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Galerkin Formulations with Greville Quadrature Rules for Isogeometric Shell Analysis: Higher Order Elements and Locking

by

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I am not sure when I first met Peter Wriggers, but it was no later than 1983/84 when he was a postdoc at UC Berkeley. During that time, he and Juan Simo jointly developed the first consistent tangent operator for contact, a great step forward in Computational Contact Mechanics. Peter returned to Berkeley in 1988 and came to Stanford quite often to work with Juan. Over the years, I interacted with Peter in a multitude of professional capacities and enjoyed and valued his friendship. I visited him in Hannover in 2009 and, with his assistant, İlker Temizer, we initiated the first applications of IGA to contact problems. (Tom Hughes)

Abstract We propose new Greville quadrature schemes that asymptotically require only four in-plane points for Reissner-Mindlin (RM) shell elements and nine in-plane points for Kirchhoff-Love (KL) shell elements in B-spline and NURBS-based isogeometric shell analysis, independent of the polynomial degree of the elements. For polynomial degrees 5 and 6, the approach delivers high accuracy, low computational cost, and alleviates membrane and transverse shear locking.

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1 Introduction

In this paper we address the problem of creating shell finite elements within the Isogeometric Analysis (IGA) paradigm, which, in structural mechanics, amounts to employing the same kinematic description (i.e., specification of the displacement field) as that being utilized in the definition of geometry emanating from a Computer Aided Design (CAD) representation [1]. We focus on B-splines and NURBS, as these CAD technologies dominate industrial usage. IGA offers a fundamental advantage in shell modeling, namely, precise, or even exact, geometric representation, and this is no doubt important as it is well known that even small geometric imperfections can significantly affect results in thin shell buckling, indicating numerical approximations of geometry may also be a primary source of error. Nevertheless, there are still major barriers to creating effective IGA shell elements, and these are shared by traditional finite element methods.

A primary concern is "locking phenomena," specifically, transverse shear locking and membrane-bending locking. Transverse shear locking is not a consequence of curved shell geometry; it is present as well for flat plate and straight beam models. In the development of shell finite elements, the main challenge to overcome, and a remaining open problem, is membrane-bending locking. It is apparent that curved, higher-order, traditional shell elements have not distinguished themselves heretofore because curvature is the root cause of membrane-bending coupling, hence locking. It is no wonder that in industrial software there is a heavy reliance on the lowest-order, four-node, quadrilateral shell elements, despite their inherently low accuracy, because they are typically flat, or almost flat, and minimize membrane-bending coupling within elements thereby. What we would like to have are simple, straightforward, IGA shell elements that would be candidates for inclusion in industrial scale, commercial general-purpose computer programs.

We have pursued a study that starts with the most direct "primal" formulations of shell finite elements, and adheres to the finite element analysis orthodoxy of using high-enough accurate Gauss quadrature rules to ensure stability of the stiffness and mass matrices. Just as in the case of traditional finite elements, there are no exact quadrature rules for non-affine element geometries. So, sufficiently accurate Gauss rules are generally accepted as about the best one can do. We have investigated Reissner-Mindlin (RM) shell theory [2, 3] and Kirchhoff-Love (KL) shell theory [4], which precludes transverse shear deformation and is "rotation free," only requiring displacement degrees of freedom, unlike RM elements, which additionally require rotation or director fields. Our study focused on maximally smooth B-splines and NURBS elements of polynomial order $p = 2, 3, 4, 5$, and 6 for RM theory, and $p =$ 3, 4, 5, and 6 for KL theory. The in-plane Gauss point patterns used involve $(p + 1)^2$ points per Bézier element. Based on previous studies, we anticipated severe locking to occur for lower orders of p and mitigation of locking for higher orders of p, and indeed this was the case [5]. For orders $p = 5$ and 6, we found promising results for all tests considered. It seems higher-order elements cure a multitude of ills, but, of course, the obvious drawback is the computational cost associated with the very large number of Gauss quadrature points per element. These orders of p may seem high, but that is probably due to lingering perceptions emanating from experience with classical finite element analysis. With one control point per element, the order of smooth spline elements is asymptotically the same as $p = 1$ in traditional finite element analysis. Given these observations, it seems that the cases $p = 5$ and 6 might provide robust capabilities of the type desired if, and only if, the cost of quadrature could be reduced to an acceptable level, independent of p.

