Oden Institute REPORT 19-07

May 2019

The Double Adaptivity Paradigm (How to circumvent the discrete inf-sup conditions of Babuška and Brezzi)

by

Leszek Demkowicz, Thomas Führer, Norbert Heuer, and Xiaochuan Tian

Oden Institute for Computational Engineering and Sciences
The University of Texas at Austin
Austin, Texas 78712

Reference: Leszek Demkowicz, Thomas Führer, Norbert Heuer, and Xiaochuan Tian, "The Double Adaptivity Paradigm (How to circumvent the discrete inf-sup conditions of Babuška and Brezzi) ," Oden Institute REPORT 19-07, Oden Institute for Computational Engineering and Sciences, The University of Texas at Austin, May 2019.

THE DOUBLE ADAPTIVITY PARADIGM

(How to circumvent the discrete inf-sup conditions of Babuška and Brezzi)

Leszek Demkowicz[a], Thomas Führer[b], Norbert Heuer[b] and Xiaochuan Tian[c]

[a]Oden Institute, The University of Texas at Austin
[b]Dept. of Mathematics, Pontificia Universidad Catolica de Chile
[c]Dept. of Mathematics, The University of Texas at Austin

Abstract

We present an efficient implementation of the double adaptivity algorithm of Cohen, Dahmen and Welpert within the setting of the Petrov-Galerkin Method with Optimal Test Functions, for both classical and the ultraweak variational formulations, and a standard Galerkin Finite Element (FE) technology. We use a 1D convection-dominated diffusion problem as an example but the presented ideas apply to virtually any well posed system of first order PDEs including singular perturbation problems.

1 Introduction

Duality pairings. Let U, V be two Hilbert spaces. A bilinear (sesquilinear in the complex case) form $b(u,v),\ u \in U, v \in V$ is called a *duality pairing* if the following relations hold:

$$\|u\|_U = \|b(u,\cdot)\|_{V'} = \sup_{v\in V} \frac{|b(u,v)|}{\|v\|_V} \quad \text{and} \quad \|v\|_V = \|b(\cdot,v)\|_{U'} = \sup_{u\in U} \frac{|b(u,v)|}{\|u\|_U}\,. \tag{1.1}$$

In particular, the bilinear form is *definite*, i.e.

$$b(u,v) = 0 \quad \forall v \in V \quad \Rightarrow \quad u = 0 \qquad \text{and} \qquad b(u,v) = 0 \quad \forall u \in u \quad \Rightarrow \quad v = 0\,.$$

The definition is motivated with the standard duality pairing, where $V = U'$, $b(u,v) = \langle u, v\rangle := v(u)$, and the (induced) norm in the dual space is defined by:

$$\|v\|_{U'} := \sup_{u\in U} \frac{|\langle u,v\rangle|}{\|u\|_U}\,.$$

For non-trivial examples of duality pairings for trace spaces of the exact sequence energy spaces, see [10]. As in the case of the classical duality pairing, any definite bilinear (sesquilinear) form *can be made in a duality pairing* if we equip V with the norm induced by the norm on U or, vice versa, space U with the norm induced by the norm on V. In other words, if we equip V with the norm induced by norm $\|u\|_U$ and use it to induce a norm on U, we recover the original norm on U,

$$\|v\|_V := \sup_{u\in U} \frac{|b(u,v)|}{\|u\|_U} \qquad \Rightarrow \qquad \sup_{v\in V} \frac{|b(u,v)|}{\|v\|_V} = \|u\|_U\,.$$

Petrov-Galerkin method with optimal test functions Consider an abstract variational problem:

$$\left\{ \begin{array}{l} u \in U \\ b(u,v) = l(v)\,, \quad v \in V \end{array} \right. \tag{1.2}$$

where U is a Hilbert trial space, and V is a Hilbert test space, $l \in V'$, and $b(u,v)$ is a bilinear (sesquilinear) form satisfying the inf-sup condition,

$$\inf_{u\in U} \sup_{v\in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma > 0 \qquad \Leftrightarrow \qquad \sup_{v\in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U \quad \forall u \in U\,.$$

For simplicity, we will assume that b is definite, i.e.

$$b(u,v) = 0 \quad \forall u \in U \qquad \Rightarrow \qquad v = 0\,.$$

By the Banach-Babuška-Nečas Theorem (see [20], Thm. 6.6.1) the problem is well-posed, i.e. it possesses a unique solution u that depends continuously on l.

Petrov-Galerkin (PG) method assumes existence of finite dimensional trial and test spaces,

$$U_h \subset U, \quad V_h \subset V, \quad \dim U_h = \dim V_h$$

and formulates a finite-dimensional *approximate problem*:

$$\left\{ \begin{array}{l} u_h \in U_h \\ b(u_h,v_h) = l(v_h)\,, \quad v_h \in V_h\,. \end{array} \right. \tag{1.3}$$

With the approximate trial and test spaces of equal dimension, the approximate problem translates into a system of linear algebraic equations to solve. If the choice of approximate trial space U_h is dictated by *approximability*, i.e. we use approximate functions that have a chance to approximate well the exact solution, the approximate test space is usually selected in an ad hoc way. In particular, very often it is completely uninformed about the choice of the test norm. This is especially the case when $V = U$ and the choice $V_h = U_h$, leading to the classical Bubnov-Galerkin method, is feasible.

The situation is completely different in the approach proposed by Cohen, Dahmen and Welpert [8] where the original problem is embedded into a mixed problem,

$$\left\{ \begin{array}{lll} \psi \in V\,, u \in U & & \\ (\psi, v)_V + b(u,v) & = l(v) & v \in V \\ b(\delta u, \psi) & = 0 & \delta u \in U\,. \end{array} \right. \tag{1.4}$$

Function $\psi \in V$ is identified as the *Riesz representation of the residual*:

$$(\psi, v)_V = l(v) - b(u,v) \quad v \in V$$

and, on the continuous level, is zero. Obviously, both formulations deliver the same solution u. This is no longer true on the approximate level. The *Ideal Petrov-Galerkin Method with Optimal Test Functions* seeks an approximate solution $\widetilde{u}_h \in U_h$ along with the corresponding exact (Riesz representation of) residual $\psi^h \in V$ that solves the semi-discrete mixed problem:

$$\left\{\begin{array}{lll} \psi^h \in V\,, \widetilde{u}_h \in U_h & & \\ (\psi^h, v)_V + b(\widetilde{u}_h, v) & = l(v) & v \in V \\ b(\delta u_h, \psi^h) & = 0 & \delta u_h \in U_h\,. \end{array}\right. \tag{1.5}$$

The name of the method refers to the fact that the mixed problem is equivalent to the original PG scheme with approximate test space $V_h = TU_h$ where

$$T\,:\,U \to V\,, \qquad (Tu, v)_V = b(u, v) \quad v \in V\,.$$

In other words, $T = R_V^{-1}B$ where $B\,:\,U \to V'$ is the operator generated by form $b(u, v)$, and R_V is the Riesz operator generated by the test inner product $(\psi, v)_V$. The ideal PG method with optimal test functions delivers orthogonal projection $\widetilde{u}_h$ in the norm induced by the test norm (called the *energy norm* in the original contribution [11]). Note two critical points: a/ the approximate solution $\widetilde{u}_h$ comes with the residual ψ^h which provides a perfect a-posteriori error estimate enabling adaptivity in the trial space, b/ if we use the test norm induced by the trial norm, the ideal PG scheme will deliver orthogonal projection in the trial norm that coincides now with the "energy norm" [22]. We shall call the test norm induced by the trial norm on U, the *optimal test norm*, and denote it by $\|v\|_{V_{\text{opt}}}$.

Obviously, we cannot code the ideal PG method. We need to approximate space V with some subspace $V_h \subset V$ as well. The ultimate approximate problem reads as follows.

$$\left\{\begin{array}{lll} \psi_h \in V_h\,, u_h \in U_h & & \\ (\psi_h, v_h)_V + b(u_h, v_h) & = l(v_h) & v_h \in V_h \\ b(\delta u_h, \psi_h) & = 0 & \delta u_h \in U_h\,. \end{array}\right. \tag{1.6}$$

This is the *Practical PG Method with Optimal Test Functions*. Brezzi's theory tells us that we have to satisfy now two discrete inf-sup conditions. The *inf-sup in kernel* is trivially satisfied because of the presence of the test inner product. The discrete LBB condition,

$$\sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{\|v_h\|_V} \geq \gamma \|u_h\|_U\,, \quad u_h \in U_h\,,$$

coincides with the discrete Babuška condition for the original problem but *it is much easier now to be satisfied as we can employ test spaces of larger dimension*:

$$\dim V_h >> \dim U_h\,.$$

The *Discontinuous* Petrov-Galerkin method [11] employs variational formulations with *discontinuous* (broken, product) test spaces, and the standard way to guarantee the satisfaction of the discrete LBB condition is to use *enriched* test spaces with order $r = p + \Delta p$ where p is the polynomial order of the trial space.

With broken test space, the Gram matrix corresponding to the inner product becomes block diagonal, and the (Riesz representation of) residual ψ_h can be eliminated at the element level. The satisfaction of discrete LBB condition can be assured by constructing *local* Fortin operators [16, 18, 5, 15]. For standard elliptic problems, we arrive typically at condition $\Delta p \geq N$ for N-dimensional problems. The local construction of the Fortin operator is equivalent to the satisfaction of the discrete LBB condition *on the element level* which in turn implies the satisfaction of the global condition. Alternatively, one can construct global Fortin operators that yield much sharper estimates for a minimum enrichment Δp but the proofs get much more technical [6].

Double adaptivity. The revolutionary idea of Cohen, Dahmen and Welpert [8] was to propose to determine an optimal discrete test space V_h using adaptivity. After all, the fully discrete mixed problem (1.6) is supposed to be an approximation of the semi-discrete mixed problem (1.5). Both problems share the same discrete trial space U_h and the task is to determine a good approximation $\psi_h \in V_h$ to the ideal $\psi^h \in V$ in terms of the test norm. This, hopefully, should guarantee that the corresponding ultimate discrete solution $u_h \in U_h$ approximates well the ideal discrete solution $\widetilde{u}_h \in U_h$ as well. We arrive at the concept of the double adaptivity algorithm described below.

Given error tolerances $\mathrm{tol}_U, \mathrm{tol}_V$ for the trial and test mesh, we proceed as follows.

```
Set initial trial mesh $U_h$
do
  (re)set the test mesh $V_h$ to coincide with the trial mesh $U_h$
  do
    solve the problem on the current meshes
    estimate error $\mathrm{err}_V := \|\psi^h - \psi_h\|_V$ and compute norm $\|\psi_h\|_V$
    if $\mathrm{err}_V/\|\psi_h\|_V < \mathrm{tol}_V$ exit the inner (test) loop
    adapt the test mesh $V_h$ using element contributions of $\mathrm{err}_V$
  enddo
  compute trial norm of the solution $\|u_h\|_U$
  if $\|\psi_h\|_V/\|u_h\|_U < \mathrm{tol}_U$ STOP
  use element contributions to $\|\psi_h\|_V$ to refine the trial mesh
enddo
```

A few comments are in place. By setting the test mesh to trial mesh, we mean the mesh and the corresponding data structure. The corresponding trial and test energy spaces may be different, dependent upon the variational formulation. There are two main challenges in implementing the method. The first one is on the coding side. As the logic of the double adaptivity calls for two independent meshes, developing an adaptive code in this context seems to be very non-trivial. We have resolved this problem successfully by using *pointers* in our Fortran 90 hp FE codes. With a very little investment, the pointer technology allows for converting an adaptive code supporting one mesh into a code supporting two or more *independent meshes*.

The second challenge lies in developing a reliable a-posteriori error estimation technique for the inner (test) adaptivity loop. After several unsuccessful attempts we have converged to a duality technique described here. It is in the context of the duality-based error estimation that the *ultraweak variational formulation* distinguishes itself from other formulations as we hope to communicate it in this paper.

Scope of this paper. We hope to convince the reader that the presented ideas are very general and can be implemented in any space dimension but, in this work, we illustrate them with a baby 1D example of *convection dominated diffusion* (the *confusion* problem). We do strive though for the range of small viscosities all the way to $\epsilon = 10^{-7}$, a value relevant for compressible Navier-Stokes equations. Building a fully adaptive methodology for this class of problems with a double precision (only) has always been a challenge. Section 2 introduces the model problem and illustrates the ideas of optimal test norm and double adaptivity for the classical variational formulation (Principle of Virtual Work). In Section 3 we discuss the same ideas for the ultraweak variational formulation and present the main idea of this paper - the a-posteriori error estimation for the inner adaptivity loop based on the duality theory. In Section 4 we return to the classical variational formulation to extend the duality error estimation technique for this case as well. The order of the presentation reflects our learning process and technical difficulties as well. We conclude the main body of the presentation with personal comments in Section 5. Throughout the whole presentation, we intertwine the presented theory immediately with numerical examples. As the presented methodology is far from the classical finite element algorithms, this seems to be the best way to communicate effectively the new concepts. The main body of the paper is complemented with two appendices. In Appendix 1 we present a stability analysis for the first order system of Broersen and Stevenson, and in Appendix 2 we offer a detailed derivation of dual problems for both variational formulations proving in particular that there is no duality gap at the continuous level.

