Oden Institute REPORT 19-14

August 2019

Construction of DPG Fortin Operators Revisited

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Reference: Leszek Demkowicz, "Construction of DPG Fortin Operators Revisited," Oden Institute REPORT 19-14, Oden Institute for Computational Engineering and Sciences, The University of Texas at Austin, August 2019.

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Abstract

We construct a general family of DPG Fortin operators for the exact energy spaces defined on a tetrahedral element.

1 Introduction

Petrov-Galerkin method with optimal test functions. Consider a general variational problem,

$$\left\{\begin{array}{l} u \in U \\ b(u,v) = l(v) \quad v \in V \end{array}\right. \tag{1.1}$$

where U, V are Hilbert trial and test spaces, $b(u,v)$ is a continuous bilinear form satisfying the inf-sup condition,

$$\sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U$$

and $l \in V'$ satisfies the compatibility condition,

$$l(v) = 0 \quad v \in V_0 := \{v \in V : b(u,v) = 0 \quad \forall u \in U\}\,.$$

By the Babuška-Nečas Theorem [11], Thm. 6.6.1, the problem is well posed.

Petrov-Galerkin discretization of (1.1) introduces discrete trial and test spaces $U_h \subset U$, $V_h \subset V$ *of equal dimension*, and approximates (1.1) with its discrete counterpart,

$$\left\{\begin{array}{l} u_h \in U_h \\ b(u_h,v_h) = l(v_h) \quad v_h \in V_h\,. \end{array}\right. \tag{1.2}$$

If a discrete inf-sup condition is satisfied,

$$\sup_{v_h \in V_h} \frac{|b(u_h,v_h)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U$$

then, by Babuška Theorem [1], the discrete problem is well-posed as well, and we have the a-priori error estimate,

$$\underbrace{\|u-u_h\|_U}_{\text{approximation error}} \leq \frac{\|b\|}{\gamma_h} \underbrace{\inf_{w_h \in U_h} \|u-w_h\|_U}_{\text{the best approximation error}}\,.$$

Unfortunately, the continuous inf-sup condition does not imply the discrete one, and coming up with a stable pair U_h, V_h of equal dimension for $U \neq V$ may be challenging.

The *Petrov-Galerkin Method with Optimal Test Functions* [7, 6],starts by replacing problem (1.1) with an equivalent mixed formulation,

$$\left\{\begin{array}{lll} \psi \in V,\, u \in U & & \\ (\psi, \delta v)_V + b(u, \delta v) & = l(\delta v) & \delta v \in V \\ b(\delta u, \psi) & = 0 & \delta u \in U \end{array}\right. \tag{1.3}$$

where the additional unknown ψ is (the Riesz representation of) the residual and, on the continuous level, is equal zero. Instead of discretizing the original problem, we discretize now the equivalent mixed problem,

$$\left\{\begin{array}{lll} \psi_h \in V_h,\, u_h \in U_h & & \\ (\psi_h, \delta v_h)_V + b(u_h, \delta v_h) & = l(\delta v_h) & \delta v_h \in V \\ b(\delta u_h, \psi_h) & = 0 & \delta u_h \in U_h\,. \end{array}\right. \tag{1.4}$$

The Brezzi theory [3] calls for the satisfaction of two inf-sup conditions. The discrete *inf-sup in kernel condition* is trivially satisfied due to the coercivity of the test inner product. The discrete *LBB condition* coincides now with the original discrete Babuška condition with one important difference - V_h need not be of the same dimension as U_h. With a sufficiently large space V_h, the discrete inf-sup condition is easily satisfied. The classical way of proving the discrete inf-sup condition is to construct the so-called Fortin operator [2],

$$\begin{array}{rl} \Pi\,:\, V \ni v & \rightarrow \Pi v \in V_h \\ \|\Pi v\|_V & \leq C_F \|v\|_V \\ b(\delta u_h, \Pi v - v) & = 0 \quad \delta u_h \in U_h\,. \end{array} \tag{1.5}$$

With the existence of the Fortin operator, the continuous inf-sup condition implies its discrete counterpart with $\gamma_h = \gamma / C_F$. Obviously, we want the continuity constant C_F to be small.

Discontinuous Petrov-Galerkin (DPG) method with optimal test functions. In the DPG method, we enlarge the test space V_h to a broken test space $V_h(\mathcal{T}_h)$ at the expense of introducing yet additional unknowns - Lagrange multipliers, the so-called traces $\hat{u}_h \in \hat{U}_h$ defined on the mesh skeleton. For *localizable* test inner products, the Gram matrix corresponding to $(\psi_h, v_h)_V$ becomes block-diagonal, and the residual ψ_h is eliminated at the element level. The ultimate global price for stability is the introduction of the additional unknows - the traces. The broken counterpart of (1.1) looks as follows.

$$\left\{\begin{array}{l} u \in U,\, \hat{u} \in \hat{U} \\ b(u, v) + \langle \hat{u}, v \rangle_{\Gamma_h} = l(v) \quad v \in V_h(\mathcal{T}_h) \end{array}\right. \tag{1.6}$$

where the bracket represents additional terms defined on the mesh skeleton. It has been shown in [4] that the broken variational formulation is well-posed and it inherits the stability of the original problem with same order stability constants.

The abstract conditions for the Fortin operator in context of the DPG method look as follows.

$$\begin{aligned}
\Pi\,:\, V(\mathcal{T}_h) \ni v &\to \Pi v \in V_h(\mathcal{T}_h) \\
\|\Pi v\|_{V(\mathcal{T}_h)} &\le C_F \|v\|_{V(\mathcal{T}_h)} \\
b(u_h, v - \Pi v) + \langle \hat{u}_h, v - \Pi v\rangle_{\Gamma_h} &= 0 \quad \forall u_h \in U_h,\, \hat{u}_h \in \hat{U}_h\,.
\end{aligned}$$

Construction of Fortin operators for conforming test spaces is challenging. Value of the operator – Πv, has to land in the (conforming) discrete test space which suggests the use of techniques used in the construction of interpolation operators : taking values at vertices, edge and face averages etc. However, the Fortin operator has to be defined on the *whole energy space*, and these operations are illegal for general members of such spaces.

