Oden Institute REPORT 20-02

January 2020

Lecture Notes on Maxwell Equations in a Nutshell

by

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Reference: L. Demkowicz, "Lecture Notes on Maxwell Equations in a Nutshell," Oden Institute REPORT 20-02, Oden Institute for Computational Engineering and Sciences, The University of Texas at Austin, January 2020.

Lecture Notes on

MAXWELL EQUATIONS IN A NUTSHELL

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Abstract

We follow the historical path to walk through electrostatics and magnetostatics to Maxwell equations in three one hour lectures.

Key words: Maxwell equations, Partial Differential Equations

AMS subject classification: 65N30, 35L15

Acknowledgment

Patience of first year CSEM students for "drinking from a hydrant" is greatly appreciated.

1 Introduction

These notes represent three one hour lectures on electromagnetics and Maxwell equations that I have delivered for the first year graduate students in our Computational Engineering, Science and Mathematics (CSEM) program, within the class on Mathematical Modeling. Any decent graduate class on electromagnetics takes two semesters and, obviously, the three lectures cannot replace it. Nevertheless, I have attempted to accomplish the following goals:

- following the historical development, present the core of the intellectual effort that led to Maxwell equations;
- illuminate the analogy between electrostatics and magnetostatics concepts;
- emphasize the distributional character of charge and currents leading to the understanding of Maxwell equations in the distributional sense;
- formulate a couple of boundary-value problems corresponding to high school physics scenarios.

I hope that my attempt will not scare away the newcomers from a systematic study of this fascinating subject.

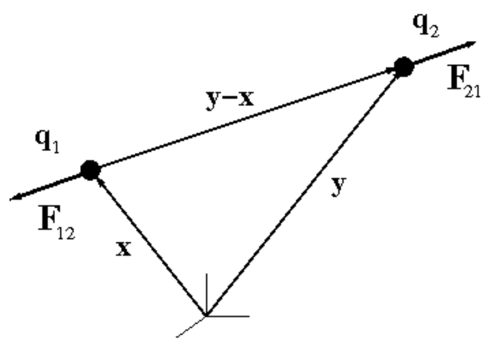

Figure 1: Coulomb's Law.

2 Electrostatics

Charges and Coulomb's Law. Almost a hundred years after Sir Isaac Newton published his *Law of Universal Gravitational Attraction* in *Philosophial Naturalis Principia Mathematica* [1687], in 1775, Charles Coulomb, a French engineer and colonel, came up with its full analogue that set up foundations for electrostatics. Corresponding to the concept of mass in Newton's mechanics, is the concept of *charge*. Like for mass, we can think of point, line, surface and volume charges. A common framework for the different charge distributions was provided by the *theory of distributions* developed almost two centuries later. Charge is measured in *Coulombs* [C] [1]. The smallest unit of charge is carried by a single electron and it is worth $1.6\ 10^{-19}$C.

Assume you have two point charges q_1, q_2 at positions $\boldsymbol{x}, \boldsymbol{y}$. The Coulomb's Law provides the formula for the force exerted on charge q_1 by charge q_2:

$$\boldsymbol{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|} = \frac{q_1 q_2}{4\pi\epsilon_0|\boldsymbol{x}-\boldsymbol{y}|^3}(\boldsymbol{x}-\boldsymbol{y}) \tag{2.1}$$

where ϵ_0 is the *permittivity of the free space*,

$$\epsilon_0 = 8.854\ 10^{-12} \approx \frac{1}{36\pi}10^{-9}[\frac{\mathrm{C}^2}{\mathrm{Nm}^2}]\,.$$

The force $\boldsymbol{F}_{21}$ exerted by charge q_1 on charge q_2 is opposite, $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$. Fig. 1 illustrates the law for two positive charges when the force is repulsive. The formula remains valid for charges of arbitrary sign. If the point charges are replaced with line, surface or volume charges, we use the superposition to compute the total force exerted by one group of charges on another one,

$$\boldsymbol{F}_{12} = \int\int \frac{q_1 q_2}{4\pi\epsilon_0|\boldsymbol{x}-\boldsymbol{y}|^3}(\boldsymbol{x}-\boldsymbol{y})\, dq_1\, dq_2\,. \tag{2.2}$$

The integral type depends upon the charge kind. For line charges we use line integrals, for surface charges we use surface integrals, for volume charges we use volume integrals. A unified framework is provided by the theory of distributions that leads to viewing the charge as a distribution.

[1] In SI, Coulomb is a derived unit, we will introduce it later.

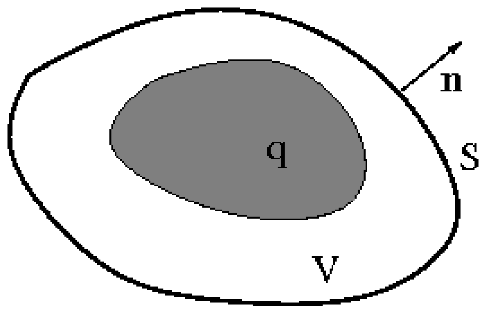

Figure 2: Gauss' Law for Electrostatics.

Assignment 1: Compute the total force exerted on point charge q by a line charge distribution with a constant charge density q_l.

Electric Field. Given a point *test charge* q, and a charge distribution Q, we define the *electric field* created by charge Q as [2]:

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q} \quad [\frac{\mathrm{N}}{\mathrm{C}} = \frac{\mathrm{V}}{\mathrm{m}}] \tag{2.3}$$

where $\boldsymbol{F}$ represents the total force exerted by charge Q on the test charge, given by the Coulomb's Law. The red color indicates equivalent units (in Volts per meter) that will be derived *a posteriori*.

Electric Flux is defined by the formula,

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} \quad [\frac{\mathrm{C}}{\mathrm{m}^2}] \,. \tag{2.4}$$

Assignment 2: Prove the *Gauss Law for Electrostatics*. Let S be an *arbitrary surface* surrounding a distributed volume charge ρ. Then

$$\int_S \boldsymbol{D} \cdot \boldsymbol{n} \, dS = Q := \int_V \rho \, dV \tag{2.5}$$

where $\boldsymbol{n}$ is the outward normal unit vector to surface S, see Fig. 2. Use integration by parts (the Gauss' Law) to derive the corresponding pointwise form of the law:

$$\mathrm{div} \boldsymbol{D} = \rho \,.$$

Note that the law remains valid for all kind of charges. The equation above has to be understood then in the distributional sense.

[2] Units printed in red color will be introduced later (derived units).

