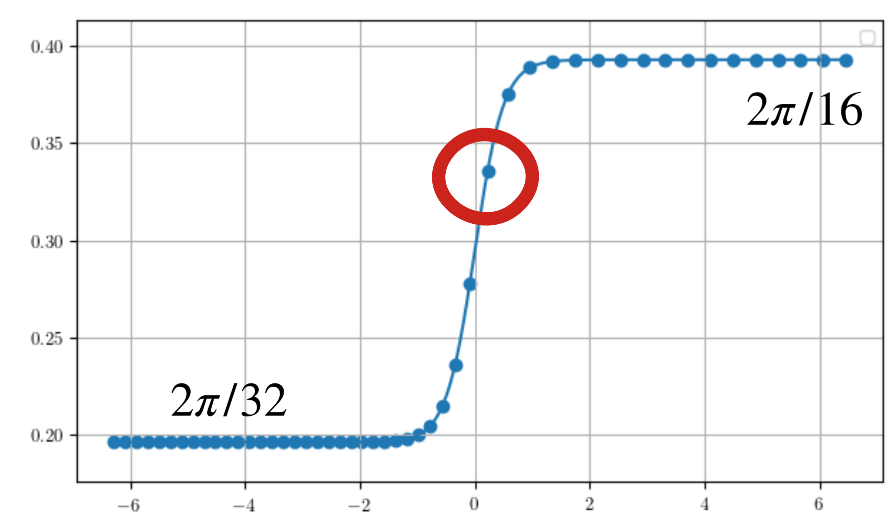
$$\frac{Du_i}{Dt} = -\partial_i p + \nu \partial_j \partial_j u_i + C_1 \Delta^{4/3} \varepsilon^{1/3} \partial_j \partial_j u_i + C_2 \Delta^2 (2 S_{lm} S_{lm})^{1/2} (B_2 - B_1 B_1) u_i$$