With broken test spaces, the global conformity is not an issue, and we can settle for a local construction of the Fortin operator:

$$\begin{aligned}
\Pi\,:\, V(K) \ni v &\to \Pi v \in V_h(K) \\
\|\Pi v\|_{V(K)} &\le C_F \|v\|_{V(K)} \\
b_K(u_h, \Pi v - v) + \langle \hat{u}_h, \Pi v - v\rangle_{\partial K} &= 0 \quad \forall u_h \in U_h,\, \hat{u}_h \in \hat{U}_h
\end{aligned} \tag{1.7}$$

where $V(K)$ denotes the test space on element K, and $V_h(K)$ denotes its discrete counterpart. Clearly, satisfaction of the local conditions implies immediately satisfaction of the global conditions as well. The main point in the construction of the Fortin operator is to use operations that are well-defined on the whole energy space. The finite-dimensionality of the range and Uniform Boundedness Theorem imply then automatically the continuity of the operator, see Exercise 2. We also want the continuity constant to be at least a) independent of element size h and, possibly, b) independent of polynomial order p. As the Fortin constant enters the ultimate stability constant for the DPG method, we also want it to be as small as possible.

Construction of the Fortin operator involves the original bilinear form and the skeleton term resulting from breaking the test space and, therefore, is problem dependent. However, if we restrict ourselves to standard test spaces: $H^1, H(\text{curl}), H(\text{div})$ (with standard norms), and make a simplifying assumption about the material data to be element-wise constant, one can strive for constructing general Fortin operators that will serve all problems satisfying the simplifying assumptions. This was done in [8, 4]. In what follows, we will generalize ideas from [9]. For an example of a non-local Fortin operator, see [5].

We will restrict ourselves to affine tetrahedral elements.

The motivation for the construction comes from the ultraweak (UW) variational formulation for two model problems. The first one is the classical diffusion-convection-reaction problem:

$$\left\{\begin{aligned}
-\text{div}\sigma + cu &= f && \text{in } \Omega \\
a^{-1}\sigma - \nabla u + a^{-1}bu &= 0 && \text{in } \Omega \\
u &= u_0 && \text{on } \Gamma_u \\
\sigma\cdot n &= \sigma_0 && \text{on } \Gamma_\sigma\,.
\end{aligned}\right.$$

An element K contribution to the bilinear form in the UW variational formulation is:

$$b_K((\sigma,u,\hat{\sigma}\cdot n,\hat{u}),(\tau,v)) = (\sigma,\nabla v + a^{-1}\tau)_K + (u, cv + \operatorname{div}\tau + (a^{-1}b)\cdot\tau)_K - \langle\hat{\sigma}\cdot n, v\rangle_{\partial K} - \langle\hat{u},\tau\cdot n\rangle_{\partial K}$$

where, consistently with the logic of using the first Nèdèlec exact sequence spaces for discretization we have,

$$\begin{aligned} & u \in \mathcal{P}^{p-1}(K), \sigma \in \mathcal{P}^{p-1}(K)^3 \\ & \hat{u} \in \gamma(\mathcal{P}^p(K)) =: \mathcal{P}^p_c(\partial K) \\ & \hat{\sigma}\cdot n \in \gamma_n(\mathcal{RT}^p(K)) =: \mathcal{P}^{p-1}_d(\partial K)\,. \end{aligned}$$

After integration by parts,

$$b_K((\sigma,u,\hat{\sigma}\cdot n,\hat{u}),(\tau,v)) = (a^{-1}\sigma - \nabla u + a^{-1}bu,\tau)_K + (-\operatorname{div}\sigma + cu, v)_K + \langle\sigma\cdot n - \hat{\sigma}\cdot n, v\rangle_{\partial K} + \langle u-\hat{u},\tau\cdot n\rangle_{\partial K}\,.$$

This leads to the following orthogonality requirements for the Fortin operators.

$$\begin{aligned} (\psi,\Pi^{\mathrm{grad}}v - v) &= 0 \qquad \psi\in\mathcal{P}^{p-1}(K) \\ \langle\phi,\Pi^{\mathrm{grad}}v - v\rangle_{\partial K} &= 0 \qquad \phi\in\mathcal{P}^{p-1}_d(\partial K)\,. \end{aligned} \tag{1.8}$$

$$\begin{aligned} (\psi,\Pi^{\mathrm{div}}\tau - \tau) &= 0 \qquad \psi\in\mathcal{P}^{p-1}(K)^3 \\ \langle\phi,(\Pi^{\mathrm{div}}\tau - \tau)\cdot n\rangle_{\partial K} &= 0 \qquad \phi\in\mathcal{P}^{p}_c(\partial K)\,. \end{aligned} \tag{1.9}$$

Our second example deals with the UW formulation for three-dimensional Maxwell equations,

$$\left\{\begin{array}{lll} E,H\in L^2(\Omega),\ \hat{E}_t,\hat{H}_t\in H^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) & & \\ (\frac{1}{\mu}E,\nabla_h\times F) + \langle n\times\hat{E}_t, F_t\rangle_{\Gamma_h} + i\omega(H,F) & = 0 & F\in H(\mathrm{curl},\mathcal{T}_h) \\ (H,\nabla_h\times G) + \langle n\times\hat{H}_t, G_t\rangle_{\Gamma_h} - ((\sigma+i\omega\epsilon)E,G) & = (J^{\mathrm{imp}},G) & G\in H(\mathrm{curl},\mathcal{T}_h) \\ \hat{E}_t & = E_{0,t} & \text{on } \Gamma_E \\ \hat{H}_t & = H_{0,t} & \text{on } \Gamma_H\,. \end{array}\right.$$

Recalling that approximate $E,H\in\mathcal{P}^{p-1}(K)^3$, and approximate $\hat{E}_t,\hat{H}_t$ belong to the tangential trace of Nèdèlec space $\mathcal{N}^p(K)$, we arrive at the orthogonality conditions for the Fortin operator,

$$\begin{aligned} (\psi,\Pi^{\mathrm{curl}}F - F)_K &= 0 \qquad \psi\in\mathcal{P}^{p-1}(K)^3 \\ \langle n\times\phi,\Pi^{\mathrm{curl}}F - F\rangle_{\partial K} &= 0 \qquad \phi\in\gamma_t\mathcal{N}^p \end{aligned} \tag{1.10}$$

where $\gamma_t\mathcal{N}^p(K)$ denotes the image of tangential trace operator of $\mathcal{N}^p(K)$.

