Single-variable formulations and isogeometric discretizations for shear deformable beams

by

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Abstract

We present numerical formulations of Timoshenko beams with only one unknown, the bending displacement, and it is shown that all variables of the beam problem can be expressed in terms of it and its derivatives. We develop strong and weak forms of the problem. The strong form of the problem involves the fourth derivative of the bending displacement, whereas the symmetric weak form involves, somewhat surprisingly, third and second derivatives. Based on these, we develop isogeometric collocation and Galerkin formulations, that are completely locking-free and involve only half the degrees of freedom compared to standard Timoshenko beam formulations. Several numerical tests are presented to demonstrate the performance of the proposed formulations.

Keywords: Timoshenko beam, shear-deformable, locking-free, isogeometric, collocation, finite elements

1. Introduction

The two major theories for structural analysis of beams are the Bernoulli-Euler and Timoshenko theories. In the Bernoulli-Euler theory cross sections are assumed to remain straight and normal to the mid-axis during deformation, which implies that the rotation of a cross section can be obtained from the derivative of the deflection. This assumption corresponds to neglecting shear deformation and is valid for thin beams. In the Timoshenko theory cross sections are also assumed to remain straight but not necessarily normal to the mid-axis, which accommodates shear deformation. As a consequence, rotation and deflection are typically considered independent variables.

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In this paper, we present a relation between deflection and rotation for Timoshenko beams which leads to formulations with only one variable. In particular, we split the displacement into a bending and a shear part, and show that these two parts are related, which allows expressing all derived variables, such as rotation and shear strain, in terms of the bending displacement. The idea of splitting the displacement of Timoshenko beams into a bending and a shear part can be found in papers from the early days of beam finite elements; see, for example, Kapur [13]. In [13], however, no relation between the bending and shear part is established. Instead, displacement and rotation are discretized independently for both the bending and shear part, resulting in four unknown fields, which eventually yields a formulation with twice the number of degrees of freedom compared to a standard formulation.

In the present paper, we derive a single differential equation for the Timoshenko beam problem with the bending displacement as the only unknown. This differential equation is form-identical to the one for a Bernoulli-Euler beam model but fully accounts for shear deformation. Similar approaches have been presented by Li [16] and Falsone et al. [8]. In both papers, the Timoshenko beam problem is reduced to a single equation in terms of one variable. These approaches do not utilize the split into bending and shear terms, but the resulting equations are essentially equivalent to the one presented in this paper. However, Li [16] does not develop a weak formulation and does not pursue a discrete formulation, or numerical calculations. Falsone et al. [8] develop a weak form and a corresponding finite element formulation, but their weak form is not symmetric and produces boundary terms at element level, which need to be coupled between elements. Furthermore, an additional change of basis at element boundaries is required in order to guarantee continuity of the displacement between elements. In the present paper, we establish a weak form which is both symmetric and without boundary terms at element level, but involves both second and third derivatives of the bending displacement. Due to the isogeometric basis functions adopted in this paper, suitable degree and continuity of the bending displacement are automatically guaranteed. In addition to the Galerkin formulation, we develop an isogeometric collocation method which utilizes the strong form of the problem. Since there is only one unknown variable, the formulations employ half the number of degrees of freedom compared to standard Timoshenko beam formulations, and, at the same time, are completely locking-free by construction. Numerical examples demonstrate the validity and efficiency of the presented methods. In the authors’ view, the formulation provides a new paradigm for the development of bending elements for structural analysis.

2. Governing equations of the Timoshenko beam model

We consider a straight beam with planar deformation, as depicted in Figure 1. Cross sections are assumed to remain straight during deformation, but not necessarily perpendicular to the beam axis due to shear deformability. The transverse displacement
of a cross section is denoted by \( w \), the total rotation by \( \varphi \), and the shear deformation by \( \gamma \). All variables are functions of the coordinate \( x \) and a prime symbol \( (\cdot)' \) indicates a derivative with respect to \( x \), i.e., \( (\cdot)' = d(\cdot)/dx \).

\[ \varphi = -w' + \gamma \]

Figure 1: Beam model and kinematic variables.