We endeavored to reduce the number of quadrature points to be substantially less than full Gauss quadrature. Greville abscissae, which are in one-to-one correspondence with the control points (i.e., nodes), represent a "one-point" quadrature rule in the sense that there is only one quadrature point per control point. This was our first attempt, but in Galerkin formulations of shell theories it was not effective. However, we found that Greville abscissae were effective, if we redefined the space that determined the Greville abscissae to include, in addition to the basis functions, all the derivatives appearing in the weak form of the problem. To be specific, in the case of maximally smooth RM elements, to determine the Greville abscissae, we used the larger space of p^{th}-order splines that are C^{p-2} continuous. Note that this is one order less continuity than for maximally smooth p^{th}-order splines, which are C^{p-1} continuous. For maximally smooth KL elements, we used the still larger space of p^{th}-order splines that are C^{p-3} continuous. In both cases, we then solve linear, moment fitting equations in each parametric direction to obtain the weights, and then the two-dimensional quadrature points and weights are generated by a simple tensor product of the one-dimensional quantities. This results in, asymptotically, four in-plane quadrature points per RM shell element and nine in-plane quadrature points per KL shell element, which are fewer than those required by full Gauss quadrature for all the cases considered, and substantially fewer in the higher-order cases, with concomitant reductions in computational cost. The accuracy of the Greville rules is found to be commensurate with full Gauss quadrature.

2 Greville quadrature

2.1 Definition of Greville quadrature

Numerical integration of a univariate function, $f(x)$, can be written as

$$\int_{\hat{I}} f \,\mathrm{d}\hat{I} \approx \sum_{I=1}^{n} f(x_I) w_I, \tag{1}$$

where f is the integrand, $\hat{I}$ is the integral domain, $\{x_I\}_{I=1}^n$ are the n quadrature points, and $\{w_I\}_{I=1}^n$ are the corresponding weights. Given a univariate p-degree ($p \geq 2$) B-spline basis $\{N_I\}_{I=1}^n$ with an open knot vector $\Xi = \left\{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\right\}$, we propose a way to determine the quadrature points and weights as follows: the Greville abscissae $\{x_I\}_{I=1}^n$, where $x_I = \frac{1}{p}(\xi_{I+1} + \xi_{I+2} + \cdots + \xi_{I+p})$, are chosen to be the quadrature points, and the weights $\{w_I\}_{I=1}^n$ are determined so that the quadrature rule can exactly integrate all linear combinations of the univariate B-spline basis $\{N_I\}_{I=1}^n$. This can be accomplished by solving the following moment fitting system of equations

$$\begin{bmatrix} \int_{\hat{I}} N_1(\xi)\mathrm{d}\xi \\ \int_{\hat{I}} N_2(\xi)\mathrm{d}\xi \\ \vdots \\ \int_{\hat{I}} N_n(\xi)\mathrm{d}\xi \end{bmatrix} = \begin{bmatrix} N_1(x_1) & N_1(x_2) & \cdots & N_1(x_n) \\ N_2(x_1) & \ddots & \cdots & N_2(x_n) \\ \vdots & & \vdots & \vdots \\ N_n(x_1) & \cdots & \cdots & N_n(x_n) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \tag{2}$$

where the left-hand side contains the moments, which are computed exactly using full Gauss quadrature. As the Greville abscissae are taken as quadrature points, we refer to this quadrature rule as the Greville quadrature. Figure 1 shows the Greville quadrature points and weights for a univariate quadratic B-spline basis associated with the knot vector $\Xi = \{0, 0, 0, 1, 2, 3, 4, 4, 4\}$. Note that the Greville quadrature

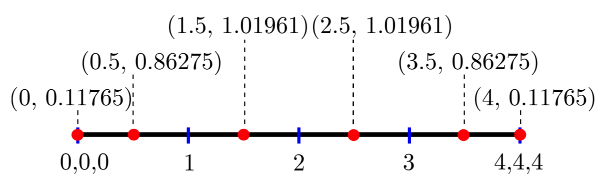


Fig. 1: Greville quadrature points and weights for a quadratic B-spline basis with knot vector $\Xi = \{0, 0, 0, 1, 2, 3, 4, 4, 4\}$. Red dots denote the locations of the quadrature points and $(\cdot, \cdot)$ indicates $(x_i, w_i), i = 1, 2, \cdots, 6$.

points and weights are calculated with respect to the global parametric domain of the patch. To utilize the method in existing FEA routines we can easily map these quadrature points into a parent element coordinate system through an affine mapping.