Electrostatic Potential. Let ρ be a distributed charge. We have,

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\, d\boldsymbol{y} \\
&= \frac{1}{4\pi\epsilon_0}\int \rho \left(-\nabla_{\boldsymbol{x}}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\right) d\boldsymbol{y} \\
&= -\nabla_{\boldsymbol{x}} \underbrace{\frac{1}{4\pi\epsilon_0}\int \frac{\rho}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\boldsymbol{y}}_{\text{electrostatic potential } V} = -\nabla_{\boldsymbol{x}} V\,.
\end{aligned} \tag{2.6}$$

The concept of the electrostatic potential is in line with the concept of a conservative (potential) force in mechanics. Let AB denote an arbitrary curve from point A to point B, parametrized with

$$\boldsymbol{r} = \boldsymbol{r}(\xi),\ \xi \in (a,b)\,.$$

The work done by electric field $\boldsymbol{E}$ on (a particle with) a unit charge is equal to the difference of potentials, and it is *independent* of the path.

$$\begin{aligned}
\int_{AB} \boldsymbol{E}\cdot d\boldsymbol{r} &= \int_a^b \left(\boldsymbol{E}\cdot\frac{d\boldsymbol{r}}{d\xi}\right) d\xi \\
&= -\int_a^b \left(\frac{\partial V}{\partial x}\frac{dx}{d\xi} + \frac{\partial V}{\partial y}\frac{dy}{d\xi} + \frac{\partial V}{\partial z}\frac{dz}{d\xi}\right) d\xi \\
&= -\int_a^b \frac{d}{d\xi}[V(x(\xi),y(\xi),z(\xi))]\, d\xi \\
&= V(\boldsymbol{r}(a)) - V(\boldsymbol{r}(b)) = V(A) - V(B)\,.
\end{aligned}$$

The value of potential at A, called the *voltage* at A, represents the work done *against* the field $\boldsymbol{E}$ to bring a unit charge from infinity to point A, and it is measured in *Volts*,

$$\mathrm{V} = \frac{\mathrm{N}}{\mathrm{C}}\mathrm{m}\,.$$

which implies the alternative, derived unit V/m for the electric field.

Free and Bound Charge. Conservation of Free Charge. So far we have talked just about charge. In practice we distinguish between the *free charge* representing a cloud of free, moving electrons, and the *bound charge* that represents electrons that cannot change their location. From now on, we shall use symbol ρ for the density of the free charge only, and denote the density of bound charge by ρ^b. The motion of free charge is subjected to a standard conservation law. Let V denote an arbitrary (control) volume. We have,

$$\frac{d}{dt}\int_V \rho\, dV + \int_{\partial V} J_n\, dS = 0 \tag{2.7}$$

where J_n stands for a *flux of free charge* across boundary ∂V. Cauchy's argument (see the corresponding derivation of stress tensor from the postulated existence of stress vector) leads to the conclusion that there exists a vector field $\boldsymbol{J}$ such that $J_n = \boldsymbol{J} \cdot \boldsymbol{n}$ where $\boldsymbol{n}$ denotes the outward normal unit to ∂V. Vector field $\boldsymbol{J}$ is called the *current density vector*, and it is measured in another unit - the Ampere.

$$\boldsymbol{J} \quad [\underbrace{\frac{\mathrm{C}}{\mathrm{s}}}_{\mathrm{A}} \frac{1}{\mathrm{m}^2} = \frac{\mathrm{A}}{\mathrm{m}^2}]$$

In the SI system, the Ampere is selected to be the fundamental unit, and both Coulomb and Volt are represented in terms of it. Integration by parts leads to the conservation of charge in a differential form - the so called *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \mathrm{div} \boldsymbol{J} = 0 \tag{2.8}$$

The equation is again to be understood in the distributional sense.

Ohm's Law. This is our first constitutive law. For a large class of *conductors*, the current density vector is proportional to the electric field,

$$\boldsymbol{J} = \sigma \boldsymbol{E}, \quad \sigma \left[\frac{\mathrm{A}}{\mathrm{m}^2} \frac{\mathrm{m}}{\mathrm{V}} = \underbrace{\frac{\mathrm{A}}{\mathrm{V}}}_{S} \frac{1}{\mathrm{m}} = \frac{\mathrm{S}}{\mathrm{m}} \right] \tag{2.9}$$

The conductivity σ is measured in another derived unit - the siemen [S]. For non-homogeneous materials, $\sigma = \sigma(x)$, for more complicated, anisotropic materials, it is replaced with a tensor.

Electric Dipole. Imagine a negative unit charge $-q$ located at the origin of a Cartesian system and a positive unit charge q located at a point $d\boldsymbol{e}$ where $\boldsymbol{e}$ is a unit vector. Compute the scalar potential corresponding to the two charges and pass to a limit with $d \to 0$ keeping the product $p = qd$ fixed (then, obviously, $q \to \infty$).

$$\begin{aligned} V(\boldsymbol{r}) &= \lim_{d \to 0} \left(\frac{q}{4\pi\epsilon_0 |\boldsymbol{r} - d\boldsymbol{e}|} - \frac{q}{4\pi\epsilon_0 |\boldsymbol{r}|} \right) \\ &= \frac{p}{4\pi\epsilon_0} \lim_{d \to 0} \frac{1}{d} \left(\frac{1}{|\boldsymbol{r} - d\boldsymbol{e}|} - \frac{1}{|\boldsymbol{r}|} \right) \\ &= -\frac{1}{4\pi\epsilon_0} \boldsymbol{p} \cdot \nabla \left(\frac{1}{r} \right) = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{|\boldsymbol{r}|^3} \end{aligned}$$

where $\boldsymbol{p} = p\boldsymbol{e}$ is identified as the *dipole moment*.

Polarization. Dielectrics. Electrons and protons form miniature dipoles that, without any electric field present, are directed randomly. Under the action of an electric field $\boldsymbol{E}$, the dipoles get "organized" (see cartoon 3) forming a *polarization vector* field with density

$$\boldsymbol{P} = \frac{\text{dipole moment}}{\text{unit volume}} .$$

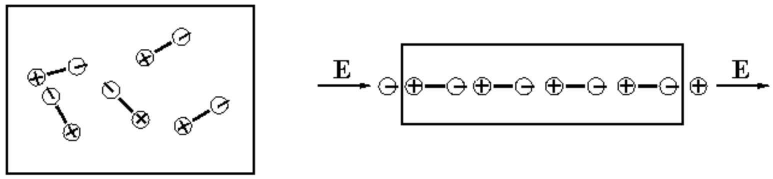

Figure 3: Electric polarization.

The process is known as *polarization*, and it characterizes *dielectrics*. The constitutive law for dielectrics states that the polarization vector is proportional to the electric field,

$$\boldsymbol{P} = \chi_e \epsilon_0 \boldsymbol{E} \tag{2.10}$$

where χ_e is the *electric susceptibility*. Note that

$$\nabla_x \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} = -\nabla_y \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,.$$

Integration by parts,

$$\begin{aligned} V(\boldsymbol{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \boldsymbol{P}(\boldsymbol{y})\cdot(-\nabla_x \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|})\,d\boldsymbol{y} \\ &= \frac{1}{4\pi\epsilon_0}\int_V \boldsymbol{P}(\boldsymbol{y})\cdot(\nabla_y \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|})\,d\boldsymbol{y} \\ &= \frac{1}{4\pi\epsilon_0}\int_V (-\mathrm{div}\boldsymbol{P}(\boldsymbol{y}))\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}d\boldsymbol{y} + \frac{1}{4\pi\epsilon_0}\int_{\partial V}(\boldsymbol{P}\cdot\boldsymbol{n})\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}dS\,, \end{aligned}$$

leads to the observation that the field corresponding to a dipole field is equivalent to a field created by the *bound charge* $\rho_b = -\mathrm{div}\boldsymbol{P}$. The boundary term with the surface charge distribution is "absorbed" in ρ_b understood in the distributional sense.