Energy pile up due to resolution inhomogeneity

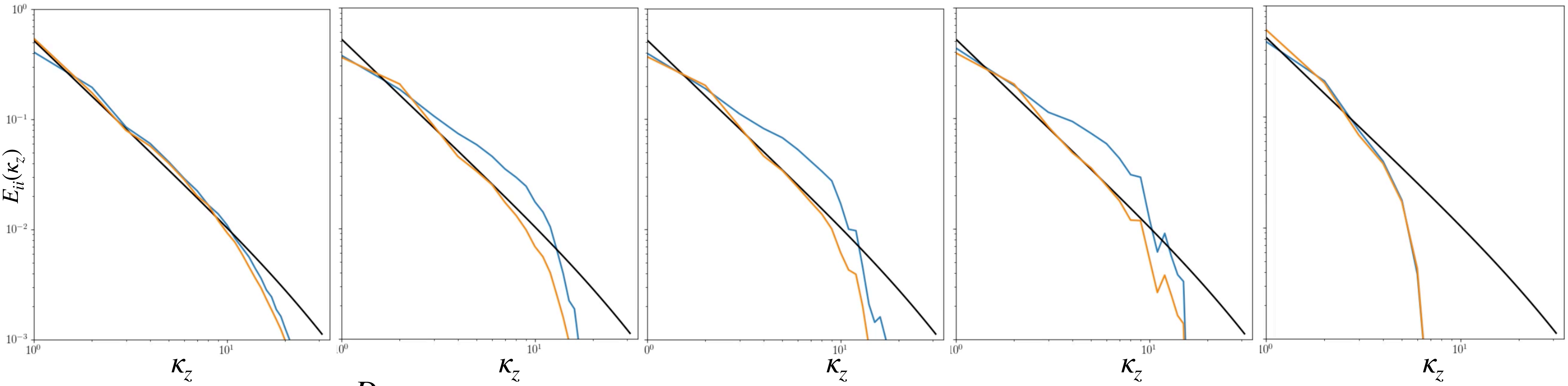


$$\varepsilon = 1 \;,\; \nu = 0 \;,\; \langle u \rangle = 0$$

$\Delta(x)$



Fine $(\Delta = 2\pi/64)$ → **Coarse** $(\Delta = 2\pi/16)$

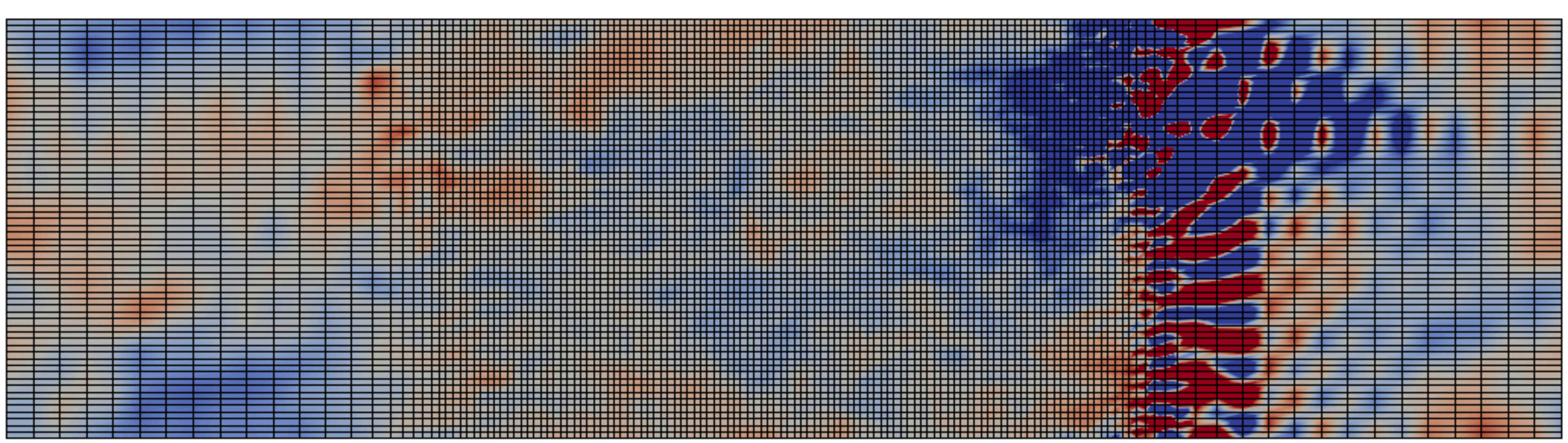


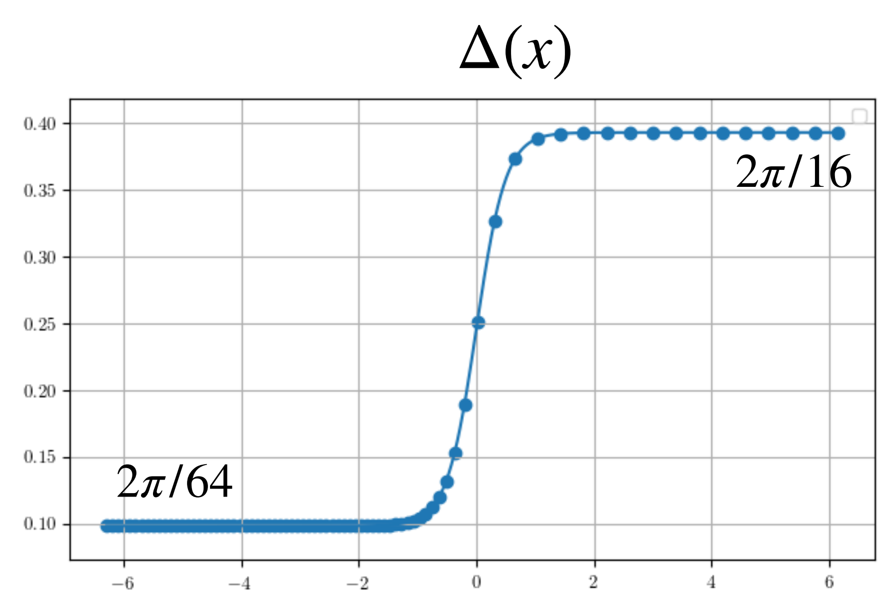
$$\frac{Du_i}{Dt} = -\partial_i p + \nu\partial_j\partial_j u_i + C_1\Delta^{4/3}\varepsilon^{1/3}\partial_j\partial_j u_i + C_2\Delta^2(2S_{lm}S_{lm})^{1/2}(B_2 - B_1B_1)u_i$$

$$\frac{Du_i}{Dt} = -\partial_i p + \nu\partial_j\partial_j u_i + C_1\Delta^{4/3}\varepsilon^{1/3}\partial_j\partial_j u_i + \left(C_2\Delta^2(2S_{lm}S_{lm})^{1/2} + C_3\Delta\,\left|u_k\frac{\partial\Delta}{\partial x_k}\right|\right)(B_2 - B_1B_1)u_i$$