2.1. Kinematics

Assuming small displacements, the relation among the kinematic variables is given by the following equation:

\[ \gamma = w' + \varphi \] (1)

2.2. Constitutive equations

For the constitutive equations we define the bending stiffness \( K_b = EI \) and the shear stiffness \( K_s = \alpha GA \), where \( E \) denotes the Young’s modulus, \( G \) the shear modulus, \( A \) the cross-sectional area, and \( I \) the second moment of inertia. The factor \( \alpha \) is a shear correction factor classically used to account for the non-linear distribution of shear stresses over the thickness. Within this paper we consider cross-sectional and material parameters to be constant throughout the beam’s length. However, no restrictions are made on the cross-sectional shape. The bending moment \( M \) and the shear force \( Q \) are obtained from curvature, \( \varphi' \), and shear strain as follows:

\[ M = K_b \varphi' \] (2)

\[ Q = K_s \gamma \] (3)

2.3. Equilibrium equations

As external load, we consider a distributed transverse load \( f \), while distributed moment loads are not considered. The equilibrium equations are then given by:

\[ M' = Q \] (4)

\[ Q' = -f \] (5)
where equation (4) represents moment equilibrium and (5) transverse force equilibrium.

2.4. Conventional formulation

Substituting the constitutive equations into the equilibrium equations and expressing the shear deformation in terms of displacement and rotation, we obtain the standard form of the differential equations for the Timoshenko beam:

\[ K_b \phi'' - K_s (\phi + w') = 0 \quad (6) \]
\[ K_s (\phi' + w'') = -f \quad (7) \]

which is a system of two equations in the two unknowns \( \phi \) and \( w \).

In the following we will show that the differential equations for the Timoshenko beam problem can be reduced to a single equation in one unknown variable. Firstly, we derive a formulation where the rotation \( \phi \) is the only unknown. Secondly, we present a formulation with a newly introduced displacement variable as the only unknown. This formulation is then used as basis for deriving weak forms and numerical solution schemes.

2.5. Single-variable formulation in terms of the rotation

Substituting the expression for \( w' \) given by (6) into the equilibrium equation (7), we obtain:

\[ K_b \phi''' = -f \quad (8) \]

Equation (8) is the differential equation for the Timoshenko beam with \( \phi \) as the only unknown variable.

The transverse displacement is obtained by integrating equation (6):

\[ w = \int_{0}^{x} -\phi \, dx + \frac{K_b}{K_s} \phi' + c \quad (9) \]

where \( c \) is an integration constant. As can be seen, the expression for \( w \) contains the integral of the unknown variable \( \phi \). This poses a complication with respect to discrete numerical solution schemes and prevents a straightforward implementation. However, we present it here in order to point out that the differential equations of the Timoshenko beam model can be reduced to a single equation in one unknown even with the standard rotation variable, without introducing anything new.

Clearly, it would be desirable to express everything in terms of the displacement \( w \) instead of the rotation \( \phi \), which, however, does not seem feasible. In the following section, we introduce an additional displacement variable which facilitates expressing the problem in terms of one unknown variable without integrals appearing in the equations.
2.6. Single-variable formulation in terms of the bending displacement

Starting from equation (9), we split the displacement $w$ into two parts $w_b$ and $w_s$ as follows:

$$w = w_b + w_s$$ (10)

$$w_b = \int_0^x -\varphi \, dx + c$$ (11)

$$w_s = \frac{K_b}{K_s} \varphi'$$ (12)

Differentiating the expressions above we obtain:

$$w' = w_b' + w_s'$$ (13)

$$w_b' = -\varphi$$ (14)

$$w_s' = \frac{K_b}{K_s} \varphi''$$ (15)

Substituting (13) and (14) into (1) yields:

$$w_s' = \gamma$$ (16)

Given relations (14) and (16) we can interpret $w_b$ as the bending part and $w_s$ as the shear part of the displacement $w$. Furthermore, by substituting (14) into (12) we find a relation between $w_s$ and $w_b$:

$$w_s = -\frac{K_b}{K_s} w_b''$$ (17)

such that we can rewrite (10) expressing $w$ in terms of $w_b$ only:

$$w = w_b - \frac{K_b}{K_s} w_b''$$ (18)

