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Polynomial splines of non-uniform degree on triangulations: Combinatorial bounds on the dimension

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Abstract

For $\mathcal{T}$ a planar triangulation, let $\mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{m}}(\mathcal{T})$ denote the space of bivariate splines on $\mathcal{T}$ such that $f \in \mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{m}}(\mathcal{T})$ is $C^{\boldsymbol{r}(\tau)}$ smooth across an interior edge τ and, for triangle σ in $\mathcal{T}$, $f|_\sigma$ is a polynomial of total degree at most $\boldsymbol{m}(\sigma) \in \mathbb{Z}_{\geq 0}$. The map $\boldsymbol{m} : \sigma \mapsto \mathbb{Z}_{\geq 0}$ is called a non-uniform degree distribution on the triangles in $\mathcal{T}$, and we consider the problem of computing (or estimating) the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{m}}(\mathcal{T})$ in this paper. Using homological techniques, developed in the context of splines by Billera (1988), we provide combinatorial lower and upper bounds on the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{m}}(\mathcal{T})$. When all polynomial degrees are sufficiently large, $\boldsymbol{m}(\sigma) \gg 0$, we prove that the number of splines in $\mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{m}}(\mathcal{T})$ can be determined exactly. In the special case of a constant map $\boldsymbol{m}$, the lower and upper bounds are equal to those provided by Mourrain and Villamizar (2013).

Keywords: splines, triangulations, mixed polynomial degrees, mixed smoothness, dimension formula

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1. Introduction

Polynomial splines on triangulations and quadrangulations have myriad applications and are ubiquitous, especially, in the fields of computer aided geometric design and computational mechanics. Meaningful use of splines for these purposes requires the construction and analysis of a suitable set of basis functions for the spline spaces. In turn, this necessitates the computation or estimation of their dimension which, following the definition of smooth splines, depends on an interplay between geometry, topology and combinatorics.

In 1D, the problem is tractable and dimension formulas for the spline spaces follow from classical arguments. Moreover, a set of optimal basis functions called B-splines can be efficiently built and possess several properties useful for both geometric design and approximation. Over the years a a rich and mature mathematical theory associated to B-splines and their tensor-product extensions has been developed [24, 15] alongside several locally refinable extensions [25, 9, 7]. Such splines on quadrangulations have found extensive use in the fields of geometric design and, more recently, isogeometric analysis [11] where smooth splines are used to numerically solve partial differential equations.

On the other hand, efficient and robust software exist for triangulating arbitrary domains, and splines on triangulations [12] — the focus of this paper — have also been widely studied and employed for geometric modeling, data interpolation and isogeometric analysis. C^0 splines are easy to study; higher orders of regularity, on the other hand, pose a problem that is significantly more challenging. A dimension formula for spline spaces on triangulations with global smoothness requirements was first conjectured in [27]. A lower bound for general planar triangulations was presented in [22], and it was shown to equal the exact dimension for triangulations containing a single interior vertex. Later, [1, 2] showed that equality also holds for spline spaces where the polynomial degree is large enough compared to the required regularity. These advancements were made with the help of Bernstein–Bézier techniques; e.g., [12].

A radically different solution was presented in [3] by employing the tools of homological algebra. Building a short exact sequence of chain complexes, [3] showed that the spline space dimension is

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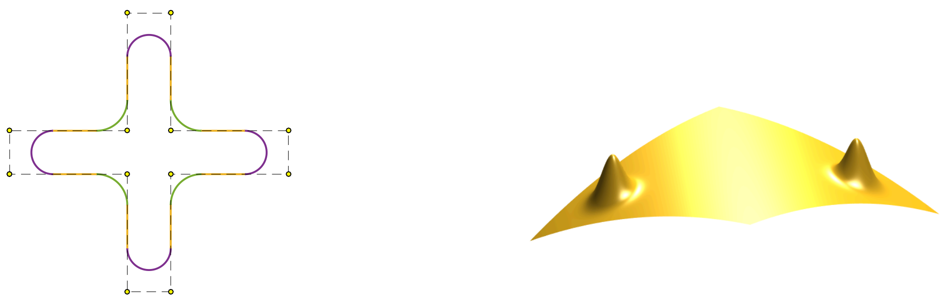


(a) (b)

Figure 1: The advantages of using a non-uniform degree spline framework for the purposes of geometric modeling and isogeometric analysis are underscored when the geometry/solution of interest possesses local features. Figure (a) reproduces an example from [28] that shows how a smooth curve of varying complexity can be efficiently represented using only a few control points by employing non-uniform degree spline spaces. Figure (b) shows an example of a bivariate surface which could benefit from similar flexibility; cf. Example 8.5 in Section 8. The surface has been obtained by superimposing two Gaussian peaks on a quadratic surface.

equal to the dimension of a graded piece of a particular homology module. A modification that improved upon [3] and yielded simpler chain complexes was presented in [20] and used to prove that the dimension formula derived by [2] holds in sufficiently high degree. In particular, this demonstrated that the homological algebra approach of the former agrees with the Bernstein–Bézier approach of the latter. These modified chain complexes were further studied, for instance, in [8, 16, 14]. A generalization to mixed orders of smoothness was presented in [8] by studying ideals generated by mixed power of linear forms in two variables. The upper bound provided in [14] improved upon the one from [23]; an alternate proof of the results from [1] was also given. Please see [17, 18] for an introduction to the usage of homology and graded algebras in the study of splines.

All existing results, including the ones discussed above, only consider uniform degree splines – for a fixed $m \in \mathbb{Z}_{\geq 0}$, a spline's restriction to a triangular face is allowed to be any arbitrary polynomial of total degree $\leq m$. In this paper, we relax this restriction and consider the case of non-uniform degree splines. The phrase "non-uniform degrees" refers to the flexibility of choosing polynomial degrees in a non-uniform manner on the triangles of $\mathcal{T}$. Let us briefly pin down the notion of such splines; the notation will be rigorously introduce in Section 2. For a triangulation $\mathcal{T}$ of a planar domain $\Omega \subset \mathbb{R}^2$, let $\mathcal{T}_2$ and $\mathring{\mathcal{T}}_1$ denote the sets of triangles and interior edges of $\mathcal{T}$, respectively. Denoting the vector space of bivariate polynomials of total degree $\leq m$ with $\mathbb{P}_m$, the space of non-uniform degree splines on $\mathcal{T}$ is defined as

$$\mathcal{S}(\mathcal{T}) = \Big\{ \Omega \xrightarrow{f} \mathbb{R} \ : \ \forall \sigma \in \mathcal{T}_2 \ \ f|_\sigma \in \mathbb{P}_{m_\sigma} \ , \ m_\sigma \in \mathbb{Z}_{\geq 0} \ , \text{ and } \\ \forall \tau \in \mathring{\mathcal{T}}_1 \ \ f \text{ is } C^{\boldsymbol{r}(\tau)} \text{ smooth across } \tau \ , \ \boldsymbol{r}(\tau) \in \mathbb{Z}_{\geq 0} \Big\} \ .$$

In general, the above spline space will allow for local polynomial degree adaptivity. For the special choice of $m_\sigma = \overline{m} \in \mathbb{Z}_{\geq 0}$ for all $\sigma \in \mathcal{T}_2$, the above space reduces to the uniform degree spline spaces usually studied [3, 21, 14].

The above notion of splines incorporates polynomial degree adaptivity and is complementary to the well-known idea of mesh-size adaptive (or locally refinable) splines. The latter idea has been the focus of many works and, using tools of homological algebra, has been recently analyzed in [19]. The setting of polynomial degree adaptivity, on the other hand, has only been studied thus far in the univariate setting [26, 13, 28] and the tensor-product multivariate setting [28]. It is intuitively clear that non-uniform degree splines combined with local mesh refinements can be tremendously powerful in the contexts of both geometric modeling and isogeometric analysis. The resulting flexibility would particularly allow design of complex shapes with fewer control points, i.e., cleaner and simpler designs; while for isogeometric analysis the same would lead to more efficient analysis. Figure 1 presents examples of intended applications of such splines.

Therefore, in this paper, we embark upon the study of non-uniform degree spline spaces on triangulations and use tools from homological algebra for computing (or estimating) their dimension. Dimension formulas for univariate (and tensor product) non-uniform degree spline spaces can be derived from classical arguments. However, dimension formulas in the multivariate setting do not follow from any direct generalization of the univariate results. We therefore approach the problem in its own right. Several assumptions underlying the frameworks presented in [8, 14] do not hold in this setting, and the corresponding tools need to be reformulated and generalized to account for polynomial degree adaptivity. We do so and use them to derive combinatorial upper and lower bounds on the spline space dimension. Furthermore, we show that, in sufficiently high degree — i.e., when $m_\sigma \gg 0$ for all $\sigma \in \mathcal{T}_2$ — the dimension of the spline space can be determined exactly. In the special case of a constant map $\boldsymbol{m}$, the lower and upper bounds are equal to those provided by [14]. The following is an overview of the paper.

In Section 2, preliminary concepts regarding planar triangulations are introduced and spline spaces of non-uniform degrees and mixed smoothness on a triangulation $\mathcal{T}$ are defined. Section 2.2 homogenizes the problem by embedding $\mathcal{T}$ in $\mathbb{R}^3$ and forming its cone with the vertex at the origin. Doing so allows us to equate the dimension of the spline space to the dimension of a graded piece of a particular homology module in a short exact sequence of chain complexes, and Section 3 elaborates upon the reasoning. Combinatorial formulas for vector spaces that will feature in the final dimension formula are presented in Section 4; in particular, more general forms of the results from [8] are presented. Sections 5 and 6 analyze several components of the short exact sequence introduced in Section 3. Section 7 collects the main results of our paper. Lower and upper bounds on the spline space dimension are provided, and it is shown that for sufficiently high polynomial degrees the dimension can be determined exactly. Several examples are presented in Section 8 before concluding the paper.

Macaulay2 [10], written by Mike Stillman and Dan Grayson, includes a package called AlgebraicSplines written by Michael DiPasquale. Given a simplicial complex as input, this package can compute the dimension of a polynomial spline space of uniform degree and uniform smoothness on the complex. We have augmented this package's functionality in this particular context, enabling it to compute the spline space dimension in the presence of non-uniform degrees and mixed orders of smoothness. All examples presented in Section 8 have been verified using this modified package, called SimplicialMDSplines ("MD" stands for mixed degree), and the Macaulay2 scripts used for the same can be downloaded from the first author's website.

2. Splines on planar triangulations

In this section we present the required notation for triangulations $\mathcal{T}$ and polynomial spline spaces. We will proceed in the setting of non-uniform degree splines of mixed smoothness.

Given $\Omega \subset \mathbb{R}^2$, denote with $\mathcal{T}$ its triangulation, a 2-dimensional simplicial complex. The faces, edges and vertices of the triangulation are collected in sets $\mathcal{T}_2, \mathcal{T}_1$ and $\mathcal{T}_0$, respectively. Edges of $\mathcal{T}$ are called interior edges if they intersect $\mathring{\Omega}$ and boundary edges otherwise. The set of interior edges will be denoted by $\mathring{\mathcal{T}}_1$. Similarly, if a vertex is in $\mathring{\Omega}$ it will be called an interior vertex and a boundary vertex otherwise. The set of interior vertices will be denoted by $\mathring{\mathcal{T}}_0$.

We will assign the faces, edges and vertices of $\mathcal{T}$ consistent orientations and equivalently call them 2-, 1-, and 0-cells, respectively. The link of $\square \in \mathcal{T}_1 \cup \mathcal{T}_0$ is defined as the set

$$\mathrm{lk}(\square) := \{\diamond \in \mathcal{T}_1 \cup \mathcal{T}_0 \ : \ \square \cup \diamond \text{ is an } i\text{-cell in } \mathcal{T}\} ,$$

where $\square\cup\diamond$ is defined to be the simplex formed by the vertices in $\square$ and $\diamond$. For example, if $\mathcal{T}_1 \ni \tau = \gamma\gamma'$, then $\gamma' \in \mathrm{lk}(\gamma)$. Similarly, if a face σ has corner vertices $\gamma, \gamma', \gamma''$ and $\tau = \gamma'\gamma''$, then $\tau \in \mathrm{lk}(\gamma)$. Finally, an i-dimensional simplicial complex will be called strongly connected if for any two i-cells $\square, \square'$, there is a sequence of i-cells $\square = \square_1, \cdots, \square_j = \square'$ such that for each $k < j$, $\square_k \cap \square_{k+1}$ is an $(i-1)$-cell of $\mathcal{T}$.

Assumption 2.1. The domain Ω is simply connected. Furthermore, $\mathcal{T}$ and links of all 0- and 1-cells of $\mathcal{T}$ are strongly connected complexes.

Remark 2.2. By convention, each **Assumption** introduced will be in effect for the entirety of the text following it.

2.1. Smooth polynomial splines

We will now define the space of smooth polynomial splines on the planar triangulations introduced above. We start by defining polynomial degree deficits on the faces of the triangulation and orders of smoothness across its edges.

