On Globally Defined Solutions of the Generalized CLM Equation

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Introduction

• We consider *Euler's equation*, which models the motion of inviscid fluids. Given

$$u = \text{velocity}, \quad p = \text{pressure}, \quad \rho = \text{density}$$

we have

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho}\nabla p = 0$$
$$\nabla \cdot u = 0$$

• Determining if solutions to the three dimensional Euler's equation exist globally in time or blow up in finite time is an active area of research. Blow up occurs when the gradient of the velocity becomes infinitely large. Beale, Kato, and Majda (1984) showed that this is equivalent to the curl of the velocity, the vorticity, tending towards infinity.

• To make the problem more tractable, Constantin, Lax and Majda (1985) proposed an analogous one dimensional model vorticity equation:

$$\frac{dw}{dt} = H(w)w$$

where $w = \text{curl } v$ and H represents the Hilbert Transform, defined by

$$Hw(x) = \frac{1}{\pi}\text{ P.V.}\int \frac{w(y)}{(x-y)}\, dy.$$

• Gregorio (1989) modified this equation by adding a convection term, vw_x. Okamoto *et al.* (2008) generalized this model by placing a constant in front of this term to study its effect on the behavior of solutions. We investigate Okamoto's equation, the *generalized model vorticity equation*:

$$w_t + avw_x = wHw, \qquad \text{where } v_x = Hw$$

Goal

We hope to numerically and analytically examine the behavior of solutions of the one dimensional model; specifically, we want to study how the convection term can effect solutions for positive a values.

Known Results

• The parameter a determines the behavior of the solution by controlling the influence of the convection term. It is known that

$$a \le 0 \implies \text{blow-up}$$
$$a \to \infty \implies \text{no blow-up}$$

• It is believed that there exists a $0 < \gamma < 1$ such that for all $a < \gamma$, solutions blow up in finite time and for all $a \ge \gamma$, solutions exist globally. Okamoto conjectured that this bifurcation occurs around $\gamma = 0.6$.

• Assuming $w \in C_c^\infty(\mathbb{R})$ odd, Cordoba *et al.* (2010) rewrote the PDE as

$$\frac{\partial Hw(0,t)}{\partial t} = \frac{1}{2}(Hw(0,t))^2 - a(H(vw_x))(0,t)$$

and showed $H(vw_x) > 0$ to prove blow up for $a \le 0$.

Numerical Methods

• Spectral Method

— Assuming solutions are periodic with form

$$w(x,t) = \sum_{n=-\infty}^{\infty} w_n(t)e^{inx},$$

we obtain the following ODE for $k \in [-N, N]$

$$w_k'(t) = -i\sum_{n=-N}^{N}\text{sgn}(k-n)w_n(t)w_{k-n}(t) + a\Big(\sum_{n=-N}^{N}\frac{in}{|k-n|}w_n(t)w_{k-n}(t) - v(-\pi)ikw_k(t)\Big)$$

and use Matlab's ODE45 to solve for the coefficients over time.

• Finite Difference Method

— We use an upwinding scheme for w_x, Matlab's Hilbert Transform function for Hw, and the following first order approximation for w_t:

$$w_t = \frac{w(t+\Delta t, x) - w(t,x)}{\Delta t}$$

Accuracy of Methods

The L^2 norm, used to measure distance between solutions, is defined below:

$$||w_{N_1} - w_{N_2}||_2 = \Big(\int_{-\pi}^{\pi}|w_{N_1} - w_{N_2}|^2\, dx\Big)^{\frac{1}{2}}$$

• L^2 Convergence of Methods for $w_0(x) = \sin(x+\pi)$, $a = 0.7$, $t_f = 2$

Spectral Method

N	Error
4	0.338244128
16	9.0001735 $\times 10^{-5}$
32	2.460683576 $\times 10^{-7}$
64	2.3573881324 $\times 10^{-9}$