2 Classical Variational Formulation

We shall consider the following 1D model *confusion* problem.

$$\left\{ \begin{array}{l} u(0) = u(1) = 0 \\ -\epsilon u'' + u' = f \quad \text{in } (0,1)\,. \end{array} \right. \tag{2.7}$$

In all numerical experiments, we shall use a single example with $f = 1$ for which the exact solution is readily available.

Multiplying the equation with test functions v vanishing at the end-points, integrating over the domain, and integrating by parts, we arrive at the *classical variational formulation*:

$$\left\{ \begin{array}{l} u \in H_0^1(0,1) \\ \epsilon(u',v') + (u',v) = (f,v) \quad v \in H_0^1(0,1)\,. \end{array} \right. \tag{2.8}$$

As usual, the parenthesis (u, v) denote the $L^2(0,1)$ inner product, with the corresponding L^2-norm denoted by $\|u\|$ without any symbol for the space.

Derivation of the optimal test norm. The formulation admits a *symmetric functional setting*, i.e. the trial and test spaces are the same. For the trial space we will employ the H_0^1-norm, i.e. the H^1-seminorm,

$$\|u\|_U := \|u'\| \,.$$

The reason for this choice is two-fold: a/ the orthogonal projection in this norm delivers *exact values at the vertex nodes*, i.e. the approximate solution $\widetilde{u}_h$ interpolates the exact solution u at vertex nodes, b/ for this trial norm, we can derive analytically the corresponding optimal test norm. Recall the definition of the optimal test norm,

$$\|v\|_{V_{\rm opt}} = \|b(\cdot, v)\|_{U'} = \|u_v\|_U$$

where u_v is the Riesz representation of functional $b(\cdot, v)$, i.e. it solves the variational problem:

$$\left\{ \begin{array}{l} u_v \in H_0^1(0,1) \\ ((\delta u)', u_v') = (\delta u, u_v)_U = b(\delta u, v) = \epsilon((\delta u)', v') + ((\delta u)', v) \quad \delta u \in H_0^1(0,1)\,. \end{array} \right.$$

Consequently, u_v satisfies the equation:

$$-u_v'' = -\epsilon v'' - v' \,.$$

Integrating,

$$u_v' = \epsilon v' + v - C$$

and integrating over $(0,1)$, with the help of BCs, we obtain: $C = \int_0^1 v$. This leads to the formula for the optimal test norm,

$$\|v\|^2_{V_{\rm opt}} = \|u_v'\|^2 = \|\epsilon v' + v - \int_0^1 v\|^2 = \epsilon^2 \|v'\|^2 + \|v\|^2 - (\int_0^1 v)^2 \,.$$

If we can use this test norm in the double adaptivity algorithm, the PG method with optimal test functions should deliver a vertex interpolant of u. Note that the optimal test norm includes the global term and therefore is not *localizable*, i.e. it cannot be used in the DPG setting where we work with a *broken* $H^1(\mathcal{T}_h)$ test space.

The concept of the optimal test norm and the importance of resolving the residual are illustrated in Fig. 1. In all cases we use the same trial mesh of five quadratic elements and refine the test mesh starting with test mesh coinciding with the trial mesh. For test mesh coinciding with the trial mesh, ψ_h vanishes, and the solution coincides with the standard Bubnov-Galerkin solution. Note that this is true for *any test norm*. In other words, the choice of the test norm matters only if $\dim V_h > \dim U_h$. The second row shows the approximate solution obtained with a test mesh of elements of order $p = 4$, and the corresponding approximate residual ψ_h. Oscillations have disappeared but the solution is still far from the H_0^1-projection.

The third row presents the analogous results for the test mesh with $p = 7$. With this test mesh, the exact residual has been resolved and the method delivers the H^1_0-projection, as promised. Finally, the last row shows the effect of refining globally the last test mesh. The residual is now better resolved but refining the test mesh has little effect on the approximate solution. Clearly, the extra refinement of the test mesh is not necessary.

Solution of the mixed problem. The algebraic structure of the mixed problem looks as follows.

$$\begin{pmatrix} G & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \psi \\ u \end{pmatrix} = \begin{pmatrix} l \\ 0 \end{pmatrix} \tag{2.9}$$

where ψ and u denote now the vectors of degrees-of-freedom (d.o.f.), B is the rectangular stiffness matrix corresponding to the original problem, and G is the Gram matrix corresponding to the test inner product. Due to the presence of the non-local term in the optimal test product, Gram matrix is fully populated (dense) which seems to necessitate the use of a dense matrix solver, a clear death sentence for the whole methodology. Fortunately, the global term represents a rank one contribution only and the use of standard banded matrix solvers (including classical frontal solver w/o pivoting) is still possible. In order to see that, we rewrite first system (2.9) using indices,

$$\left\{ \begin{array}{lll} (\bar{G}_{ik} + d_i d_k)\psi_k + B_{il} u_l & = l_i & \quad i = 1, \ldots, m \\ B_{kj}\psi_k & = 0 & \quad j = 1, \ldots, n \,. \end{array} \right.$$

With e_k, $k = 1, \ldots, n$ and g_l, $l = 1, \ldots, m$ denoting the global basis functions for the trial and test spaces, the corresponding matrices are defined as follows.

$$\begin{aligned} \bar{G}_{ik} &= \int_0^1 \epsilon^2 g_i' g_k' + g_i g_k \\ d_i &= \int_0^1 g_i \\ B_{il} &= \int_0^1 \epsilon e_l' g_i' + e_l' g_i \\ l_i &= \int_0^1 f g_i \,. \end{aligned}$$

Denote now $d_k \psi_k =: \lambda$ and solve the mixed system without the global term contribution to the Gram matrix twice, first for the original load, and second time with load d_i,

$$\left\{ \begin{array}{ll} \bar{G}_{ik}\psi^1_k + B_{il} u^1_l & = l_i \\ B_{kj}\psi^1_k & = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} \bar{G}_{ik}\psi^2_k + B_{il} u^2_l & = d_i \\ B_{kj}\psi^2_k & = 0 \end{array} \right. . \tag{2.10}$$

Clearly, by superposition, the solution to the original problem is:

$$\psi_k = \psi^1_k - \lambda \psi^2_k, \qquad u_l = u^1_l - \lambda u^2_l \,.$$

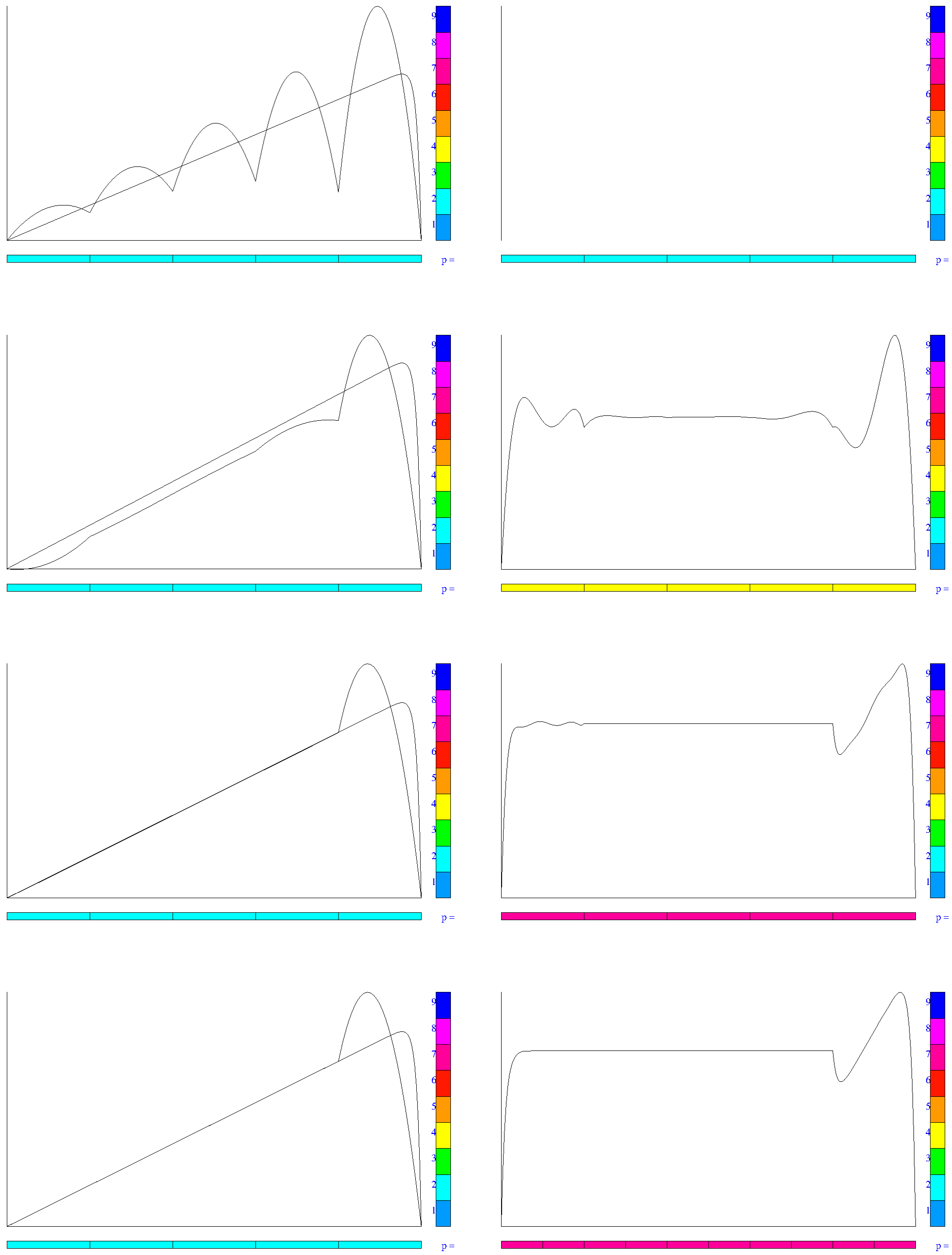

Figure 1: $\epsilon = 10^{-2}$. Left: Evolution of the approximate solution on a trial mesh of five quadratic elements corresponding to different test meshes. Right: The test mesh with the corresponding approximate residual ψ_h.

The constant λ is determined by solving the linear equation:

$$\lambda = d_k\psi_k = d_k(\psi_k^1 - \lambda\psi_k^2) \qquad \Rightarrow \qquad \lambda = \frac{d_k\psi_k^1}{1 + d_k\psi_k^2}\,.$$

The solution of the original system of equations is obtained thus by solving the band matrix mixed system (2.10) with two load vectors (using e.g. a frontal solver), computing constant λ, and using the superposition as explained above. The technique is known as *Sherman-Morrison formula.*

Other Boundary Conditions. The formula for the optimal test norm (and the presence of the global term) is a consequence of the particular boundary conditions (BCs). Consider another (physically meaningful) set of BCs:

$$\begin{aligned} -\epsilon u' + u &= 0 \qquad \text{at} \quad x = 0 \\ u &= 0 \qquad \text{at} \quad x = 1\,. \end{aligned} \tag{2.11}$$

We have now,

$$U = V := \{v \in H^1(0,1) \,:\, v(1) = 0\}$$

and the bilinear form is given by:

$$b(u,v) = \epsilon(u',v') - (u,v')\,.$$

Note that the transport term has now been integrated by parts as well. Employing the same trial inner product, determination of the optimal test norm is reduced to the solution of the variational problem:

$$\begin{cases} u_v \in U \\ ((\delta u)', u_v') = (\delta u, u_v)_U = b(\delta u, v) = \epsilon((\delta u)', v') - (\delta u, v') \quad \delta u \in U\,. \end{cases}$$

This is equivalent to the following BVP,

$$\begin{aligned} -u_v'' &= -\epsilon v'' - v' && \text{in } (0,1) \\ -u_v' &= -\epsilon v' && \text{at } x = 0 \\ u_v &= 0 && \text{at } x = 1\,. \end{aligned}$$

Integrating the first equation and utilizing the BC at $x = 0$, we obtain

$$-u_v' = -\epsilon v' - v + v(0)\,.$$

This leads to the following formula for the optimal test norm.

$$\|v\|^2_{V_{\text{opt}}} = \|u_v'\|^2 = \|\epsilon v' + v - v(0)\|^2 \tag{2.12}$$

The global term is now missing.