2 Auxiliary Results

We will need a few fundamental results on polynomial spaces defined on a tetrahedron. The first four lemmas deal with bubble spaces.

Lemma 1
Let $\mathcal{P}_0^{p+3}(K)$ *denote the subspace of* $\mathcal{P}^{p+3}(K)$ *of* H^1 *bubbles on element* K*. Let* $u \in \mathcal{P}_0^{p+3}(K)$*, and*

$$(\psi, u)_K = 0 \quad \forall \psi \in \mathcal{P}^{p-1}(K)\,.$$

Then $u = 0$ *and, consequently,*

$$\inf_{u \in \mathcal{P}_0^{p+3}(K)} \sup_{\psi \in \mathcal{P}^{p-1}(K)} \frac{|(\psi, u)_K|}{\|\psi\|\,\|u\|} \geq \beta > 0\,.$$

As spaces $\mathcal{P}_0^{p+3}(K)$ *and* $\mathcal{P}^{p-1}(K)$ *are of equal dimension, the order of spaces in the inf-sup condition can be reversed,*

$$\inf_{\psi \in \mathcal{P}^{p-1}(K)} \sup_{u \in \mathcal{P}_0^{p+3}(K)} \frac{|(\psi, u)_K|}{\|u\|\,\|\psi\|} \geq \beta > 0\,.$$

▮

Proof: Function u must be of the form:

$$u = \lambda_0 \ldots \lambda_3\, v$$

where $\lambda_i,\ i = 0, \ldots, 3$ are affine coordinates, and $v \in \mathcal{P}^{p-1}(K)$. Choosing $\psi = v$ gives

$$(\psi, u)_K = \int_K \lambda_0 \ldots \lambda_3 v^2 = 0 \quad \Rightarrow \quad v = 0 \quad \Rightarrow \quad u = 0\,.$$

The result implies that the supremum

$$\sup_{\psi \in \mathcal{P}^{p-1}(K)} \frac{|(\psi, u)_K|}{\|\psi\|}$$

defines a norm on u, and the inf-sup condition follows then from the equivalence of norms in a finite dimensional space. ▮

The following result can be found in [10].

Lemma 2
Let $\mathcal{RT}_0^{p+1}(K)$ *denote the subspace of* $\mathcal{RT}^{p+1}(K)$ *of* $H(\mathrm{div})$ *bubbles on element* K*. Let* $\tau \in \mathcal{RT}_0^{p+1}(K)$*, and*

$$(\psi, \tau)_K = 0 \quad \forall \psi \in \mathcal{P}^{p-1}(K)^d\,.$$

Then $\tau = 0$ *and, consequently,*

$$\inf_{\tau \in \mathcal{RT}_0^{p+1}(K)} \sup_{\psi \in \mathcal{P}^{p-1}(K)^d} \frac{|(\psi, \tau)_K|}{\|\psi\|\,\|\tau\|} \geq \beta > 0\,.$$

As spaces $\mathcal{RT}_0^{p+1}(K)$ and $\mathcal{P}^{p-1}(K)^d$ are of equal dimension, the order of spaces in the inf-sup condition can be reversed,

$$\inf_{\psi\in\mathcal{P}^{p-1}(K)^d}\ \sup_{\tau\in\mathcal{RT}_0^{p+1}(K)} \frac{|(\psi,\tau)_K|}{\|\tau\|\,\|\psi\|} \geq \beta > 0\,.$$

∎

Proof: It is sufficient to prove the result for the master tetrahedron. Choosing $\psi = \nabla u$, $u \in \mathcal{P}^p(K)$ and integrating by parts, we obatin,

$$0 = (\nabla u, \tau)_K = -(u, \operatorname{div}\tau)_K\,.$$

As u and $\operatorname{div}\tau$ are both of order p, this implies that $\operatorname{div}\tau = 0$. This implies that τ is a curl of an element of Nèdèlec space $\mathcal{N}^p(K)$ and, in particular, it must be a polynomial of order p, i.e. $\tau \in \mathcal{P}^p(K)^d$. As τ satisfies the homogeneous normal BC, there must exist $\psi_i \in \mathcal{P}^{p-1}(K)$ such that

$$\tau_i = \xi_i\psi_i\,.$$

Testing with such a ψ gives,

$$\int_K \tau\psi = \int_K \sum_i \xi_i|\psi_i|^2 = 0 \quad \Rightarrow \quad \psi = 0 \quad \Rightarrow \quad \tau = 0\,.$$

The result implies that the supremum

$$\sup_{\psi\in\mathcal{P}^{p-1}(K)^d} \frac{|(\psi,\tau)_K|}{\|\psi\|}$$

defines a norm on τ, and the inf-sup condition follows then from the equivalence of norms in a finite dimensional space. ∎

Lemma 3

Let $\mathcal{N}_0^{p+2}(K)$ denote the subspace of $\mathcal{N}^{p+2}(K)$ of $H(\mathrm{curl})$ bubbles defined on tetrahedron K. Let $F \in \mathcal{N}_0^{p+2}(K)$,and

$$(\psi, F)_K = 0 \quad \forall\psi \in \mathcal{P}^{p-1}(K)^3\,.$$

Then $F = 0$ and, consequently,

$$\inf_{F\in\mathcal{N}_0^{p+2}(K)}\ \sup_{\psi\in\mathcal{P}^{p-1}(K)^3} \frac{|(\psi,F)_K|}{\|\psi\|\,\|F\|} = \beta > 0\,.$$

As spaces $\mathcal{N}_0^{p+2}(K)$ and $\mathcal{P}^{p-1}(K)^3$ are of equal dimension, the order of space in the inf-sup condition can be reversed,

$$\inf_{\psi\in\mathcal{P}^{p-1}(K)^3}\ \sup_{F\in\mathcal{N}_0^{p+2}(K)} \frac{|(\psi,F)_K|}{\|\psi\|\,\|F\|} = \beta > 0\,.$$