For a bivariate B-spline or NURBS basis, the quadrature points and weights are efficiently obtained through a simple tensor product of the corresponding univariate quantities.

2.2 Greville quadrature for shells

The Greville quadrature proposed in Section 2.1 lays down a general framework for determining quadrature points and weights, i.e., preselecting the Greville points as the quadrature points and then generating the quadrature weights by solving a moment fitting equation system. However, for a specific isogeometric Galerkin formulation, a proper integration accuracy is necessary to ensure that the resulting linear equation system is stable and accurate. By construction, the Greville quadrature rule can exactly integrate all B-spline basis functions $\{N_I\}_{I=1}^n$ adopted in (2). Therefore, one can easily control the quadrature accuracy by using specific B-spline bases to build the quadrature rule. In this section, we propose different B-spline bases to build quadrature rules for KL and RM shells.

Assuming the highest order of derivatives in the Galerkin formulation is k and the univariate B-spline basis along one of the parametric directions in the Galerkin formulation is $\{N_I^p\}_{I=1}^n$ with knot vector $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, the Greville quadrature rules for analysis should be constructed in a way such that all basis functions $\{N_I^p\}_{I=1}^n$ and their derivatives of order less than or equal to k are integrated exactly. In other words, (2) should be satisfied for all functions in $\left\{N_{I,m}^p \mid 1 \leq I \leq n, 0 \leq m \leq k\right\}$. Notice that these functions are equivalent to a set of new B-spline basis functions $\{\tilde{N}_I^p\}_{I=1}^{\tilde{n}}$, with knot vector $\tilde{\Xi}$ obtained by increasing the multiplicity of each interior knot of Ξ by k. It is preferable to use $\{\tilde{N}_I^p\}_{I=1}^{\tilde{n}}$ to build the Greville quadrature rules, because, in this way, we can avoid calculating the derivatives of the B-spline basis functions $\{N_I^p\}_{I=1}^n$ and the quadrature points are naturally the Greville quadrature points calculated from the knot vector $\tilde{\Xi}$. In what follows, we will use the notations

$$\mathcal{S}_0^p = \{N_I^p\}_{I=1}^n \quad \text{and} \quad \mathcal{S}_k^p = \{\tilde{N}_I^p\}_{I=1}^{\tilde{n}}, \quad k \in \{1, 2\}, \tag{3}$$

to indicate different B-spline bases.

According to the rules given above, for KL shells, the quadrature rule along one direction will be constructed with $\mathcal{S}_2^p$, and for RM shells, it will be constructed with $\mathcal{S}_1^p$. A two-dimensional quadrature rule is simply the tensor product of two one-dimensional quadrature rules as mentioned in Section 2.1. To distinguish these two quadrature rules for KL and RM shells, we will refer to them as GREVI-K and GREVI-R, respectively, hereafter. For a cubic B-spline basis with knot vector $\Xi = \{0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4\}$, the one-dimensional quadrature points and weights for GREVI-K and GREVI-R are illustrated in Figure 2. It is clear that the GREVI-R and GREVI-K rules, asymptotically, only involve two and three quadrature points in each parametric direction per element, respectively, regardless of the basis degrees. Consequently, only four and nine in-plane quadrature points are required for RM and KL shell elements.

Remarks.

1. The Greville quadrature weights are not always positive for an arbitrary knot vector. For example, if a knot interval of Ξ is extremely small compared to adjacent intervals, it is possible for the GREVI-K and GREVI-R quadrature rules to exhibit negative weights locally. Quadrature rules with negative weights are prone to instability and not preferred in engineering analysis. In this work we confine ourselves to uniform knot vectors. With uniform knot vectors we only see negative weights for the GREVI-K rule with $p = 4$. How to effectively remove the negative weights for arbitrary knot vectors is non-trivial and will be addressed in future work.