3 Ultraweak Variational Formulation

In this section we develop a duality based error estimation methodology for another variational formulation of the confusion problem - the *ultraweak (UW) variational formulation*. Contrary to the classical formulation discussed in Section 2, the ultraweak formulation extends naturally to multiple space dimensions. The derivation of the UW formulation starts by rewriting the original problem as a system of first order equations. This involves introducing a new variable and can be done in more than one way. Thus, perhaps, we should talk not about *the* but *an* UW formulation.

In these notes, we will use the formulation advocated by Broersen and Stevenson [3, 4]. Consider a multidimensional version of the confusion problem,

$$\left\{\begin{aligned} u &= 0 && \text{on } \Gamma \\ -\epsilon\Delta u + \beta\cdot\nabla u &= f && \text{in } \Omega\,. \end{aligned}\right. \tag{3.13}$$

We begin by rewriting the second order problem as a system of first order equations,

$$\left\{\begin{aligned} u &= 0 && \text{on } \Gamma \\ \sigma - \epsilon^{\frac{1}{2}}\nabla u &= 0 && \text{in } \Omega \\ -\epsilon^{\frac{1}{2}}\operatorname{div}\sigma + \beta\cdot\nabla u &= f && \text{in } \Omega\,. \end{aligned}\right.$$

The first equation defines the auxiliary variable - a scaled viscous flux. A rationale between splitting the diffusion constant ϵ in between the two equations is motivated with the goal of dealing with bigger numbers than ϵ.

We can rewrite the system using the formalism of closed operators theory. Introducing the first order operator and its L^2-adjoint,

$$\begin{aligned} &\mathsf{u} := (\sigma, u) \in D(A) := H(\operatorname{div},\Omega)\times H_0^1(\Omega) \subset (L^2(\Omega))^N \times L^2(\Omega) \overset{'}{=} \boldsymbol{L}^2(\Omega) \\ &A\,:\,D(A) \to \boldsymbol{L}^2(\Omega)\,, A\mathsf{u} = A(\sigma,u) := (\sigma - \epsilon^{\frac{1}{2}}\nabla u, -\epsilon^{\frac{1}{2}}\operatorname{div}\sigma + \beta\cdot\nabla u) \\ &\mathsf{v} := (\tau, v) \in D(A^*) = D(A) \\ &A^*\,:\,D(A^*) \to \boldsymbol{L}^2(\Omega)\,, A^*\mathsf{v} = A^*(\tau,v) = (\tau + \epsilon^{\frac{1}{2}}\nabla v, \epsilon^{\frac{1}{2}}\operatorname{div}\tau - \operatorname{div}(\beta v)) \end{aligned} \tag{3.14}$$

we can rewrite the problem in a concise form as:

$$\left\{\begin{aligned} &\mathsf{u} \in D(A) \\ &A\mathsf{u} = \mathsf{f} \end{aligned}\right.$$

where $\mathsf{f} = (0, f)$.

Multiplying the equation with a test function $\mathsf{v} = (\tau, v)$ and integrating by parts, we obtain the UW formulation:

$$\left\{\begin{aligned} &\mathsf{u} \in \boldsymbol{L}^2(\Omega) \\ &(\mathsf{u}, A^*\mathsf{v}) = (\mathsf{f}, \mathsf{v}) \quad \mathsf{v} \in D(A^*)\,. \end{aligned}\right. \tag{3.15}$$

Computing the optimal test norm for the UW formulation is straightforward,

$$\|\mathsf{v}\|_{V_{\mathrm{opt}}} = \sup_{\mathsf{u}} \frac{|(\mathsf{u}, A^*\mathsf{v})|}{\|\mathsf{u}\|} = \|A^*\mathsf{v}\| \,. \tag{3.16}$$

The optimal test norm coincides simply with the *adjoint norm*. Note that the result is independent of considered BCs. The ideal PG method with this norm delivers the L^2-projection of the exact solution. For reasons that will become clear in a moment, we will compute with a related *graph adjoint norm*:

$$\|\mathsf{v}\|^2_{V_{\mathrm{qopt}}} = \|A^*\mathsf{v}\|^2 + \alpha\|\mathsf{v}\|^2 \tag{3.17}$$

with a scaling coefficient α. We have frequently called it the *quasi-optimal test norm*, hence the notation. If A is bounded below with a constant γ independent of ϵ [1], the adjoint graph norm with $\alpha = 1$ is *robustly* equivalent with the adjoint norm. We no longer deliver the L^2-projection but a solution close to it, *uniformly in* ϵ. If γ depends upon ϵ, we can still retain the robust equivalence of adjoint and adjoint graph norms by using a scaling constant α of order γ^2. We have been able to show that for the 1D confusion problem with BCs (2.11), the Broersen-Stevenson operator is indeed bounded below robustly in ϵ, see Appendix 1. Unfortunately, we have not been able to show the same result for the case of pure Dirichlet BC.

Duality theory. We discuss now the main idea of this contribution - the a-posteriori error estimation and adaptivity for the inner loop problem based on the classical duality theory [14]. We offer here a sketch of the main ideas with a more detailed presentation and proof that there is no duality gap at the continuous level in Appendix2.

We begin by noticing that the semi-discrete mixed problem is equivalent to the constrained minimization (primal) problem:

$$\inf_{\substack{\psi \in D(A^*) \\ A^*\psi \in U_h^\perp}} \underbrace{\frac{1}{2}\|A^*\psi\|^2 + \alpha\frac{1}{2}\|\psi\|^2 - (\mathsf{f}, \psi)}_{=:J(\psi)} \tag{3.18}$$

where the $U_h^\perp$ denotes the $L^2(\Omega)$-orthogonal component of trial space U_h. Introducing the auxiliary variable,

$$\sigma = A^*\psi \,,$$

we turn the problem into a saddle point problem,

$$\inf_{\sigma\in U_h^\perp} \inf_{\psi\in D(A^*)} \sup_{\phi\in D(A)} \left\{ \frac{1}{2}\|\sigma\|^2 + \alpha\frac{1}{2}\|\psi\|^2 - (\mathsf{f}, \psi) + (A^*\psi - \sigma, \phi) \right\} \,.$$

Switching the minima with the maximum and computing them explicitly, we obtain,

$$\sigma = \phi^\perp, \qquad \alpha\psi = \mathsf{f} - A\phi \,.$$

[1]Recall that the adjoint A^* is then bounded below with the same constant.

Substituting into the Lagrangian, we obtain the dual problem,

$$\sup_{\phi} \underbrace{-\frac{1}{2}\|\phi^{\perp}\|^2 - \frac{1}{2\alpha}\|\mathsf{f} - A\phi\|^2}_{=:J^*(\phi)} \tag{3.19}$$

Solving approximately the primal and dual problems for ψ_h and ϕ_h, we use again the duality gap $2(J(\psi_h) - J^*(\phi_h))$ to estimate the error in the energy norms,

$$\left.\begin{array}{l} \frac{1}{\alpha}\left\{\alpha\|A^*(\psi-\psi_h)\|^2 + \|\psi-\psi_h\|^2\right\} \\ \frac{1}{\alpha}\left\{\alpha\|\phi^{\perp}-\phi_h^{\perp}\|^2 + \|A(\phi-\phi_h)\|^2\right\} \end{array}\right\} \leq 2(J(\psi_h) - J^*(\phi_h))$$

where the duality gap can be expressed as an integral of the consistency terms,

$$2(J(\psi_h) - J^*(\phi_h)) = \frac{1}{\alpha}\int \alpha(A^*\psi_h - \sigma_h)^2 + (\alpha\psi_h - (\mathsf{f} - A\phi_h))^2\,. \tag{3.20}$$

Can we pass with $\alpha \to 0$? Clearly, for small α, the dual problem approached the least squares method for the original problem, and the least squares term dominates the duality gap. The two problems disconnect, and the duality gap is no longer a meaningful estimate for neither primal nor the dual problem. This is consistent with the well known fact that the duality theory for linear elastostatic requires the maximization over stress fields satisfying the equilibrium equations. In our case, we would need to maximize over ϕ_h satisfying the equation $A\phi = \mathsf{f}$. There is only one such a ϕ - the solution to our problem. In conclusion, *we have to compute with finite* α.

Numerical experiments. We start with a moderate value of $\epsilon = 10^{-2}$ to illustrate the algorithm. Our original trial mesh consists of five cubic elements, and the starting test mesh is (re)set to the trial mesh but with elements of one order higher. Note that by the order of elements we mean always the order for the H^1-conforming elements. This means that effectively we approximate σ and u with piece-wise quadratics, and the two components of residual ψ with piece-wise quartic elements. Raising the initial order of test functions is related to the use of a classical frontal solver w/o pivoting. For $p = 1$, and trial and test meshes of equal order, we encounter a zero pivot in the very first element. The tolerance for the outer and inner loop adaptivity is set to 1 and 5 percent, respectively. We use the Dörfler refinement strategy with 1 and 25 percent factors. The first inner loop iterations (a total of 9) are presented in Figures 2 and 3. The solutions seem to evolve very little but the a-posteriori error estimate evolves from a 162 to 4.7 percent of error, see Table 1. Note that the ultimate discrete solution is *not* the L^2-projection of the exact solution. This is a consequence of using the adjoint graph norm rather than the adjoint norm.

The evolution of "trusted" trial solutions along with the corresponding resolved residual is shown in Fig.4. In order to solve the problem with the requested 1 percent of accuracy, the algorithm has performed five outer loop iterations. The corresponding evolution of the error and inner loop duality error estimates is shown in Table 1.

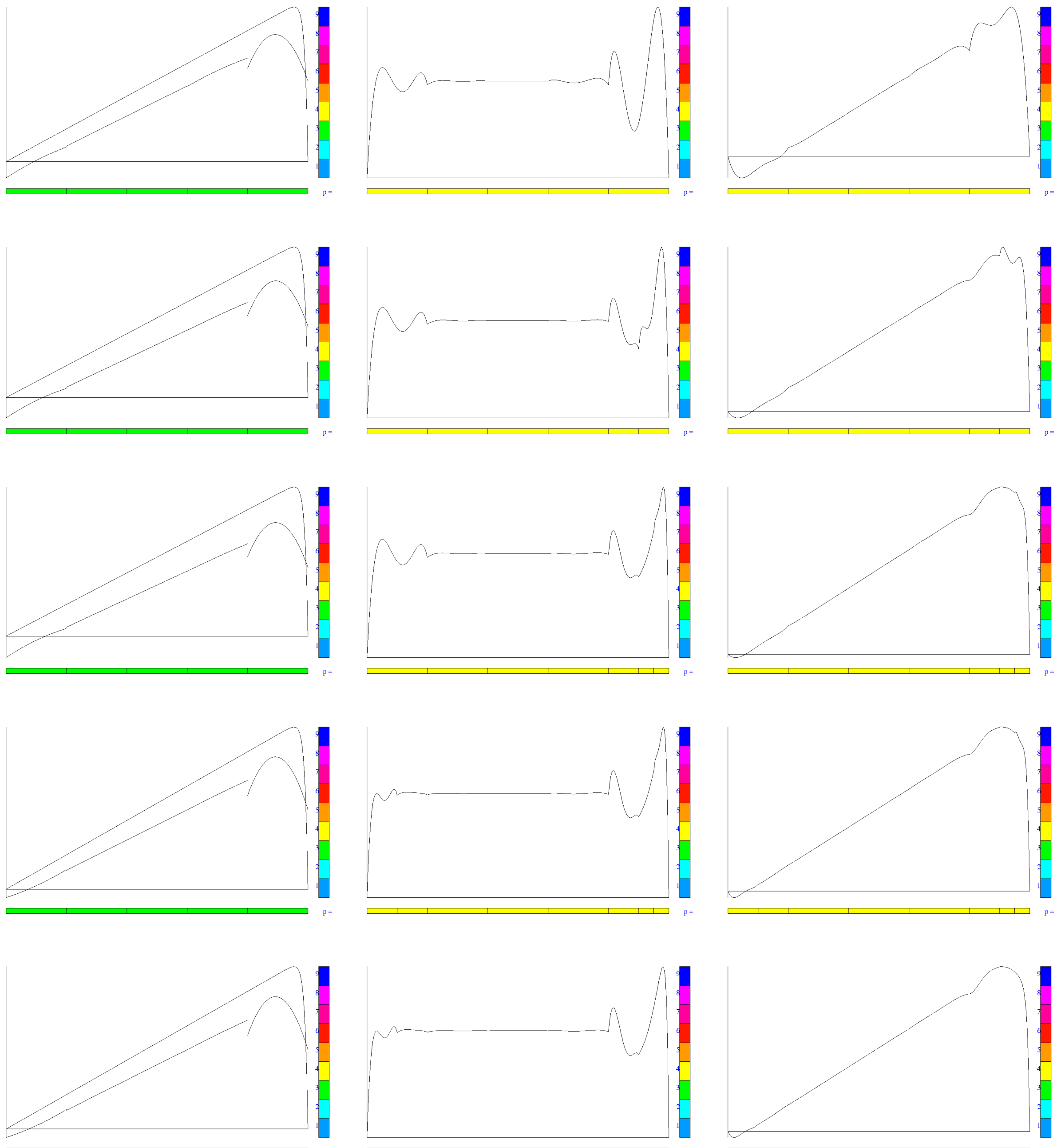

Figure 2: UW formulation, $\epsilon = 10^{-2}$, first inner loop, iterations 1-5. Left: Evolution of the approximate solution u_h on a trial mesh of five cubic elements corresponding to different test meshes. Middle: The test mesh with the corresponding u component of approximate residual ψ_h. Right: The test mesh with the corresponding v component of solution to the dual problem.