Definition 2.3 (Degree deficit distribution)**.** A degree deficit distribution on $\mathcal{T}$ is a map

$$\begin{aligned}\boldsymbol{\Delta m} \ : \ &\mathcal{T}_2 \to \mathbb{Z}_{\geq 0} \ , \\ &\sigma \mapsto \Delta m_\sigma \ ,\end{aligned}$$

such that $\exists \sigma \in \mathcal{T}_2,\ \Delta m_\sigma = 0$. The face deficits induce deficits on edges and vertices and these are defined as below,

$$\Delta m_\tau := \min_{\tau \subset \partial\sigma} \Delta m_\sigma \ , \quad \Delta m_\gamma := \min_{\gamma \in \partial\sigma} \Delta m_\sigma \ .$$

The map $\boldsymbol{\Delta m}$ will help specify the maximum total degree of polynomials on a face $\sigma \in \mathcal{T}_2$. Let us first denote with $\Delta \underline{m}$ the largest degree deficit specified by $\boldsymbol{\Delta m}$,

$$\Delta \underline{m} := \max_{\sigma \in \mathcal{T}_2} \Delta m_\sigma \ .$$

Then, given $\overline{m} \in \mathbb{Z}_{\geq 0}$, the polynomial degree associated to σ will be denoted as m_σ and is defined as follows, where we have used $\underline{m} := \overline{m} - \Delta \underline{m}$,

$$\underline{m} \leq m_\sigma := \overline{m} - \boldsymbol{\Delta m}(\sigma) \leq \overline{m} \ . \tag{2.1}$$

Denote with $R := \mathbb{R}[s,t]$ the polynomial ring with coefficients in $\mathbb{R}$. We define $R_m \subset R$ as the $\mathbb{R}$-linear vector space of polynomials of total degree exactly equal to m spanned by the monomials $s^i t^j$, $i + j = m$, $i, j \in \mathbb{Z}_{\geq 0}$. If m is negative, then $R_m := 0$. To any $\sigma \in \mathcal{T}_2$, we will associate the vector space $R_\sigma := \oplus_{m=0}^{m_\sigma} R_m$. This induces an association of edges $\tau \in \mathcal{T}_1$ and vertices $\gamma \in \mathcal{T}_0$ with the spaces $R_\tau := \oplus_{m=0}^{m_\tau} R_m$ and $R_\gamma := \oplus_{m=0}^{m_\gamma} R_m$, respectively, where m_τ and m_γ are given by

$$m_\tau := \max_{\tau \subset \partial\sigma} m_\sigma = \overline{m} - \Delta m_\tau \ , \qquad m_\gamma := \max_{\gamma \in \partial\sigma} m_\sigma = \overline{m} - \Delta m_\gamma \ .$$

Definition 2.4 (Smoothness distribution)**.** A smoothness distribution on $\mathcal{T}$ is a map

$$\begin{aligned}\boldsymbol{r} \ : \ &\mathring{\mathcal{T}}_1 \to \mathbb{Z}_{\geq 0} \ , \\ &\tau \mapsto \boldsymbol{r}(\tau) \ .\end{aligned}$$

The map $\boldsymbol{r}$ will help us define the smoothness across all interior edges, i.e., for $\tau \in \mathring{\mathcal{T}}_1$, splines will be required to be at least $C^{\boldsymbol{r}(\tau)}$ smooth across τ.

Definition 2.5 (Spline space on $\mathcal{T}$)**.** Given triangulation $\mathcal{T}$, degree deficit and smoothness distributions $\boldsymbol{\Delta m}$ and $\boldsymbol{r}$, respectively, and $\overline{m} \in \mathbb{Z}_{\geq 0}$, we define the spline space $\mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{\Delta m},\overline{m}}(\mathcal{T})$ as

$$\mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{\Delta m},\overline{m}} \equiv \mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{\Delta m},\overline{m}}(\mathcal{T}) := \Bigg\{ f \ : \ \forall \sigma \in \mathcal{T}_2 \ \ f|_\sigma \in R_\sigma \ , \\ \forall \tau \in \mathring{\mathcal{T}}_1 \ \ f \text{ is } C^{\boldsymbol{r}(\tau)} \text{ across } \tau \Bigg\} \ .$$

We will use the following algebraic characterization of the smoothness of a piecewise polynomial function; its proof can be found in [4], for instance.

Lemma 2.6. *Let σ and σ' be two d-dimensional polyhedra in $\mathbb{R}^d$ such their intersection $\sigma \cap \sigma'$ is a $(d-1)$-dimensional polyhedron τ. A piecewise polynomial function equaling p on σ and p' on σ', $p, p' \in \mathbb{R}[x_1, \ldots, x_d]$, is at least r times continuously differentiable across τ if and only if*

$$p - p' \in \left(l^{r+1}\right) \ ,$$

where $\left(l^{r+1}\right)$ is the ideal generated by l^{r+1} and l is the linear polynomial vanishing on τ.

2.2. Homogenized problem

As in [3, 8, 14], we approach the problem of finding the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$ by homogenizing the problem. First, we embed the triangulation $\mathcal{T}$ in the hyperplane $\{(s,t,w)|w=1\} \subset \mathbb{R}^3$; let $\hat{\mathcal{T}}$ denote the cone of the embedded triangulation formed with the vertex at the origin. The sets $\hat{\mathcal{T}}_i$ are obtained by forming cones of i-cells with the vertex at the origin, $i = 0,1,2$. We denote with $S := R[w] = \mathbb{R}[s,t,w]$ the extension of ring R by the homogenizing variable w. The associated vector space of homogeneous polynomials of total degree exactly m is denoted as S_m ($\cong R_{\leq m}$).

Definition 2.7 (Module of splines on $\hat{\mathcal{T}}$)**.** We define $C^{\boldsymbol{r}}(\hat{\mathcal{T}})$ as

$$C^{\boldsymbol{r}}(\hat{\mathcal{T}}) := \Big\{ f \ : \ \forall \widehat{\sigma} \in \hat{\mathcal{T}}_2 \ \ f|_{\widehat{\sigma}} \in S \ , \qquad \forall \widehat{\tau} \in \mathring{\hat{\mathcal{T}}}_1 \ \ f \text{ is } C^{\boldsymbol{r}(\tau)} \text{ across } \widehat{\tau} \Big\} \ .$$

Let $p = p(s,t) \in R$ be of degree m. Its homogenization $\widehat{p} \in S$ is defined as

$$\widehat{p} := w^m p(s/w, t/w) \ .$$

Similarly, its homogenization in degree $m' \geq m$ is defined as $w^{m'-m}\widehat{p}$. Conversely, for $p \in S$, we define its dehomogenized counterpart $\check{p} \in R$ as

$$\check{p} \equiv p(1) := p(s,t,1) \ .$$

The next result presents useful properties concerning (de)homogenization and proofs can be found in [4], for instance.

Proposition 2.8. *Consider $p_1, p_2 \in R$ and the principal ideal $\mathfrak{J} = \left(l^{r+1}\right)$ where l is a linear polynomial in R.*

(a) $\widehat{p_1 p_2} = \widehat{p}_1 \widehat{p}_2$*;*

(b) $\check{\widehat{p}}_1 = p$*;*

(c) $\widehat{\mathfrak{J}} := \left(\widehat{f} \ : \ f \in \mathfrak{J}\right) = \left(\widehat{l}^{r+1}\right)$.

Let $l_\tau \in S$ denote the homogenization (in degree 1) of the linear polynomial that vanishes on τ, and denote the ideal generated by $l_\tau^{\boldsymbol{r}(\tau)+1}$ with $\mathfrak{J}^{\boldsymbol{r}}(\tau) := \left(l_\tau^{\boldsymbol{r}(\tau)+1}\right) \subset S$. The following effectively restates Lemma 2.6 albeit using notation of the homogenized setup.

Lemma 2.9. *A function f lies in $C^{\boldsymbol{r}}(\hat{\mathcal{T}})$ if and only if, for every pair of faces σ and σ' that share a common edge τ,*

$$f|_{\widehat{\sigma}} - f|_{\widehat{\sigma}'} \in \mathfrak{J}^{\boldsymbol{r}}(\tau) \ .$$

Definition 2.10 (Restricted spline module on $\hat{\mathcal{T}}$)**.** We define $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}}) \subseteq C^{\boldsymbol{r}}(\hat{\mathcal{T}})$ as

$$\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}}) := \Big\{ f \ : \ \forall \widehat{\sigma} \in \hat{\mathcal{T}}_2 \ \ f|_{\widehat{\sigma}} \in \left(w^{\Delta m_\sigma}\right) \ , \qquad \forall \widehat{\tau} \in \mathring{\hat{\mathcal{T}}}_1 \ \ f \text{ is } C^{\boldsymbol{r}(\tau)} \text{ across } \widehat{\tau} \Big\} \ .$$

It was shown in [4] that $C^{\boldsymbol{r}}(\hat{\mathcal{T}})$ is a finitely generated, torsion free S-module and forms a graded algebra. The same can be said for $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$, and the following results make the reasoning explicit.

Proposition 2.11. *$\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$ is a finitely generated, torsion free S-module.*

Proof. For arbitrary $f, f' \in \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$ and $p \in S$, we have $f + f' \in \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$ and $pf \in \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$. Since S is a Noetherian integral domain, $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$ is finitely generated as it is a submodule of the finitely generated S-module $C^{\boldsymbol{r}}(\hat{\mathcal{T}})$, and it is torsion free as 0 is its only associated prime. ■

Proposition 2.12. *$\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$ is a graded $\mathbb{R}$-algebra.*

Proof. Consider an arbitrary $f \in \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$. Let f_m denote the homogeneous component of f of degree m. For two faces σ and σ' that share an edge τ, from Lemma 2.9 $f|_{\widehat{\sigma}} - f|_{\widehat{\sigma}'}$ is a member of a homogeneous ideal and, therefore, so is its homogeneous component of degree m, $f_m|_{\widehat{\sigma}} - f_m|_{\widehat{\sigma}'}$ [5]. This implies that $f_m \in \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$, making $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$ a graded algebra. ■

As in [3, 4], splines in $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$ of degree exactly $\overline{m}$ can be equivalently understood as the splines in $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$ that have been homogenized (in degree $\overline{m}$), and the following $\mathbb{R}$-vector space isomorphism holds.

Theorem 2.13.

$$\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})_{\overline{m}} \cong \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} \; .$$

Proof. We approach the proof as in [4] and build an isomorphism between $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})_{\overline{m}}$ and $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$. In fact, this map is simply the $\mathbb{R}$-linear dehomogenization map

$$\begin{aligned} \check{} \; : \; & \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})_{\overline{m}} \to \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} \; , \\ & f \mapsto \check{f} = s(1) \; . \end{aligned}$$

We will first show that the map is well defined and, thereafter, that it is a bijection. In the following, we assume that σ and σ' are any two faces that share an edge τ.

If $f \in \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})_{\overline{m}}$, then $f|_{\widehat{\sigma}} - f|_{\widehat{\sigma}'}$ lies in $\mathfrak{J}^{\boldsymbol{r}}(\tau)$. Then, $f|_{\widehat{\sigma}}(1) - f|_{\widehat{\sigma}'}(1) = (f|_{\widehat{\sigma}} - f|_{\widehat{\sigma}'})(1)$ lies in the ideal generated by $l_\tau(1)^{\boldsymbol{r}(\tau)+1}$. Therefore, $\check{f} \in \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$ and the map is well defined.

To show that the map is an injection, observe that if $\check{f} \equiv 0$, then $f|_{\widehat{\sigma}}(1) \equiv 0$ for all faces σ. This implies that $f|_{\widehat{\sigma}}$ is a multiple of $(w-1)$ for all σ. Since f is a homogeneous spline, this is not possible and we must have $f \equiv 0$.

Next, let $f \in \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$ and denote with $\widehat{f}$ its homogenization in degree $\overline{m}$. Since $f|_\sigma$ has degree at most m_σ, it is clear that $\widehat{f}|_{\widehat{\sigma}} \in \left(w^{\Delta\boldsymbol{m}(\sigma)}\right) \cap S_{\overline{m}}$. Moreover, by Lemma 2.6 and Proposition 2.8, $\widehat{f}|_{\widehat{\sigma}} - \widehat{f}|_{\widehat{\sigma}'}$ lies in $\mathfrak{J}^{\boldsymbol{r}}(\tau)$. From Lemma 2.9, this implies that $\widehat{f}$ is a member of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})_{\overline{m}}$. Since $\widehat{f}(1) = f$, the dehomogenization map is a surjection between the spaces. ■

Theorem 2.13 shows that to analyze the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$, we can instead focus on the $\overline{m}^{th}$ graded piece of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$, i.e., $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})_{\overline{m}}$, and we proceed in this direction.

3. Topological chain complexes

In this section we will describe the tools from homology that we will use for computing the dimensions of graded pieces of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m}}(\hat{\mathcal{T}})$. For $\square \in \mathcal{T}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_0$, define $S_\square := \left(w^{\Delta m_\square}\right)$. We also define $\mathfrak{J}_\tau := \mathfrak{J}^{\boldsymbol{r}}(\tau) \cap S_\tau$, $\tau \in \mathring{\mathcal{T}}_1$ and use it to define $\mathfrak{J}_\gamma := \sum_{\gamma \in \partial\tau} \mathfrak{J}_\tau$, $\gamma \in \mathring{\mathcal{T}}_0$. The $\overline{m}^{th}$ graded pieces of the above will be denoted in the usual manner as $S_{\square,\overline{m}}$ and $\mathfrak{J}_{\square,\overline{m}}$.