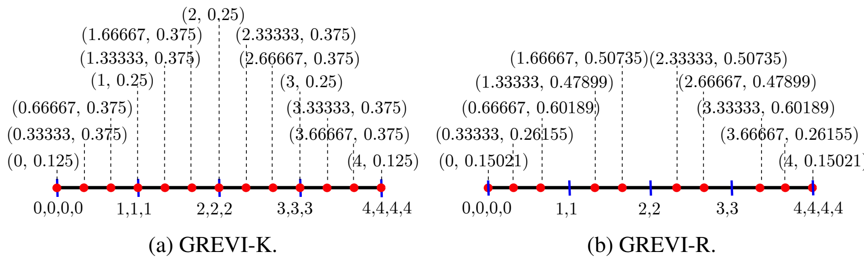


Fig. 2: Quadrature points and weights of GREVI-K and GREVI-R for a cubic B-spline basis with knot vector $\Xi = \{0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4\}$. Red dots denote the locations of the quadrature points, and $(\cdot, \cdot)$ indicates the global quadrature point and weight pair (x_i, w_i).

2. For $p = 2$, the multiplicities of the interior knots of the resulting knot vector $\tilde{\Xi}$ will be three for GREVI-K. Therefore, each element is an independent Bézier patch and the quadrature rule needs to be determined on the element level through (2). The resulting quadrature points will be distributed by the Simpson's rule, and unfortunately the two coincident quadrature points at the element interface can not be combined into one point due to the discontinuous second order derivatives [6]. As a result, the number of quadrature points will be the same as for full Gauss quadrature and thus we will not explore the case of $p = 2$ for GREVI-K further. □

2.3 Scordelis-Lo roof

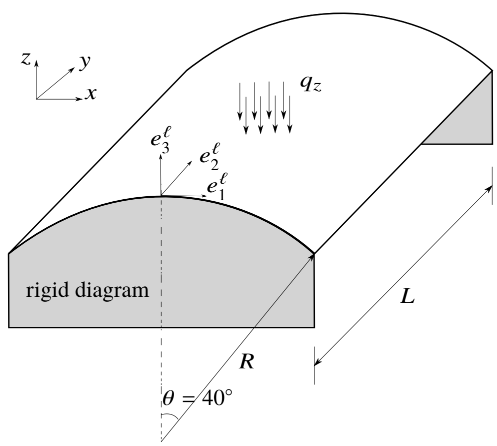
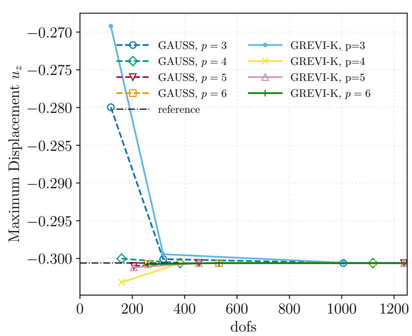


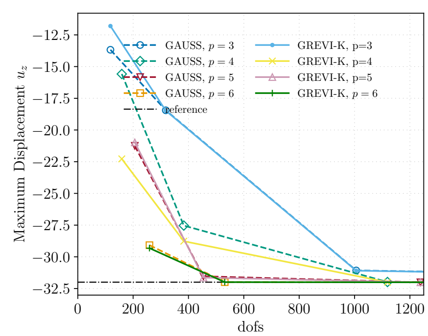
Fig. 3: Schematic for the Scordelis-Lo roof problem.

The Scordelis-Lo roof problem is part of the so-called shell obstacle course [7] and tests a shell element's ability to handle both membrane and bending modes. An 80° arc of a cylinder with radius $R = 25$, length $L = 50$, and thickness $t = 0.25$ or 0.025 is supported on each end by a rigid diaphragm. It is loaded with its own weight $q_z = 90$. The material has Young's Modulus, $E = 4.32 \times 10^8$, and Poisson's ratio $\nu = 0$. Figure 3 shows the problem setup. The initial mesh of the *whole* model consists of 4×4 maximally smooth elements.