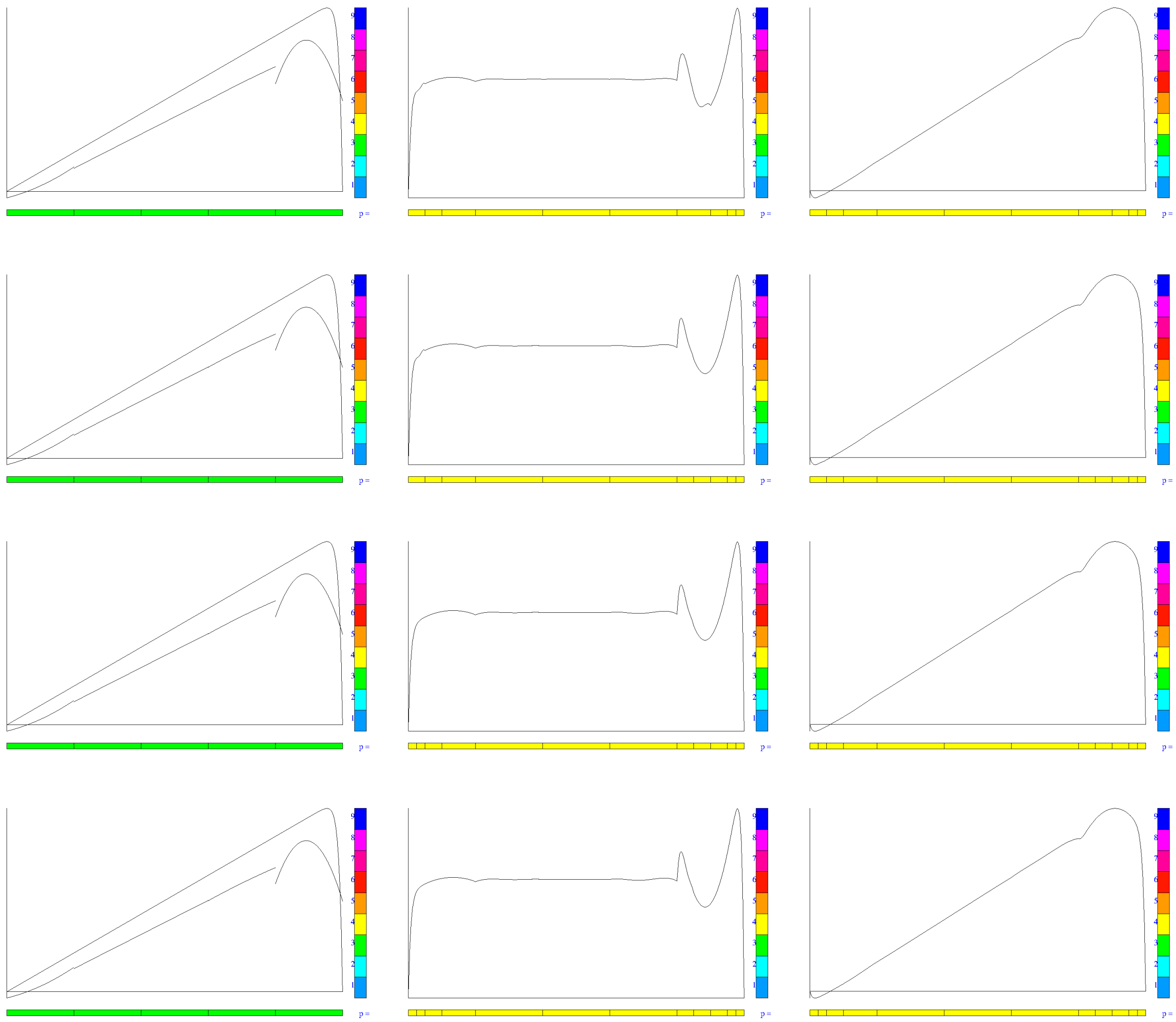

Figure 3: UW formulation, $\epsilon = 10^{-2}$, first inner loop, iterations 6-9. Left: Evolution of the approximate solution u_h on a trial mesh of five cubic elements corresponding to different test meshes. Middle: The test mesh with the corresponding u component of approximate residual ψ_h. Right: The test mesh with the corresponding v component of solution to the dual problem.

Conclusions at this point? 1/ Number of inner loop iterations decreases with the outer loop iterations. 2/ Residual for the unresolved solution has a significant variation not only in the boundary layer but also at the inflow. At the end, the residual around the inflow becomes insignificant, note lack of refinements at the inflow in the last test mesh. Philosophically, we need to think of a new residual after each trial mesh refinement. If we decide to keep the test mesh from the previous inner loop iterations, we need to implement unrefinements as well.

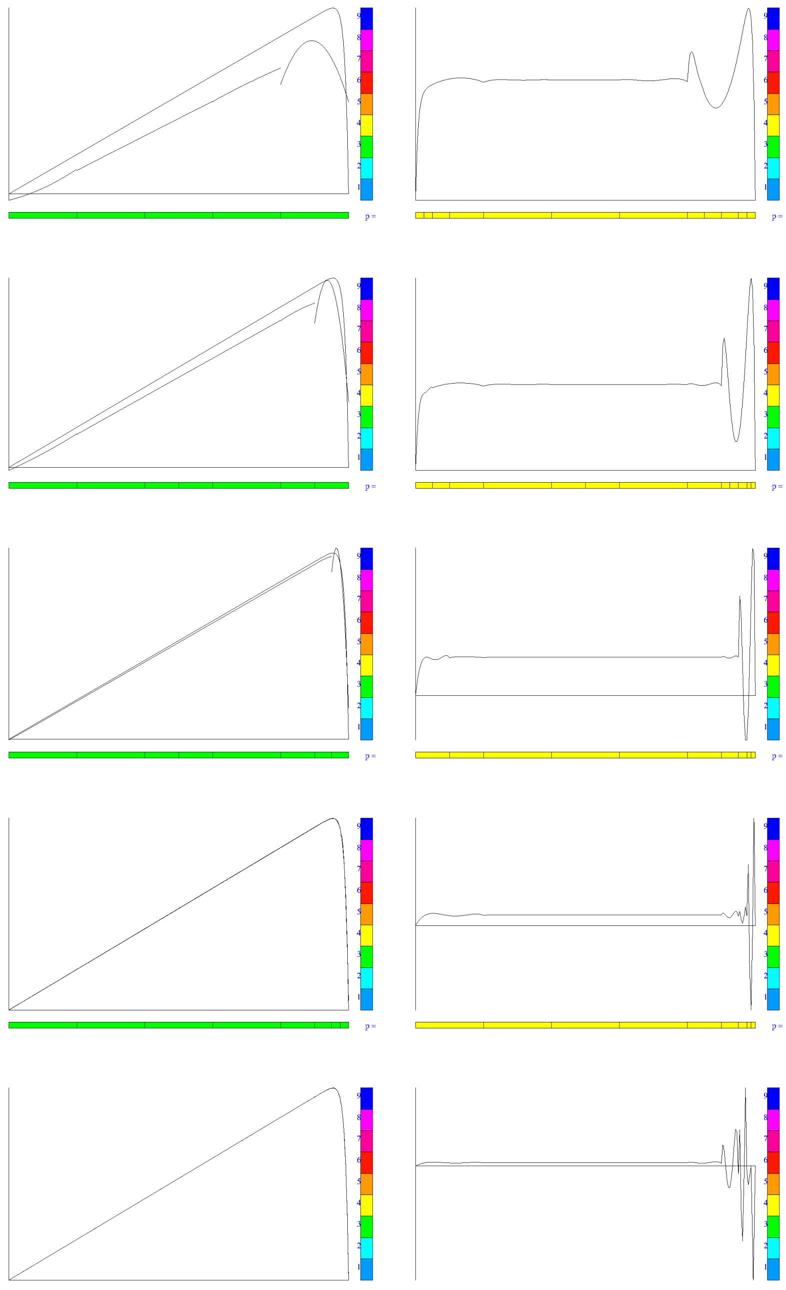

Figure 4: UW formulation, $\epsilon = 10^{-2}$, outer loop, iterations 1-5. Left: Evolution of the approximate solution u_h. Right: The test mesh with the corresponding resolved u component of approximate residual ψ_h.

Pushing the code. We have been able to solve the problem for $\epsilon = 10^{-6}$ but we failed for $\epsilon = 10^{-7}$. The number of inner loop iterations increased significantly with smaller ϵ, and in the end, the inner loop

1	50.6	162.4	76.8	35.9	23.6	14.4	9.0	5.6	4.7
2	27.8	106.3	34.3	20.1	10.3	7.2	5.0	2.7	
3	10.9	59.7	12.1	8.5	4.2				
4	2.6	31.0	4.6						
5	0.4	21.4	16.4	9.7	4.3				

Table 1: UW formulation, $\epsilon = 10^{-2}$. Column 1: Outer loop iteration number. Column 2: Error (residual) estimate for the "trusted" solution. Column 3 and next: evolution of inner loop a-posteriori error estimate.

$\epsilon = 10^{-6}$	33	32	32	31	31	31	30	33	47	45	42	39	37	37	38	41	30	14		
$\epsilon = 10^{-7}$	42	41	41	40	40	40	40	40	40	52	57	56	53	50	47	46	45	46	48	*

Table 2: UW formulation. Number of inner loop iterations for the extreme values of the viscosity constant. The star indicates no convergence

iterations did not converge. We have implemented a number of energy identities which should be satisfied and the code stopped passing those tests. Note that the duality gap estimate *has to decrease* with any mesh refinements. This stopped being the case in the end of the last run. Clearly, we have lost the precision.

We were a bit more lucky using continuation in ϵ. Starting with $\epsilon = 10^{-2}$, we run the double adaptivity algorithm. Upon a convergence, we restarted the algorithm with $\epsilon_{\text{new}} = \epsilon_{\text{old}}/2$ and the initial trial mesh obtained from the previous run. Except for the last couple of cases, number of inner loop iterations dramatically decreased (did not exceed 10) and, in the end, the smallest value of ϵ for which we have been able to solve the problem, was $\epsilon = 3.81410^{-8}$.

3.1 Controlling the Solution Error

The *ideal PG* method (with infinite-dimensional test space) inherits the inf-sup condition from the continuous level. In other words, the operator $B : U \to V'$ generated by the bilinear form $b(u, v)$ is bounded below. This implies that the error $u - \widetilde{u}_h$ is controlled by the residual,

$$\gamma \|u - \widetilde{u}_h\|_U \leq \|l - B\widetilde{u}_h\|_{V'} = \|\psi^h\|_V .$$

Once the residual converges to zero, so must the error, at the same rate. The inner adaptivity loop guarantees that we approximate the (Riesz representation of) residual ψ^h within a required tolerance with ψ_h. But coming with ψ_h is only the approximation u_h of $\widetilde{u}_h$. How do we know that u_h converges to $\widetilde{u}_h$? Can we estimate the difference $\widetilde{u}_h - u_h$? The problem deals again with a mixed problem albeit somehow special - space U_h is finite-dimensional. An attempt to use Brezzi theory makes little sense as it calls for a discrete LBB inf-sup condition which is precisely what we are trying to circumvent.

This is where the duality theory comes to the rescue again. Please visit Appendix 2 to find the critical piece of information: the ideal approximate solution $\widetilde{u}_h$ coincides with the L^2-projection $\widetilde{w}_h$ of the solution

ϕ of the dual problem. The primal problem is a standard [2] mixed problem but the dual problem is a (double) minimization problem. The duality gap used to estimate the error in solution to the primal problem, estimates also the error in the solution of the dual problem,

$$\|A(\phi - \phi_h)\|^2 + \alpha\|\phi^\perp - \phi_h^\perp\|^2 \le 2(J(\psi_h) - J^*(\phi_h)) =: \text{est} \, .$$

Operator A is bounded below,

$$\beta\|\phi - \phi_h\| \le \|A(\phi - \phi_h)\|$$

which implies that

$$\beta^2\|\phi - \phi_h\|^2 \le \text{est} \, .$$

This implies the bound for the projection as well,

$$\beta^2\|\widetilde{\mathsf{w}}_h - \mathsf{w}_h\| \le \beta^2\|\phi - \phi_h\|^2 \le \text{est} \, . \tag{3.21}$$

In conclusion, if we believe in proofs, we should use w_h and not u_h as our final (numerical) solution of the problem. To illustrate the point, we present approximate solution u_h and projection w_h (second components) at the beginning and at the end of the first inner loop for the problem presented in this section, see Fig. 5. As we can see, with an unresolved residual, the two functions are significantly different. However, once the residual has been resolved (error tolerance = 5%), the two solutions are indistinguishable.