3.1. Definitions

In the following, faces σ, edges τ, and vertices γ will generate S-modules with the indexed generators denoted, respectively, by $[\sigma], [\tau]$ and $[\gamma]$. We will assume that all oriented 2-cells have been assigned a counter-clockwise orientation. For $\tau \in \mathcal{T}_1$ with end points $\gamma, \gamma' \in \mathcal{T}_0$, the associated generator will be denoted as $[\tau] = [\gamma\gamma']$, with $[\gamma'\gamma] = -[\gamma\gamma']$ defining the oppositely oriented edge. In the following sections we will only be interested in homology relative to $\partial\Omega$. Therefore, we will assume that $[\tau] = 0$ and $[\gamma] = 0$ when $\tau \subset \partial\Omega$ and $\gamma \in \partial\Omega$, respectively.

Consider the orientation function, $\varepsilon_{\theta,\phi} \equiv \varepsilon([\theta],[\phi])$, that takes as inputs one n-dimensional cell $[\theta]$, and one $(n-1)$-dimensional cell, $[\phi]$, and returns

- 0 if $\theta \cap \phi = \emptyset$,
- -1 if $\theta \cap \phi = \phi$ and the orientation endowed by $[\theta]$ upon its boundary is incompatible with orientation of $[\phi]$, and,

- $+1$ if $\theta \cap \phi = \phi$ and the orientation endowed by $[\theta]$ upon its boundary is compatible with orientation of $[\phi]$.

We will consider the usual boundary maps defined as

$$\partial([\sigma]) = \sum_{\tau \in \mathring{\mathcal{T}}_1} \varepsilon_{\sigma,\tau}[\tau] \;, \qquad \partial([\tau]) = \sum_{\gamma \in \mathring{\mathcal{T}}_0} \varepsilon_{\tau,\gamma}[\gamma] \;, \qquad \partial([\gamma]) = 0 \;.$$

3.2. Topological complexes

Consider an element $p = \sum_\sigma [\sigma] p_\sigma$ of the S-module $\oplus_{\sigma \in \mathcal{T}_2} [\sigma] S_\sigma$. We can express its image under the action of ∂ as

$$\partial \left(\sum_{\sigma \in \mathcal{T}_2} [\sigma] p_\sigma \right) = \sum_{\tau \in \mathring{\mathcal{T}}_1} [\tau] \left(\sum_{\sigma \in \mathcal{T}_2} \varepsilon_{\sigma,\tau} p_\sigma \right) \;.$$

It is clear that $\sum_\sigma \varepsilon_{\sigma,\tau} p_\sigma \in S_\tau$. Since p also needs to be in smoothness class $C^{\boldsymbol{r}}$, we require

$$\forall \tau \in \mathring{\mathcal{T}}_1 \;, \quad \sum_{\sigma \in \mathcal{T}_2} \varepsilon_{\sigma,\tau} p_\sigma \in \mathfrak{J}_\tau \;.$$

Then, $\mathcal{S}^{\boldsymbol{r}}_{\Delta \boldsymbol{m}}(\hat{\mathcal{T}})$ contains all splines s such that their polynomial pieces satisfy the above requirement, with $p_\sigma = s|_{\widehat{\sigma}}$. In other words,

$$\mathcal{S}^{\boldsymbol{r}}_{\Delta \boldsymbol{m}}(\hat{\mathcal{T}}) = \ker\left(\overline{\partial}\right) \;,$$

where $\overline{\partial}$ is the map

$$\overline{\partial} : \bigoplus_{\sigma \in \mathcal{T}_2} [\sigma] S_\sigma \to \bigoplus_{\tau \in \mathring{\mathcal{T}}_1} [\tau] S_\tau / \mathfrak{J}_\tau \;.$$

The above map is obtained by composing ∂ with the natural quotient map. In light of the above reasoning, we consider the following short exact sequence of chain complexes of S-modules as the object of our analysis, with the top homology of the complex $\mathcal{Q}$ corresponding to $\mathcal{S}^{\boldsymbol{r}}_{\Delta \boldsymbol{m}}(\hat{\mathcal{T}})$.

$$\begin{array}{lccccccc}
 & & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \\
\mathcal{I} : & 0 & \longrightarrow & \bigoplus_{\tau \in \mathring{\mathcal{T}}_1} [\tau] \mathfrak{J}_\tau & \longrightarrow & \bigoplus_{\gamma \in \mathring{\mathcal{T}}_0} [\gamma] \mathfrak{J}_\gamma & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C} : & \bigoplus_{\sigma \in \mathcal{T}_2} [\sigma] S_\sigma & \longrightarrow & \bigoplus_{\tau \in \mathring{\mathcal{T}}_1} [\tau] S_\tau & \longrightarrow & \bigoplus_{\gamma \in \mathring{\mathcal{T}}_0} [\gamma] S_\gamma & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{Q} : & \bigoplus_{\sigma \in \mathcal{T}_2} [\sigma] S_\sigma & \longrightarrow & \bigoplus_{\tau \in \mathring{\mathcal{T}}_1} [\tau] S_\tau / \mathfrak{J}_\tau & \longrightarrow & \bigoplus_{\gamma \in \mathring{\mathcal{T}}_0} [\gamma] S_\gamma / \mathfrak{J}_\gamma & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & &
\end{array} \tag{3.1}$$

Considering only the $\overline{m}^{th}$ graded pieces of the S-modules above would yield a short exact sequence of chain complexes of $\mathbb{R}$-vector spaces. In particular, from Theorem 2.13, $\mathcal{S}^{\boldsymbol{r}}_{\Delta \boldsymbol{m}, \overline{m}}$ is isomorphic to the top homology of $\mathcal{Q}_{\overline{m}}$,

$$\mathcal{S}^{\boldsymbol{r}}_{\Delta \boldsymbol{m}, \overline{m}} \cong \mathcal{S}^{\boldsymbol{r}}_{\Delta \boldsymbol{m}}(\hat{\mathcal{T}})_{\overline{m}} = \ker\left(\overline{\partial}\right)_{\overline{m}} = H_2(\mathcal{Q})_{\overline{m}} \;. \tag{3.2}$$

3.3. Summary of approach

From Equation (3.2), the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$ can be computed using the Euler characteristic of $\mathcal{Q}$,

$$\begin{aligned}\chi(\mathcal{Q})_{\overline{m}} &= \sum_{\sigma\in\mathcal{T}_2} \dim_{\mathbb{R}} S_{\sigma,\overline{m}} - \sum_{\tau\in\mathring{\mathcal{T}}_1} \dim_{\mathbb{R}} S_{\tau,\overline{m}}/\mathfrak{J}_{\tau,\overline{m}} + \sum_{\gamma\in\mathring{\mathcal{T}}_0} \dim_{\mathbb{R}} S_{\gamma,\overline{m}}/\mathfrak{J}_{\gamma,\overline{m}}\ , \\ &= \dim_{\mathbb{R}} H_2(\mathcal{Q})_{\overline{m}} - \dim_{\mathbb{R}} H_1(\mathcal{Q})_{\overline{m}} + \dim_{\mathbb{R}} H_0(\mathcal{Q})_{\overline{m}}\ .\end{aligned} \tag{3.3}$$

Therefore, our objective is the computation (or estimation) of $\dim_{\mathbb{R}} H_1(\mathcal{Q})_{\overline{m}} - \dim_{\mathbb{R}} H_0(\mathcal{Q})_{\overline{m}}$. Using the fact that the vertical columns in Equation (3.1) are short exact sequences, we obtain the long exact sequence

$$0 \to H_2(\mathcal{C}) \to H_2(\mathcal{Q}) \to H_1(\mathcal{I}) \to H_1(\mathcal{C}) \to H_1(\mathcal{Q}) \to H_0(\mathcal{I}) \to H_0(\mathcal{C}) \to H_0(\mathcal{Q}) \to 0\ . \tag{3.4}$$

Assumption 3.1. The mesh $\mathcal{T}$ is such that $H_1(\mathcal{C})_{\overline{m}} = 0$.

Roughly, the above assumption rules out those $\Delta\boldsymbol{m}$ that imply polynomial degree adaptivity patterns that contain "holes" or are "ring-like"; cf. Proposition 5.4 for the precise implication of this assumption. Then, by exactness of the sequence in Equation (3.4), we obtain

$$\dim_{\mathbb{R}} H_1(\mathcal{Q})_{\overline{m}} - \dim_{\mathbb{R}} H_0(\mathcal{Q})_{\overline{m}} = \dim_{\mathbb{R}} H_0(\mathcal{I})_{\overline{m}} - \dim_{\mathbb{R}} H_0(\mathcal{C})_{\overline{m}}\ .$$

Thus, we obtain the following result from Theorem 2.13 from Equations (3.2) and (3.3).

Theorem 3.2.

$$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} = \chi(\mathcal{Q})_{\overline{m}} + \dim_{\mathbb{R}} H_0(\mathcal{I})_{\overline{m}} - \dim_{\mathbb{R}} H_0(\mathcal{C})_{\overline{m}}\ .$$

An alternate expression for the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$ can be obtained by using the first part of the long exact sequence from Equation (3.4) and the Euler characteristic of $\mathcal{I}$,

$$\begin{aligned}\chi(\mathcal{I})_{\overline{m}} &= -\sum_{\tau\in\mathring{\mathcal{T}}_1} \dim_{\mathbb{R}} \mathfrak{J}_{\tau,\overline{m}} + \sum_{\gamma\in\mathring{\mathcal{T}}_0} \dim_{\mathbb{R}} \mathfrak{J}_{\gamma,\overline{m}}\ , \\ &= -\dim_{\mathbb{R}} H_1(\mathcal{I})_{\overline{m}} + \dim_{\mathbb{R}} H_0(\mathcal{I})_{\overline{m}}\ .\end{aligned}$$

Theorem 3.3.

$$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} = \dim_{\mathbb{R}} H_2(\mathcal{C})_{\overline{m}} + \dim_{\mathbb{R}} H_0(\mathcal{I})_{\overline{m}} - \chi(\mathcal{I})_{\overline{m}}\ .$$

4. Dimensions of relevant vector spaces

Before progressing to the analysis of the ingredients in Theorems 3.2 and 3.3, we collect combinatorial formulas for dimensions of relevant vector spaces that appear in the previous section. The first result follows directly from the definitions and is presented below.

Proposition 4.1.

$$\begin{aligned}\dim_{\mathbb{R}} S_{\sigma,\overline{m}} &= \binom{m_\sigma + 2}{2}\ , \\ \dim_{\mathbb{R}} S_{\tau,\overline{m}}/\mathfrak{J}_{\tau,\overline{m}} &= \binom{m_\tau + 2}{2} - \binom{m_\tau - \boldsymbol{r}(\tau) + 1}{2}\ .\end{aligned}$$

Only a characterization of graded pieces of the ideal $\mathfrak{J}_\gamma$ remains. This is an ideal generated by mixed powers of homogeneous linear forms. We will use results proved in [8] (Lemmas 4.2—4.4 below) for treatment of these, albeit in a slightly more general form; see Lemma 4.5 below. Let us first set up the problem and present some preliminary results that help with the final solution.

Let $A := \mathbb{R}[x_1, x_2]$, and let $h_1, \dots, h_k \in A$ be k pairwise linearly independent homogeneous linear forms. Denote with $\boldsymbol{c} := (c_1, \dots, c_k) \in \mathbb{Z}^k_{>0}$ an exponent vector, $c_1 \leq \cdots \leq c_k$. If, for all $2 \leq i \leq k-1$,

$$c_{i+1} \leq \frac{\sum_{j=1}^{i} c_j - i}{i-1}\ , \tag{4.1}$$

then $\boldsymbol{c}$ will be called a *minimal exponent vector*. This is because, as stated in the next result, this means that the linear forms $h_1, \dots, h_k$ form a minimal generating set for the ideal $(h_1^{c_1}, \dots, h_k^{c_k}) \subseteq A$.

Lemma 4.2 (Geramita and Schenck [8]). *For $2 \leq i \leq k-1$,*

$$h_{i+1}^{c_{i+1}} \notin (h_1^{c_1}, \ldots, h_i^{c_i}) \Leftrightarrow (c_1, \ldots, c_{i+1}) \text{ is minimal} .$$

Lemma 4.3 (Geramita and Schenck [8]). *Let $\boldsymbol{c}$ be a minimal exponent vector and $J = (h_1^{c_1}, \ldots, h_k^{c_k}) \subseteq A$. Then the dimension of the vector space J_m is given by*

$$\dim_{\mathbb{R}} J_m = \dim_{\mathbb{R}} J \cap A_m = \sum_{i=1}^{k} \dim_{\mathbb{R}} A(-c_i)_m - \alpha \dim_{\mathbb{R}} A(-\Omega-1)_m - \beta \dim_{\mathbb{R}} A(-\Omega)_m ,$$

where

$$\Omega := \left\lfloor \frac{\sum_{i=1}^{k} c_i - k}{k-1} \right\rfloor + 1 , \qquad \alpha := \sum_{i=1}^{k} c_i + (1-k)\Omega , \qquad \beta := k - 1 - \alpha .$$

If $k = 1$, then we define $\Omega = \alpha = 0$.

Consider now $B := A[x_3] = \mathbb{R}[x_1, x_2, x_3]$, and let $h_1, \ldots, h_k \in B$ be k homogeneous linear forms.