Gauss Law for Dielectrics. Utilizing the relation between the polarization vector $\boldsymbol{P}$ and bound charge ρ_b, and the Gauss Law (for the free space) we obtain,

$$\mathrm{div}(\epsilon_0 \boldsymbol{E}) = \rho + \rho_b = \rho - \mathrm{div}\boldsymbol{P}\,.$$

This leads to an update of the Gauss Law for general dielectric materials,

$$\mathrm{div}(\epsilon_0\boldsymbol{E} + \boldsymbol{P}) = \mathrm{div}(\epsilon_0(1+\chi_e)\boldsymbol{E}) = \mathrm{div}(\epsilon\boldsymbol{E}) = \rho\,. \tag{2.11}$$

Coefficient $\epsilon_r = 1 + \chi_e$ is known as the *relative permittivity*, $\epsilon = \epsilon_r\epsilon_0$ is the *permittivity*, and $\boldsymbol{D} = \epsilon\boldsymbol{E}$ is the *electric flux* $\boldsymbol{D}$.

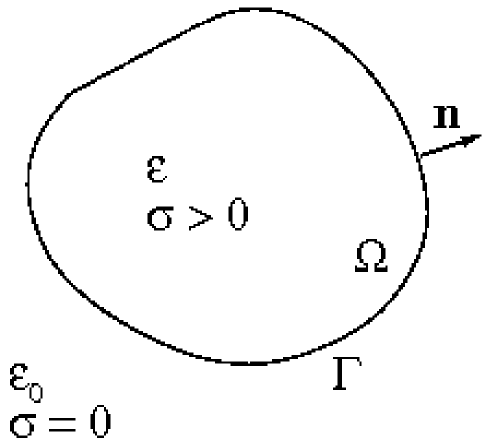

Figure 4: Example of an electrostatics BVP.

Example of a Boundary-Value Problem (BVP). A free charge Q has been put into a conductor occupying a bounded domain Ω (see Fig. 4) surrounded by the free space. Determine potential V, electric field $\boldsymbol{E}$, and the resulting distribution of free charge ρ and bound charge ρ_b. Assume that all fields have reached a steady state, i.e. they are independent of time.

Continuity equation (conservation of charge) understood in the distributional sense provides a BVP for the potential inside of the domain Ω.

$$\nabla \cdot \boldsymbol{J} = \nabla \cdot (\sigma \boldsymbol{E}) = -\nabla \cdot (\sigma \nabla V) = 0 \text{ in } \Omega$$

and

$$\boldsymbol{J} \cdot \boldsymbol{n} = -\sigma \nabla V \cdot \boldsymbol{n} = -\sigma \frac{\partial V}{\partial n} = 0 \text{ on } \Gamma\,.$$

The solution is a constant potential $V = V_0$ and a zero electric field $\boldsymbol{E} = \boldsymbol{0}$ inside of the conductor. The Gauss' Law implies that the free charge inside of the conductor is zero.

Energy considerations lead to a regularity assumption on potential V that is assumed to be continuous. Consider an *exterior* BVP for the potential V in the free space implied by the Gauss' Law.

$$\begin{cases} -\nabla \cdot (\epsilon_0 \nabla V) = 0 & \text{in } \mathbb{R}^3 - \bar{\Omega} \\ V = 1 & \text{on } \Gamma \\ V = 0 & \text{at } \infty\,. \end{cases}$$

The problem is well posed and it has a unique solution. Linearity of the problem implies that the actual potential is equal to the product of the unknown constant potential V_0 inside of the conductor and solution V. The distributional understanding of the Gauss' law provides a formula for the surface free charge,

$$(\rho^S =)\rho = \epsilon_0 \boldsymbol{E}^{out} \cdot \boldsymbol{n} - \underbrace{\epsilon \boldsymbol{E}^{in} \cdot \boldsymbol{n}}_{=0} = \epsilon_0 \boldsymbol{E}^{out} \cdot \boldsymbol{n} = -\epsilon_0 V_0 \frac{\partial V}{\partial n}$$

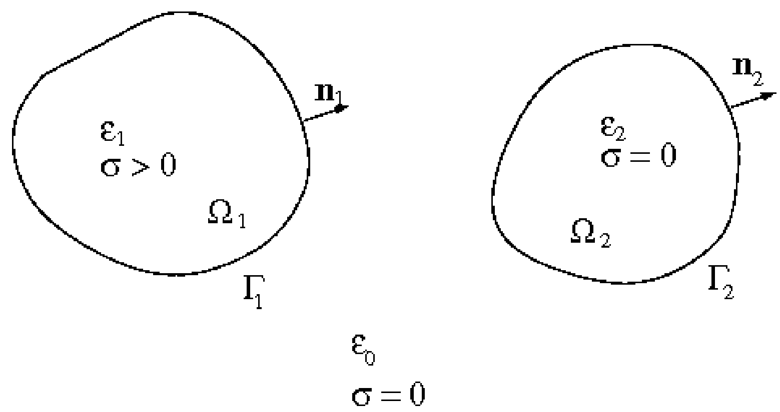

Figure 5: An electrostatics BVP.

where V denotes the solution of the exterior BVP. The total free surface charge must equal the impressed charge brought to the conductor from outside,

$$-\epsilon_0 V_0 \int_\Gamma \frac{\partial V}{\partial n} = Q$$

which provides the closing equation for the unknown potential V_0. The bound charge can be computed from the Gauss' Law for the free space understood in the distributional sense. Equation

$$\rho + \rho_b = \mathrm{div}(\epsilon_0 \boldsymbol{E})$$

must be satisfied in both interior and exterior domains and it implies that $\rho_b = 0$ is zero there. On boundary Γ we must have,

$$\rho + \rho_b = \epsilon_0 (\boldsymbol{E}^{out} - \boldsymbol{E}^{in}) \cdot \boldsymbol{n} = 0$$

which implies that there is no surface bound charge, too.

Assignment 3: Discuss a slightly more complicated scenario shown in Fig. 5. Determine the distribution of free and bound charge. Use the example to explain why a charged amber rod attracts pieces of paper even though there is no (impressed) charge brought to the paper.

Relaxation Time for Conductors. The conservation of charge and Gauss' Electric Law remain valid for general Maxwell equations in a dynamic regime. Combining equations (2.8) and (2.11), we obtain an ordinary differential equation for free charge density ρ,

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = 0\,.$$

Given an initial condition ρ_0, we obtain,

$$\rho = \rho_0 e^{-\frac{t}{\tau}}$$

where $\tau = \epsilon/\sigma$ is the *relaxation time* needed for the charge density to decrease by a factor of $e^{-1} \approx 0.37$. For copper, $\tau = 1.510^{-19}$ s, but for quartz $\tau \approx 10$ days.