Now, all variables of the beam model can be expressed in terms of the new displacement variable $w_b$ and its derivatives, as summarized in the following:

$$w = w_b - \frac{K_b}{K_s} w_b''$$ (19)

$$\varphi = -w_b'$$ (20)

$$\gamma = -\frac{K_b}{K_s} w_b'''$$ (21)

$$M = -K_b w_b'$$ (22)

$$Q = -K_b w_b'''$$ (23)
and the governing equation for the Timoshenko beam problem can be written as:

$$K_b w'''' = f \quad (24)$$

The differential equation (24) is of fourth order and four boundary conditions are necessary to complete the specification of the boundary value problem. The boundaries of the beam are denoted by $\Gamma = \{0\} \cup \{l\}$, with $l$ being the beam length. Furthermore, $\Gamma_w, \Gamma_\varphi, \Gamma_M, \Gamma_Q$ indicate the boundaries with prescribed $w, \varphi, M, \text{and } Q$, respectively. The boundary conditions can then be formulated as follows, with barred symbols indicating the prescribed boundary values:

$$w_b - \frac{K_b}{K_s} w'_b' = \bar{w} \quad \text{on } \Gamma_w \quad (25)$$
$$- w'_b = \bar{\varphi} \quad \text{on } \Gamma_\varphi \quad (26)$$
$$K_b w''_b' = \pm \bar{M} \quad \text{on } \Gamma_M \quad (27)$$
$$K_b w'''_b = \pm \bar{Q} \quad \text{on } \Gamma_Q \quad (28)$$

where, in (27)-(28), the plus sign refers to $x = 0$, the minus sign refers to $x = l$. At each boundary, we have to prescribe two boundary conditions, one on either $w$ or $Q$ and one on either $\varphi$ or $M$, i.e., $\Gamma_w \cap \Gamma_Q = \emptyset$ and $\Gamma_\varphi \cap \Gamma_M = \emptyset$, where $\emptyset$ denotes the empty set. To ensure the well-posedness of the problem, the displacement must be prescribed on at least one of the boundaries, $\Gamma_w \neq \emptyset$, while $\Gamma_\varphi, \Gamma_M, \Gamma_Q$ can be empty sets.

With equations (24)-(28) the strong form of the Timoshenko beam problem can then be stated as follows: Given a distributed load $f$ and prescribed boundary values $\bar{w}, \bar{\varphi}, \bar{M}, \bar{Q}$, find $w_b$ such that (24) is satisfied on $]0, l[$ and (25)-(28) are satisfied on the boundaries.

This formulation is rotation-free with only one unknown variable, yet shear deformability is fully accounted for. It is interesting to note that equation (24) is form-identical to the differential equation of the Bernoulli-Euler beam problem, with $w_b$ replacing $w$. For precise terminology sake, we will denote $w_b$ as the bending displacement and $w$ as the total displacement. The fact that the definition of the total displacement (19) contains both $w_b$ and its second derivative has some interesting consequences on the development of the weak form, which will be presented in the next section.

It is important to note that a zero displacement boundary condition does not imply that both $w_b$ and $w_s$ are zero at the boundary, but that their sum is zero, i.e., $w_b + w_s = 0$. This fact makes it clear that this formulation is not the same as solving a Bernoulli-Euler beam problem and adding the shear deformation calculated from the bending deformation in a post-processing step.
2.7. Development of a weak form

Based on the strong form presented in the previous section, we develop a weak form by multiplying equation (24) with a test function and integrating the equation over the domain $[0, l]$:

$$
\int_0^l K_b w_b''' \left( \tilde{w}_b - \frac{K_b}{K_s} \tilde{w}_b' \right) \, dx = \int_0^l f \left( \tilde{w}_b - \frac{K_b}{K_s} \tilde{w}_b' \right) \, dx
$$

(29)