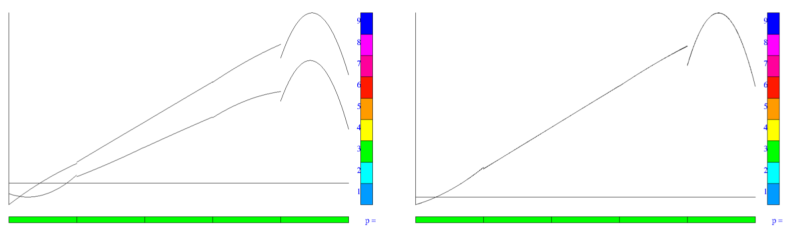

Figure 5: UW formulation, $\epsilon = 10^{-2}$, first inner loop. Second components of projection w_h (top), and approximate solution u_h (bottom). Left: at the beginning of the inner loop. Right: at the end of the loop.

Remark 1 If boundness below constant β depends upon ϵ then, unfortunately, bound (3.21) is *not* robust in ϵ, even if we choose α to be of order β^2. ∎

4 Dual Formulation and Inner Loop Adaptivity for Primal Variational Formulation

In this section we return to the primal formulation of our 1D confusion problem and develop two dual problems that provide a basis for the a-posteriori error estimation and inner loop adaptivity. As we will see,

[2] Originating from a constrained minimization problem.

developing the dual formulations is much less straightforward than for the UW formulation and the adjoint graph norm.

We begin by noticing that the semi-discrete mixed problem is equivalent to the constrained minimization problem:

$$\inf_{\psi \in V^\perp} \frac{1}{2}\|\epsilon\psi'\|^2 + \frac{1}{2}\|\psi - \int_0^1 \psi\|^2 - (f, \psi)\,, \tag{4.22}$$

where

$$V^\perp := \{v \in V : b(\delta u_h, v) = 0 \quad \forall \delta u_h \in U_h\}\,.$$

Note that through integration by parts,

$$\epsilon(u_h', v') + (u_h', v) = 0 \quad \Leftrightarrow \quad (\epsilon u_h' - u_h, v') = 0 \quad \forall u_h \in U_h\,.$$

The orthogonality of v to the trial space through the bilinear form can thus be interpreted in terms of orthogonality of v' to a special finite dimensional space. We will see in the next two different dual formulations slightly different interpretations of this orthogonality. Please refer to Appendix 2 for a more detailed derivation, where the strong duality is shown.

4.1 Dual Formulation I

Introducing finite dimensional space,

$$W_h = \{\epsilon u_h' - u_h \,:\, u_h \in U_h\}\,, \tag{4.23}$$

and the notation $\bar{\psi}$ for $\int_0^1 \psi$, we reformulate the semi-discrete mixed problem as the constrained minimization problem,

$$\inf_{\substack{\psi \in H_0^1(0,1) \\ \psi' \in W_h^\perp}} \underbrace{\frac{1}{2}\|\epsilon\psi'\|^2 + \frac{1}{2}\|\psi - \bar{\psi}\|^2 - (f, \psi)\,,}_{=:J(\psi)} \tag{4.24}$$

where the $W_h^\perp$ denotes the L^2-orthogonal part of W_h. The derivation follows now similar lines to those for the UW formulation. First, we turn the minimization problem into an inf-sup problem,

$$\inf_{\psi \in H_0^1(0,1)} \inf_{\substack{\sigma \in L^2(0,1) \\ \sigma \in W_h^\perp}} \sup_{\phi \in H^1(0,1)} \frac{1}{2}\|\sigma\|^2 + \frac{1}{2}\|\psi - \bar{\psi}\|^2 - (f, \psi) + (\epsilon\psi' - \sigma, \phi)\,.$$

We replace now inf sup with sup inf, and perform minimizations in σ and ψ, leading to the duality relations:

$$\sigma = \phi^\perp, \quad \psi - \bar{\psi} = f + \epsilon\phi'\,.$$

Now since the average of $\psi - \bar{\psi}$ is zero, this forces ϕ to satisfy the periodic boundary condition:

$$\phi(1) - \phi(0) = -\frac{1}{\epsilon}\bar{f}\,.$$

The dual problem is thus given by

$$\sup_{\substack{\phi \in H^1(0,1) \\ \phi\big|_0^1 = -\frac{1}{\epsilon}\bar{f}}} \underbrace{-\frac{1}{2}\|\phi^\perp\|^2 - \frac{1}{2}\|f + \epsilon\phi'\|^2}_{=:J^*(\phi)} . \tag{4.25}$$

4.2 Dual Formulation II

The second dual formulation is realized by doing a slight change of the finite dimensional space W_h. Instead of (4.23), we now define

$$W_h = \{\epsilon u_h' - u_h \,:\, u_h \in U_h\} \oplus \mathbb{R} , \tag{4.26}$$

Noticing that the homogeneous Dirichlet boundary condition for u implies that u' is L^2-orthogonal to constants, we can rewrite the semi-discrete mixed problem into

$$\inf_{\substack{\psi \in H^1(0,1), \psi(0)=1 \\ \psi' \in W_h^\perp}} \underbrace{\frac{1}{2}\|\epsilon\psi'\|^2 + \frac{1}{2}\|\psi - \bar{\psi}\|^2 - (f,\psi)}_{=:J(\psi)} . \tag{4.27}$$

Now turning the minimization problem into a inf-sup problem, we have

$$\inf_{\substack{\psi \in H^1(0,1) \\ \psi(0)=0}} \quad \inf_{\substack{\sigma \in L^2(0,1) \\ \sigma \in W_h^\perp}} \quad \sup_{\substack{\phi \in H^1(0,1) \\ \phi(1)=0}} \quad \frac{1}{2}\|\sigma\|^2 + \frac{1}{2}\|\psi - \bar{\psi}\|^2 - (f,\psi) + (\epsilon\psi' - \sigma, \phi) .$$

Now with the same process of finding the first dual formulation, we arrive at the second dual formulation

$$\sup_{\substack{\phi \in H^1(0,1) \\ \phi(0) = \frac{1}{\epsilon}\bar{f},\ \phi(1)=0}} \underbrace{-\frac{1}{2}\|\phi^\perp\|^2 - \frac{1}{2}\|f + \epsilon\phi'\|^2}_{=:J^*(\phi)} . \tag{4.28}$$

Note that $\phi^\perp$ is the L^2-orthogonal part of ϕ to the finite dimensional space W_h, where we have slightly different definitions of W_h in the two dual formulations.

Numerical experiments. We present numerical results for the first dual problem only. All results were obtained with the same tolerances as for the UW formulation: 5% for the inner loop, and 1% error for the outer loop. As the trial norm is now stronger, these tolerances seem perhaps to be a bit too strict. Visually, one observes perfect resolution of the solution already for bigger errors. Figures 6 and 7 present the outer loop iterations for $\epsilon = 10^{-2}$. As we can see, at least visually, we could have stopped after just four iterations. Notice how the (resolved!) residual changes with the trial mesh. Note also that the solution of the dual problem is rather smooth and gets smoother with iterations. This indicates that the duality gap estimate is a perfect estimate for the error in residual.

Table 3 presents number of inner loop iterations for different values of ϵ. The smallest value for which we have been able to solve the problem, was $\epsilon = 10^{-6}$ and it was clearly on the round-off limit as the inner

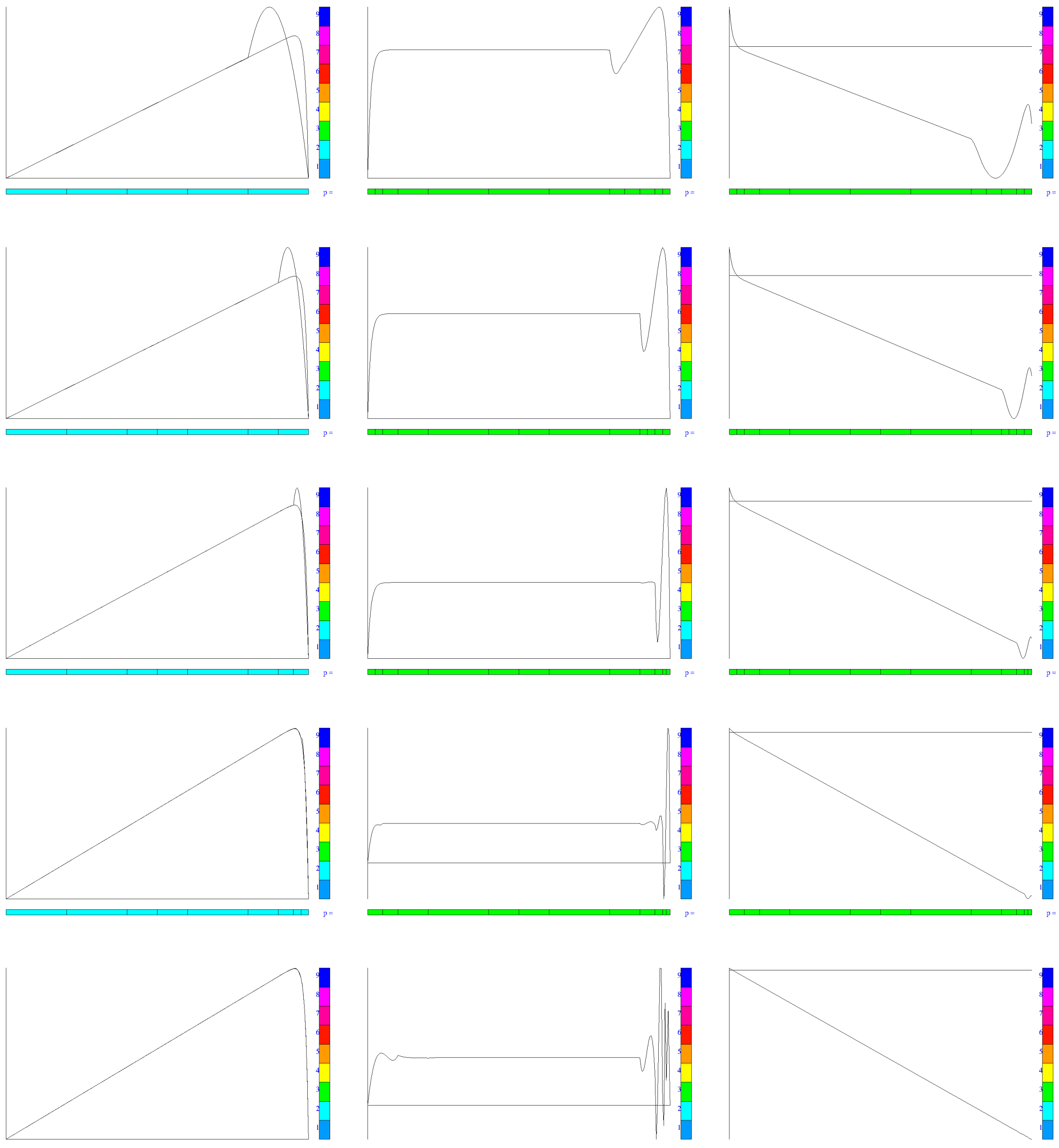

Figure 6: Primal formulation, $\epsilon = 10^{-2}$, outer loop, iterations 1-5. Left: Evolution of the approximate solution u_h. Middle column: the test mesh with the corresponding resolved residual. Right: The test mesh with the corresponding solution of the dual problem.