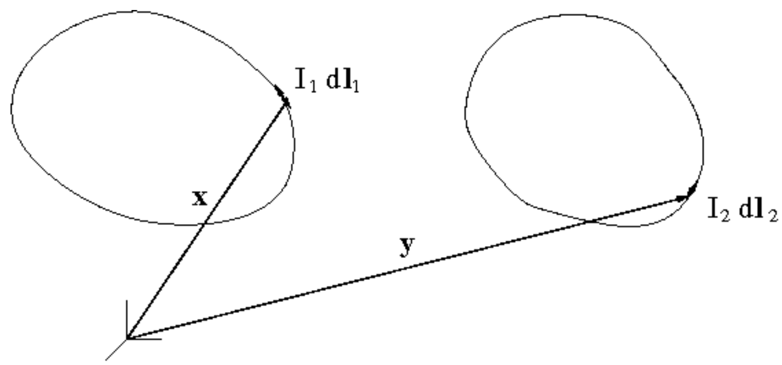

Figure 6: Ampère's Force Law

3 Magnetostatics

In electrostatics charges are stationary and there are no currents. In magnetostatics currents are steady, i.e. constant in time. Essential in understanding the magnetostatics is the concept of *current element* $I d\boldsymbol{l}$ that corresponds to the concept of charge in electrostatics. Contrary to charge though, the current element must be part of a line, surface or volume current and cannot be understood pointwise.

Ampère's Force Law (1820). Consider two current elements illustrated in Fig. 6. The force exerted on current element $I_1 d\boldsymbol{x}$ by current element $I_2 d\boldsymbol{y}$ is given by the famous law discovered by André-Marie Ampère,

$$d\boldsymbol{F}_{12} = \frac{\mu_0}{4\pi} \frac{I_1 d\boldsymbol{x} \times \left(I_2 d\boldsymbol{y} \times \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|} \right)}{|\boldsymbol{x}-\boldsymbol{y}|^2} \tag{3.12}$$

where μ_0 is the *free space permeability*,

$$\mu_0 = 4\pi 10^{-7} \quad [\frac{\mathrm{N}}{\mathrm{A}^2} = \frac{\mathrm{h}}{\mathrm{m}}] \,.$$

Notice the full analogy with the Coulomb's Law. The total force exerted on current loop 1 by current loop 2 is:

$$\boldsymbol{F}_{12} \quad = \int I_1 d\boldsymbol{x} \times \underbrace{\mu_0 \int \frac{I_2 d\boldsymbol{y} \times (\boldsymbol{x}-\boldsymbol{y})}{4\pi |\boldsymbol{x}-\boldsymbol{y}|^3}}_{\text{magnetic flux } \boldsymbol{B}} \,.$$

The justification of defining the magnetic flux in this way comes from experiments that confirm an identical behavior of a current loop being placed in a field created by a permanent magnet. In other words, the effect of a current loop 2 on current loop 1 is the same as the effect of the magnet.

Assignment 4: Check that the total force $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$ but demonstrate by means of a counterexample that, in general, $d\boldsymbol{F}_{12} \neq -d\boldsymbol{F}_{21}$. Follow the steps:

1. Recall (or prove) the fundamental vector identity,

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{A}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}\,. \tag{3.13}$$

2. Use the identity to split the Ampère's force into two parts,

$$(I_1 d\boldsymbol{x} \cdot \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}) I_2 d\boldsymbol{y} - (I_1 d\boldsymbol{x} \cdot I_2 d\boldsymbol{y}) \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,.$$

3. Use the fact that

$$\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} = -\nabla_{\boldsymbol{x}} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}$$

and the Stokes' Theorem to show that the integral of the first term vanishes.

Assignment 5: Consider two infinitely long straight conductors (cables) shown in Fig. 7. Use the Ampère's force law

- to show that $d\boldsymbol{F}_{12}$, the force exerted by current I_2 on current I_1 is horizontal (the only non-zero component is the y_2 component, and it is positive),
- to compute the force *per unit length* of conductor carrying I_1,

$$\frac{\mu_0}{2\pi b} I_1 I_2\,. \tag{3.14}$$

Hint: Integrate first in y_2 and then switch to θ shown in the figure. Notice that the force is attractive. What will happen if one (both) of the currents changes direction? Notice that the force exterted by cable one on cable two matches that exterted by cable two on cable one even though the two cables *do not form close loops*.

Biot-Savart Law (Superposition of Steady Currents). Very soon after the discovery of Ampère, French mathematicians, Jean-Baptiste Biot and Félix Savart generalized the Ampère Law to arbitrary steady currents. In the following formula, $\boldsymbol{J}(\boldsymbol{y})$ may be a line, surface or volume current with the corresponding line, surface or volume integral. In other words, similarly to charges, currents are assumed to be distributions. The general formula for the magnetic flux reads as follows.

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{x}) &= \mu_0 \int \frac{\boldsymbol{J}(\boldsymbol{y}) \times (\boldsymbol{x}-\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}\, d\boldsymbol{y} \\ &= -\mu_0 \int \boldsymbol{J}(\boldsymbol{y}) \times \nabla_x \left(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\right) d\boldsymbol{y} \\ &= -\nabla_x \times \underbrace{\mu_0 \int \frac{\boldsymbol{J}(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\, d\boldsymbol{y}}_{\text{vector potential } \boldsymbol{A}}\,. \end{aligned} \tag{3.15}$$

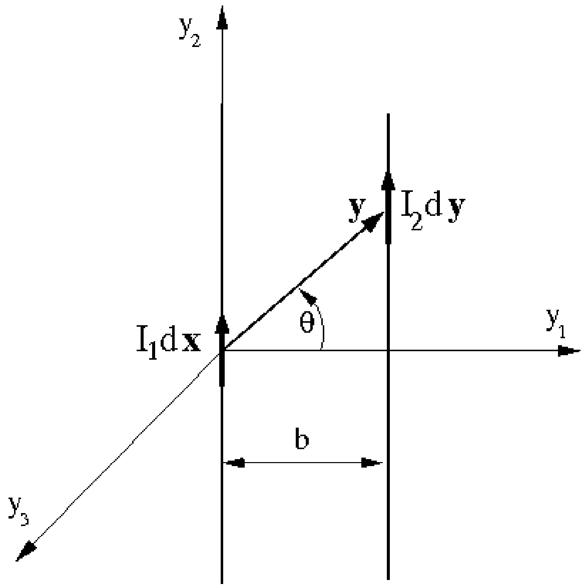

Figure 7: Parallel conductors

Integration by parts,

$$
\begin{aligned}
-\int_V \boldsymbol{J}(\boldsymbol{y}) \times \nabla_x \left(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right) \, d\boldsymbol{y} &= \int_V \boldsymbol{J}(\boldsymbol{y}) \times \nabla_y \left(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right) \, d\boldsymbol{y} \\
&= \int_V (\nabla \times \boldsymbol{J})(\boldsymbol{y}) \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} + \int_{\partial V} (\boldsymbol{n} \times \boldsymbol{J}) \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \, dS \,,
\end{aligned}
$$

leads to the observation that only currents with $\mathrm{curl}\boldsymbol{E} \neq \mathbf{0}$ (in the distributional sense) produce a magnetic flux.