As can be seen, the test function consists of a function $\tilde{w}_b$ and its second derivative. This approach is inspired by the principal of virtual work, where a virtual displacement is usually used as test function. This is also the case here, but the virtual displacement $\tilde{w}$ is represented in terms of the virtual bending displacement $\tilde{w}_b$ in the same manner as in (19). It also may be interpreted as a Petrov-Galerkin formulation since the weighting function is a linear functional of $\tilde{w}_b$. The boundary conditions for the test function are:

$$
\tilde{w}_b - \frac{K_b}{K_s} \tilde{w}_b'' = 0 \quad \text{on } \Gamma_w
$$

(30)

$$
\tilde{w}_b' = 0 \quad \text{on } \Gamma_\varphi
$$

(31)

which are the homogeneous counterparts to the specification of $\tilde{w}$ on $\Gamma_w$ and $\tilde{\varphi}$ on $\Gamma_\varphi$. This test function might seem rather unusual at first, but it provides important advantages as shown in the following. The two terms on the left side of (29) are integrated by parts separately. The first is integrated twice and the second only once, such that in both cases trial and test functions appear with the same derivative order:

$$
\int_0^l K_b w_b''' \tilde{w}_b' \, dx - \int_0^l K_b w_b''' \frac{K_b}{K_s} \tilde{w}_b'' \, dx =
$$

$$
- \int_0^l K_b w_b'' \tilde{w}_b' \, dx + K_b w_b'' \tilde{w}_b|_{\Gamma_b}|_0 + \int_0^l K_b w_b'' \frac{K_b}{K_s} \tilde{w}_b'' \, dx - K_b w_b'' \frac{K_b}{K_s} \tilde{w}_b''|_{\Gamma_b}|_0
$$

$$
\int_0^l K_b w_b'' \tilde{w}_b'' \, dx - K_b w_b'' \tilde{w}_b|_{\Gamma_b}|_0 + K_b w_b'' \tilde{w}_b|_{\Gamma_b}|_0 + \int_0^l K_b w_b'' \frac{K_b}{K_s} \tilde{w}_b''' \, dx - K_b w_b'' \frac{K_b}{K_s} \tilde{w}_b'''|_{\Gamma_b}|_0
$$

(32)

For essential boundary conditions, i.e., those on $\Gamma_w$ and $\Gamma_\varphi$, the boundary terms in (32) vanish due to (30)-(31), and the boundary conditions are strongly enforced by (25)-(26). For natural boundary conditions, i.e., those on $\Gamma_M$ and $\Gamma_Q$, the definitions in (27)-(28) allow writing the boundary terms in a unified manner for both $x = 0$ and $x = l$:

$$
-K_b w_b'' \tilde{w}_b|_{\Gamma_b}|_0 = \bar{M} \tilde{w}_b|_{\Gamma_M}
$$

(33)

$$
K_b w_b'' \left( \tilde{w}_b - \frac{K_b}{K_s} \tilde{w}_b' \right)|_{\Gamma_\varphi}_0 = - \bar{Q} \left( \tilde{w}_b - \frac{K_b}{K_s} \tilde{w}_b' \right)|_{\Gamma_\varphi}_0
$$

(34)
The natural boundary terms are brought on the right hand side of equation (29) and the weak form finally reads as:

\[
\int_0^l K_b \tilde{w}_b'' \tilde{w}_b'' \, dx + \int_0^l \frac{K_b^2}{K_s} \tilde{w}_b''' \tilde{w}_b''' \, dx = \int_0^l f (\tilde{w}_b - \frac{K_b}{K_s} \tilde{w}_b') \, dx - \bar{M} \tilde{w}_b' \big|_{\Gamma_M} \\
+ \bar{Q} \left( \tilde{w}_b - \frac{K_b}{K_s} \tilde{w}_b' \right) \bigg|_{\Gamma_Q}
\]

(35)

As can be seen, the weak form is of third order and symmetric. It is also worth noting that the corresponding strong form is of fourth order, and not of sixth order as one might expect for a weak form with squares of third derivatives. These facts stem from the special choice of test function which includes \( \tilde{w}_b \) and its second derivative. It is also interesting to note that equation (35) represents the classical equilibrium of virtual work. The two terms on the left hand side are the bending and shear parts of the internal virtual work. On the right hand side, we can identify the contributions from transversal load times displacement on the domain, external boundary moments times boundary rotation, and external boundary forces times boundary displacement.