Energy pile up causes numerical instability near resolution change with just subgrid stress model

Mean convection example:

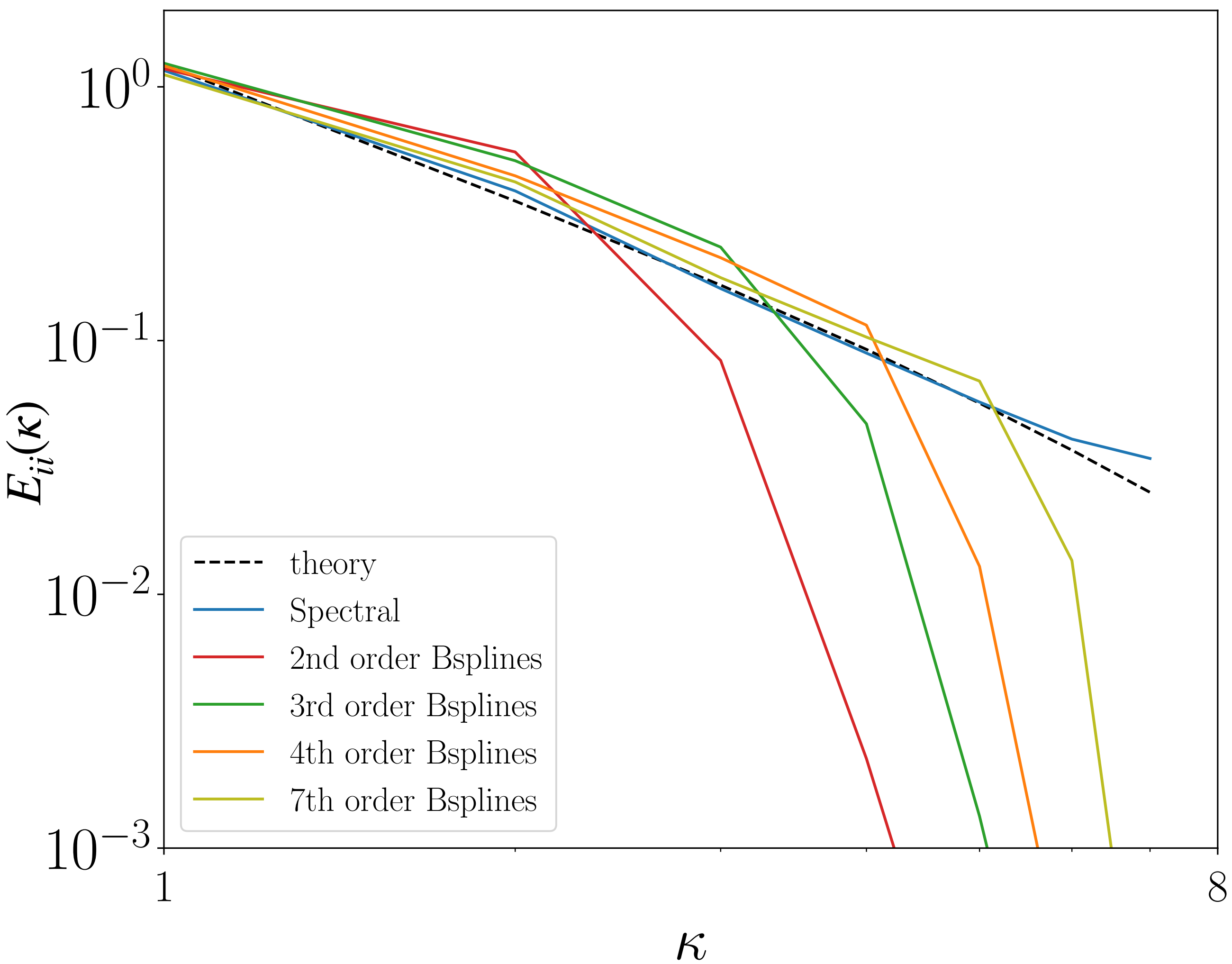




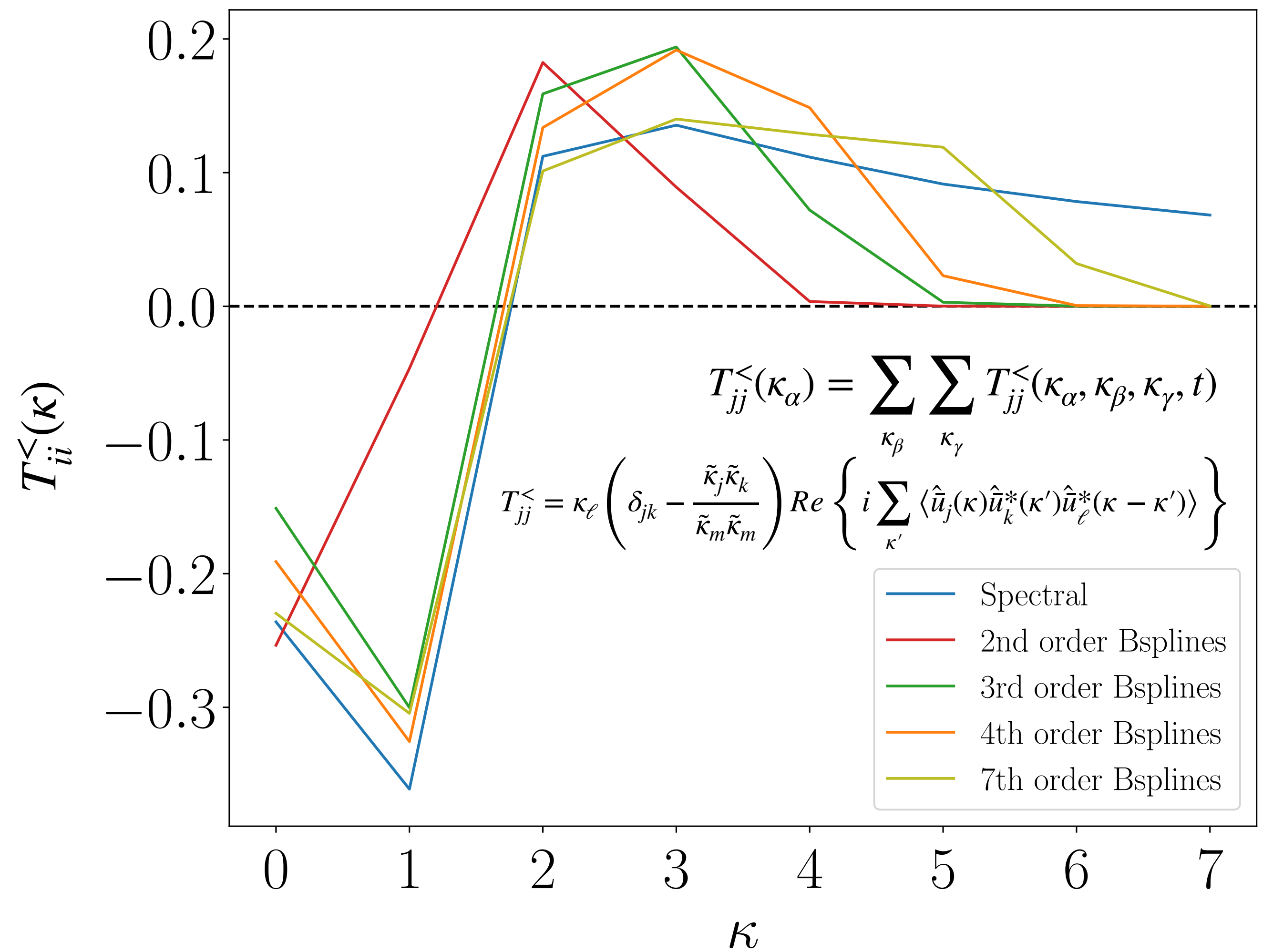
Statistical Effects of Dispersion Error

Consider homogeneous, isotropic turbulence with uniform resolution and a mean convection velocity:

1D energy spectra in direction of convection



1D resolved transfer spectra in direction of convection



- Energy spectra in the direction of convection are deficient in scales that experience significant dispersion error.
- Corresponding transfer spectra tend to zero in poorly resolved scales.
- An energy pile up occurs in well resolved mode as energy is not transferred to larger wavenumber.
- Phenomenon scales with both convection velocity and order of accuracy of numerics.