Finally, another integration by parts establishes that the divergence of vector potential is zero.

$$
\begin{aligned}
(\nabla_x \cdot \boldsymbol{A})(\boldsymbol{x}) &= \mu_0 \int_V \boldsymbol{J}(\boldsymbol{y}) \cdot \nabla_x \left(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right) \, d\boldsymbol{y} = -\mu_0 \int_V \boldsymbol{J}(\boldsymbol{y}) \cdot \nabla_y \left(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right) \, d\boldsymbol{y} \\
&= \mu_0 \int_V (\nabla_y \cdot \boldsymbol{J}) \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \, dV + \mu_0 \int_{\partial V} (\boldsymbol{J} \cdot \boldsymbol{n}) \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \, dS = 0
\end{aligned}
\tag{3.16}
$$

since, for steady currents, $\frac{\partial \rho}{\partial t} = 0$.

Ampère's Law for Magnetostatics. We are ready now to derive a differential relation between the magnetic flux and currents.

$$\begin{aligned}(\nabla \times \boldsymbol{B})(\boldsymbol{x}) &= -\nabla \times \nabla \times \boldsymbol{A} \\ &= -\Delta \boldsymbol{A} + \nabla \underbrace{(\nabla \cdot \boldsymbol{A})}_{=0} \\ &= -\Delta_x \left(\mu_0 \int \frac{\boldsymbol{J}(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \, d\boldsymbol{y} \right) \\ &= \mu_0 \int \boldsymbol{J}(\boldsymbol{y}) \underbrace{\left(-\Delta_x \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right)}_{\delta(\boldsymbol{y}-\boldsymbol{x})} \, d\boldsymbol{y} \\ &= \mu_0 \boldsymbol{J}(\boldsymbol{x}) \, .\end{aligned}$$

The resulting Ampère's equation for magnetostatics,

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \, , \tag{3.17}$$

is to be understood again in the distributional sense.

Assignment 6: Consider scenario depicted in Fig. 8. A horizontal circular loop of radius a with center at the origin is carrying current I. Use the spherical coordinates to show that the magnetic vector potential at a point $\boldsymbol{x}$ is given by the formula:

$$\boldsymbol{A} = \frac{\mu_0 I a^2 \sin\psi}{4|\boldsymbol{x}|^2} \boldsymbol{e}_\theta + \text{ higher order terms in } a \tag{3.18}$$

where $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\psi$ denote the unit vectors of spherical system of coordinates.

Magnetic Dipole. Introducing a vector $\boldsymbol{M} = \pi a^2 I \boldsymbol{e}_z$ and passing to the limit with $a \to 0$ keeping $M = \pi a^2 I$ constant, we obtain a *magnetic dipole*. Consistently with (3.18), the magnetic vector potential of the magnetic dipole is given by the formula,

$$\boldsymbol{A} = \mu_0 \boldsymbol{M} \times \nabla_x \left(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right) \, . \tag{3.19}$$

Magnetic Polarization. Analogously to electric polarization vector $\boldsymbol{P}$, we introduce now a *magnetic polarization vector* $\boldsymbol{M}$, representing a density of magnetic dipoles per unit volume. Integration by parts,

$$\begin{aligned}\boldsymbol{A} &= \mu_0 \int_V \boldsymbol{M}(\boldsymbol{y}) \times \nabla_x \left(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right) \, dV \\ &= -\mu_0 \int_V \boldsymbol{M}(\boldsymbol{y}) \times \nabla_y \left(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right) \, dV \\ &= -\mu_0 \int_V (\nabla \times \boldsymbol{M}) \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \, dV - \mu_0 \int_{\partial V} (\boldsymbol{n} \times \boldsymbol{M}) \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \, dS \, ,\end{aligned}$$

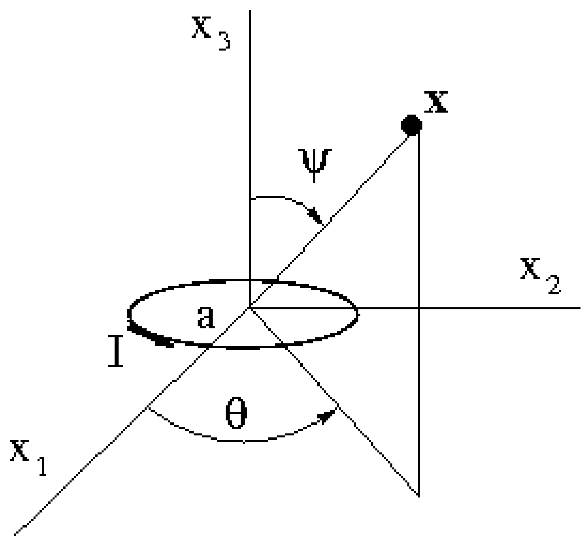

Figure 8: A current loop.

leads to the observation that an equivalent magnetic potential is generated by the so-called *bound current* $\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$. The equivalent bound currents are again understood in the distributional sense, as they include surface currents as well.

Magnetic Field. Separating currents into free currents $\boldsymbol{J}$ and bound currents $\boldsymbol{J}_b$, and recalling the Ampère's Law,

$$\frac{1}{\mu_0}\nabla \times \boldsymbol{B} = \boldsymbol{J} + \boldsymbol{J}_b\,,$$

we introduce the notion of *magnetic field*,

$$\nabla \times (\underbrace{\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}}_{\text{magnetic field } \boldsymbol{H}}) = \boldsymbol{J}\,. \tag{3.20}$$

Consequently,

$$\begin{aligned} \boldsymbol{B} &= \mu_0 \boldsymbol{H} + \boldsymbol{M} \\ &= \mu_0 \boldsymbol{H} + \mu_0 \chi_m \boldsymbol{H} \\ &= \mu_0 (1 + \chi_0) \boldsymbol{H} = \mu \boldsymbol{H} \end{aligned}$$

where χ_m denotes the *magnetic susceptibility*, $\mu_r = 1 + \chi_m$ is the *relative permeability* and $\mu = \mu_0 \mu_r$ is the permeability of a specific material.

With the newly introduced constitutive laws and the concept of magnetic field, the Ampère's Law for magnetostatics takes the form,

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}\,. \tag{3.21}$$

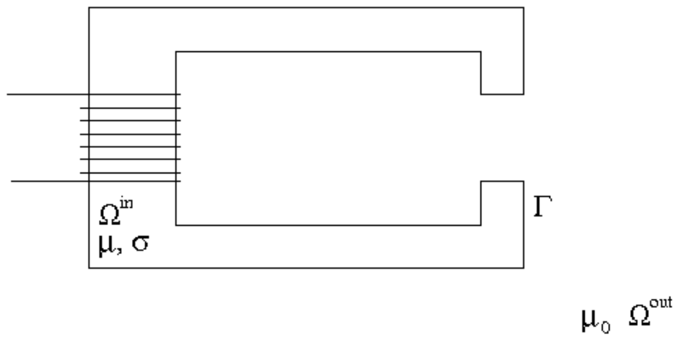

Figure 9: An electromagnet.