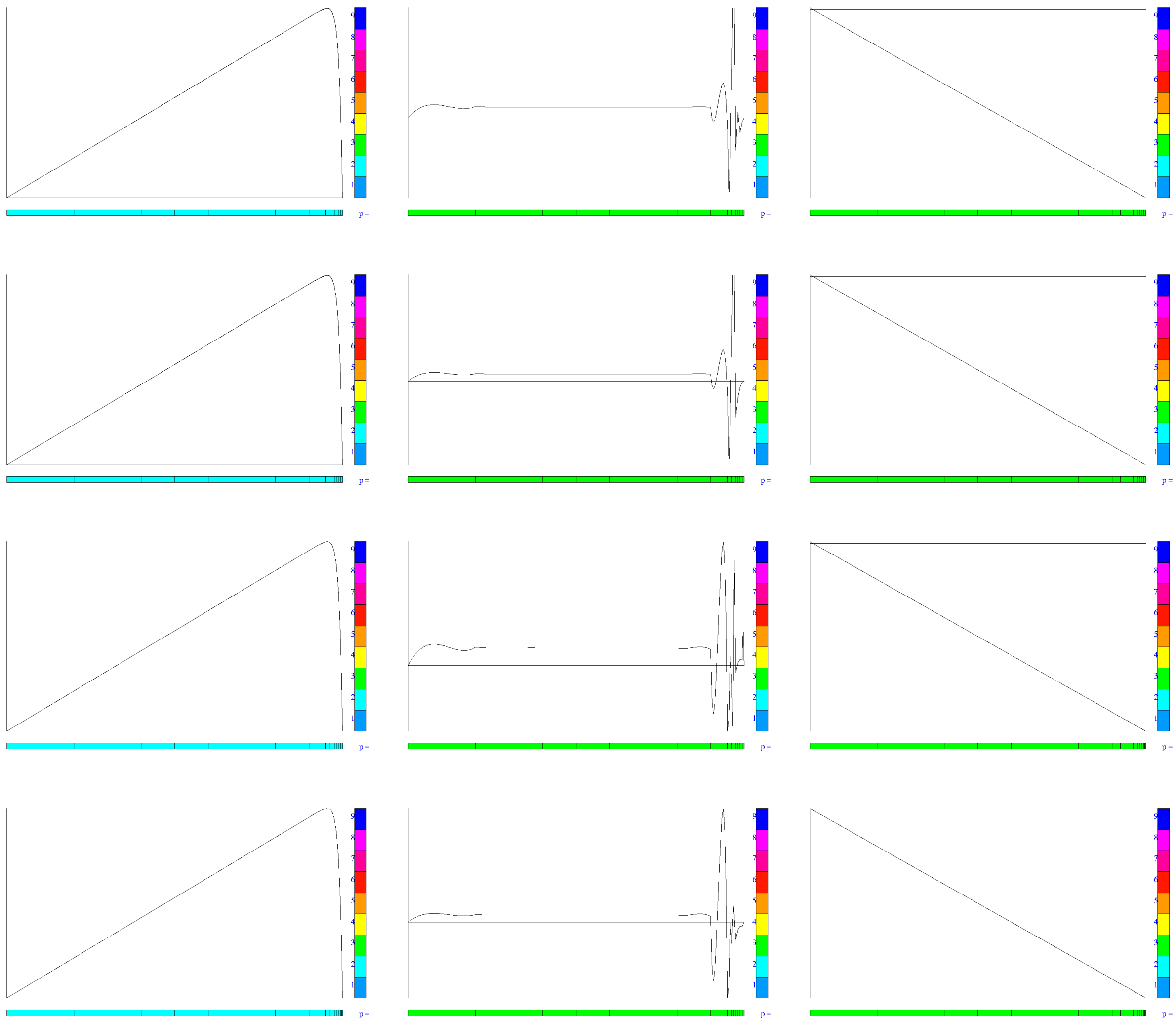

Figure 7: Primal formulation, $\epsilon = 10^{-2}$, outer loop, iterations 6-9. Left: Evolution of the approximate solution u_h. Middle column: the test mesh with the corresponding resolved residual. Right: The test mesh with the corresponding solution of the dual problem.

loop convergence was not always monotone. As expected, the number of inner loop iterations grows with smaller ϵ as one has to resolve smaller scales in the residual. Also, as the trial mesh gets refined, the number of inner loop iterations *monotonically decreases*. Note that we did not observe such a systematic decrease for the UW formulation.

Controlling the Solution Error. Controlling the solution error for the classical formulation follows the arguments from Section 3.1 although is again a bit more complicated than for the UW formulation. We shall

Outer iter	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\epsilon = 10^{-2}$	8	7	6	4	4	4	3	3	3																			
$\epsilon = 10^{-3}$	20	20	18	15	14	12	8	7	6	5	5	4	4															
$\epsilon = 10^{-4}$	38	37	36	35	33	32	31	29	24	19	10	10	9	8	8	7												
$\epsilon = 10^{-5}$	58	57	56	55	54	53	52	50	49	48	46	43	35	27	15	13	12	11	10									
$\epsilon = 10^{-6}$	78	77	72	71	66	65	60	59	54	53	48	47	46	44	43	42	41	40	38	36	30	22	12	11	10	10	9	8

Table 3: Primal formulation. Number of inner loop iterations for different values of the viscosity constant.

discuss the first dual problem only. The key starting point is again the relation between the solution to the ideal PG problem $\widetilde{u}_h$ and the auxiliary variable $\widetilde{w}_h$ in the exact dual problem,

$$\epsilon \widetilde{u}_h' - \widetilde{u}_h = \widetilde{w}_h \,.$$

This relation defines also the discrete space W_h present in the dual formulation. Recall that $\widetilde{w}_h$ is the L^2-projection of the exact solution to the dual problem ϕ onto the discrete space W_h. Similarly, after discretization of the dual problem, w_h is the L^2-projection of the approximate solution ϕ_h onto space W_h. The duality gap estimate,

$$\|\phi^\perp - \phi_h^\perp\|^2 + \|\epsilon(\phi - \phi_h)'\|^2 = 2(J^*(\phi) - J^*(\phi_h)) \le \text{est}\,,$$

and Poincaré inequality,

$$C\|\phi\| \le \epsilon\|\phi'\|, \qquad \phi \in H^1(0,1),\, \phi(0) = \phi(1)\,,$$

imply that we control the error in ϕ,

$$C^2\|\phi - \phi_h\| \le \text{est}\,.$$

Poincaré constant depends upon ϵ so the estimate is not robust in ϵ. Note, however, that we actually need a different Poincaré-like bound,

$$C^2\|\phi\|^2 \le \|\phi^\perp\|^2 + \|\epsilon\phi'\|^2\,.$$

so the linear dependence of C upon ϵ may be very pessimistic.

Control of error $\|\phi - \phi_h\|$ translates in the control of error in $\|\widetilde{w}_h - w_h\|$,

$$\|\widetilde{w}_h - w_h\| \le \|\phi - \phi_h\|\,.$$

Finally, let $u_h \in U_h$ be the unique function defining w_h - the approximation of $\widetilde{w}_h$. We have the relation,

$$\epsilon(\widetilde{u}_h - u_h)' - (\widetilde{u}_h - u_h) = \widetilde{w}_h - w_h\,.$$

Multiply both sides with $-(\widetilde{u}_h - u_h)$, integrate over interval $(0,1)$ and use periodic BC to obtain,

$$\|\widetilde{u}_h - u_h\|^2 = -\int_0^1 (\widetilde{w}_h - w_h)(\widetilde{u}_h - u_h)$$

which implies the final estimate,

$$\|\widetilde{u}_h - u_h\| \leq \|\widetilde{w}_h - w_h\| \, .$$

Concluding, function u_h corresponding to w_h can be viewed as the final approximate solution with the error control implied by the duality arguments.

5 Conclusions

The double adaptivity idea is fascinating for at least two reasons. First of all, it sends a clear message that the discrete stability conditions of Babuška and Brezzi are sufficient but not necessary for convergence. This was emphasized for the first time by Bänsch, Morin and Nochetto [1] in context of Stokes problem and Uzawa's algorithm. Cohen, Dahmen and Welpert [8] developed the idea into the mixed problem stabilization of an arbitrary variational problems and applied to the confusion problem.

Secondly, the ultraweak variational formulation allows for determining the optimal and quasi-optimal test norms for *any well-posed* system of first order PDEs. In particular, it circumvents the need for an elaborate stability analysis that we performed for the confusion problem [12, 7] that formed a foundation for our DPG methodology based on the so-called *robust test norms*.

The problem has been a challenge for the first two authors for the last four years and I (LD) will allow myself for several rather personal comments. First of all, I would like to thank not only the official co-authors but several of my friends for the patience of listening to me bragging about the subject for the last four years. The project started with post-lunch discussions with Renato da Silva who was visiting ICES about four years ago. A bit later, Norbert Heuer visited ICES as well, and I convinced him to start a more serious research on the subject. At that point we were stuck with the DPG technology, i.e. the use of discontinuous (broken) test spaces. Although the duality argument was on our mind, we did not manage to develop the duality argument in context of DPG method and so we settled with an explicit a-posteriori error estimate. The estimate involved several constants that required solving eigenvalue problems dependent on several parameters: diffusion constant, advection coefficient (in 2D), element size. The work was very academic. Solving those eigenvalue problems was a challenge as they turned out to be extremely unstable, both LAPACK library and inverse power iterations failed for smaller values of the parameters. We were forced to "cheat" by extrapolating the reliable values. As a consequence, the a-posteriori error estimate stopped being reliable and the double adaptivity did not converge except for relatively large values of $\epsilon \geq 10^{-4}$ in 1D, and $\epsilon \geq 10^{-2}$ in 2D (sic!). After presenting our work at several meetings starting with the *Second Workshop on Minimum Residual Methods* at Delft, I failed to complete the computations and the paper was left unfinished. Besides the round off error issues, I was struggling with coding, one code had to serve two masters - the trial and the test space. In the classical Brezzi's methodology, the two FE spaces remain related, using the enriched test spaces in DPG with a fixed Δp is a good example. In the double adaptivity algorithm, the two spaces want to be really *independent of each other*. Working with a single FE data structure forced me to hack the code and this also got out of hand at some point. I gave up...

I restarted the idea of double adaptivity in context of *non-local problems* discussed since last Fall with Xiaochuan Tian and Leng Yu. Broken test spaces make little sense in context of non-local problems, and I started looking more seriously for means of supporting two completely independent mesh data structures at the same time. This is where my students - Socratis Petrides and Stefan Henneking, taught me the idea of *pointers in Fortran 90*. In literary few days, I was able to convert my standard 1D hp code into a code supporting two meshes. This is not a small deal, as our 1D code is a stepping stone to both 2D and 3D codes, it uses exactly the same logic, data structures, even routines have the same names. Over the years, we have optimized our hp data structures that reduce now to just two arrays (of objects): initial mesh elements ELEMS and nodes array NODES. All I had to do is to set up two copies of those arrays: ELEMS_U and NODES_U for the trial mesh, ELEMS_V and NODES_V for the test mesh, with the original arrays ELEMS and NODES becoming now pointers. If you point to the trial mesh, all implemented data structure algorithms operate on on the trial mesh, if you point them to the test mesh arrays, they operate on the test meshes. This has been really a breakthrough on the coding side. Otherwise, coding with two meshes requires a few minor changes. For instance, instead of a single loop through elements, we have formally a double loop but we exit if the elements do not overlap. An element in a standard code is replaced with the intersection of trial and test mesh elements. These are very minor changes and one learns how to code in this environment very quickly.

The standard duality gap technology used in this paper requires the use of *conforming elements*. You can guarantee that $J(u_h) \geq J(u)$ only if u_h comes from the energy space U on which J is defined on. With the double mesh technology there is no need for using broken spaces and this condition is now satisfied. I introduced Thomas who visited with me last December to the project, and we restarted thinking about the double adaptivity algorithm again. In meantime, in my discussions with Xiaochuan and Leng, we realized that in 1D we could also determine the optimal test norm for the standard variational formulation. Within the conforming elements for both trial and test meshes, the idea of using duality for error estimation came in. First for the ultraweak formulation, and then, with help of Xiaochuan, for the classical formulations as well.

It is more likely possible to generalize the technology to the Banach setting, and we already are making fast progress in this direction with Banach DPG experts: Ignacio Muga and Chris van der Zee.

In the optimal test norm business, we are trading solution of a convection-dominated problem with diffusion ϵ, for the solution a reaction-dominated problem with diffusion ϵ^2 using standard Galerkin. Clearly, robustness of the a-posteriori error estimate is the main issue here, see e.g. [17, 21] and the literature therein. I have used the duality for error estimation and adaptivity many years ago [9, 13, 19] but never in context of singular perturbation problems. The use of duality theory arguments for the a-posteriori error estimation seems to be recently on a rise, see [2] and the literature therein. In this context, the fact that we are able to solve our reaction-dominated problem with $\epsilon^2 = 10^{-14}$ is really extraordinary.

It remains to try out the method in 2D and 3D and we plan to do it as soon as possible.

Acknowledgments. L. Demkowicz was partially supported with NSF grant No. 1819101, AFRL grant No. FA9550-17-1-0090 and Army grant No. W911NF-15-1-0552. T. Führer and N. Heuer are supported by CONICYT through FONDECYT projects 11170050 and 1190009. X. Tian is partially supported with NSF grant No. 1819233.

References

[1] E. Bänsch, P. Morin, and R.H. Nochetto. An adaptive Uzawa FEM for the Stokes problem: Convergence without the inf-sup condition. *SIAM J. Numer. Anal.*, 40(4):1207–1229, 2002.

[2] S. Bartels and M. Milicevic. Primal-dual estimators for a posteriori error analysis of nonsmooth minimization problems. Feb. 2019. arXiv:1902.03967v1[math.NA].

[3] D. Broersen and R. A. Stevenson. A robust Petrov-Galerkin discretisation of convection-diffusion equations. *Comput. Math. Appl.*, 68(11):1605–1618, 2014.