Lemma 4.4 (Geramita and Schenck [8]). *Let $\boldsymbol{c}$ be a minimal exponent vector and $J = (h_1^{c_1}, \ldots, h_k^{c_k}) \subseteq B$. If all h_i only involve the variables x_1 and x_2, the dimension of the vector space J_m is given by*

$$\dim_{\mathbb{R}} J_m = \dim_{\mathbb{R}} J \cap B_m = \sum_{i=1}^{k} \dim_{\mathbb{R}} B(-c_i)_m - \alpha \dim_{\mathbb{R}} B(-\Omega-1)_m - \beta \dim_{\mathbb{R}} B(-\Omega)_m ,$$

with Ω, α, β as defined in Lemma 4.3.

Let us now present a more general form of this last result. Assume that $\boldsymbol{c} = (c_1, \ldots, c_k)$ is now an exponent vector that is not necessarily minimal, but still has non-decreasing entries, and let $(e_1, \ldots, e_k) \in \mathbb{Z}_{\geq 0}^k$ be any vector; define $E := \{e_1, \ldots, e_k\}$. Our main object of interest now is the homogeneous ideal

$$J = (x_3^{e_1} h_1^{c_1}, \ldots, x_3^{e_k} h_k^{c_k}) \subseteq B ,$$

where the homogeneous linear forms h_i only involve the variables x_1 and x_2.

Let $\boldsymbol{I} := \{1, \ldots, k\}$. Given a fixed $e \in E$, let $\boldsymbol{I}^e := \{i_1, \ldots, i_{a(e)}\}$, $i_1 < \cdots < i_{a(e)}$, be the maximal subset of $\boldsymbol{I}$ such that $e_i < e$ for all $i \in \boldsymbol{I}^e$. Next, recursively define the following ideals in rings A and B (we allow h_i to denote linear forms in both A and B as they only involve variables x_1 and x_2),

$$\begin{aligned} B &\supseteq J^0 := (h_i^{c_i} \ : \ i \in \boldsymbol{I}^\infty) , \\ A &\supseteq J^1 := \left(h_i^{c_i} \ : \ i \in \boldsymbol{I}^{e^1} \text{ for } e^1 := \max E\right) , \\ A &\supseteq J^j := \left(h_i^{c_i} \ : \ i \in \boldsymbol{I}^{e^j} \text{ for } e^j := \max E \backslash \{e^1, \ldots, e^{j-1}\}\right) , \quad j = 2, \ldots, d , \end{aligned} \tag{4.2}$$

where we assume that d is the integer such that the set $E \backslash \{e^1, e^2, \ldots, e^{d-1}, e^d\}$ contains exactly one element denoted e^{d+1}. For any given e^j, $\left\{h_i^{c_i} \ : \ i \in \boldsymbol{I}^{e^j}\right\}$ may not form a minimal generating set for J^j. However, it can be easily pruned by removing pairwise linear dependencies between the forms h_i, $i \in \boldsymbol{I}^{e^j}$ and forming a minimal generating set using Lemma 4.2.

Once minimal generating sets have been formed for ideals J^j, dimensions of their m^{th} graded pieces can be found using Lemma 4.3 for $j > 0$ and Lemma 4.4 for $j = 0$. Note that the next result collapses onto Lemma 4.4 when $E = \{0\}$.

Lemma 4.5. *With the above setup, the dimension of the vector space J_m is given by*

$$\dim_{\mathbb{R}} J_m := \dim_{\mathbb{R}} J \cap B_m = \dim_{\mathbb{R}} J^0 \cap B(-e^1)_m + \sum_{l=1}^{d} \sum_{e=e^{l+1}}^{e^l-1} \dim_{\mathbb{R}} J^l \cap A(-e)_m .$$

Proof. We can equivalently write J_m as

$$J_m = \sum_{i=1}^{k} x_3^{e_i} h_i^{c_i} B(-c_i - e_i)_m \ .$$

Since $B = A[x_3]$, the claim follows upon using $B_m = \oplus_{i=0}^{m} x_3^i A_{m-i}$ and grading J_m by powers of x_3. ■

Remark 4.6. The above result inherits the generality of Lemmas 4.3 and 4.4 — none of them depend on the particular linear forms chosen as long as they are pairwise linearly independent.

By definition, $\mathfrak{J}_\gamma$ is of the same form as J in Lemma 4.5, and we can therefore use the above result to compute its dimension. Let $\boldsymbol{I}_\gamma$ be the index set for edges containing γ and define $d_\gamma := \#\boldsymbol{I}_\gamma$. We assume that $\boldsymbol{I}_\gamma = \left\{i_1, \dots, i_{d_\gamma}\right\}$ such that $\boldsymbol{r}(\tau_{i_1}) \leq \cdots \leq \boldsymbol{r}(\tau_{i_{d_\gamma}})$. Define the k-tuple $\boldsymbol{c}$ and e_j, $j = 1, \dots, d_\gamma$, as

$$e_j := \Delta m_{\tau_{i_j}} \ , \qquad \boldsymbol{c} := \Big(\boldsymbol{r}(\tau_{i_1}) + 1, \dots, \boldsymbol{r}(\tau_{i_{d_\gamma}}) + 1\Big) \ .$$

Finally, with $E := \{e_1, \cdots, e_{d_\gamma}\}$, define the integers $e^1, \dots, e^{d+1}$ and the ideals $\mathfrak{J}^0, \dots, \mathfrak{J}^d$ analogously to Equation (4.2) and Lemma 4.5, where the homogeneous linear forms are now chosen to be $l_{\tau_{i_1}}, \dots, l_{\tau_{i_{d_\gamma}}}$. The dimension of $\mathfrak{J}_{\gamma,\overline{m}}$ can then be computed to be

$$\dim_{\mathbb{R}} \mathfrak{J}_{\gamma,\overline{m}} = \dim_{\mathbb{R}} \mathfrak{J}^0 \cap S(-e^1)_{\overline{m}} + \sum_{l=1}^{d} \sum_{e=e^{l+1}}^{e^l - 1} \dim_{\mathbb{R}} \mathfrak{J}^l \cap R(-e)_{\overline{m}} \ . \tag{4.3}$$

Example 4.7. Consider a vertex γ where 5 edges $\tau_1, \dots, \tau_5$ meet. Let the homogeneous linear forms corresponding to the 5 edges in $\Gamma(\gamma)$ be $l_1, \dots, l_5$ and assume they are pairwise linearly independent. Let the smoothness requirements imposed on the edges be

$$\boldsymbol{r}(\tau_1) = 3 \ , \ \ \boldsymbol{r}(\tau_2) = 5 \ , \ \ \boldsymbol{r}(\tau_3) = \boldsymbol{r}(\tau_4) = \boldsymbol{r}(\tau_5) = 6 \ ,$$

so that $\boldsymbol{c} = (4, 6, 7, 7, 7)$. Let $\Delta\boldsymbol{m}$ be such that

$$e_1 = \Delta m_{\tau_1} = 3 \ , \ \ e_2 = e_3 = \Delta m_{\tau_2} = \Delta m_{\tau_3} = 1 \ , \ \ e_4 = e_5 = \Delta m_{\tau_4} = \Delta m_{\tau_5} = 0 \ ,$$

thereby implying that $E = \{0, 1, 3\}$. Let us illustrate the computation of the dimension of $\mathfrak{J}_{\gamma,\overline{m}}$, where

$$\mathfrak{J}_\gamma = \sum_{i=1}^{5} w^{e_i} l_i^{c_i} S(-c_i - e_i) \ .$$

Using similar notation as in Lemma 4.5 and Equation (4.3), we see that $d = 2$ and

$$\boldsymbol{I}^\infty = \{1, 2, 3, 4, 5\} \ ; \qquad e^1 = 3 \ , \ \boldsymbol{I}^{e^1} = \{2, 3, 4, 5\} \ ; \qquad e^2 = 1 \ , \ \boldsymbol{I}^{e^2} = \{4, 5\} \ ; \qquad e^3 = 0 \ .$$

Thus, from Lemma 4.2, the ideals of interest are minimally generated as given below,

$$S \supseteq \mathfrak{J}^0 = \left(l_1^4, l_2^6, l_3^7, l_4^7\right) \ , \quad R \supseteq \mathfrak{J}^1 = \left(l_2^6, l_3^7, l_4^7, l_5^7\right) \ , \quad R \supseteq \mathfrak{J}^2 = \left(l_4^7, l_5^7\right) \ .$$

Using Equation (4.3), the dimension of $\mathfrak{J}_{\gamma,\overline{m}}$ is found to be

$$\dim_{\mathbb{R}} \mathfrak{J}_{\gamma,\overline{m}} = \dim_{\mathbb{R}} \mathfrak{J}^0 \cap S(-3)_{\overline{m}} + \sum_{e=1}^{2} \dim_{\mathbb{R}} \mathfrak{J}^1 \cap R(-e)_{\overline{m}} + \dim_{\mathbb{R}} \mathfrak{J}^2 \cap R_{\overline{m}} \ .$$

Everything on the right can be computed using Lemmas 4.3 and 4.4, and we confirm using Macaulay2 that $\dim_{\mathbb{R}} \mathfrak{J}_{\gamma,\overline{m}}$ is correctly computed to be $4, 13, 27$ and 42 for $\overline{m} = 7, 8, 9$ and 10, respectively.

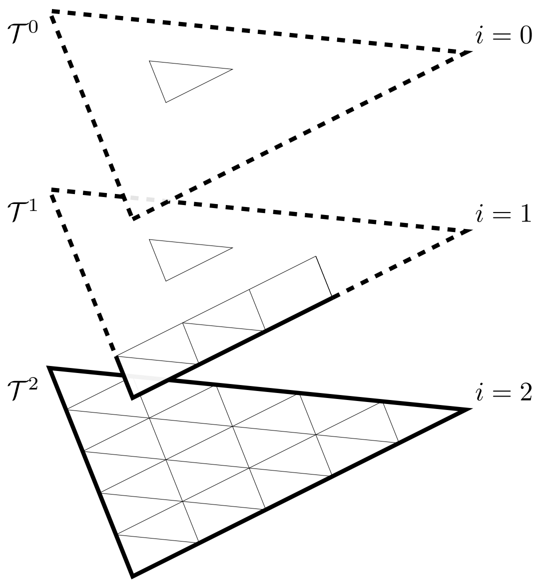


Figure 2: An example illustrating the concept of active triangulations introduced in Definition 5.2. The degree deficit on all triangles in $\mathcal{T}_2^2 \backslash \mathcal{T}_2^1$ is 2, while for triangles in $\mathcal{T}_2^1 \backslash \mathcal{T}_2^0$ it is equal to 1; the degree deficit on all triangles in $\mathcal{T}_2^0$ is equal to 0.

5. Homology of $\mathcal{C}$

In this section we collect main results characterizing the homology of the chain complex $\mathcal{C}$. Recall the notation introduced in Equation (2.1).

Proposition 5.1.

$$\dim_{\mathbb{R}} H_2(\mathcal{C})_{\overline{m}} = \dim_{\mathbb{R}} \bigcap_{\sigma \in \mathcal{T}_2} S_{\sigma,\overline{m}} = \binom{\underline{m}+2}{2} .$$

Proof. Let $p = \sum_\sigma [\sigma] p_\sigma$, $p_\sigma \in S_{\sigma,\overline{m}}$, be in the kernel of ∂, i.e.,

$$0 = \partial(p) = \sum_{\tau \in \mathring{\mathcal{T}}_1} [\tau] \sum_{\sigma \in \mathcal{T}_2} \varepsilon_{\sigma,\tau} p_\sigma \Leftrightarrow \forall \tau \in \mathring{\mathcal{T}}_1 \ , \ \sum_{\sigma \in \mathcal{T}_2} \varepsilon_{\sigma,\tau} p_\sigma = 0 \ .$$

If any σ and σ' share an edge τ, $\varepsilon_{\sigma,\tau} = -\varepsilon_{\sigma',\tau}$. Therefore, we must have $p_\sigma = p_{\sigma'}$ for every σ and σ' that share an edge. By Assumption 2.1, all p_σ must correspond to the same global polynomial. ■

In order to analyze $H_1(\mathcal{C})_{\overline{m}}$ and $H_0(\mathcal{C})_{\overline{m}}$, we introduce the concept of the *active part of the triangulation $\mathcal{T}$ with respect to an integer i*; Figure 2 visually introduces this notion. Recall that $\Delta\underline{m}$ is the largest degree deficit specified by $\boldsymbol{\Delta m}$.

Definition 5.2 (Active triangulation)**.** The active triangulation $\mathcal{T}^i$ with respect to $i \in \mathbb{Z}_{\geq 0}$, $0 \leq i \leq \Delta\underline{m}$, is defined as:

- $\mathcal{T}_2^i \subset \mathcal{T}_2$ such that $\sigma \in \mathcal{T}_2^i \overset{\text{def.}}{\Longleftrightarrow} i \geq \Delta m_\sigma$,
- $\mathcal{T}_1^i \subset \mathcal{T}_1$ defined as the set of all edges contained in the union $\cup_{\sigma \in \mathcal{T}_2^i} \partial\sigma$, and,
- $\mathcal{T}_0^i \subset \mathcal{T}_0$ defined as the set of all vertices contained in the union $\cup_{\tau \in \mathcal{T}_1^i} \partial\tau$.

The domain of this active triangulation is defined as the union $\Omega^i = \cup_{\sigma \in \mathcal{T}_2^i} \sigma \subset \mathbb{R}^2$.