Example 3: Consider scenario shown in Fig. 9. Given an impressed free surface current $\boldsymbol{J}^{imp}$ representing the coil, determine magnetic field $\boldsymbol{H}$, vector potential $\boldsymbol{A}$, magnetic flux $\boldsymbol{B}$, and bound currents $\boldsymbol{J}_b$.

The magnetostatics problem has to be solved in the whole space. The starting point is provided by the Ampère's Law (3.21). The equation has to be understood in the sense of distributions. With the free current reduced to the impressed current only[3] , $\boldsymbol{J} = \boldsymbol{J}^{imp}$, this implies that $\nabla \times \boldsymbol{H} = \boldsymbol{0}$ in both Ω, Ω^e and the jump condition,

$$[\boldsymbol{n} \times \boldsymbol{H}] = \boldsymbol{J}^{imp} \quad \text{on } \Gamma \,. \tag{3.22}$$

Notice that the prescribed impressed current must be tangent to the surface. Similarly as electrostatics problems are naturally formulated in terms of the scalar potential (voltage), it is natural to formulate the magnetostatics problems in terms of the vector potential $\boldsymbol{A}$. We have,

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} = -\frac{1}{\mu}\nabla \times \boldsymbol{A} \,.$$

Equation (3.16) provides a *gauge condition* on $\boldsymbol{A}$ making it unique. In the end, we obtain the following

[3]OK, this is tricky. The impressed current, representing a stationary movement of charges in a coil, according to the Coloumb law, does produce an electric field $\boldsymbol{E}$. In the free space however, $\sigma = 0$, so $\boldsymbol{J} = \sigma\boldsymbol{E} = \boldsymbol{0}$. In the electromagnet, $\sigma > 0$ and, according to our discussion in Example 1, the electric field *inside* of the conductor vanishes, so $\boldsymbol{J} = \boldsymbol{0}$ there as well. Can you explain why there is no free surface current on Γ as well?

Boundary-Value Problem for the magnetic potential $\boldsymbol{A}$.

$$\left\{\begin{array}{rll}\nabla\times(\frac{1}{\mu}\nabla\times\boldsymbol{A}^{in}) & =\boldsymbol{0} & \text{in } \Omega \\ \nabla\times(\frac{1}{\mu_0}\nabla\times\boldsymbol{A}^{out}) & =\boldsymbol{0} & \text{in } \Omega^c \\ \boldsymbol{n}\times(\boldsymbol{A}^{out}-\boldsymbol{A}^{in}) & =\boldsymbol{0} & \text{on } \Gamma \\ \boldsymbol{n}\times(\frac{1}{\mu_0}\nabla\times\boldsymbol{A}^{out}-\frac{1}{\mu}\nabla\times\boldsymbol{A}^{in}) & =-\boldsymbol{J}^{imp} & \text{on } \Gamma \\ \operatorname{div}\boldsymbol{A} & =0 & \text{in } \Omega\cup\Omega^c \\ \boldsymbol{n}\cdot(\boldsymbol{A}^{out}-\boldsymbol{A}^{in}) & =0 & \text{on } \Gamma \\ \boldsymbol{A} & =\boldsymbol{0} & \text{at } \infty\end{array}\right. \tag{3.23}$$

Condition $(3.23)_3$ reflects the assumption that magnetic field is a function (a regular distribution). Conditions $(3.23)_5$,$(3.23)_6$ reflect the fact that gauge condition is understood in the distributional sense. Notice that conditions $(3.23)_3$ and $(3.23)_6$ imply that the vector potential is globally continuous, consistently with its physical interpretation.

The curl-curl problem is typical for magnetostatics and Maxwell's equations in general. One can prove that the problem above has a unique solution. Once the magnetic vector potential is known, we can compute the corresponding magnetic flux $\boldsymbol{B}=\nabla\times\boldsymbol{A}$, magnetic field $\boldsymbol{H}=\boldsymbol{B}/\mu$, and the bound currents $\boldsymbol{J}_b=(\nabla\times\boldsymbol{B})/\mu_0-\boldsymbol{J}^{imp}$.

4 Maxwell Equations

Faraday's Law (1831). A decade after the discovery of Ampère, English physicist - Michael Faraday, discovered that a *time varying* magnetic flux will induce a current in a loop placed in the field. The discovery led to a new relation between the magnetic flux vector and electric field, see Fig. 10,

$$\int_c \boldsymbol{E}\cdot d\boldsymbol{l}=-\frac{d}{dt}\int_S \boldsymbol{B}\cdot\boldsymbol{n}\,dS\,. \tag{4.24}$$

Application of Stoke's Theorem leads to the differential form of the Faraday's law,

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}\,.$$

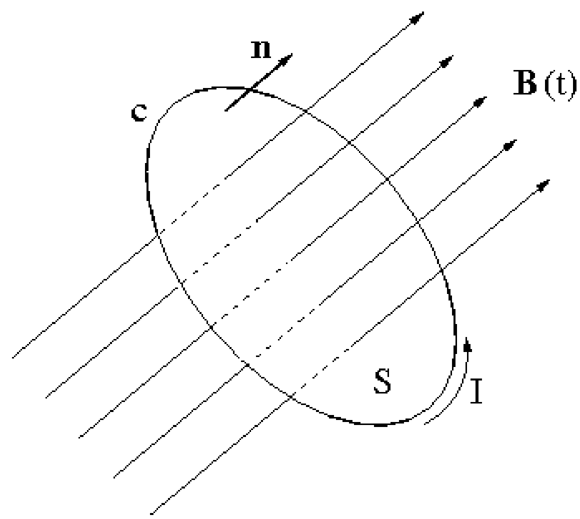

Figure 10: Faraday's Law.