[4] D. Broersen and R. P. Stevenson. A Petrov-Galerkin discretization with optimal test space of a mild-weak formulation of convection-diffusion equations in mixed form. *IMA J. Numer. Anal.*, 35(1):39–73, 2015.

[5] C. Carstensen, L. Demkowicz, and J. Gopalakrishnan. Breaking spaces and forms for the DPG method and applications including Maxwell equations. *Comput. Math. Appl.*, 72(3):494–522, 2016.

[6] C. Carstensen and F. Hellwig. Low order discontinuous Petrov-Galerkin finite element methods for linear elasticity. *SIAM J. Num. Anal.*, 54(6):3388–3410, 2016.

[7] J. Chan, N. Heuer, Tan Bui-Thanh B., and L. Demkowicz. A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms. *Comput. Math. Appl.*, 67(4):771–795, 2014.

[8] A. Cohen, W. Dahmen, and G. Welper. Adaptivity and variational stabilization for convection-diffusion equations. *ESAIM Math. Model. Numer. Anal.*, 46(5):1247–1273, 2012.

[9] L. Demkowicz. On some results concerning the reciprocal formulation for the Signorini's problem. *Comput. Math. Appl.*, 8(1):57–74, 1982.

[10] L. Demkowicz. Lecture notes on Energy Spaces. Technical Report 13, ICES, 2018.

[11] L. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov-Galerkin methods. Part II: Optimal test functions. *Numer. Meth. Part. D. E.*, 27:70–105, 2011. See also ICES Report 2009-16.

[12] L. Demkowicz and N. Heuer. Robust DPG method for convection-dominated diffusion problems. *SIAM J. Num. Anal*, 51:2514–2537, 2013. see also ICES Report 2011/13.

[13] L. Demkowicz and M. Swierczek. An adaptive finite element method for a class of variational inequalities. In W. Kosinsky et all., editor, *Selected Problems of Modern Continuum Theory*. Pitagora Editrice, Bologna, 1987. Proceedings of the Italian-Polish Meeting in Bologna, June 3-6, 1987.

[14] I. Ekeland and R. Témam. *Convex Analysis and Variational Problems*. North Holland, 1976.

[15] T. Führer and N. Heuer. Fully discrete DPG methods for the Kirchhoff-Love plate bending model. *Comput. Methods Appl. Mech. Engrg.*, 343:550–571, 2019.

[16] J. Gopalakrishnan and W. Qiu. An analysis of the practical DPG method. *Math. Comp.*, 83(286):537–552, 2014.

[17] N. Heuer and M. Karkulik. A robust DPG method for singularly perturbed reaction-diffusion problems. *SIAM J. Numer. Anal.*, 55(3):1218–1242, 2017.

[18] S. Nagaraj, S. Petrides, and L. Demkowicz. Construction of DPG Fortin operators for second order problems. *Comput. Math. Appl.*, 74(8):1964–1980, 2017.

[19] J.T. Oden, L. Demkowicz, R. Rachowicz, and T.A. Westermann. Toward a universal hp adaptive finite element strategy. Part 2: A posteriori error estimation. *Comput. Methods Appl. Mech. Engrg.*, 77:113–180, 1989.

[20] J.T. Oden and L.F. Demkowicz. *Applied Functional Analysis for Science and Engineering*. Chapman & Hall/CRC Press, Boca Raton, 2018. Third edition.

[21] I. Smears and M. Vohralík. Simple and robust equilibrated flux a posteriori estimates for singularly perturbed reaction–diffusion problems. HAL Preprint 01956180, submitted for publication, 2018.

[22] J. Zitelli, I. Muga, L. Demkowicz, J. Gopalakrishnan, D. Pardo, and V. Calo. A class of discontinuous Petrov-Galerkin methods. Part IV: Wave propagation problems. *J. Comp. Phys.*, 230:2406–2432, 2011.

1 Appendix: Stability Analysis for a 1D System of Broersen and Stevenson

In this appendix we analyze systematically the task of constructing a first order reformulation of the second order convection-dominated diffusion problem in such a way that the resulting first order operator is robustly bounded below. We show that, for a specific 1D problem, the formulation of Broersen and Stevenson is indeed optimal.

Consider the following model problem,

$$\left\{\begin{aligned} -\epsilon u' + u &= 0 \quad \text{at } 0\\ u &= 0 \quad \text{at } 1\\ -\epsilon u'' + u' &= f \quad \text{in } (0,1)\end{aligned}\right. \tag{1.29}$$

with $\epsilon << 1$.

We ask the following question: *How do we reformulate the problem as a first order system in such a way that the corresponding operator is bounded below with a constant independent of ϵ ?* More precisely, the question involves two issues: how to define an auxiliary unknown σ, and how to weight the equation defining σ ? If we assume the weight to be in the form of ϵ^α then, obviously, we are looking for the biggest value of exponent α.

As we intend to keep u as an unknown, the necessary condition is that the L^2-norm of solution to (1.29) must be uniformly bounded by $\|f\|$. This is indeed the case. Introduce a test function v,

$$v(x) := \int_x^1 u(s)\, ds \quad v' = -u, \quad v(1) = 0 \tag{1.30}$$

Poincaré inequality implies that

$$\|v\| \lesssim \|u\| \tag{1.31}$$

where $\lesssim$ denotes a bound with a constant independent of ϵ. We now multiply equation (1.29) with v, and integrate over $(0,1)$. Next we integrate the left-hand side by parts to obtain:

$$-\int_0^1 (-\epsilon u' + u)(-u) = \|u\|^2 + \frac{\epsilon}{2}|u(0)|^2 \tag{1.32}$$

Schwarz inequality and (1.31) imply that

$$\|u\|^2 \lesssim \|f\|\,\|u\| \tag{1.33}$$

which implies the robust bound for $\|u\|$.

In turn, multiplying (1.29) with u, integrating over $(0,1)$, and integrating by parts, we obtain,

$$\epsilon\|u'\|^2 + \frac{1}{2}|u(0)|^2 \leq \|f\|\,\|u\| \lesssim \|f\|^2 \tag{1.34}$$

where we have again used Schwarz inequality, and the robust bound for $\|u\|$ to estimate the right-hand side. This yields the robust estimate:

$$\epsilon^{1/2}\|u'\| \lesssim \|f\|\,. \tag{1.35}$$

As the flux is supposed to be also bounded uniformly by $\|f\|$, this suggests the definition:

$$\sigma = \epsilon^{1/2} u'\,. \tag{1.36}$$

We replace now the original second order equation with the first order system,

$$\left\{\begin{array}{rll} -\epsilon^{1/2}\sigma + u & = 0 & \text{at } 0 \\ u & = 0 & \text{at } 1 \\ \sigma - \epsilon^{1/2}u' & = g & \text{in } (0,1) \\ -\epsilon^{1/2}\sigma' + u' & = f & \text{in } (0,1) \end{array}\right. \tag{1.37}$$

We point out that equation (1.36) can be weighted with any power of ϵ without any effect on estimates for $\|u\|$ and $\|\sigma\|$ by $\|f\|$. Eliminating σ from the first order system, we obtain now a modified version of the second order problem:

$$\left\{\begin{array}{rll} -\epsilon u' + u & = \epsilon^{1/2} g & \text{at } 0 \\ u & = 0 & \text{at } 1 \\ -\epsilon u'' + u' & = f + \epsilon^{1/2} g' & \text{in } (0,1) \end{array}\right. \tag{1.38}$$

Notice that weighting equation (1.36) with a power of ϵ would have resulted in different powers of ϵ in front of the g terms in problem (1.38). The unit weight, however, turns out to be optimal, as we can bound $\|u\|$ and $\epsilon^{1/2}\|u'\|$ robustly by $\|f\|$ and $\|g\|$. We need to consider only the bound in terms of $\|g\|$, as we already have the estimate in terms of $\|f\|$. Indeed, integrating both sides of the equation and using the first boundary condition, we obtain:

$$-\epsilon u' + u = \epsilon^{1/2} g\,. \tag{1.39}$$

Multiplying equation (1.39) with u, integrating over $(0,1)$ etc., we obtain the estimate,

$$\frac{\epsilon}{2}|u(0)|^2 + \|u\|^2 \le \epsilon^{1/2}\|g\|\,\|u\| \tag{1.40}$$

which yields the robust estimate,

$$\|u\| \le \epsilon^{1/2}\|g\| \le \|g\|\,. \tag{1.41}$$

Notice that the estimate for $\|u\|$ is stronger than we need. Multiplying equation (1.39) with $-u'$ and proceeding in the usual way, we obtain,

$$\epsilon\|u'\|^2 + \frac{1}{2}|u(0)|^2 \le \epsilon^{1/2}\|g\|\,\|u'\| \tag{1.42}$$

which yields the optimal estimate for the derivative,

$$\epsilon^{1/2}\|u'\| \lesssim \|g\|\,. \tag{1.43}$$

The last inequality and $(1.37)_3$ yield now the optimal estimate for σ,

$$\|\sigma\| \lesssim \|g\|\,. \tag{1.44}$$

Discussion: The possibility of turning the original second order problem into a robust Friedrichs system using L^2-setting seems to be conditional. First of all, we must have a robust L^2 bound for $\|u\|$ in the original second order problem. Secondly, we need a robust bound for a power of ϵ times $\|u'\|$. This is possible for the case of the considered boundary condition but *it is not true* for Dirichlet BCs: $u(0) = u(1) = 0$. Our previous work shows that we can control then the derivative only in a weighted L^2-norm. Does it imply perhaps that the definition of σ should then include the weight ? Or should we use weighted L^2-spaces ? Finally, the last point is that the estimation in terms of g can be used to figure out the optimal weighting of the first equation without any guesswork.

2 Appendix: Derivation of the Dual Problems

We offer here a more detailed derivation of the dual problems discussed in the main text.

UW variational formulation. Let H_A, H_{A^*} be the energy graph spaces,

$$\begin{aligned} H_A(\Omega) &:= \{\mathsf{u} \in L^2(\Omega) \,:\, A\mathsf{u} \in L^2(\Omega)\} \\ H_{A^*}(\Omega) &:= \{\mathsf{u} \in L^2(\Omega) \,:\, A^*\mathsf{u} \in L^2(\Omega)\}\,, \end{aligned}$$

and let B^* be the boundary operator associated with operator A^*,

$$(A^*\mathsf{u},\mathsf{v}) = (\mathsf{u}, A\mathsf{v}) + \langle B^*\mathsf{v},\mathsf{u}\rangle, \qquad \mathsf{u} \in D(A^*), \mathsf{v} \in H_A(\Omega)\,.$$

In our case,

$$D(A) = D(A^*) = H(\mathrm{div},\Omega)\times H^1(\Omega), \qquad \langle B^*\mathsf{v},\mathsf{u}\rangle = {}_{-\frac{1}{2}}\langle \tau_n, \epsilon^{\frac{1}{2}} u\rangle_{\frac{1}{2}} + {}_{\frac{1}{2}}\langle v, \epsilon^{\frac{1}{2}}\sigma_n - \beta_n\, u\rangle_{-\frac{1}{2}}$$

where $\sigma_n = \sigma\cdot n$, etc. Due to density of H_A in $L^2(\Omega)$, with $\sigma \in L^2(\Omega), \mathsf{u} \in D(A^*)$, we have:

$$\sup_{\mathsf{v}\in L^2(\Omega)} (A^*\mathsf{u}-\sigma,\mathsf{v}) = \sup_{\mathsf{v}\in H_A} (A^*\mathsf{u}-\sigma,\mathsf{v}) = \begin{cases} 0 & \text{if } A^*\mathsf{u} = \sigma \\ +\infty & \text{otherwise.} \end{cases}$$

Consequently,

$$\begin{aligned} &\inf_{\substack{\psi \in D(A^*) \\ A^*\psi \in U_h^\perp}} \underbrace{\frac{1}{2}\|A^*\psi\|^2 + \alpha\frac{1}{2}\|\psi\|^2 - (\mathsf{f},\psi)}_{=:J(\psi)} \\ = &\inf_{\substack{\sigma \in L^2(\Omega) \\ \sigma \in U_h^\perp}} \inf_{\psi\in D(A^*)} \sup_{\phi\in H_A} \left\{\frac{1}{2}\|\sigma\|^2 + \alpha\frac{1}{2}\|\psi\|^2 - (\mathsf{f},\psi) + (A^*\psi - \sigma,\phi)\right\} \\ = &\inf_{\substack{\sigma \in L^2(\Omega) \\ \sigma \in U_h^\perp}} \inf_{\psi\in D(A^*)} \sup_{\phi\in H_A} \left\{\frac{1}{2}\|\sigma\|^2 + \alpha\frac{1}{2}\|\psi\|^2 - (\mathsf{f},\psi) + (\psi, A\phi) + \langle B^*\phi,\psi\rangle - (\sigma,\phi)\right\} = (^*) \end{aligned} \tag{2.45}$$