∎

Proof: Again, it is sufficient to consider the master tetrahedron. Let $F \in \mathcal{N}_0^{p+2}(K)$. Let $\psi \in \mathcal{P}^p(K)^3$. Then

$$(\psi, \nabla \times F)_K = (\nabla \times \psi, F)_K = 0\,.$$

As the curl operator sets $H(\mathrm{curl})$ bubbles into $H(\mathrm{div})$ bubbles, Lemma 2 proves that $\nabla \times F = 0$ and, in particular, $F \in \mathcal{P}^{p+1}(K)^3$. Any $H(\mathrm{curl})$ bubble on the master tetrahedron must be of the form:

$$F = (\phi_1 \xi_2 \xi_3, \phi_2 \xi_1 \xi_3, \phi_3 \xi_1 \xi_2)$$

with some scalar factors ϕ_i. As F is of order $p+1$, ϕ_i must be of order $p-1$. Selecting $\psi = (\phi_1, \phi_2, \phi_3)$, we conclude that $F = 0$. The rest of the reasoning is the same as in the proof of Lemma 2. ∎

In order to cope with boundary terms, we will also need a 2D equivalent of Lemma 3.

Lemma 4

Let $\mathcal{N}_0^{p+1}(K)$ denote the subspace of $\mathcal{N}^{p+1}(K)$ of $H(\mathrm{curl})$ bubbles on the master triangle K. Let $F \in \mathcal{N}_0^{p+1}(K)$, and

$$(\psi, F)_K = 0 \quad \forall \psi \in \mathcal{P}^p(K)^2\,.$$

Then $F = 0$ and , consequently

$$\inf_{F \in \mathcal{N}_0^{p+1}(K)} \sup_{\psi \in \mathcal{P}^{p-1}(K)^2} \frac{|(\psi, F)_K|}{\|\psi\| \, \|F\|} = \beta > 0\,.$$

As spaces $\mathcal{N}_0^{p+1}(K)$ and $\mathcal{P}^{p-1}(K)^2$ are of equal dimension, the order of space in the inf-sup condition can be reversed,

$$\inf_{\psi \in \mathcal{P}^{p-1}(K)^2} \sup_{F \in \mathcal{N}_0^{p+2}(K)} \frac{|(\psi, F)_K|}{\|\psi\| \, \|F\|} = \beta > 0\,.$$

∎

Proof: The result follows directly from the 2D version of Lemma 2 and the relation between the two 2D exact sequences. See also Exercise 4. ∎

The next three lemmas deal with polynomial spaces satisfying the orthogonality constraints necessary for Fortin operators. We will upgrade slightly the orthogonality assumptions $(1.10)_2$ replacing them with:

$$\begin{aligned} (\psi, \Pi^{\mathrm{curl}} F - F)_K &= 0 \qquad \psi \in \mathcal{P}^{p-1}(K)^3 \\ \langle n \times \phi, \Pi^{\mathrm{curl}} F - F \rangle_{\partial K} &= 0 \qquad \phi \in \gamma_t(\mathcal{P}^p(K)^3) \end{aligned} \tag{2.11}$$

Lemma 5
Let $F \in H(\mathrm{curl}, K)$ satisfy the constraints:

$$\begin{aligned} (\psi, F)_K &= 0 \qquad \forall \psi \in \mathcal{P}^{p-1}(K)^3 \\ \langle n \times \phi, F \rangle_{\partial K} &= 0 \qquad \forall \phi \in \mathcal{P}^p(K)^3 \,. \end{aligned} \tag{2.12}$$

Then $\mathrm{curl}\, F$ *satisfies the constraint:*

$$(\chi, \mathrm{curl}\, F)_K = 0 \quad \forall \chi \in \mathcal{P}^p(K)^3 \tag{2.13}$$

which, in turn, implies,

$$\langle \eta, \mathrm{curl}\, F \cdot n \rangle_{\partial K} = 0 \quad \forall \eta \in \mathcal{P}^{p+1}(K) \,. \tag{2.14}$$

Conversely, let $F \in H(\mathrm{curl}, K)$ satisfy (2.13). Then, there exists $u \in \mathcal{P}^{p+2}(K)$ such that

$$\begin{aligned} (\psi, F + \nabla u)_K &= 0 \qquad \forall \psi \in \mathcal{P}^{p-1}(K)^3 \text{ and,} \\ \langle n \times \phi, F + \nabla u \rangle_{\partial K} &= 0 \qquad \forall \phi \in \mathcal{P}^p(K)^3 \,. \end{aligned} \tag{2.15}$$

∎

Proof: Taking $\psi = \mathrm{curl}\, F$ in $(2.12)_1$, and utilizing $(2.12)_2$ gives (2.13). Use $\chi = \nabla \eta$ in (2.13) to obtain (2.14).

Let $F \in H(\mathrm{curl}, K)$ now satisfy (2.13). It is sufficient to show $(2.15)_1$. The second property follows from the first one with $\psi = \nabla \phi$ and (2.13). We view (2.13) as an overdetermined variational problem for ∇u and, in the spirit of the DPG method, we consider the mixed problem:

$$\left\{ \begin{array}{lll} \psi \in \mathcal{P}^{p-1}(K)^3, \; u \in \mathcal{P}^{p+2}(K) & & \\ (\psi, \delta\psi)_K + (\nabla u, \delta\psi)_K & = -(F, \delta\psi) & \delta\psi \in \mathcal{P}^{p-1}(K)^3 \\ (\nabla \delta u, \psi)_K & = 0 & \delta u \in \mathcal{P}^{p+2}(K) \,. \end{array} \right.$$

We claim that the constraint for ψ is equivalent to $\psi = \mathrm{curl}\, F_0$ where $F_0 \in \mathcal{P}^p(K))^3$ with a zero tangential trace. Sufficiency follows from integration by parts. To show necessity, we test first with $\delta u \in \mathcal{P}_0^{p+2}$ to obtain,

$$(\underbrace{\mathrm{div}\, \psi}_{\in \mathcal{P}^{p-2}(K)} , \delta u) = 0 \,.$$

Taking $\delta u = \mathrm{div}\, \psi \, \lambda_0 \dots \lambda_3$ where λ_i, $i = 0, \dots, 3$ are affine coordinates, we conclude that $\mathrm{div}\, \psi = 0$. Testing next with a general δu, we obtain,

$$0 = (\nabla \delta u, \psi)_K = \langle u, \psi \cdot n \rangle_{\partial K} \,.$$

Taking $u = (\psi \cdot n) \lambda_i \lambda_j \lambda_k$ on each $[ijk]$ face, we conclude that $\psi \cdot n = 0$ on ∂K. Consequently, there exists a vector potential $F_0 \in \mathcal{P}^p(K)^2$ with zero tangential trace such that $\psi = \mathrm{curl}\, F_0$.