Maxwell's Equations (1856). Another quarter of century later, James Clerk Maxwell, a Scottish mathematician and theoretical physicist, upgraded the Ampère's equation [4] to its transient version:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \underbrace{\frac{\partial \boldsymbol{D}}{\partial t}}_{\text{missing term}}$$

and formulated the final system of what we call today *Maxwell Equations*.

$$\left\{\begin{array}{rll} \nabla \times \boldsymbol{E} & = -\dfrac{\partial}{\partial t}(\mu \boldsymbol{H}) & \text{Faraday's Law} \\ \nabla \times \boldsymbol{H} & = \boldsymbol{J}^{imp} + \underbrace{\sigma \boldsymbol{E}}_{\boldsymbol{J}} + \dfrac{\partial}{\partial t}(\epsilon \boldsymbol{E}) & \text{Ampère's Law} \\ \nabla \cdot (\mu \boldsymbol{H}) & = 0 & \text{Gauss' Magnetic Law} \\ \nabla \cdot (\epsilon \boldsymbol{E}) & = \rho^{imp} + \rho & \text{Gauss' Electric Law} \\ \dfrac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} & = 0 & \text{Continuity Equation} \end{array}\right. \tag{4.25}$$

The system is accompanied by appropriate boundary and initial conditions, and it is solved for the electric field $\boldsymbol{E}$, magnetic field $\boldsymbol{H}$, and density of free charge ρ. The impressed current and charge are given, and must satisfy the continuity equation as well. The equations are non independent. In order to see it, consider

[4]More precisely, Ampère- Maxwell Equation.

the time-harmonic version of the equations,

$$
\left\{
\begin{array}{rll}
\nabla \times \boldsymbol{E} & = -i\omega(\mu \boldsymbol{H}) & \text{Faraday's Law} \\
\nabla \times \boldsymbol{H} & = \boldsymbol{J}^{imp} + \underbrace{\sigma \boldsymbol{E}}_{\boldsymbol{J}} + i\omega(\epsilon \boldsymbol{E}) & \text{Ampère's Law} \\
\nabla \cdot (\mu \boldsymbol{H}) & = 0 & \text{Gauss' Magnetic Law} \\
\nabla \cdot (\epsilon \boldsymbol{E}) & = \rho^{imp} + \rho & \text{Gauss' Electric Law} \\
i\omega\rho + \nabla \cdot \boldsymbol{J} & = 0 & \text{Continuity Equation.}
\end{array}
\right.
\tag{4.26}
$$

Taking divergence of both sides in the Faraday's Law, we obtain the the Gauss' Magnetic Law (premultiplied by $i\omega$ factor). Similarly, taking divergence of both sides of the Ampère's Law, and utilizing the continuity equation, we obtain the Gauss' Electric Law (again premultiplied by $i\omega$ factor). Obeying this dependence is critical in establishing stable discretization schemes, see [1]. Notice that this interdependence disappears in the static case, where the Faraday and Ampère equations decouple from each other. The Gauss' Laws and the conservation of charge provide then the necessary closing equations. Understanding the degeneration of Maxwell equations when $\omega \to 0$, is critical in modeling electromagnetics problems.

Assignment 7 (Plane wave): Consider the time-harmonic Maxwell equations in the free space and assume that the electric field depends only upon a single coordinate, say x_1. Use the Maxwell equations to derive the closed form solution (a plane wave) of the Maxwell equations. Observe that, at any point in the space, $\boldsymbol{E} \cdot \boldsymbol{H} = 0$. Note that, contrary to what you might have learned in high school, this property is not true in general, e.g. for spherical waves (waves that depend only upon spherical coordinate r).

5 Summary

A quote from Wikepedia: " In 1931, on the centennial of Maxwell's birthday, Einstein himself described Maxwell's work as the 'most profound and the most fruitful that physics has experienced since the time of Newton.' Einstein kept a photograph of Maxwell on his study wall, alongside pictures of Michael Faraday and Isaac Newton."

Understanding the development of electromagnetics that took time from Coulomb, through Ampère and Faraday to Maxwell, is critical for understanding of science in general. The subject is not easy, it requires a combination of strong math skills and sound foundations in physics. The discussed, concise from of the Maxwell equations is due to Oliver Heaviside who published it in 1884, recasting original Maxwell's mathematical analysis form of twenty equations with twenty unknowns. One might say that it took 28 years for the contemporary scientists to understand the Maxwell's work.

I have not provided in this note any examples of BVPs for the full set of Maxwell equations. If you are interested, see e.g. [6, 4, 5, 2, 3].

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6 Appendix: Distributions

Distributions are *functionals* that act on test functions. Given a domain $\Omega \subset \mathbb{R}^3$, we introduce the space of test fumctions,

$$C_0^\infty(\Omega) := \{\phi \in C^\infty(\Omega) \ : \ \text{supp}\phi \text{ is bounded and contained in } \Omega\} \tag{6.27}$$

where supp ϕ, denoting the *support* of function ϕ, is obtained by closing the set over which function ϕ does not vanish,

$$\text{supp } \phi = \overline{\{\boldsymbol{x} \in \Omega \ : \ \phi(\boldsymbol{x}) \neq 0\}}$$

The test functions satisfy thus two important properties, ther are different from zero only in a bounded set and they vanish along with all derivatives as we approach boundary $\partial\Omega$. The second assertion follows from the fact that the support, being a *closed set* is contained in *open* set Ω.

Assignment A1: Explain more precisely why a test function and all its derivatives vanish on $\partial\Omega$.

Functionals operating on test functions are called *distributions*. The are classified into two groups: *regular* and *irregular* distributions. Regular distributions are generated by $L^1_{loc}(\Omega)$ (locally integrable) functions. A Lebesgue measurable (complex-valued) function defined on Ω is *locally integrable*, if for any point $\boldsymbol{x} \in \Omega$, there exists a ball B centered at $\boldsymbol{x}$ with a radius ϵ contained in Ω, $B = B(\boldsymbol{x}, \epsilon) \subset \Omega$, such that $u \in L^1(B)$, i.e. $\int_B |u| < \infty$.

Assignment A2: Show that $u \in L^1_{loc}(\Omega)$ if an only if, $\int_K |u| < \infty$ for any compact (closed and bounded) subset K of Ω.

Any distribution that is not regular is called *irregular*. Using language of electrostatics, volume charges:

$$\phi \to \int_V \rho_V \phi \, dV$$

define regular charges, but surface, line and point charges:

$$\phi \to \int_S \rho_S \phi \, dS, \quad \phi \to \int_l \rho_l \phi \, dl, \quad \phi \to \rho_P \phi(\boldsymbol{x}_P)$$

define irregular distribution. The distribution prescribing for a test function ϕ its value at point $\boldsymbol{x}_0$ is known as *Dirac's delta* and denote by $\delta_{\boldsymbol{x}_0}$,

$$\delta_{\boldsymbol{x}_0} \ : \ C_0^\infty(\Omega) \ni \phi \to \phi(\boldsymbol{x}_0) \in \mathbb{C}$$

Dirac's delta corresponds thus to a point charge. To simplify the notation we drop indices V, S, l, P indicating the type of charge. We shall use also the bracket (duality pairing) notation to indicate the action of a general distribution L on test function ϕ,

$$\phi \to < L, \phi >$$

A regular distribution generated by an L^1_{loc}-function u, will be denoted by R_u,

$$< R_u, \phi > = \int_\Omega u\phi$$

If function u generating a regualar distribution is differentiable in the classical sense, the integration by parts implies that

$$\int_\Omega \frac{\partial u}{\partial x_i}\phi = -\int_\Omega u\frac{\partial \phi}{\partial x_i} = - < R_u, \frac{\partial \phi}{\partial x_i} >$$

This prompts defining the derivative of an *arbitrary distribution* by passing the differentiation to the test function,

$$< \frac{\partial L}{\partial x_i}, \phi >:= - < L, \frac{\partial \phi}{\partial x_i} >$$

The following simple example is crucial in understanding the concept of differentiating distributions.