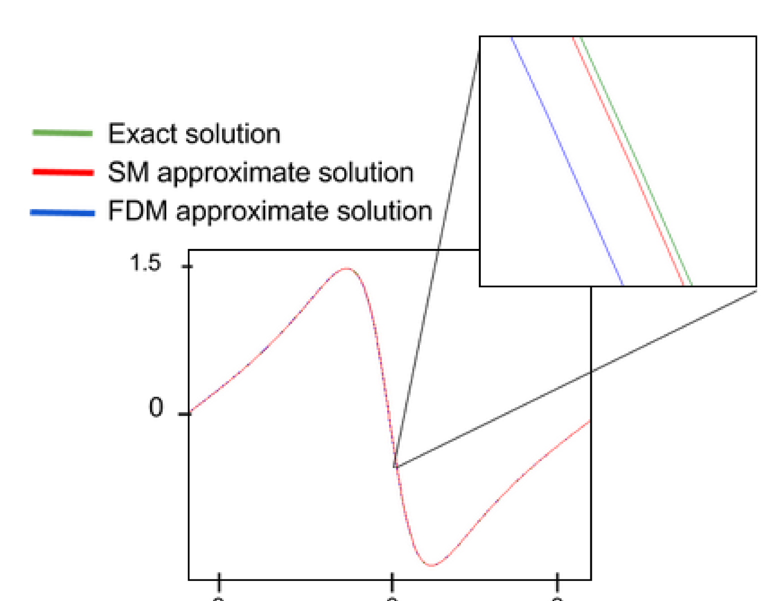
Error $= ||w_{128} - w_N||_2$

FD Method

N_x	Error
300	0.070999625
600	0.034846144
1200	0.016603790
2400	0.007591044

Error $= ||w_{128} - w_{N_x}||_2$

• Comparison with exact solution of CLM equation ($a = 0$)



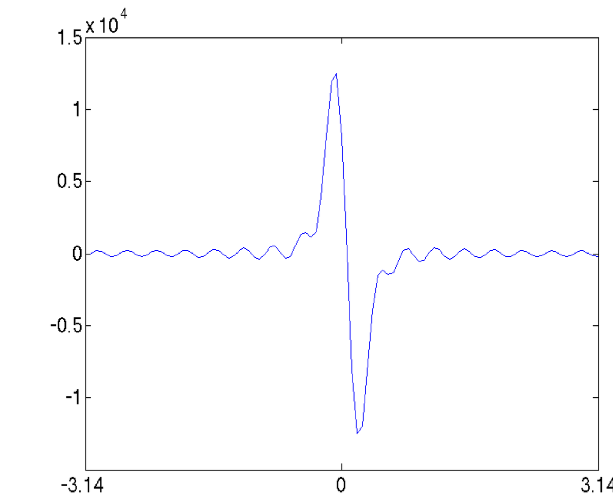
$$||w_{exact} - w_{sm}||_2 = 0.0011$$
$$||w_{exact} - w_{fd}||_2 = 0.0099$$

$$w_{exact} = \frac{2sin(x+\pi)}{(2 - tH(sin(x+\pi))^2 + t^2 sin^2(x+\pi)}$$

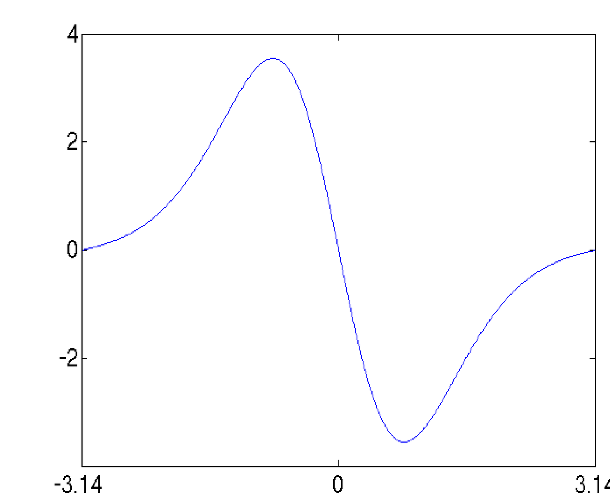
Bifurcation Results

• Given a fixed final time, t_f, we use a binary search method to find the largest a for which blow up occurs. Our hope is that the a values will converge to some γ as we increase t_f, producing graphs like the following.