We test now the first equation in the mixed problem with $\delta\psi = \psi$. The assumption on F implies that

$$\|\psi\|_K^2 = 0 \quad \Rightarrow \quad \psi = 0\,.$$

Consequently, equation (2.15)$_1$ is satisfied. Note that the LBB condition is easily satisfied so the mixed problem is well-posed. ∎

Lemma 6
Let $\tau \in H(\mathrm{div}, K)$ *satisfy the constraints:*

$$\begin{aligned} (\psi, \tau)_K &= 0 \qquad \forall \psi \in \mathcal{P}^{p-1}(K)^3 \\ \langle \phi, \tau \cdot n \rangle_{\partial K} &= 0 \qquad \forall \phi \in \mathcal{P}^p(K)\,. \end{aligned} \tag{2.16}$$

Then $\mathrm{div}\,\tau$ *satisfies the constraint:*

$$(\chi, \mathrm{div}\,\tau)_K = 0 \quad \forall \chi \in \mathcal{P}^p(K)\,. \tag{2.17}$$

Conversely, let $\tau \in H(\mathrm{div}, K)$ *satisfy (2.17). Then, there exists* $F \in \mathcal{N}^{p+1}(K)$ *such that*

$$\begin{aligned} (\psi, \tau + \mathrm{curl}\,F)_K &= 0 \qquad \forall \psi \in \mathcal{P}^{p-1}(K)^3 \textit{ and,} \\ \langle \phi, (\tau + \mathrm{curl}\,F) \cdot n \rangle_{\partial K} &= 0 \qquad \forall \phi \in \mathcal{P}^p(K)\,. \end{aligned} \tag{2.18}$$

∎

Proof: Taking $\psi = \nabla\phi$ in (2.16)$_1$ and utilizing (2.16)$_2$ gives (2.17).

Let τ satisfy (2.17). In the same way as in the proof of. Lemma 5, consider the mixed problem:

$$\left\{ \begin{array}{lll} \psi \in \mathcal{P}^{p-1}(K)^3,\ F \in \mathcal{N}^{p+1}(K) & & \\ (\psi, \delta\psi)_K + (\mathrm{curl}\,F, \delta\psi)_K & = -(\tau, \delta\psi)_K & \delta\psi \in \mathcal{P}^{p-1}(K)^3 \\ (\mathrm{curl}\,\delta F, \psi) & = 0 & \delta F \in \mathcal{N}^{p+1}(K) \end{array} \right.$$

We claim that ψ satisfies the constraint iff $\psi = \nabla u,\ u \in \mathcal{P}^p_0(K)$. The sufficiency follows from integration by parts. In order to prove necessity, we first test with $\delta F_0 \in \mathcal{N}^{p+2}(K)$ with zero tangential trace. We obtain,

$$(\delta F_0, \underbrace{\mathrm{curl}\,\psi}_{\in \mathcal{P}^{p-2}(K)^3})_K = 0$$

and, by Lemma 3, $\mathrm{curl}\,\psi = 0$. Testing next with a general F and using Lemma 4, we conclude that $\gamma_t\psi = 0$ on ∂K. Consequently, there exists a $u \in \mathcal{P}^p_0(K)$ such that $\psi = \nabla u$. Testing with ψ in the first equation and utilizing assumption on F, we obtain $\psi = 0$. By Lemma 3, the problem is well posed and with $\psi = 0$ we obtain the desired orthogonality property. ∎

Lemma 7
Let $u \in H^1(K)$ satisfy the constraints:

$$\begin{aligned} (\psi, u)_K &= 0 \qquad \forall \psi \in \mathcal{P}^{p-1}(K) \\ \langle \phi \cdot n, u \rangle_{\partial K} &= 0 \qquad \forall \phi \in \mathcal{P}^p(K)^3 \,. \end{aligned} \tag{2.19}$$

Then ∇u satisfies the constraint:

$$(\chi, \nabla u)_K = 0 \quad \forall \chi \in \mathcal{P}^p(K)^3 \tag{2.20}$$

which, in turn, implies,

$$\langle n \times \eta, \nabla u \rangle_{\partial K} = 0 \quad \forall \eta \in \mathcal{P}^{p+1}(K)^3 \,. \tag{2.21}$$

Conversely, let $u \in H^1(K)$ satisfy (2.20). Then, there exists a constant c such that

$$\begin{aligned} (\psi, u + c)_K &= 0 \qquad \forall \psi \in \mathcal{P}^{p-1}(K) \text{ and,} \\ \langle \phi \cdot n, u + c \rangle_{\partial K} &= 0 \qquad \forall \phi \in \mathcal{P}^p(K)^3 \,. \end{aligned} \tag{2.22}$$

$\blacksquare$

Proof: See Exercise 1. $\blacksquare$

3 Construction of Fortin Operators for DPG Problems

3.1 Π^{div} Fortin Operator.