Example of a ditributional derivative. In order to understand the difference between the classical and the distributional derivative, partition interval $(0,l)$ into two subintervals $(0,x_0)$ and (x_0,l), and consider a function $u(x)$ specified by two different branches,

$$u(x) = \begin{cases} u_1(x), & x \in (0,x_0), \\ u_2(x), & x \in (x_0,l) \; . \end{cases}$$

We assume that $u_1 \in C^1(0,x_0]$, and $u_2 \in C^1[x_0,l)$. Let now $\phi \in C_0^\infty(0,l)$ be an arbitrary test function. We have,

$$\begin{aligned} \int_0^l u\phi' \, dx &= \int_0^{x_0} u_1\phi' \, dx + \int_{x_0}^l u_2\phi' \, dx \\ &= -\int_0^{x_0} u_1'\phi \, dx + u_1(x_0)\phi(x_0) - \int_{x_0}^l u_2'\phi \, dx - u_2(x_0)\phi(x_0) \; . \end{aligned}$$

If we introduce a function $w(x)$,

$$w(x) = \begin{cases} u_1'(x), & x \in (0,x_0) \\ u_2'(x), & x \in (x_0,l) \end{cases} \quad ,$$

we can rewrite our result in the single integral form,

$$-\int_0^l u\phi' \, dx = \int_0^l w\phi \, dx + [u(x_0)]\phi(x_0) \; . \tag{6.28}$$

Here $[u(x_0)]$ denotes the (possible) jump of function u at interface x_0, $[u(x_0)] = u_2(x_0) - u_1(x_0)$. If function u is continuous at x_0, the second term is gone, and function w is the distributional derivative of u. The fact that function w has not been defined at interface x_0, does not matter since the Lebesgue integral is insensitive to changes of the integrand on subsets of measure zero, and any countable set of points *is of measure zero*. In other words, it is sufficient to define the distributional derivative only up to a subset of measure zero. Thus a function that consists even of an infinite (but countable) number of C^1 branches may not be differentiable at the interface points but, as long as it is globally continuous, it will be differentiable in the distributional sense.

If function u is discontinuous at x_0, the second has to be interpreted in terms of Dirac's delta. A mathematically precise statement is,

$$\frac{d}{dx}R_u = R_w + [u(x_0)]\delta_{x_0} \,,$$

For the *Heaviside function*,

$$u(x) = \begin{cases} 0 & x \in (0, x_0) \\ 1 & x \in (x_0, l) \,, \end{cases}$$

the distributional derivative is equal to the Dirac delta.

The notion of distributional derivative is generalized to higher order derivatives and arbitrary differential operators. We have

$$\begin{aligned} < \nabla \cdot \boldsymbol{E}, \phi > &:= - < \boldsymbol{E}, \nabla\phi > \\ < \nabla \times \boldsymbol{E}, \boldsymbol{\phi} > &:= < \boldsymbol{E}, \nabla \times \boldsymbol{\phi} > \\ < \nabla u, \boldsymbol{\phi} > &:= - < u, \nabla \cdot \boldsymbol{\phi} > \end{aligned}$$

Here $\boldsymbol{E}, u$ denote arbitrary distributions. In a particular case, when $\boldsymbol{E}, u$ are L^1_{loc} functions, a precise statement should use $R_{\boldsymbol{E}}, R_u$ (regualr distributions corresponding to functions $\boldsymbol{E}, u$) on the left-hand side. Notice that, for the second and third operator, we are using vector-valued test functions. The "missing" minus sign in the second case is *not* a misprint. Use the elementary integration by parts formula to justify the definitions above.

Assignment A3: Consider scenario depicted in Fig. 11. Domain Ω has been split into subdomains Ω_1, Ω_2 with an interface Γ. Assume $\boldsymbol{E}, u$ are L^1_{loc} functions consisting of two regular (C^1) branches defined over the two subdomains with a possible jump across the interface Γ. Derive the following formulas for the distributional derivatives:

$$\begin{aligned} < \nabla R_u, \phi > &= < R_{\nabla u}, \boldsymbol{\phi} > + \int_\Gamma [u] \underbrace{\boldsymbol{\phi} \cdot \boldsymbol{n}}_{=:\phi_n} \\ < \nabla \times R_{\boldsymbol{E}}, \boldsymbol{\phi} > &= < R_{\nabla\times\boldsymbol{E}}, \phi > + \int_\Gamma [\boldsymbol{n} \times \boldsymbol{E}]\phi \\ < \nabla \cdot R_{\boldsymbol{E}}, \phi > &= < R_{\nabla\cdot\boldsymbol{E}}, \phi > + \int_\Gamma [\boldsymbol{E} \cdot \boldsymbol{n}]\phi \end{aligned}$$

Note the following practical implications of these formulas:

- If distributional grad of a function u is itself a function then u must be globally continuous. Otherwise, ∇u (understood in the distributional sense) includes a (surface) Dirac's contribution.
- Similarly, if distributional curl of a function $\boldsymbol{E}$ is a function then the tangential component of $\boldsymbol{E}$ must be continuous across the interface but the normal need not.
- Finally, if distributional div of a function $\boldsymbol{E}$ is a function then the normal component of $\boldsymbol{E}$ must be continuous across the interface but the tangential need not.

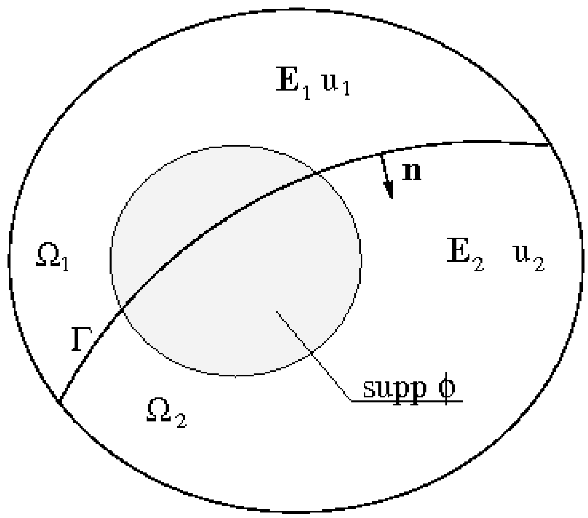

Figure 11: Calculation of distributional derivatives.

- Equation

$$\nabla \cdot \boldsymbol{E} = \rho$$

understood in the distributional sense includes not only the information that the equation holds in the subdomains where $\boldsymbol{E}$ is differentiable in the classical sense (function ρ equals then the classical divergence of $\boldsymbol{E}$) but also that the jump of the normal component of $\boldsymbol{E}$ across a possible interface matches a possible surface contribution,

$$[\boldsymbol{E} \cdot \boldsymbol{n}] = \rho_\Gamma$$

In other words, distribution ρ may include both regular and irregular contributions,

$$< \rho, \phi >= \sum_{i=1,2} \int_{\Omega_i} u_i \phi \, d\boldsymbol{x} + \int_\Gamma \rho_\Gamma \phi \, dS$$

Similar interpretation holds for the other differential operators.