The symbols for interior edges, vertices etc. are all appended with a superscript of i when talking about the active triangulation with respect to i. Note that "interior" will always mean interior with respect to Ω. The above definition has been motivated by the fact that for $i, j \in \mathbb{Z}_{\geq 0}$, $i + j = m$, monomial $s^i t^j w^{\overline{m}-m}$ is in $S_{\sigma,\overline{m}}$ (resp. $S_{\tau,\overline{m}}$ and $S_{\gamma,\overline{m}}$) only for $\sigma \in \mathcal{T}_2^{\overline{m}-m}$ (resp. $\tau \in \mathcal{T}_1^{\overline{m}-m}$ and $\gamma \in \mathcal{T}_0^{\overline{m}-m}$). Of course, for $m \leq \underline{m}$, $\mathcal{T}^{\overline{m}-m} = \mathcal{T}$.

Definition 5.3 (Number of relative holes in Ω^i)**.** We define π^i to be the number of linearly independent, non-trivial cycles in Ω^i relative to $\partial\Omega^i \cap \partial\Omega$,

$$\pi^i := \operatorname{rank}\left(H_1(\Omega^i, \partial\Omega^i \cap \partial\Omega)\right) .$$

Proposition 5.4.

$$\dim_{\mathbb{R}} H_1(\mathcal{C})_{\overline{m}} = \sum_{m=\underline{m}+1}^{\overline{m}} (m+1)\pi^{\overline{m}-m} .$$

Proof. The entire kernel of $\partial : \bigoplus_{\tau\in\mathring{\mathcal{T}}_1} [\tau]S_{\tau,\overline{m}} \to \bigoplus_{\gamma\in\mathring{\mathcal{T}}_0} [\gamma]S_{\gamma,\overline{m}}$ can be generated by ($\mathbb{R}$-linear combinations of) c^m, $0 \le m \le \overline{m}$, of the form $c^m = [\phi^m]s^i t^j w^{\overline{m}-m}$, where

- $i, j \in \mathbb{Z}_{\ge 0}$, $i + j = m$;
- $[\phi^m] = \sum_{\tau\in\mathring{\mathcal{T}}_1} [\tau]o_\tau$, $o_\tau \in \mathbb{Z}$, is a relative cycle, i.e., $\partial[\phi^m] = 0$; and,
- $o_\tau \neq 0 \Rightarrow s^i t^j w^{\overline{m}-m} \in S_{\tau,\overline{m}}$.

Then, for fixed i, j, we only need to see how many such c^m are linearly independent and not nullhomologous. It is sufficient to check if there exists some $d^m = (\sum_{\sigma\in\mathcal{T}_2}[\sigma]o_\sigma)s^i t^j w^{\overline{m}-m}$ such that $\partial(d^m) = c^m$, where

- $o_\sigma \in \mathbb{Z}$; and,
- $o_\sigma \neq 0 \Rightarrow s^i t^j w^{\overline{m}-m} \in S_{\sigma,\overline{m}}$.

Since $o_\sigma \neq 0 \Rightarrow \sigma \in \mathcal{T}_2^{\overline{m}-m}$, it is clear that c^m is not nullhomologous in $H_1(\mathcal{C})$ iff $[\phi^m]$ is not nullhomologous in $H_1(\Omega^{\overline{m}-m}, \partial\Omega^{\overline{m}-m} \cap \partial\Omega)$. For $i+j = m$, each such homology class $[\phi^m]$ contributes to $(m+1)$ homology classes in $H_1(\mathcal{C})$. The claim follows upon recalling that $\pi^i = 0$ for $i \ge \Delta\underline{m}$ from Assumption 2.1. ∎

Remark 5.5. Following Proposition 5.4, Assumption 3.1 implies that all π^i must be 0 for all $0 \le i \le \Delta\underline{m}$. In other words, none of the domains Ω^i are allowed to have any holes in homology relative to $\partial\Omega \cap \partial\Omega^i$.

Definition 5.6 (Number of relative connected components in Ω^i)**.** We define N^i to be the number of connected components in Ω^i relative to $\partial\Omega^i \cap \partial\Omega$,

$$N^i := \operatorname{rank}\left(H_0(\Omega^i, \partial\Omega^i \cap \partial\Omega)\right) .$$

Proposition 5.7.

$$\dim_{\mathbb{R}} H_0(\mathcal{C})_{\overline{m}} = \sum_{m=\underline{m}+1}^{\overline{m}} (m+1)N^{\overline{m}-m} .$$

Proof. For $0 \le i + j \le \overline{m}$, all $[\gamma]s^i t^j w^{\overline{m}-i-j}$, $\gamma \in \mathring{\mathcal{T}}_0$, $s^i t^j w^{\overline{m}-i-j} \in S_{\gamma,\overline{m}}$, are in the kernel of ∂. Let vertex γ_0, edges $\tau_1, \ldots, \tau_k \in \mathring{\mathcal{T}}_1$ and $o_1, \ldots, o_k \in \mathbb{Z}$ be such that

- $\forall l \in \{1, \ldots, k\}$, $s^i t^j w^{\overline{m}-i-j} \in S_{\tau_l,\overline{m}}$; and,
- $[\gamma] = [\gamma_0] + \partial\left(\sum_{l=1}^k [\tau_l]o_l\right)$.

Let $i + j = m$. Then, $[\gamma]s^i t^j w^{\overline{m}-m}$ is nullhomologous only if (a) $\gamma_0 \in \partial\Omega$ and (b) γ and γ_0 belong to the same connected component of $\Omega^{\overline{m}-m}$. Otherwise, $[\gamma_0]s^i t^j w^{\overline{m}-m}$ would be a generator of $H_0(\mathcal{C})$ corresponding to the particular connected component of $\Omega^{\overline{m}-m}$ that γ_0 belongs to. For each $s^i t^j w^{\overline{m}-m}$, the number of such generators is exactly equal to $N^{\overline{m}-m}$. In particular, the number of generators for a fixed $i + j = m$ is equal to $(m+1)N^{\overline{m}-m}$, and the claim follows. ∎

6. Homology of $\mathcal{I}$

In this section we collect results on the characterization and computation of the homology of $\mathcal{I}$. We start by providing a lower bound on the dimension of $\partial\left(\underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\tau]\mathfrak{J}_{\tau,\overline{m}}\right)$, and we follow the recipe employed in [14]. For $\gamma\in\mathring{\mathcal{T}}_0$, consider the map ϕ defined as

$$\begin{aligned}\phi\ :\ \underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\tau]\mathfrak{J}_{\tau,\overline{m}} &\to \underset{\gamma\in\mathring{\mathcal{T}}_0}{\oplus}\ \underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\gamma|\tau]S_{\tau,\overline{m}}\\ [\tau] &\mapsto \sum_{\gamma\in\tau}\varepsilon_{\tau,\gamma}[\gamma|\tau]\ ,\end{aligned}$$

and the map ψ defined as

$$\begin{aligned}\psi\ :\ \underset{\gamma\in\mathring{\mathcal{T}}_0}{\oplus}\ \underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\gamma|\tau]S_{\tau,\overline{m}} &\to [\gamma]\mathfrak{J}_{\gamma,\overline{m}}\\ [\gamma|\tau] &\mapsto \begin{cases}[\gamma]\ , & \gamma\in\tau\\ 0\ , & \text{otherwise}\end{cases}\ ,\end{aligned}$$

where $[\gamma|\tau]$ is a half-edge element, with $[\gamma|\tau]:=0$ when $\varepsilon_{\tau,\gamma}=0$ or when $\gamma\in\partial\Omega$.

Lemma 6.1.

$$\partial=\psi\circ\phi\ .$$

Definition 6.2 (Interior vertex ordering)**.** An injective map $\iota:\mathring{\mathcal{T}}_0\to\mathbb{N}$ is called an ordering of the interior vertices of the triangulation $\mathcal{T}$. For $\gamma\in\mathring{\mathcal{T}}_0$, define $\Gamma_\iota(\gamma)$ as the set of interior edges connecting γ to γ' such that either $\gamma'\in\partial\Omega$ or $\iota(\gamma)>\iota(\gamma')$. We will abuse the notation by saying $\gamma\succ\gamma'$ when $\iota(\gamma)>\iota(\gamma')$.

We will assume that ι, once chosen, is fixed and will omit it from all notation hereafter. Next, define $\tilde{\mathfrak{J}}_{\gamma,\overline{m}}:=\sum_{\tau\in\Gamma(\gamma)}\mathfrak{J}_{\tau,\overline{m}}$, and consider the map

$$\begin{aligned}\delta\ :\ \underset{\gamma\in\mathring{\mathcal{T}}_0}{\oplus}\ \underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\gamma|\tau]S_{\tau,\overline{m}} &\to \underset{\gamma\in\mathring{\mathcal{T}}_0}{\oplus}\ \underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\gamma|\tau]S_{\tau,\overline{m}}\\ [\gamma|\tau] &\mapsto \begin{cases}[\gamma|\tau]\ , & \tau\in\Gamma(\gamma)\\ 0\ , & \text{otherwise}\end{cases}\ .\end{aligned}$$

Define $\tilde{\partial}:=\psi\circ\delta\circ\phi$. By construction,

$$\dim_{\mathbb{R}}\underset{\gamma\in\mathring{\mathcal{T}}_0}{\oplus}[\gamma]\tilde{\mathfrak{J}}_{\gamma,\overline{m}}=\dim_{\mathbb{R}}\tilde{\partial}\left(\underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\tau]\mathfrak{J}_{\tau,\overline{m}}\right)\le\dim_{\mathbb{R}}\partial\left(\underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\tau]\mathfrak{J}_{\tau,\overline{m}}\right)\ , \tag{6.1}$$

with each interior edge τ contributing to exactly one $\tilde{\mathfrak{J}}_{\gamma,\overline{m}}$.

Proposition 6.3.

$$0\le\dim_{\mathbb{R}}H_1(\mathcal{I})_{\overline{m}}\le\sum_{\tau\in\mathring{\mathcal{T}}_1}\dim_{\mathbb{R}}\mathfrak{J}_{\tau,\overline{m}}-\sum_{\gamma\in\mathring{\mathcal{T}}_0}\dim_{\mathbb{R}}\tilde{\mathfrak{J}}_{\gamma,\overline{m}}\ .$$

Proof. Since $H_1(\mathcal{I})=\ker(\partial)$, and $\underset{\tau\in\mathring{\mathcal{T}}_1}{\oplus}[\tau]\mathfrak{J}_\tau\cong\ker(\partial)\oplus\operatorname{im}(\partial)$, the claim follows from Equation (6.1). ■

Corollary 6.4.

$$0\le\dim_{\mathbb{R}}H_0(\mathcal{I})_{\overline{m}}\le\sum_{\gamma\in\mathring{\mathcal{T}}_0}\dim_{\mathbb{R}}\mathfrak{J}_{\gamma,\overline{m}}-\sum_{\gamma\in\mathring{\mathcal{T}}_0}\dim_{\mathbb{R}}\tilde{\mathfrak{J}}_{\gamma,\overline{m}}\ .$$

Remark 6.5. The dimension of $\tilde{\mathfrak{J}}_{\gamma,\overline{m}}$ can be found by following the same set of steps that led to Equation (4.3) albeit by defining $\boldsymbol{I}_\gamma$ to be the index set of only those edges that are contained in $\Gamma(\gamma)$.

Alternate lower bounds can be derived from the following observation: since the vector spaces of the complex $\mathcal{I}_{\overline{m}}$ are subspaces of the vector spaces of $\mathcal{C}_{\overline{m}}$, if a representative of a homology class in $H_0(\mathcal{C})_{\overline{m}}$ is a member of $\oplus[\gamma]\mathfrak{J}_{\gamma,\overline{m}}$, then it must correspond to a homology class in $H_0(\mathcal{I})_{\overline{m}}$ as it can not be in the image of $\partial : \oplus[\tau]\mathfrak{J}_{\tau,\overline{m}} \to \oplus[\gamma]\mathfrak{J}_{\gamma,\overline{m}}$.

Proposition 6.6. *Given $m \geq \underline{m}+1$, let each connected component of $\Omega^{\overline{m}-m}$ that does not intersect the boundary contain a vertex γ such that $l_\gamma^m w^{\overline{m}-m} \in \mathfrak{J}_{\gamma,\overline{m}}$ for any homogeneous linear form l_γ corresponding to a linear polynomial vanishing on γ. Then*

$$\dim_{\mathbb{R}} H_0(\mathcal{I})_{\overline{m}} \geq \dim_{\mathbb{R}} H_0(\mathcal{C})_{\overline{m}}\ .$$

Proof. For some $m \geq \underline{m}+1$, consider a connected component of $\Omega^{\overline{m}-m}$ that does not intersect the boundary. Then, there exists a vertex γ in this connected component such that, upon translating γ to the origin without loss of generality, we can find $(m+1)$ homogeneous linear forms l_γ,

$$l_\gamma^m = \sum_{i+j=m} a_{ij} s^i t^j\ ,$$

such that the coefficient matrix $[a_{ij}]$ is full rank. This implies that we can find representatives in $\mathfrak{J}_{\gamma,\overline{m}}$ of each of the $(m+1)$ homology classes for degree m used in the proof of Proposition 5.7. ■

Corollary 6.7. *If the statement of sufficiency in Proposition 6.6 holds, then*

$$\dim_{\mathbb{R}} H_1(\mathcal{I})_{\overline{m}} \geq \sum_{\tau\in\mathring{\mathcal{T}}_1} \dim_{\mathbb{R}} \mathfrak{J}_{\tau,\overline{m}} - \sum_{\gamma\in\mathring{\mathcal{T}}_0} \dim_{\mathbb{R}} \mathfrak{J}_{\gamma,\overline{m}} + \dim_{\mathbb{R}} H_0(\mathcal{C})_{\overline{m}}\ .$$

Remark 6.8. Equation (4.3) can be used to verify if the statement of sufficiency in Proposition 6.6 holds. One only needs to check if the dimension of $\mathfrak{J}_{\gamma,\overline{m}}$ changes after adding $l_\gamma^m w^{\overline{m}-m}$ to its set of generators. In particular, let $m \geq \underline{m}+1$ be the smallest degree for which $\gamma \in \Omega^{\overline{m}-m}$ belongs to a connected component not intersecting $\partial\Omega$, and let $r = \max\{\boldsymbol{r}(\tau)\ :\ \gamma \in \tau \in \mathring{\mathcal{T}}_1^{\overline{m}-m}\}$. Then, from Equation (4.1), the statement of sufficiency will always be satisfied for such a γ if $m \geq 2r+1$.