We begin with the construction of the Π^{div} Fortin operator. The idea is to construct first operator $\hat{\Pi}^{\mathrm{div}}$ on master tetrahedron $\hat{K}$, and then use the $H(\mathrm{div})$ pullback map T to extend it to an arbitrary affine element K,

$$\hat{\Pi}^{\mathrm{div}} \tau := F^{-1} \hat{\Pi}^{\mathrm{div}} F \tau \,.$$

Similarly to the interpolation error estimates, the scaling properties of pullback maps imply that we should have the commuting diagram:

$$\begin{array}{ccc} H(\mathrm{div}, K) & \xrightarrow{\mathrm{div}} & L^2(K) \\ \Pi^{\mathrm{div}} \downarrow & & P \downarrow \\ V^{p+1} & \xrightarrow{\mathrm{div}} & Y^p \end{array} \tag{3.23}$$

where V^{p+1} is the enriched test $H(\mathrm{div})$-space, $Y^p = \mathrm{div}\, V^{p+1}$, and P is a Fortin operator for the L^2 space. In other words, divergence of $\Pi^{\mathrm{div}} \tau$ should depend only upon the divergence of function τ. Given that $y^p := P^{\mathrm{div}} \tau$ must satisfy constraints (2.17), we are naturally led to the definition of y^p through the constrained minimization problem:

$$\| y^p - \underbrace{\mathrm{div}\, \tau}_{=:y} \| \to \min_{y^p \in Y^p} \qquad \text{subject to constraint } (2.17) \,. \tag{3.24}$$

The constraint leads also to the minimum assumption on the enriched L^2 test space:

$$\mathcal{P}^p \subset Y^p \, .$$

Note that, for the minimal space, $Y^p = \mathcal{P}^p(K)$, operator P reduces to the L^2-projection.

Once we have defined $Y^p = \operatorname{div} \tau^{p+1}$, $\tau^{p+1} := \Pi^{\text{div}} \tau$, we proceed with a second minimization problem to define τ^{p+1} itself.

$$\left\{ \begin{array}{l} \|\tau^{p+1} - \tau\| \to \min\limits_{\tau^{p+1} \in V^{p+1}} \quad \text{subject to constraints } (2.16), \text{ and the constraint on divergence,} \\ \operatorname{div} \tau^{p+1} = y^p \, . \end{array} \right. \tag{3.25}$$

It follows from Lemma 6 that the problem is well-posed, provided we satisfy the minimum assumption on the enriched $H(\text{div})$ test space:

$$\mathcal{RT}^{p+1}(T) \subset V^{p+1}$$

and the divergence maps V^{p+1} *onto* space Y^p. The assumptions and Lemma 6 guarantee that there exists a function $\tau^{p+1} \in V^{p+1}$ satisfying the constraints, i.e., the set over which we set up the minimization problem is non-empty.

We can offer an alternate argument based on mixed problems theory. The constrained minimization problem leads to the equivalent mixed problem:

$$\left\{ \begin{array}{lll} \tau^{p+1} \in V^{p+1}, \, \psi \in \mathcal{P}^{p-1}(K)^3, \, \phi \in \mathcal{P}^p_c(\partial K), \, \chi \in Y^p_0 & & \\ (\tau^{p+1}, \delta\tau) + (\psi, \delta\tau)_K + \langle \phi, \delta\tau \rangle_{\partial K} + (\chi, \operatorname{div} \delta\tau) & = (\tau, \delta\tau) & \delta\tau \in V^{p+1} \\ (\delta\psi, \tau^{p+1})_K & = (\delta\psi, \tau)_K & \delta\psi \in \mathcal{P}^{p-1}(K)^3 \\ \langle \delta\phi, \tau^{p+1} \cdot n \rangle_{\partial K} & = \langle \delta\phi, \tau \cdot n \rangle_{\partial K} & \delta\phi \in \mathcal{P}^p_c(\partial K) \\ (\delta\chi, \operatorname{div} \tau^{p+1}) & = (\delta\chi, \operatorname{div} \tau) & \delta\chi \in Y^p_0 \end{array} \right. \tag{3.26}$$

where Y^p_0 is the subspace of Y^p satisfying constraints (2.17) . We need to check the two Brezzi inf-sup conditions. The *inf-sup in kernel condition* is satisfies trivially since the form is coercive. The proof of LBB condition follows the logic of Exercise 3. The inf-sup condition for $b_3(\chi, \delta v) := (\chi, \operatorname{div} \delta v)$ follows from Lemma 6 and coercivity of the form. The inf-sup condition for $b_2(\psi, \delta v) := (\psi, \delta v)$ follows from Lemma 2, and the inf-sup condition for $b_1(\phi, \delta v) = \langle \phi, \delta v \cdot n \rangle$ follows from the choice

$$\delta v \cdot n = \phi$$

on each face of the tetrahedron. Consequently, the mixed problem is well-posed. According to the result from Exercise 2, master element operator $\hat{\Pi}^{\text{div}}$ is well-defined and continuous. Finally, commuting property (3.23) implies the continuity of operator Π^{div} defined on an arbitrary affine tetrahedron K.

THEOREM 1

The operator defined by the constrained minimization problem (3.25) is well-defined and continuous,

$$\Pi^{\text{div}} : H(\text{div}, K) \to V^{p+1}, \qquad \|\Pi^{\text{div}} \tau\|_{H(\text{div},K)} \le C_{\Pi^{\text{div}}} \|v\|_{H(\text{div},K)} \, .$$

The continuity constant $C_{\Pi^{\mathrm{div}}}$ is independent of element size but it may depend upon the polynomial order p. $\blacksquare$

We conclude this section by observing the action of operator Π^{div} on a curl, i.e. for $\tau = \mathrm{curl}\, F$. It follows from the construction that $\mathrm{div}(\Pi^{\mathrm{div}}\mathrm{curl}\, F) = 0$, so the constrained minimization problem to determine τ^{p+1} simplifies to:

$$\|\tau^{p+1} - \mathrm{curl}\, F\| \to \min_{\tau^{p+1} \in V^{p+1}(\mathrm{div}_0)} \quad \text{subject to constraints } (2.16)_1 \tag{3.27}$$

where $V^{p+1}(\mathrm{div}_0)$ denotes the subspace of V^{p+1} of divergence-free functions.