The proper function space setting for (35) is \( H^3(0, l) \), the Sobolev space with three square-integrable generalized derivatives. Since \( H^3(0, l) \subset C^2_b(0, l) \), the space of continuous and bounded functions on \( (0, l) \) with first and second derivatives continuous and bounded, the force can include a Dirac measure (i.e., point values) that define continuous linear functionals on the space of continuous and bounded functions on \( (0, l) \), that is, the \( \tilde{w}_b'' \)'s. The spaces for trial and test functions are defined as follows:

\[
S = \{ w_b \mid w_b \in H^3, w_b - \frac{K_b}{K_s} w_b'' = \tilde{w} \text{ on } \Gamma_w, w_b' = -\bar{\varphi} \text{ on } \Gamma_\varphi \} \]

(36)

\[
V = \{ \tilde{w}_b \mid \tilde{w}_b \in H^3, \tilde{w}_b - \frac{K_b}{K_s} \tilde{w}_b'' = 0 \text{ on } \Gamma_w, \tilde{w}_b' = 0 \text{ on } \Gamma_\varphi \}
\]

(37)

3. Isogeometric discretization

We develop numerical schemes to solve both the weak and the strong form of the problem by a Galerkin and a collocation method, respectively. Since higher order derivatives appear in the strong as well as in the weak form, we use B-splines as basis functions for the discretized models, following the isogeometric concept [12]. Isogeometric analysis has been successfully applied in the context of thin structures, such as Bernoulli-Euler beams and rods, Kirchhoff plates, and Kirchhoff-Love shells, which necessitate \( C^1 \)-continuity in the displacements [6, 10, 15]. Since in this paper we deal with straight beams, the use of Non-Uniform Rational B-Splines, NURBS, is unnecessary. We point out that the presented method is not
inherently coupled with the isogeometric concept and any other numerical method capable of solving higher-order equations could be used as well. In the following, we give a short introduction to B-splines and discuss some properties which are important for the presented formulations.

3.1. B-splines

B-splines are smooth functions consisting of piecewise polynomials. These functions are generated from so-called knot vectors. A knot vector is a set of non-decreasing real numbers, the knots, which are coordinates in the parametric space:

\[ \{ \xi_1 = 0, ..., \xi_{n+p+1} = 1 \} \]  

where \( p \) is the polynomial degree and \( n \) is the number of basis functions. The interval \([\xi_i, \xi_{i+1}]\) is called a knot span, the interval \([\xi_1, \xi_{n+p+1}]\) is called a patch. A knot vector is said to be uniform if its knots are uniformly-spaced and non-uniform otherwise. A repeated knot value is called a multiple knot. If the first and last knots of a patch have multiplicity \( p + 1 \), the knot vector is called open. Open knot vectors have important properties, which are described below. In what follows, we always employ open knot vectors.

Given a knot vector, univariate B-splines of degree \( p \) are defined recursively as follows. For \( p = 0 \) (piecewise constants):

\[ N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \]  

for \( p \geq 1 \):

\[ N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \]

where, in (40), we adopt the convention \( 0/0 = 0 \). These functions are \( C^\infty \)-continuous inside knot spans, i.e., between two distinct knots, and \( C^{p-1} \)-continuous at a single knot. At a knot of multiplicity \( k \) the continuity is \( C^{p-k} \), i.e., by increasing the multiplicity of a knot the continuity can be decreased. Figure 2 shows an example of
a set of cubic B-spline functions determined from an open knot vector. As can be seen, the functions are not interpolatory in general except for the first and the last basis functions which are interpolatory at the ends of the parametric space, i.e., the patch.