The maximum displacement occurs on the free edge at $\frac{L}{2}$. For $t = 0.25$, the usual FEA solution converges to 0.3006 for KL shells [4] and 0.3024 for RM shells [2, 7]. For $t = 0.025$, the reference solution given in [8] is 32.0 for KL shells and we also take it as the reference solution for RM shells in this work. The maximum displacement on the free edge at $\frac{L}{2}$ is monitored and results for the KL shell are shown in Figure 4. For $p = 3$ and 4, $t = 0.25$, as shown in Figure 4a, the GREVI-K rule obtains slightly worse results than the GAUSS rule with the initial mesh, but with one refinement they almost achieve identical results that are close to the reference solution. As the degrees increase to $p = 5$ and 6, GREVI-K and GAUSS obtain nearly coincident results and, since locking is alleviated largely by higher-order bases, good results are achieved with even the initial mesh. As the shell thins, i.e., $t = 0.025$, membrane locking becomes more severe and the results converge more slowly for both quadrature rules as shown in Figure 4b. In this case, the GREVI-K rule achieves superior results for $p = 4$ while, for $p = 5$ and 6, the results obtained by GREVI-K and GAUSS hardly differ from each other.



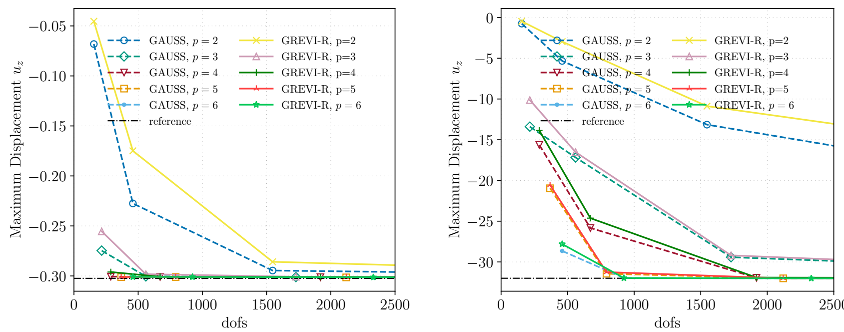
(a) t = 0.25.



(b) t = 0.025.

Fig. 4: Scordelis-Lo roof modeled as a KL shell: Convergence of the maximum displacement u_z with GAUSS and GREVI-K, degrees $p = 3$ to 6, and maximally smooth elements. The whole roof is modeled with an initial 4×4 mesh.

Since this problem is membrane dominated, transverse shear locking is not significant. Figure 5 demonstrates that for $p = 3$ to 6 the RM shell with GAUSS and GREVI-R converges in a similar way as the KL shell with GAUSS and GREVI-K shown in Figure 4. For $p = 2$, the GREVI-R rule underperforms the GAUSS rule but otherwise both rules perform about the same.



(a) t = 0.25. (b) t = 0.025.

Fig. 5: Scordelis-Lo roof modeled as an RM shell: Convergence of the maximum displacement u_z with GAUSS and GREVI-R, degrees $p = 2$ to 6, and maximally smooth elements. The whole roof is modeled with an initial 4×4 mesh.

3 Conclusions

We proposed Greville quadrature schemes for isogeometric shell analysis. The quadrature points are chosen to be Greville abscissae for B-splines and NURBS, but the spaces from which the rules emanate are unusual. The proposed method for Reissner-Mindlin (RM) shells, referred to as GREVI-R, is a Greville quadrature scheme based on p^{th}-order basis functions, but with one-order lower continuity across element interfaces than the p^{th}-order, maximally smooth, basis functions used for analysis. The proposed scheme for Kirchhoff-Love (KL) shells, referred to as GREVI-K, constructs the Greville quadrature points based on p^{th}-order basis functions, but with continuity two orders lower than the maximally smooth basis used for analysis. The quadrature weights are determined by solving linear moment fitting equations. These methods are free of rank deficiency and spurious modes. At the same time, they achieve similar accuracy as full Gauss quadrature. The proposed methods for RM and KL shell elements only involve four and nine in-plane quadrature points, respectively, per Bézier element, compared to the usual $(p+1)^2$ in-plane quadrature points for elements with standard “full” Gauss integration. As increasing the basis order does not asymptotically increase degrees-of-freedom or the number of quadrature points for higher-order basis functions, the proposed methods are efficient, robust, accurate and alleviate locking. For further details of the methodology and a comprehensive evaluation, please see our forthcoming paper [9].

4 Acknowledgments

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