3.2 Π^{curl} Fortin Operator

We follow the same logic as for the $H(\mathrm{div})$ operator starting by defining the divergence of $\Pi^{\mathrm{curl}}F$. The obvious choice is to use operator (3.27) but we have to make a small correction accounting for the orthogonality property (2.13) involving polynomials of order p, one order higher than in (3.27). Thus we seek $\tau^{p+2} := \mathrm{curl}\, \Pi^{\mathrm{curl}}F$ in the subspace of divergence-free functions from a larger space $V^{p+2} \supset \mathcal{RT}^{p+2}(K)$. In other words, we require that $\mathrm{curl}\, Q^{p+2} \supset \mathcal{P}^{p+1}(K)^3$. We have,

$$\|\tau^{p+2} - \mathrm{curl}\, F\| \to \min_{\tau^{p+2} \in \mathrm{curl}\, Q^{p+2}} \quad \text{subject to constraints } (2.13)\,. \tag{3.28}$$

We can formulate now a constrained minimization problem defining $\Pi^{\mathrm{curl}}F$,

$$\begin{aligned}
&\Pi^{\mathrm{curl}} : H(\mathrm{curl}, K) \to Q^{p+2}, \quad \Pi^{\mathrm{curl}}F := F^{p+2} \in Q^{p+2} \\
&\|F^{p+2} - F\| \to \min_{F^{p+2} \in Q^{p+2}} \ \text{subject to constraints: } (2.12) \text{ and the constraint on curl :} \\
&\mathrm{curl}\, F^{p+2} = \tau^{p+2}\,.
\end{aligned} \tag{3.29}$$

It follows from Lemma 5 that the problem is well-posed, provided we satisfy the minimum assumption on the enriched $H(\mathrm{curl})$ test space:

$$\mathcal{N}^{p+2}(K) \subset Q^{p+2}\,.$$

The constrained minimization problem above is equivalent to the mixed problem:

$$\left\{\begin{array}{lll}
F^{p+2} \in Q^{p+2},\, \psi \in \mathcal{P}^{p-1}(K)^3, \phi \in \gamma_t(\mathcal{P}^p(K)^3),\, \tau \in V_0^{p+1} & & \\
(F^{p+2}, \delta F)_K + (\psi, \delta F)_K + \langle n \times \phi, \delta F\rangle_{\partial K} + (\tau, \mathrm{curl}\delta F)_K & = (F, \delta F)_K & \delta F \in Q^{p+2} \\
(\delta\psi, F^{p+2})_K & = (\delta\psi, F)_K & \delta\psi \in \mathcal{P}^{p-1}(K)^3 \\
\langle n \times \delta\phi, F^{p+2}\rangle_{\partial K} & = \langle n \times \delta\phi, F\rangle_{\partial K} & \delta\phi \in \gamma_t(\mathcal{P}^p(K)^3) \\
(\delta\tau, \mathrm{curl} F^{p+2})_K & = (\delta\tau, \mathrm{curl} F)_K & \delta\tau \in V_0^{p+1}
\end{array}\right. \tag{3.30}$$

where V_0^{p+1} is the subspace of $\mathrm{curl}\, Q^{p+2}$ satisfying constraints (2.13). We use the same arguments as for the Π^{div} operator to prove the LBB inf-sup condition, utilizing Lemma 5, Lemma 3, and Lemma 4.

THEOREM 2

The operator defined by the constrained minimization problem (3.29) is well-defined and continuous,

$$\Pi^{\text{curl}} : H(\text{curl}, K) \to Q^{p+2}, \qquad \|\Pi^{\text{curl}} F\|_{H(\text{curl},K)} \leq C_{\Pi^{\text{curl}}} \|F\|_{H(\text{curl},K)} \,.$$

The continuity constant $C_{\Pi^{\text{curl}}}$ is independent of element size but it may depend upon the polynomial order p. ▮

We conclude this section by observing the action of operator Π^{curl} on a gradient, i.e. for $F = \nabla u$. It follows from the construction that $\text{curl}(\Pi^{\text{curl}} \nabla u) = 0$, so the constrained minimization problem to determine F^{p+2} simplifies to:

$$\|F^{p+2} - \nabla u\| \to \min_{F^{p+2} \in Q^{p+2}(\text{curl}_0)} \quad \text{subject to constraints } (2.12)_1 \tag{3.31}$$

where $Q^{p+2}(\text{curl}_0)$ denotes the subspace of Q^{p+2} of curl-free functions.

3.3 Π^{grad} Fortin Operator

By now, the reader should anticipate the construction and should be able to fill in all necessary details. We seek $F^{p+3} := \nabla \Pi^{\text{grad}} u$ in the subspace of curl-free functions from a larger space $Q^{p+3} \supset \mathcal{N}^{p+3}(K)$. In other words, we require that $\nabla W^{p+3} \supset \mathcal{P}^{p+2}(K)^3$.

$$\|F^{p+3} - \nabla u\| \to \min_{F^{p+3} \in \nabla W^{p+3}} \quad \text{subject to constraints } (2.20)\,. \tag{3.32}$$

We formulate now a constrained minimization problem defining $\Pi^{\text{grad}} u$,

$$\begin{aligned}
&\Pi^{\text{grad}} : H^1(K) \to W^{p+3}, \quad \Pi^{\text{grad}} u := u^{p+3} \in W^{p+3} \\
&\|u^{p+3} - u\| \to \min_{u^{p+3} \in W^{p+3}} \ \text{subject to constraints: } (2.19) \text{ and the constraint on gradient :} \\
&\nabla u^{p+3} = F^{p+3}\,.
\end{aligned} \tag{3.33}$$

It follows from Lemma 7 that the problem is well-posed, provided we satisfy the minimum assumption on the enriched H^1 test space:

$$\mathcal{P}^{p+3}(K) \subset W^{p+3}\,.$$

The constrained minimization problem above is equivalent to the mixed problem:

$$\left\{\begin{array}{lll}
u^{p+3} \in W^{p+3},\ \psi \in \mathcal{P}^{p-1}(K)^3, \phi \in \gamma_n(\mathcal{P}^p(K)^3),\ \tau \in Q_0^{p+2} & & \\
(u^{p+3}, \delta u)_K + (\psi, \delta u)_K + \langle \phi, \delta u \rangle_{\partial K} + (F, \nabla \delta u)_K & = (u, \delta u)_K & \delta u \in W^{p+3} \\
(\delta\psi, u^{p+3})_K & = (\delta\psi, u)_K & \delta\psi \in \mathcal{P}^{p-1}(K)^3 \\
\langle \delta\phi, u^{p+3} \rangle_{\partial K} & = \langle \delta\phi, u \rangle_{\partial K} & \delta\phi \in \gamma_n(\mathcal{P}^p(K)^3) \\
(\delta F, \nabla u^{p+3})_K & = (\delta\tau, \nabla u)_K & \delta F \in Q_0^{p+2}
\end{array}\right. \tag{3.34}$$

where Q_0^{p+2} is the subspace of ∇W^{p+3} satisfying constraints (2.20). We use the same arguments as for the Π^{div} and Π^{curl} operators to prove the LBB inf-sup condition, utilizing Lemma 7, Lemma 1, and Lemma 2.