Statistical Effects of Dispersion Error

The evolution equation for the instantaneous energy $E(\boldsymbol{\kappa},t) = \frac{1}{2}\hat{u}_j^*(\boldsymbol{\kappa},t)\hat{u}(\boldsymbol{\kappa},t)$ is

$$\left(\frac{\partial}{\partial t} + 2\nu_t \tilde{\tilde{\boldsymbol{\kappa}}}^2\right) E(\boldsymbol{\kappa},t) = -\tilde{\kappa}_j \left(\delta_{\ell i} - \frac{\tilde{\kappa}_i \tilde{\kappa}_\ell}{\tilde{\kappa}_m \tilde{\kappa}_m}\right) \mathrm{Im}\left\{\int_{\boldsymbol{\kappa}'} \hat{u}_i(\boldsymbol{\kappa},t)\hat{u}_\ell^*(\boldsymbol{\kappa}',t)\hat{u}_j^*(\boldsymbol{\kappa}-\boldsymbol{\kappa}',t)d\boldsymbol{\kappa}'\right\} + F(\boldsymbol{\kappa},t)$$

The time- and space-Fourier transformed energy transfer function is

$$\hat{T}(\boldsymbol{\kappa},\omega) = -\tilde{\kappa}_j \left(\delta_{\ell i} - \frac{\tilde{\kappa}_i \tilde{\kappa}_\ell}{\kappa^2}\right) \mathrm{Im}\left\{\int_{\boldsymbol{\kappa}'}\int_{\omega'}\int_{\omega''}\int_{\omega'''} \hat{u}_i(\boldsymbol{\kappa},\omega')\hat{u}_\ell(-\boldsymbol{\kappa}',\omega'')\hat{u}_j(\boldsymbol{\kappa}'-\boldsymbol{\kappa},\omega''') \int_t e^{i(\omega'+\omega''+\omega'''-\omega)t} dt\, d\omega' d\omega'' d\omega''' d\boldsymbol{\kappa}'\right\}$$

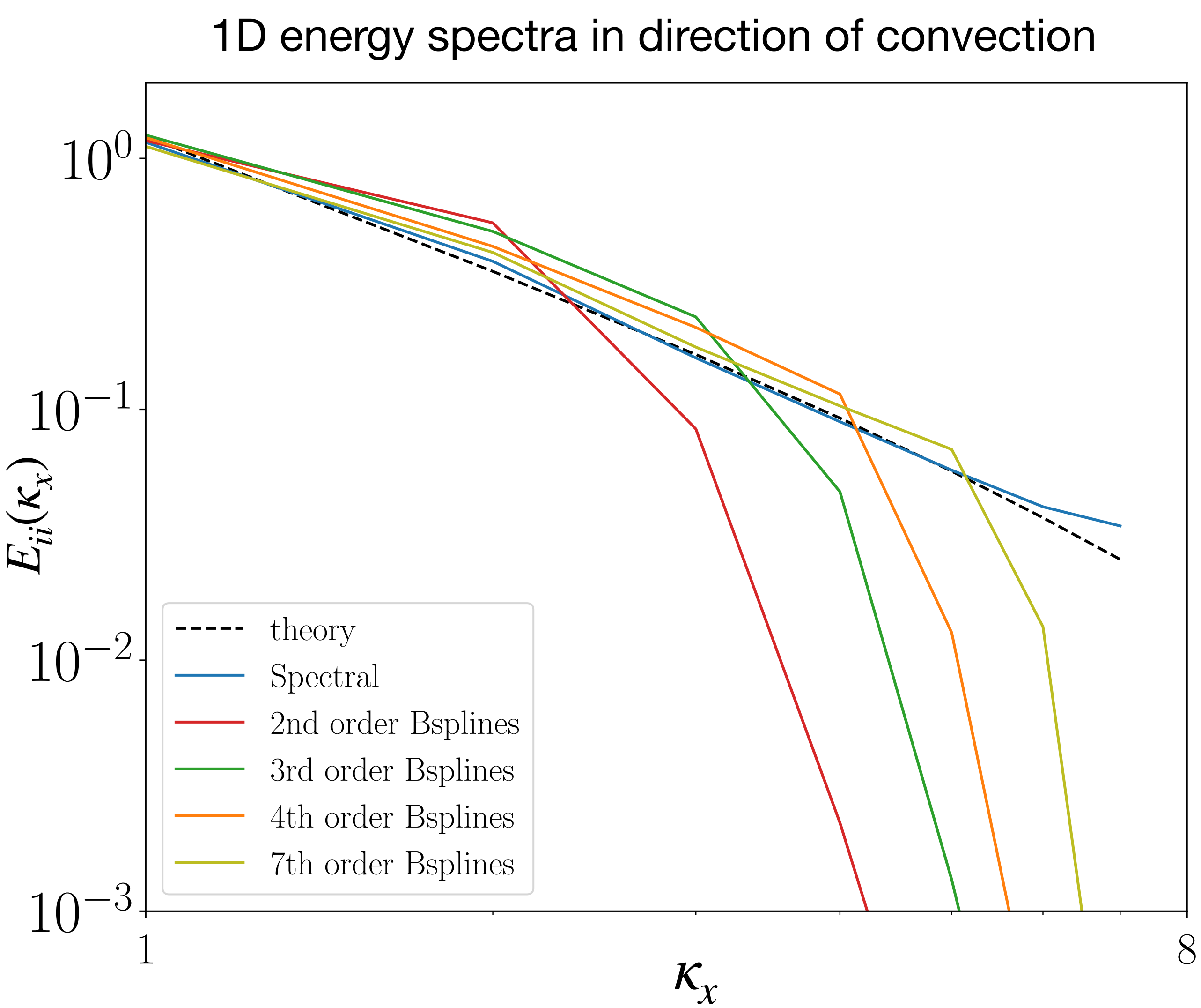
To simplify the analysis, consider triad interactions between wave modes whose frequency directly correspond to the mean convection velocity $\omega = -U_j\tilde{\kappa}_j$ (i.e., mean velocity dominates turbulent fluctuations). Then $\hat{T}(\boldsymbol{\kappa},\omega)$ becomes