7. Dimension of spline space

Our main results on the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$ are collected here and have been obtained by combining the results from previous sections. In the following sections, we will first use Theorems 3.2 and 3.3 in conjunction with the results in Sections 5 and 6 to provide bounds on the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$. Subsequently, we will show that for $\overline{m} \gg 0$ the dimension can be determined exactly.

7.1. Upper and lower bounds

Theorem 7.1.

$$0 \leq \dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} - \chi(\mathcal{Q})_{\overline{m}} + \sum_{m=\underline{m}+1}^{\overline{m}} (m+1)N^{\overline{m}-m} \leq \sum_{\gamma\in\mathring{\mathcal{T}}_0} \left(\dim_{\mathbb{R}} \mathfrak{J}_{\gamma,\overline{m}} - \dim_{\mathbb{R}} \tilde{\mathfrak{J}}_{\gamma,\overline{m}}\right)\ .$$

Proof. The claim follows from the bounds provided in Corollary 6.4. ■

Corollary 7.2. *If $\dim_{\mathbb{R}} \tilde{\mathfrak{J}}_{\gamma,\overline{m}} = \dim_{\mathbb{R}} \mathfrak{J}_{\gamma,\overline{m}}$ for all γ, then equality holds in Theorem 7.1.*

Theorem 7.3. *If the statement of sufficiency in Proposition 6.6 holds, then*

$$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} \geq \chi(\mathcal{Q})_{\overline{m}}\ .$$

Proof. The claim follows from Proposition 6.6. ■

It can be readily observed that, in the special case of uniform degrees, the upper and lower bounds on the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{0},\overline{m}}$ coincide with those presented in [14].

7.2. Dimension formula for $\overline{m} \gg 0$

It can be rigorously shown that the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$ is stable for $\overline{m} \gg 0$, but it requires an alternate approach than the one summarized in Section 3. Specifically, instead of considering the complex $\mathcal{Q}$ directly, we "break it up" into pieces that explicitly indicate the contributions that different degree deficits have toward the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$. Doing so, we can prove the main result of this section, Theorem 7.4.

Theorem 7.4.

$$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} = \chi\left(\mathcal{Q}\right)_{\overline{m}} \ , \qquad \overline{m} \gg 0 \ .$$

The proof of Theorem 7.4 follows from Lemmas 7.7 and 7.10. We start with a result that will help detail when parts of $H_0(\mathcal{I})$ vanish in sufficiently high degree.

Lemma 7.5. *For a triangle σ, let $l_{\tau_1}, l_{\tau_2}, l_{\tau_3}$ be the homogeneous linear polynomials associated to its edges τ_1, τ_2, τ_3, respectively, where τ_1 and τ_2 meet at $(0,0)$. Additionally, let $p \in S$ be a homogeneous polynomial that does not vanish at $(0,0,1)$, and consider a graded S-module $\mathfrak{M}$. Then, $\mathfrak{M}$ vanishes in sufficiently high degree if either of the following two conditions hold:*

(a) some power of each of l_1, l_2 and p annihilate $\mathfrak{M}$,

(b) some power of each of l_1, $l_2 l_3$ and w annihilate $\mathfrak{M}$.

Proof. For either part of the claim, let $\mathfrak{K}$ be the ideal generated by the three polynomials given; it can be verified that $\sqrt{\mathfrak{K}} = (s,t,w)$. Since $\mathfrak{K}$ must contain a power of its radical (S is Noetherian), $(s,t,w)^K \subset \mathfrak{K}$, $K \in \mathbb{Z}_{\geq 0}$, and thus $\mathfrak{M}$ vanishes in sufficiently high degree. ■

As mentioned earlier, let us now consider an alternate approach toward characterization of the homology of $\mathcal{Q}$. To do so, we first need some additional notation. First, we define the (shifted) ideal $\mathfrak{L}_{[i]}(-j)$ for $0 \leq i \leq \Delta\underline{m}+1$ as below,

$$\mathfrak{L}_{[i]}(-j) := \begin{cases} 0 \ , & i = \Delta\underline{m}+1 \\ w^i S(-i-j) \ , & 0 \leq i \leq \Delta\underline{m} \end{cases} .$$

Next, for $\square \in \mathcal{T}_2 \cup \mathcal{T}_1 \cup \mathcal{T}_0$ we define

$$\begin{aligned} S_{\square,\|i\|} &:= S_\square / (S_\square \cap \mathfrak{L}_{[i]}) \ , & \mathfrak{J}_{\square,\|i\|} &:= \mathfrak{J}_\square \cdot S_{\square,\|i\|} \ , \\ S_{\square,[i]} &:= S_{\square,\|i\|} \cap \mathfrak{L}_{[i-1]} \ , & \mathfrak{J}_{\square,[i]} &:= \mathfrak{J}_\square \cdot S_{\square,[i]} \ , \end{aligned} \tag{7.1}$$

where, as usual, $\mathfrak{J}_\sigma := 0$ for all $\sigma \in \mathcal{T}_2$. We then define the complex $\mathcal{Q}_{[i]}$ as

$$\bigoplus_{\sigma\in\mathcal{T}_2} [\sigma] S_{\sigma,[i]} \longrightarrow \bigoplus_{\tau\in\mathring{\mathcal{T}}_1} [\tau] S_{\tau,[i]}/\mathfrak{J}_{\tau,[i]} \longrightarrow \bigoplus_{\gamma\in\mathring{\mathcal{T}}_0} [\gamma] S_{\gamma,[i]}/\mathfrak{J}_{\gamma,[i]} \longrightarrow 0 \ ,$$

and the complex $\mathcal{Q}_{\|i\|}$ as

$$\bigoplus_{\sigma\in\mathcal{T}_2} [\sigma] S_{\sigma,\|i\|} \longrightarrow \bigoplus_{\tau\in\mathring{\mathcal{T}}_1} [\tau] S_{\tau,\|i\|}/\mathfrak{J}_{\tau,\|i\|} \longrightarrow \bigoplus_{\gamma\in\mathring{\mathcal{T}}_0} [\gamma] S_{\gamma,\|i\|}/\mathfrak{J}_{\gamma,\|i\|} \longrightarrow 0$$

The following result follows directly from these definitions.

Lemma 7.6. *The following is a short exact sequence of complexes,*

$$0 \longrightarrow \mathcal{Q}_{[i]} \longrightarrow \mathcal{Q}_{\|i\|} \longrightarrow \mathcal{Q}_{\|i-1\|} \longrightarrow 0 \ .$$

Notice that $\mathcal{Q}_{\|\Delta\underline{m}+1\|} = \mathcal{Q}$. Furthermore, it can be readily checked that $\mathcal{Q}_{\|0\|}$ is identically zero. Therefore, from Lemma 7.6, $\mathcal{Q}$ can be studied by studying the complexes $\mathcal{Q}_{[i]}$, $i = 1, 2, \ldots, \Delta\underline{m}+1$.

We do so by analyzing the following short exact sequence of chain complexes for each $1 \le i \le \Delta\underline{m}+1$.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
\mathcal{I}_{[i]} \ : & & 0 & \longrightarrow & \bigoplus_{\tau\in\mathring{\mathcal{T}}_1}[\tau]\mathfrak{J}_{\tau,[i]} & \longrightarrow & \bigoplus_{\gamma\in\mathring{\mathcal{T}}_0}[\gamma]\mathfrak{J}_{\gamma,[i]} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}_{[i]} \ : & & \bigoplus_{\sigma\in\mathcal{T}_2}[\sigma]S_{\sigma,[i]} & \longrightarrow & \bigoplus_{\tau\in\mathring{\mathcal{T}}_1}[\tau]S_{\tau,[i]} & \longrightarrow & \bigoplus_{\gamma\in\mathring{\mathcal{T}}_0}[\gamma]S_{\gamma,[i]} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{Q}_{[i]} \ : & & \bigoplus_{\sigma\in\mathcal{T}_2}[\sigma]S_{\sigma,[i]} & \rightarrow & \bigoplus_{\tau\in\mathring{\mathcal{T}}_1}[\tau]S_{\tau,[i]}/\mathfrak{J}_{\tau,[i]} & \rightarrow & \bigoplus_{\gamma\in\mathring{\mathcal{T}}_0}[\gamma]S_{\gamma,[i]}/\mathfrak{J}_{\gamma,[i]} & \rightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{7.2}
$$

Note that the morphisms above are obtained in the obvious way by composing the (restrictions of) ∂ with quotient maps. Also note that, owing to quotients by $\mathfrak{L}_{[i]}$ in Equation (7.1), some of the faces, edges and vertices will not contribute to the complexes in Equation (7.2). In fact, it can be readily verified that only the active triangulation $\mathcal{T}^{i-1}$ will participate in the above.

Now, the short exact sequence of complexes in Lemma 7.6 implies the long exact sequence

$$
0 \to H_2\left(\mathcal{Q}_{[i]}\right) \to H_2\left(\mathcal{Q}_{\|i\|}\right) \to H_2\left(\mathcal{Q}_{\|i-1\|}\right) \overset{\hat{\partial}_i}{\to} H_1\left(\mathcal{Q}_{[i]}\right) \to \cdots \to H_0\left(\mathcal{Q}_{\|i-1\|}\right) \to 0 \tag{7.3}
$$

with $\hat{\partial}_1 \equiv 0$, and

$$
\chi\left(\mathcal{Q}\right)_{\overline{m}} = \chi\left(\mathcal{Q}_{\|\Delta\underline{m}+1\|}\right)_{\overline{m}} = \sum_{i=1}^{\Delta\underline{m}+1} \chi\left(\mathcal{Q}_{[i]}\right)_{\overline{m}} .
$$

Furthermore, it can be easily shown (in manners completely analogous to the proofs of Propositions 5.4 and 5.7) that Assumption 3.1 implies that for all $1 \le i \le \Delta\overline{m}+1$,

$$
H_1(\mathcal{C}_{[i]})_{\overline{m}} = 0 \ , \qquad H_0(\mathcal{C}_{[i]})_{\overline{m}} = (\overline{m} - i + 2)N^{i-1}. \tag{7.4}
$$

Then, all of the above, in conjunction with the long exact sequence of homology implied by Equation (7.2), directly yield the following result.

Lemma 7.7.

$$
\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} - \chi\left(\mathcal{Q}\right)_{\overline{m}} = \sum_{i=1}^{\Delta\underline{m}+1} \dim_{\mathbb{R}} H_0\left(\mathcal{I}_{[i]}\right)_{\overline{m}} - \dim_{\mathbb{R}} H_0\left(\mathcal{C}_{[i]}\right)_{\overline{m}} - \dim_{\mathbb{R}} \operatorname{im} \hat{\partial}_{i,\overline{m}} .
$$

Proof. The proof follows from the above developments since the exact sequence from Lemma 7.6 implies that

$$
\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}} = \sum_{i=1}^{\Delta\underline{m}+1} \dim_{\mathbb{R}} H_2\left(\mathcal{Q}_{[i]}\right)_{\overline{m}} - \dim_{\mathbb{R}} \operatorname{im} \hat{\partial}_{i,\overline{m}} .
$$

■

Therefore, to show Theorem 7.4, we only need to prove that the right hand side in Lemma 7.7 vanishes for $\overline{m} \gg 0$. This is done by showing that for $\overline{m} \gg 0$ the map from $H_0\left(\mathcal{I}_{[i]}\right)_{\overline{m}}$ to $H_0\left(\mathcal{C}_{[i]}\right)_{\overline{m}}$ in the long exact sequence of homology implied by Equation (7.2) is an isomorphism. This will, in particular imply that both $H_1\left(\mathcal{Q}_{[i]}\right)_{\overline{m}}$ and $H_0\left(\mathcal{Q}_{[i]}\right)_{\overline{m}}$ vanish (from Equation (7.2)) and, therefore, $\operatorname{im} \hat{\partial}_{i,\overline{m}}$ vanishes (from Equation (7.3)). First, we constructively demonstrate the existence of certain vertex orderings; Lemma 7.8 is a specialized form of [21, Lemma 3.3].