THEOREM 3
The operator defined by the constrained minimization problem (3.33) is well-defined and continuous,

$$\Pi^{\text{grad}} : H^1(K) \to W^{p+3}, \qquad \|\Pi^{\text{grad}} u\|_{H^1(K)} \le C_{\Pi^{\text{grad}}} \|u\|_{H^1(K)} \,.$$

The continuity constant $C_{\Pi^{\text{grad}}}$ is independent of element size but it may depend upon the polynomial order p. ▮

4 Conclusions

The main contribution of this note lies in the proofs of Lemmas 5,6 and 7. The results presented in these lemmas can be concisely stated by claiming the exact sequence:

$$W_0^{p-1} \xrightarrow{\nabla} Q_0^p \xrightarrow{\nabla\times} V_0^{p+1} \xrightarrow{\nabla\cdot} Y_0^{p+2}$$

where the involved spaces are subspaces of exact sequence spaces W, Q, V, Y defined on element K and satisfying the constraints:

$$\begin{aligned}
W_0^{p-1} &:= \left\{ u \in W : \begin{array}{rll} (\psi, u)_K &= 0 & \psi \in \mathcal{P}^{p-1}(K) \\ \langle \phi, u \rangle_{\partial K} &= 0 & \phi \in \gamma_n(\mathcal{P}^p(K)^3) \end{array} \right\} \\
Q_0^p &:= \left\{ F \in Q : \begin{array}{rll} (\psi, F)_K &= 0 & \psi \in \mathcal{P}^p(K)^3 \\ \langle n \times \phi, F \rangle_{\partial K} &= 0 & \phi \in \gamma_t(\mathcal{P}^{p+1}(K)^3) \end{array} \right\} \\
V_0^{p+1} &:= \left\{ \tau \in Q : \begin{array}{rll} (\psi, \tau)_K &= 0 & \psi \in \mathcal{P}^{p+1}(K)^3 \\ \langle \phi, \tau \cdot n \rangle_{\partial K} &= 0 & \phi \in \gamma(\mathcal{P}^{p+2}(K)) \end{array} \right\} \\
Y_0^{p+2} &:= \{ y \in Y : (\psi, y)_K = 0 \quad \psi \in \mathcal{P}^{p+2}(K)^3 \}
\end{aligned}$$

under the assumptions:

$$\mathcal{P}^{p+3}(K) \subset W, \quad \mathcal{N}^{p+3}(K) \subset Q, \quad \mathcal{RT}^{p+3}(K) \subset V, \quad \mathcal{P}^{p+3}(K) \subset Y$$

comp. [4]. The exact sequence enables the definition of the Fortin operators using the double minimization paradigm. I have every reason to believe that the construction extends to differential forms. I am also hoping that it is general enough to be extended to elements of differents shapes.

Exercises

Exercise 1

Prove Lemma 7. *Hint:* Recall that if $\psi \in \mathcal{P}^{p-1}(K)$ with zero average then there exists a polynomial $v \in \mathcal{P}^p(K)^3$ such that $\operatorname{div} v = \psi$.

$\blacksquare$

Exercise 2

Let $A : U \to V$ be a well-defined linear operator from a Banach space U into a Banach space V.

- Argue that, for every $u \in U$, there exists a constant C_u such that

$$\|Au\|_V \leq C_u \,.$$

- Use the Uniform Boundedness Theorem to conclude that A is uniformly bounded on the unit ball, i.e. there exists a constant C such that

$$\|Au\|_V \leq C \quad \|u\|_U \leq 1 \,.$$

- Conclude that A is continuous.

$\blacksquare$

Exercise 3

Let $u = (u_1, u_2, u_3) \in U_1 \times U_2 \times U_3$ be a group variable where U_1, U_2, U_3 are Hilbert spaces. Consider a composite bilinear form,

$$b(u,v) := b_1(u_1,v) + b_2(u_2,v) + b_3(u_3,v)$$

where $v \in V$, a Hilbert test space. Define the kernel spaces

$$\begin{aligned} V_{12} &:= \{v \in V \,:\, b_1(u_1,v) + b_2(u_2,v) = 0 \quad u_1 \in U_1,\ u_2 \in U_2\} \\ V_1 &:= \{v \in V \,:\, b_1(u_1,v) = 0 \quad u_1 \in U_1\} \end{aligned}$$

and assume three inf-sup conditions:

$$\begin{aligned} \sup_{v_{12} \in V_{12}} \frac{|b_3(u_3, v_{12})|}{\|v_{12}\|_V} &\geq \gamma_3 \|u_3\|_{U_3} \\ \sup_{v_1 \in V_1} \frac{|b_2(u_2, v_1)|}{\|v_1\|_V} &\geq \gamma_2 \|u_2\|_{U_2} \\ \sup_{v \in V} \frac{|b_1(u_1, v)|}{\|v\|_V} &\geq \gamma_1 \|u_1\|_{U_1} \,. \end{aligned}$$

Show that there exists a constant $\gamma = \gamma(\gamma_1, \gamma_2, \gamma_3, \|b_1\|, \|b_2\|)$ such that,

$$\sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \left(\|u_1\|^2_{U_1} + \|u_2\|^2_{U_2} + \|u_3\|^2_{U_3} \right)^{1/2} .$$

$\blacksquare$

Exercise 4

Prove Lemma 4. $\blacksquare$

Acknowledgments. The work was partially supported with NSF grant No. 1819101 and AFRL grant No. FA9550-17-1-0090.

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