At this point, we are ready to trade the inf sup for the sup inf,

$$(^*) \;\ge \sup_{\phi\in H_A} \inf_{\substack{\sigma \in L^2(\Omega) \\ \sigma \in U_h^\perp}} \inf_{\psi\in D(A^*)} \left\{\frac{1}{2}\|\sigma\|^2 + \alpha\frac{1}{2}\|\psi\|^2 - (\mathsf{f},\psi) + (\psi, A\phi) + \langle B^*\phi,\psi\rangle - (\sigma,\phi)\right\} = (^{**})\,.$$

We plan to show *a posteriori* that, in fact, we still have the inequality above. The whole point is now that we can compute the two minimization problems *explicitly*. Minimization in σ yields,

$$\sigma = \phi^\perp \quad \Rightarrow \quad \inf_{\substack{\sigma \in L^2(\Omega) \\ \sigma \in U_h^\perp}} \frac{1}{2}\|\sigma\|^2 - (\sigma,\phi) = -\frac{1}{2}\|\phi^\perp\|^2\,.$$

Minimizing in $\psi \in D(A^*)$, we get,

$$\alpha\psi = \mathsf{f} - A\phi \quad \Rightarrow \quad \inf_{\psi \in D(A^*)} \left\{ \frac{\alpha}{2}\|\psi\|^2 - (\mathsf{f} - A\phi, \psi) + \langle B^*\phi, \psi \rangle \right\} = \begin{cases} -\frac{1}{2\alpha}\|\mathsf{f} - A\phi\|^2 & \text{if } B^*\phi = 0 \\ -\infty & \text{otherwise}\,. \end{cases}$$

Consequently, there is no chance for the equality of inf sup to sup inf, unless we restrict the maximization to $\phi \in D(A)$. We could have assumed that from the very beginning but our reasoning shows that the boundary conditions on ϕ are a *must*. In the end, we obtain the dual problem:

$$(^{**}) = \sup_{\phi \in D(A)} \underbrace{-\frac{1}{2}\|\phi^\perp\|^2 - \frac{1}{2\alpha}\|\mathsf{f} - A\phi\|^2}_{=:J^*(\phi)}\,. \tag{2.46}$$

A simple algebra and one integration by parts show that,

$$2(J(\psi) - J^*(\phi)) = \frac{1}{\alpha}\int_\Omega \{\alpha(A^*\psi - \phi^\perp)^2 + (\alpha\psi - (\mathsf{f} - A\phi))^2\}\,,$$

for *any* $\psi \in D(A^*)$ and $\phi \in D(A)$. We will demonstrate now that, if ψ is the solution of the primal minimization problem, and ϕ is the solution of the dual maximization problem, the right hand side above is equal zero, i.e. there is no duality gap on the continuous level. This is, of course, later necessary, to use the duality gap for the a-posteriori error estimation for approximate solutions. Strict convexity of primal functional and strict concavity of the dual functional imply that the minimizers of $J(\psi)$ and $-J^*(\phi)$ exist and are unique.

The solution of the primal problem satisfies the mixed problem:

$$\left\{\begin{array}{llll} \psi \in D(A^*),\ \mathsf{u}_h \in U_h & & & \\ (A^*\psi, A^*\delta\psi) + (\alpha\psi, \delta\psi) & +(\mathsf{u}_h, A^*\delta\psi) & = (\mathsf{f}, \delta\psi) & \delta\psi \in D(A^*) \\ (A^*\psi, \delta\mathsf{u}_h) & & = 0 & \delta\mathsf{u}_h \in U_h \end{array}\right. \tag{2.47}$$

where $\mathsf{u}_h \in U_h$ is the corresponding Lagrange multiplier.

The solution to the dual problem satisfies another mixed problem:

$$\left\{\begin{array}{llll} \phi \in D(A),\ \mathsf{w}_h \in U_h & & & \\ (A\phi, A\delta\phi) + \alpha(\phi, \delta\phi) & -\alpha(\mathsf{w}_\mathsf{h}, \delta\phi) & = (\mathsf{f}, A\delta\phi) & \delta\phi \in D(A) \\ -\alpha(\phi, \delta\mathsf{w}_h) & +\alpha(\mathsf{w}_h, \delta\mathsf{w}_h) & = 0 & \delta\mathsf{w}_h \in U_h\,, \end{array}\right.$$

or, in the strong form,

$$A^*A\phi + \alpha(\phi - \mathsf{w}_h) = A^*f \tag{2.48}$$

plus the BC:

$$BA\phi = B\mathsf{f} \quad \Rightarrow \quad \mathsf{f} - A\phi \in D(A^*) \tag{2.49}$$

where boundary operator B corresponds to operator A. Let now ϕ be the solution to the dual problem. Use one of the duality relations to define *a function* ψ,

$$\psi := \frac{1}{\alpha}(\mathsf{f} - A\phi)\,.$$

First of all, ψ satisfies the second duality relation. Indeed, equation (2.48) implies that

$$A^*\psi = \frac{1}{\alpha}(A^*\mathsf{f} - A^*A\phi) = \phi - \mathsf{w}_h = \phi^{\perp}\,.$$

Secondly, BC (2.49) implies that $\psi \in D(A^*)$. Finally, plugging the function ψ and $\mathsf{u}_h = \mathsf{w}_h$ into variational formulation $(2.47)_1$, we obtain,

$$\begin{aligned}(A^*\psi, A^*\delta\mathsf{u}) + (\alpha\psi, \delta\mathsf{u}) + (\mathsf{u}_h, A^*\delta\mathsf{u}) &= (\phi^{\perp}, A^*\delta\mathsf{u}) + (\mathsf{f} - A\phi, \delta\mathsf{u}) + (\mathsf{w}_h, A^*\delta\mathsf{u})\\ &= (\phi, A^*\delta\mathsf{u}) - (A\phi, \delta\mathsf{u}) + (\mathsf{f}, \delta\mathsf{u})\\ &= (\mathsf{f}, \delta\mathsf{u})\,.\end{aligned}$$

Note that the duality relation $A^*\psi = \phi^{\perp}$ implies that equation $(2.47)_2$ is satisfied as well. Consequently, uniqueness of the solution to the primal problem implies that function ψ derived from the duality relations indeed is the solution of the primal problem. *There is no duality gap*.

Classical variational formulation. In the first dual formulation, a simple algebra combined with integration by parts shows that

$$2(J(\psi) - J^*(\phi)) = \int_0^1 (\epsilon\psi' - \phi^{\perp})^2 + (\psi - \bar{\psi} - (f + \epsilon\phi'))^2 \tag{2.50}$$

where $\psi \in H_0^1(0,1)$ and ϕ is taken from $H^1(0,1)$ with periodic constraint $\phi(1) - \phi(0) = -\frac{1}{\epsilon}\bar{f}$. We will now verify that the duality gap is zero on the continuous level if ψ is the solution of the primal minimization problem, and ϕ is the solution of the dual maximization problem. Similar to the derivation in the UW formulation, we can first write down the primal and dual mixed problems. The solution of the primal problem satisfies the mixed problem:

$$\left\{\begin{array}{llll}\psi \in H_0^1(0,1),\ u_h \in U_h & & & \\ \epsilon^2(\psi', \delta\psi') + (\psi - \bar{\psi}, \delta\psi) & +b(u_h, \delta\psi) & = (f, \delta\psi) & \delta\psi \in H_0^1(0,1)\\ b(\delta u_h, \psi) & & = 0 & \delta u_h \in U_h\end{array}\right. \tag{2.51}$$

where $u_h \in U_h$ is the corresponding Lagrange multiplier.

For the dual problem (4.25), it can be resolved by turning the maximum problem into a double minimization problem,

$$\inf_{\substack{\phi \in H^1(0,1)\\ \phi(1) - \phi(0) = -\frac{1}{\epsilon}\bar{f}}} \ \inf_{w_h \in W_h} \frac{1}{2}\|\phi - w_h\|^2 + \frac{1}{2}\|f + \epsilon\phi'\|^2\,,$$

where W_h is defined in (4.23). This leads to the mixed problem:

$$\begin{cases} \phi \in H^1(0,1),\ \phi\big|_0^1 = -\frac{1}{\epsilon}\bar{f},\ w_h \in W_h \\ \epsilon^2(\phi', \delta\phi') + (\phi, \delta\phi) - (w_h, \delta\phi) \quad = -\epsilon(f, \delta\phi') \qquad \delta\phi \in H^1(0,1),\ \delta\phi\big|_0^1 = 0 \\ -(\phi, \delta w_h) + (w_h, \delta w_h) \qquad\qquad = 0 \qquad\qquad\qquad \delta w_h \in W_h\,, \end{cases} \tag{2.52}$$

which has the strong form:

$$-\epsilon^2\phi'' + \phi - w_h = \epsilon f', \quad \phi - w_h \in W_h^{\perp}$$

plus the boundary condition:

$$(\epsilon\phi' + f)\big|_0^1 = 0\,.$$

Now assume that ϕ is a solution to the dual problem, then can define $\psi = f + \epsilon\phi' + C$, for a constant C to be determined. With boundary conditions, we have the following relations:

$$\psi - \bar{\psi} = f + \epsilon\phi', \quad \psi(1) - \psi(0) = 0\,. \tag{2.53}$$

Notice that there are infinitely many ψ's that satisfy the above relations. In fact, any constant shift of a particular ψ still satisfies the relations. We can thus pick ψ such that $\psi(0) = 0$, so that $\psi \in H_0^1(0,1)$.

First, we see that the ψ and ϕ defined in this way satisfy $2(J(\psi) - J^*(\phi)) = 0$, because in addition to (2.53), we have

$$\psi' = f' + \epsilon\phi'' = \frac{1}{\epsilon}(\phi - w_h) = \frac{1}{\epsilon}\phi^{\perp}\,.$$

Next, if $u_h \in U_h$ is given such that $\epsilon u_h' - u_h = w_h$, then ψ and u_h together satisfy the primal problem (2.51) This is true because for all $\delta\psi \in H_0^1(0,1)$, we have

$$\begin{aligned} &\epsilon^2(\psi', \delta\psi') + (\psi - \bar{\psi}, \delta\psi) + \epsilon(u_h', \delta\psi') + (u_h', \delta\psi) \\ =&\epsilon(\phi - w_h, \delta\psi') + (f + \epsilon\phi', \delta\psi) + (\epsilon u_h' - u_h', \delta\psi') \\ =&\epsilon(\phi, \delta\psi') + (f + \epsilon\phi', \delta\psi) = 0\,, \end{aligned}$$

and

$$b(\delta u_h, \psi) = \epsilon(\delta u_h' - u_h', \psi') = \frac{1}{\epsilon}(w_h, \phi^{\perp}) = 0\,.$$

This show that the function ψ derived from the duality relations indeed is the solution of the primal problem. Therefore, there is no duality gap at the continuous level.

At last, the derivation for the second dual formulation is quite similar, except for dealing with different boundary conditions. We still have the formula (2.50) for the duality gap, but we take $\psi \in H^1(0,1)$ with $\psi(0) = 0$ and $\phi \in H^1(0,1)$ with $\phi(0) = \frac{1}{\epsilon}\bar{f}$ and $\phi(1) = 0$. Now the dual problem becomes

$$\begin{cases} \phi \in H^1(0,1),\ \phi(0) = \dfrac{1}{\epsilon}\bar{f},\ \phi(1) = 0,\ w_h \in W_h \\ \epsilon^2(\phi', \delta\phi') + (\phi, \delta\phi) - (w_h, \delta\phi) = -\epsilon(f, \delta\phi') \qquad \delta\phi \in H_0^1(0,1) \\ -(\phi, \delta w_h) + (w_h, \delta w_h) = 0 \qquad\qquad\qquad\qquad \delta w_h \in W_h\,, \end{cases} \tag{2.54}$$

where W_h is taken as (4.26). If ϕ is the solution to (2.54), then it satisfies in the strong form

$$-\epsilon^2\phi'' + \phi - w_h = \epsilon f', \quad \phi - w_h \in W_h^{\perp}.$$

Note that in this case we do not have additional boundary conditions. Now we again define $\psi = f + \epsilon\phi' + C$, for a constant C to be determined. Then from the boundary condition of ϕ we have $C = \bar{\psi}$, so we arrive at $\psi - \bar{\psi} = f + \epsilon\phi'$. For the same reason, we can choose ψ so that $\psi(0) = 0$. Now the other boundary condition for ψ comes from the fact that

$$\psi' = f' + \epsilon\phi'' = \frac{1}{\epsilon}(\phi - w_h) = \frac{1}{\epsilon}\phi^{\perp},$$

and since this time $\phi^{\perp}$ is orthogonal to constants by the definition of W_h so we have $\psi(1) - \psi(0) = 0$. The rest of the derivation for the second dual form goes exactly the same as for the first dual form.