Lemma 7.8 (Specialized form of Lemma 3.3 from Schenck and Stillman [21])**.** *Let $\Omega' \subset \mathbb{R}^2$ be a simply connected region and consider its triangulation $\mathcal{T}'$. Then, there exists a total order on $\mathcal{T}'_0$ so that for a vertex γ there exist k vertices $\gamma_1, \dots, \gamma_k \in \mathrm{lk}(\gamma)$ such that $\gamma_j \prec \gamma$, $j = 1, \dots, k$, and all edges $\gamma_j\gamma$ have different slopes. The integer k depends on the position of γ as below:*

(a) $k \geq 2$ for all $\gamma \in \mathring{\Omega}'$;

(b) $k \geq 1$ for all but one $\gamma \in \partial\Omega'$.

Corollary 7.9.

(a) There exists a total order on $\mathcal{T}_0 = \mathcal{T}_0^{\Delta\underline{m}}$ such that, for each $\gamma \in \mathring{\mathcal{T}}_0^{\Delta\underline{m}}$, there exist $\gamma_1, \gamma_2 \in \mathrm{lk}(\gamma)$ such that $\gamma_1, \gamma_2 \prec \gamma$ and the edges, $\gamma\gamma_1$ and $\gamma\gamma_2$ have different slopes.

(b) For $0 \leq i \leq \Delta\underline{m} - 1$, denote the j^{th} connected component of Ω^i with Ω^i_j.

- *If Ω^i_j intersects $\partial\Omega$, there exists a total order on $\mathcal{T}^i_{0,j}$ such that, for all $\gamma \in \mathring{\mathcal{T}}^i_{0,j}$, there is an edge $\tau = \gamma\gamma_1$, $\gamma_1 \prec \gamma$.*
- *If Ω^i_j does not intersect $\partial\Omega$, there exists a total order on $\mathcal{T}^i_{0,j}$ such that, for all but one $\gamma \in \mathring{\mathcal{T}}^i_{0,j}$, there exists an edge $\tau = \gamma\gamma_1$, $\gamma_1 \prec \gamma$.*

Proof. Part (a) and the second bullet in part (b) follow from Lemma 7.8, so we only need to prove the first bullet. Denote with $D^i_j(\gamma, \partial)$ the smallest number of edges in $\mathcal{T}^i_{1,j}$ that connect γ in the interior of Ω^i_j to a vertex on $\partial\Omega$. If $D^i_j(\gamma, \partial) = k > 0$ then there exists at least one adjacent vertex $\gamma' \in \mathcal{T}^i_{0,j}$ such that $D^i_j(\gamma', \partial) = k - 1$ and $\gamma\gamma'$ is an edge in $\mathcal{T}^i_{1,j}$. Order the vertices in $\mathcal{T}^i_{0,j}$ such that if $D^i_j(\gamma, \partial) > D^i_j(\gamma', \partial)$ then $\gamma \succ \gamma'$. For a fixed k, the ordering of the vertices in the set $\{\gamma \,:\, D^i_j(\gamma, \partial) = k\}$ relative to each other does not affect the claim. ■

We will now prove the claimed isomorphism $H_0(\mathcal{I}_{[i]})_{\overline{m}} \cong H_0(\mathcal{C}_{[i]})_{\overline{m}}$ for $\overline{m} \gg 0$ by generalizing the results presented in [21] to the present setting. We will use Lemma 7.5 and Corollary 7.9 for the same. Most importantly, this will imply that unlike [21] the 0-homology of $\mathcal{I}_{[i]}$ will, in general, not have finite length for $1 \leq i \leq \Delta\underline{m}$.

Lemma 7.10.

$$H_0(\mathcal{I}_{[i]})_{\overline{m}} \cong H_0(\mathcal{C}_{[i]})_{\overline{m}}\ , \qquad \overline{m} \gg 0\ .$$

Proof. For $i = \Delta\underline{m} + 1$, we are in the usual uniform degree setting in Equation (7.2). In this case, it was shown in [21, Lemma 3.2] that $H_0(\mathcal{I}_{[i]})$ vanishes in sufficiently high degree. Additionally, we know from Equation (7.4) that $H_0(\mathcal{C}_{[i]})_{\overline{m}} = 0$. Therefore, in the following we will assume $1 \leq i \leq \Delta\underline{m}$.

As mentioned in the remarks immediately following Equation (7.1), only the cells contained in active triangulation $\mathcal{T}^{i-1}$ will contribute to Equation (7.2). We will prove the claim by showing that, for $\overline{m} \gg 0$, $H_0(\mathcal{I}_{[i]})_{\overline{m}}$ surjects onto $H_0(\mathcal{C}_{[i]})_{\overline{m}}$ and that $\dim_{\mathbb{R}} H_0(\mathcal{I}_{[i]})_{\overline{m}}$ is bounded from above by $(\overline{m} - i + 2)N^{i-1} = \dim_{\mathbb{R}} H_0(\mathcal{C}_{[i]})_{\overline{m}}$. We will use Equation ($\star$) below,

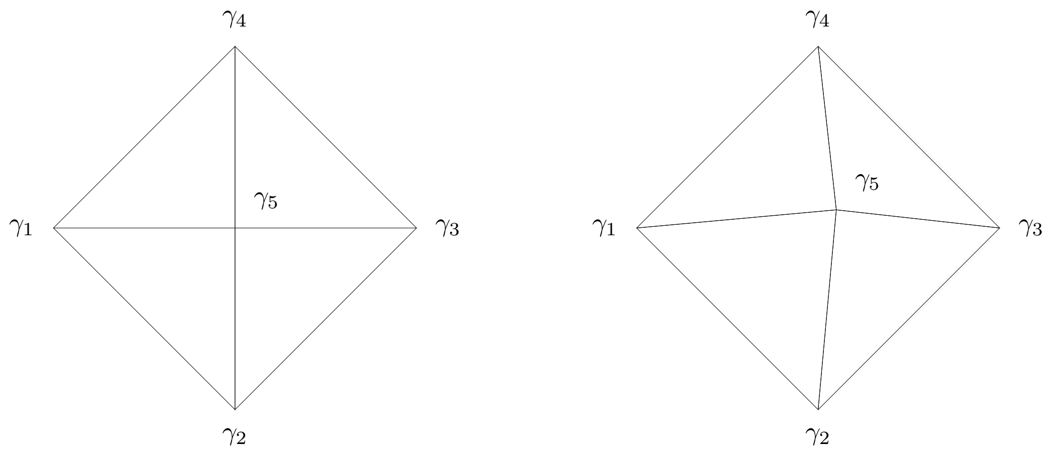
$$\overline{m} \gg 0 \Rightarrow \dim_{\mathbb{R}} \left(\mathfrak{J}_{\gamma,[i]}\right)_{\overline{m}} = \dim_{\mathbb{R}} \left(\mathfrak{L}_{[i-1]}/\mathfrak{L}_{[i]}\right)_{\overline{m}} = \overline{m} - i + 2\ . \tag{$\star$}$$

Upper bound. It is sufficient to analyze a particular connected component of Ω^{i-1}; let us denote it with Ω^{i-1}_j. Consider vertex $\gamma \in \Omega^{i-1}_j$ and an arbitrary $f \in \mathfrak{J}_{\gamma,[i]}$; $[\gamma]f$ is an element of $H_0(\mathcal{I}_{[i]})$. If $\gamma \in \partial\Omega$ then, by previously defined convention, $[\gamma] = 0$. Order the vertices of Ω^{i-1}_j as in Corollary 7.9(b).

Let γ belong to a face $\sigma \subset \Omega^{i-1}_j$ with edges $\tau_1 = \gamma\gamma_1, \tau_2 = \gamma\gamma_2$ and $\tau_3 = \gamma_2\gamma_1$, and assume that $\gamma \succ \gamma_1$. Then, in $H_0(\mathcal{I}_{[i]})$, and with $r := \max\{\boldsymbol{r}(\tau_2), \boldsymbol{r}(\tau_3)\}$, we have

$$[\gamma]fl_{\tau_1}^{\boldsymbol{r}(\tau_1)+1} = [\gamma_1]fl_{\tau_1}^{\boldsymbol{r}(\tau_1)+1}\ , \quad [\gamma]fl_{\tau_2}^{\boldsymbol{r}(\tau_2)+1} = [\gamma_2]fl_{\tau_2}^{\boldsymbol{r}(\tau_2)+1}\ , \quad [\gamma_1]fl_{\tau_3}^{\boldsymbol{r}(\tau_3)+1} = [\gamma_2]fl_{\tau_3}^{\boldsymbol{r}(\tau_3)+1}$$
$$[\gamma]fl_{\tau_2}^{r+1}l_{\tau_3}^{r+1} = [\gamma_1]fl_{\tau_2}^{r+1}l_{\tau_3}^{r+1}\ ,$$

where the fourth relation has been derived from the second and the third. Therefore, if powers of l_{τ_1} and $l_{\tau_2}l_{\tau_3}$ annihilate $[\gamma_1]\mathfrak{J}_{\gamma_1,[i]}$ in $H_0(\mathcal{I}_{[i]})$, they will also annihilate $[\gamma]\mathfrak{J}_{\gamma,[i]}$. Furthermore, by definition, any multiple of w will annihilate $[\gamma]\mathfrak{J}_{\gamma,[i]}$. Now, from Corollary 7.9(b), we also know the following:



(a) $\mathcal{T}(1)$ (b) $\mathcal{T}(2)$

Figure 3: Two triangulations containing a single interior vertex. Perturbing the interior vertex of the triangulation in (a) yields the triangulation in (b); see Example 8.1.

- If $\Omega_j^{i-1} \cap \partial\Omega \neq \emptyset$ then such a γ_1 exists for all $\gamma \in \mathring{\Omega}_j^{i-1}$.
- If $\Omega_j^{i-1} \cap \partial\Omega = \emptyset$ then such a γ_1 exists for all but one $\gamma \in \mathring{\Omega}_j^{i-1} = \Omega_j^{i-1}$.

Using Lemma 7.5(b), the contribution from Ω_j^{i-1} to $H_0(\mathcal{I}_{[i]})_{\overline{m}}$ will vanish if, for all $\gamma \in \mathring{\Omega}_j^{i-1}$, $[\gamma]\mathfrak{J}_{\gamma,[i]}$ vanishes in $H_0(\mathcal{I}_{[i]})$ in sufficiently high degree. Then, from the above reasoning and Equation ($\star$), we see that the upper bound for this contribution is n where

$$n = \begin{cases} 0\ , & \Omega_j^{i-1} \cap \partial\Omega \neq \emptyset \\ \overline{m} - i + 2\ , & \Omega_j^{i-1} \cap \partial\Omega = \emptyset \end{cases} .$$

Lower bound. From Equation ($\star$), following the same reasoning as in the proof of Proposition 6.6 and Remark 6.8, $H_0(\mathcal{I}_{[i]})_{\overline{m}}$ will contain representatives of all homology classes from $H_0(\mathcal{C}_{[i]})_{\overline{m}}$ when $\overline{m} \gg 0$. This implies that the map from $H_0(\mathcal{I}_{[i]})_{\overline{m}}$ to $H_0(\mathcal{C}_{[i]})_{\overline{m}}$ in the long exact sequence of homology implied by Equation (7.2) is a surjection. Combining this with the upper bound, the claim follows. ■

8. Examples

In this section we consider examples of non-uniform degree spline spaces on triangulations and compute bounds on their dimension using Sections 4 and 7. In particular, we present configurations where the upper and lower bounds coincide and thus equal the exact dimension. All of the following computations have been verified using Macaulay2.

It is also possible to express the smoothness condition in Lemma 2.6 in terms of relations between Bernstein–Bézier coefficients [12] and to assemble the relations in a matrix of constraints. Doing so for the entire triangulation, the null space of the full matrix of constraints can be numerically computed. The dimension of this null space will equal the dimension of $\mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}$. The computed null space can also be utilized to build non-uniform degree splines on triangulations. We have extended a Matlab codebase written for uniform degree splines by the authors of [6] for performing the above steps. Example 8.5 utilizes splines built in this manner. An analogous but more efficient approach was adopted in [28] for building univariate non-uniform degree splines by explicitly constructing a sparse null space of the matrix of constraints without solving any linear systems.

Example 8.1. Consider the triangulations shown in Figure 3; they contain a single interior vertex γ_5. Assume that we are interested in the degree deficit and smoothness distributions

$$\begin{aligned} \Delta\boldsymbol{m} &: \gamma_1\gamma_2\gamma_5 = \sigma_1 \mapsto 1\ , \quad \gamma_2\gamma_3\gamma_5 = \sigma_2 \mapsto 0\ , \quad \gamma_3\gamma_4\gamma_5 = \sigma_3 \mapsto 0\ , \quad \gamma_4\gamma_1\gamma_5 = \sigma_4 \mapsto 1\ , \\ \boldsymbol{r} &: \gamma_1\gamma_5 = \tau_1 \mapsto 1\ , \quad \gamma_2\gamma_5 = \tau_2 \mapsto 1\ , \quad \gamma_3\gamma_5 = \tau_3 \mapsto 2\ , \quad \gamma_4\gamma_5 = \tau_4 \mapsto 1\ . \end{aligned}$$

Then, Corollary 7.2 applies and we can compute the dimension of the associated spline space exactly for all $\overline{m}$; see the table below.