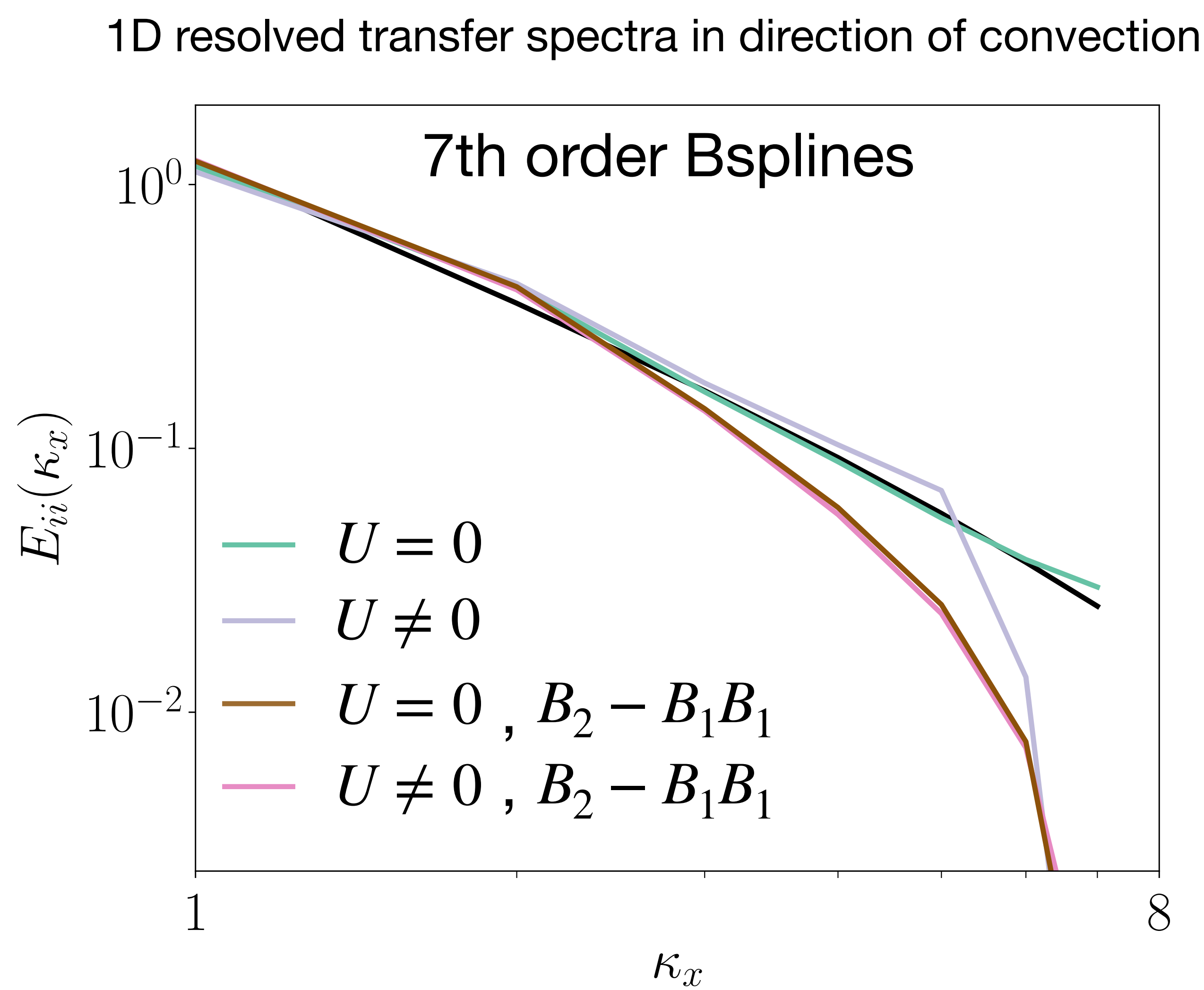
$$\hat{T}(\boldsymbol{\kappa},\omega) = -\tilde{\kappa}_j \left(\delta_{\ell i} - \frac{\tilde{\kappa}_i \tilde{\kappa}_\ell}{\kappa^2}\right) \mathrm{Im}\left\{\int_{\boldsymbol{\kappa}'} \hat{u}_i(\boldsymbol{\kappa},-\mathbf{U}\cdot\tilde{\boldsymbol{\kappa}})\hat{u}_\ell(-\boldsymbol{\kappa}',\mathbf{U}\cdot\tilde{\boldsymbol{\kappa}}')\hat{u}_j(\boldsymbol{\kappa}'-\boldsymbol{\kappa},-\mathbf{U}\cdot(\widetilde{\boldsymbol{\kappa}'-\boldsymbol{\kappa}})) \int_t e^{i(\mathbf{U}\cdot(\tilde{\boldsymbol{\kappa}}'-\tilde{\boldsymbol{\kappa}}-(\widetilde{\boldsymbol{\kappa}'-\boldsymbol{\kappa}}))-\omega)t} dt\, d\boldsymbol{\kappa}'\right\}$$