Blow up of $w(x,t)$: $a \le \gamma$



No Blow up of $w(x,t)$: $a > \gamma$



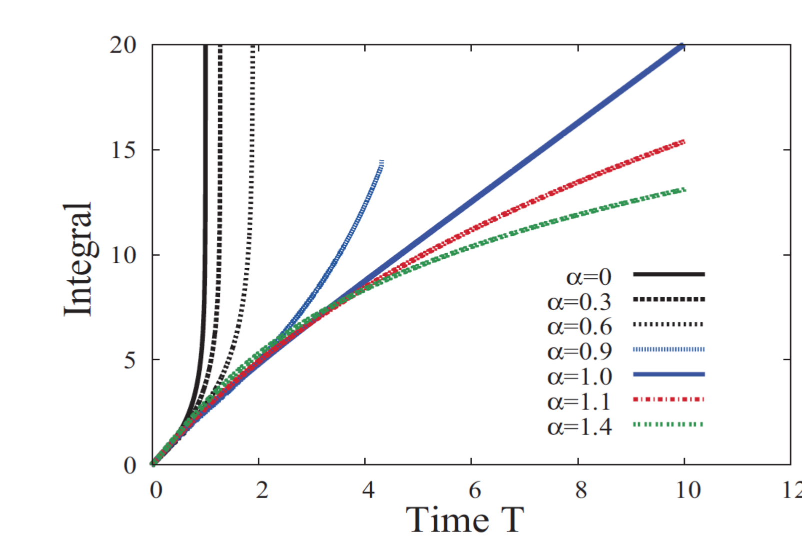
• The difficulty in approximating a:

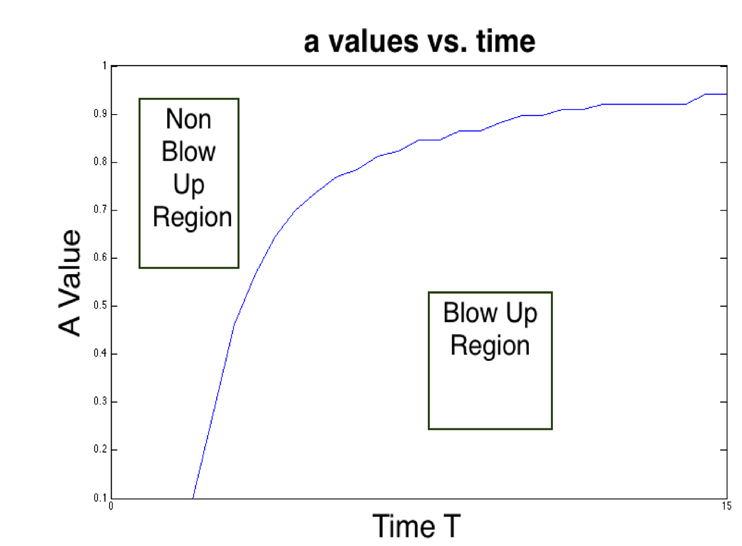
— Blow up could result from accumulation of error due to the time-step (here fixed at $\Delta t = 10^{-4}$) or truncation in the spectral method.

— Difficult to distinguish blow up solutions from solutions growing in time.

• Observations using the binary search method indicate:

— ODE45 warning and $\int_0^T ||Hw(t,\cdot)||_\infty dt > 10^{12}$ flags, produce the same intervals of a values where blow up occurs.





$$w(0,x) = \sin(x) + 2\sin(2x)$$

Analytical Attempts

• Considering Cordoba's reformulation of the PDE:

— Find conditions where $H(vw_x) < 0$ to prove blow up for a finite $a > 0$.

— Given the generalized *Ricatti equation*:

$$y'(t) = \frac{1}{2}y(t)^2 + x(t), \text{ we studied the ratio } \frac{a(H(vw_x))(0,t))}{Hw(0,t)}$$

in order to relate Cordoba's equation to a Ricatti form, where conditions for blow up are known.

• Write solutions in the form $w(t) = w_0(t) + H(w_1(t))$ and use a Fefferman-Stein decomposition to produce a system of ODEs in which to look for bifurcations.

Future Directions

• Gain a better approximation of the bifurcation value, γ.
• Rewrite Cordoba's equation into a Ricatti form.

Acknowledgements

• We would like to thank the Leadership Alliance Program.
• Research funded by NSA Grant H98230-14-1-0150.