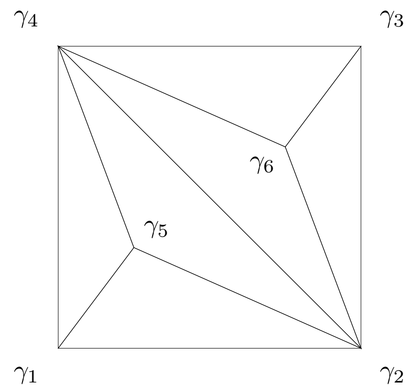


Figure 4: A mesh $\mathcal{T}(3)$ containing two interior vertices but no interior edge connecting them. The dimension of spline spaces on this triangulation can be computed exactly; see Example 8.2.

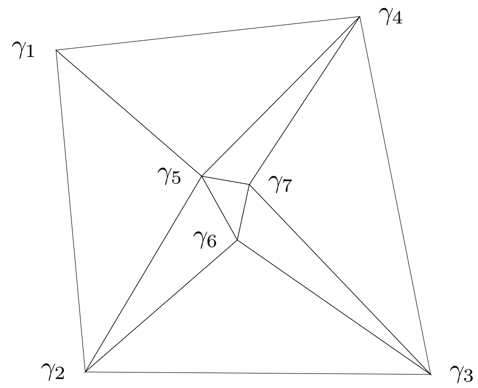


Figure 5: A mesh $\mathcal{T}(4)$ containing three interior vertices. In Example 8.3(a), an ordering of the vertices can be found such that the lower and upper bounds specified by Theorem 7.1 coincide, thus yielding the exact dimension.

$\overline{m}$	2	3	4	5
$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}(\mathcal{T}(1))$	4	9	17	30
$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}(\mathcal{T}(2))$	3	7	16	29

Example 8.2. Consider the triangulation $\mathcal{T}(3)$ shown in Figure 4 containing two interior vertices that are not connected to each other. Assume that we are interested in the degree deficit and smoothness distributions

$$\begin{aligned}
\Delta\boldsymbol{m} \ &: \ \gamma_1\gamma_2\gamma_5 = \sigma_1 \mapsto 3\ , \quad \gamma_1\gamma_5\gamma_4 = \sigma_2 \mapsto 3\ , \quad \gamma_2\gamma_4\gamma_5 = \sigma_3 \mapsto 2\ , \quad \gamma_2\gamma_6\gamma_4 = \sigma_4 \mapsto 1\ ,\\
&\quad\ \gamma_2\gamma_3\gamma_6 = \sigma_5 \mapsto 0\ , \quad \gamma_6\gamma_3\gamma_4 = \sigma_6 \mapsto 0\ ,\\
\boldsymbol{r} \ &: \ \gamma_1\gamma_5 = \tau_1 \mapsto 0\ , \quad \gamma_2\gamma_5 = \tau_2 \mapsto 0\ , \quad \gamma_4\gamma_5 = \tau_3 \mapsto 0\ , \quad \gamma_2\gamma_4 = \tau_4 \mapsto 1\ ,\\
&\quad\ \gamma_2\gamma_6 = \tau_5 \mapsto 2\ , \quad \gamma_4\gamma_6 = \tau_6 \mapsto 2\ , \quad \gamma_3\gamma_6 = \tau_7 \mapsto 3\ .
\end{aligned}$$

Again, from Corollary 7.2 we can exactly compute the spline space dimension for all degrees.

$\overline{m}$	3	4	5	6
$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\Delta\boldsymbol{m},\overline{m}}(\mathcal{T}(3))$	2	8	20	38

Example 8.3. Consider the triangulation $\mathcal{T}(4)$ shown in Figure 5 and let $\boldsymbol{r}(\tau) = 1$ for all $\tau \in \mathring{\mathcal{T}}_1$. Let the degree deficit distribution be

$$\Delta\boldsymbol{m}(\sigma) = \begin{cases} \Delta m_\sigma\ , & \sigma = \gamma_5\gamma_6\gamma_7 \\ \Delta m'_\sigma\ , & \text{otherwise} \end{cases}\ .$$

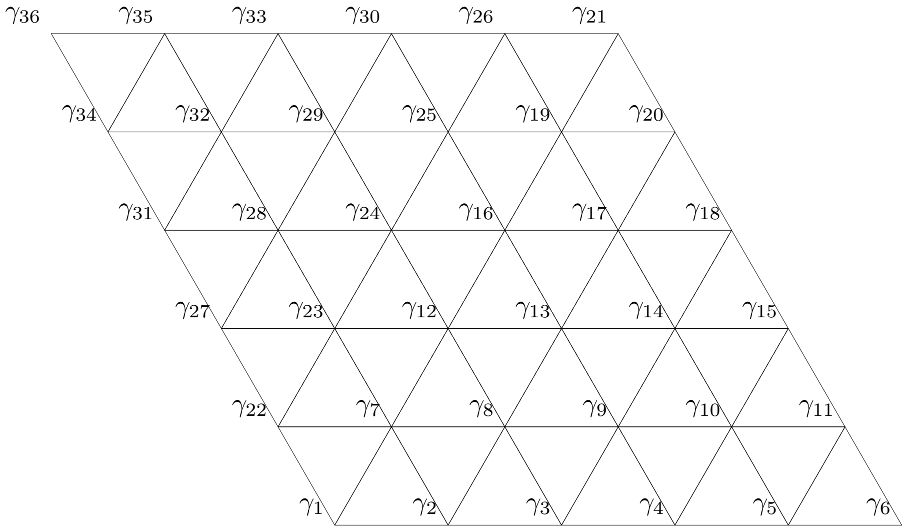


Figure 6: A uniform triangulation $\mathcal{T}(5)$ of a parallelogram-shaped domain. We compute the dimension of the space of C^2 smooth splines on this mesh in Example 8.4.

(a) Let $\Delta m_\sigma = 1$ and $\Delta m'_\sigma = 0$. Then, upon ordering the vertices as $\gamma_7 \succ \gamma_6 \succ \gamma_5$, it can be verified that $\mathfrak{J}_\gamma = \tilde{\mathfrak{J}}_\gamma$ for all γ. Using Corollary 7.2 we can compute the spline space dimension exactly for all $\overline{m}$.

$\overline{m}$	2	3	4	5
$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{\Delta m},\overline{m}}(\mathcal{T}(4))$	4	15	34	61

(b) Let $\Delta m_\sigma = 0$ and $\Delta m'_\sigma = 1$. Then, there is no ordering of the vertices for which the upper and lower bounds coincide in Theorem 7.1. Assuming that we order them as $\gamma_7 \succ \gamma_6 \succ \gamma_5$, the table below presents the computed upper and lower bounds, as well as the exact spline space dimension.

$\overline{m}$	2	3	4	5	6	7
$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{\Delta m},\overline{m}}(\mathcal{T}(4))$ (exact)	3	7	19	39	68	105
$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{\Delta m},\overline{m}}(\mathcal{T}(4))$ (estimated)	$[0,3]$	$[5,7]$	$[18,20]$	$[39,41]$	$[68,70]$	$[105,107]$

In the above table, following Remark 6.8, the lower bound has been estimated using Theorem 7.3 except for $\overline{m} = 2$. As can be observed, the dimension coincides with the lower bound ($= \chi(\mathcal{Q})_{\overline{m}}$) for all $\overline{m} \geq 5$; cf. Theorem 7.4.

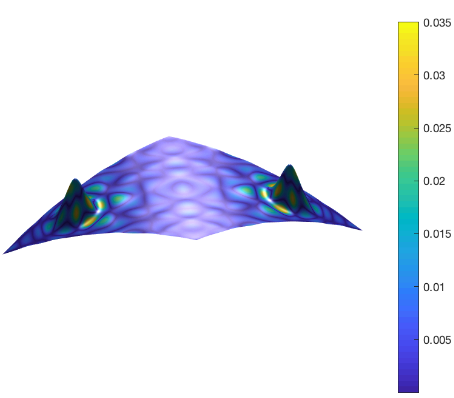
Example 8.4. Let us consider the uniform triangulation $\mathcal{T}(5)$ shown in Figure 6. Assume that we are interested in building a spline space such that the degree and smoothness distributions are

$$\begin{aligned} \boldsymbol{\Delta m} &: \sigma \mapsto \Delta m_\sigma\,, \qquad \sigma' \mapsto \Delta m_{\sigma'}\,,\\ \boldsymbol{r} &: \tau \mapsto 2\,, \end{aligned}$$

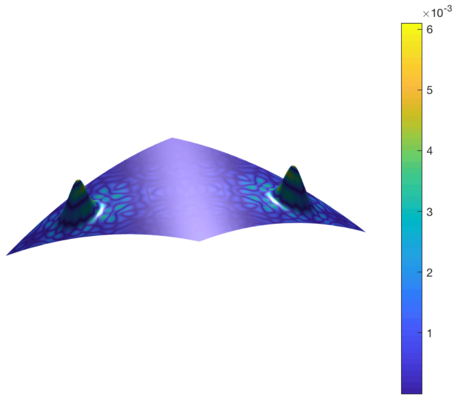
for all interior edges $\tau \in \mathring{\mathcal{T}}_1$ and such that σ is one of the faces contained in the regions bounded by either $\{\gamma_{36}, \gamma_{30}, \gamma_{27}\}$ or $\{\gamma_6, \gamma_3, \gamma_{18}\}$; σ' is allowed to be any face outside of these two regions. The dimensions of $\mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{\Delta m},\overline{m}}$ can computed exactly using the interior vertex ordering given below,

$$(\gamma_7, \gamma_{19}) \succ (\gamma_{12}, \gamma_{16}) \succ (\gamma_{24}, \gamma_{13}) \succ (\gamma_8, \gamma_{17}, \gamma_{23}, \gamma_{25}) \succ (\gamma_9, \gamma_{14}, \gamma_{28}, \gamma_{29}) \succ (\gamma_{10}, \gamma_{32})\ ,$$

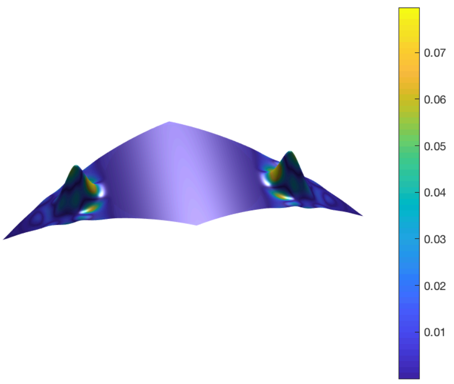
where vertices contained inside a given set of parentheses can be ordered in any manner with regards to each other.



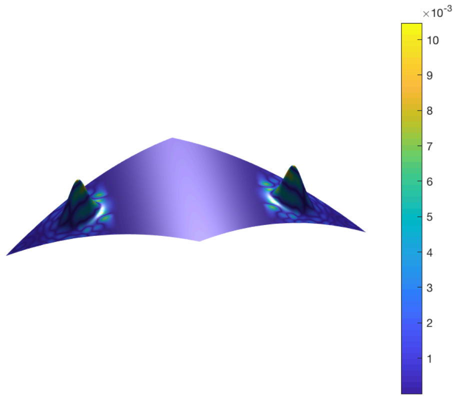
(a) $(m_\sigma, m_{\sigma'}) = (5,5)$



(b) $(m_\sigma, m_{\sigma'}) = (7,7)$



(c) $(m_\sigma, m_{\sigma'}) = (2,5)$



(d) $(m_\sigma, m_{\sigma'}) = (2,7)$

Figure 7: Using the spline spaces analyzed in Example 8.4, we approximate the surface shown in Figure 1(b) using a least squares projection. The approximated surfaces are shown above, and the color schemes correspond to the absolute value of their pointwise distance from the exact surface.

$(\Delta m_\sigma, \Delta m_{\sigma'}, \overline{m})$	$(0,0,5)$	$(0,0,7)$	$(3,0,5)$	$(5,0,7)$
$\dim_{\mathbb{R}} \mathcal{S}^{\boldsymbol{r}}_{\boldsymbol{\Delta m},\overline{m}}(\mathcal{T}(5))$	187	547	66	192

Example 8.5. For the setup used in Example 8.4, consider the problem of approximating the surface shown in Figure 1(b). It can be intuited that employing non-uniform degree splines for approximation of the surface may be more efficient than using uniform degree splines. By building spline spaces corresponding to all the different degree configurations considered in Example 8.4, we approximate the surface using a discrete least-squares projection. The results are shown in Figure 7(a,b) for the uniform degree configurations and 7(c,d) for the non-uniform degree configurations.

9. Conclusions

For the purposes of both geometric modeling and isogeometric analysis, spline spaces allowing polynomial degree adaptivity will lead to new classes of local refinement. This paper presents first steps toward the development of a theory underlying spline spaces with such flexibility. Focusing on the setting of degree adaptive splines on triangulations, we have presented combinatorial upper and lower bounds on their dimension. These bounds generalize previous approaches [3, 8, 14] that considered the setting of uniform degree splines.

Several future extensions of the theory are possible. A first direction could focus on the estimation of dimension for refinement patterns such that $H_1(\mathcal{C}) \neq 0$. Another direction of practical and theoretical interest is the case of supersmoothness across vertices as was noted in [18]. From the point of view of applications, a good set of basis functions (locally supported, non-negative, partition of unity, well conditioned etc.) needs to be constructed. While it is not known if such basis functions exist or how to build them, studying spline spaces over locally subdivided triangulations (in the spirit of Clough-Tocher/Powell-Sabin refinements) may lead to constructions that are sufficiently flexible for both geometric modeling and isogeometric analysis. The above considerations will form part of future research endeavors which will focus on formulation of constructive approaches.

Acknowledgments

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