Orthogonality implies:

$$\omega = \mathbf{U}\cdot\left(\tilde{\boldsymbol{\kappa}}' - \tilde{\boldsymbol{\kappa}} - (\widetilde{\boldsymbol{\kappa}'-\boldsymbol{\kappa}})\right)$$

Statistical Effects of Dispersion Error





Conclusion

- The presence of resolved scales whose dynamics are incorrectly represented can corrupt the flow throughout the domain.

- Understanding how these errors manifest in LES is crucial for the development of robust turbulence models.

- An a priori statistical requirement should be imposed on the flow so that the energy spectrum is sufficiently small for scales at which aliasing or dispersion are significant.

Recent Publications

R. D. Moser, S. W. Haering, and **G. R. Yalla**. "Statistical Properties of Subgrid-Scale Turbulence Models." Annual Review of Fluid Mechanics 53 (2020).

G. R. Yalla, R. D. Moser, T. Oliver, S. W. Haering, and B. Engquist. "On the Effects of Resolution Inhomogeneity in LES" arXiv preprint arXiv:2003.02355 (2020).

Thank you for listening and stay safe!

73rd Annual Meeting of the APS Division of Fluid Dynamics

Sunday–Tuesday, November 22–24, 2020; Virtual, CT (Chicago time)

Session X11: Turbulence: Modeling & Simulations (10:45am - 11:30am)

10:45 AM, Tuesday, November 24, 2020

Abstract: X11.00008 : Impacts of Numerical Discretization on Large Eddy Simulation

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