A B-spline curve of degree $p$ is a linear combination of $n$ basis functions and control points $P_i$ which are points in the physical space:

$$C(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)P_i$$  \hspace{1cm} (41)

The first derivative of a B-spline basis function is computed by the following recursive formula:

$$N'_{i,p}(\xi) = \frac{p}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) - \frac{p}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$  \hspace{1cm} (42)

Figure 3 shows the first derivatives of the cubic basis functions from Figure 2. Higher derivatives are obtained by the following formula, where $(k)$ denotes the $k$-th derivative:

$$N^{(k)}_{i,p}(\xi) = \frac{p}{\xi_{i+p} - \xi_i} N^{(k-1)}_{i,p-1}(\xi) - \frac{p}{\xi_{i+p+1} - \xi_{i+1}} N^{(k-1)}_{i+1,p-1}(\xi)$$  \hspace{1cm} (43)

Assuming $p \geq 3$, open knot vectors impose the following conditions on the basis functions and their derivatives at the boundaries. At $\xi = 0$ all basis functions except the first one vanish, and only the first derivatives of the first two functions and second derivatives of the first three functions are non-zero:

$$N_{i,p}(0) = 0, \quad i > 1$$  \hspace{1cm} (44)

$$N'_{i,p}(0) = 0, \quad i > 2$$  \hspace{1cm} (45)

$$N''_{i,p}(0) = 0, \quad i > 3$$  \hspace{1cm} (46)
Furthermore, the first basis function is interpolatory and the first derivatives of first
two functions sum to zero:

\[ N_{1,p}(0) = 1 \quad (47) \]
\[ N'_{1,p}(0) = -N'_{2,p}(0) \quad (48) \]

The same conditions apply to the basis functions at the right end of the parametric
space, as can be seen in Figures 2 and 3. Equations (44)-(48) are fundamental for
the correct imposition of essential boundary conditions, which will be explained in
detail in the next section.

3.2. Galerkin-based isogeometric analysis

In isogeometric analysis, the B-spline (or NURBS) functions are used to define
both the geometry and approximate the solution field in an isoparametric fashion.
The control point variables represent the degrees of freedom and the knot spans are
considered as elements. In this paper, numerical integration is performed by Gauss
quadrature at the element level. For more details, reference is made to [5, 11].

For the present method, we approximate the bending displacement field \( w_b \) by the
function \( w^h_b \):

\[ w^h_b(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) \hat{w}_{b,i} \quad (49) \]

where the \( \hat{w}_{b,i} \) are the control variables and \( N_{i,p} \) are the B-spline shape functions.

For deriving the element formulation, we express (49) in matrix form:

\[ w^h_b = N \hat{w}_b \quad (50) \]

where \( \hat{w}_b \) is the vector of control variables \( \hat{w}_b = [\hat{w}_{b,1}, \hat{w}_{b,2}, \ldots]^T \), and \( N \) is the row
vector of shape functions \( N = [N_{1,p}, N_{2,p}, \ldots] \). In the following, derivatives of \( w^h_b \)
with respect to \( x \) are necessary which requires consideration of the isoparametric
mapping \( x(\xi) \). Since, in this work, we only consider straight beams with a linear
parametrization, we take \( x = \xi \) and, therefore, \( (\cdot)' = d(\cdot)/dx = d(\cdot)/d\xi \). Substituting
(50) into the weak form established in Section 2.7, we can compute the stiffness
matrix \( K \) and the load vector \( f \) and write the standard system of linear algebraic
equations:

\[ K \hat{w}_b = f \quad (51) \]

With \( n_e \) denoting the number of elements, \( l_e \) the of length element \( e \), and \( A \) the
assembly operator [11], the global matrices \( K \) and \( f \) are obtained in a standard way
by assembly of the matrices at element level:

\[
K = \sum_{e=1}^{n_e} \int_{0}^{l_e} \left( N'^T K_b N'' + N'^T \frac{K_b^2}{K_s} N'' \right) \, dx
\]  

(52)

\[
f = \sum_{e=1}^{n_e} \int_{0}^{l_e} f \left( N^T - \frac{K_b}{K_s} N'' \right) \, dx - \bar{M} \left. N^T \right|_{\Gamma_M} + \bar{Q} \left. \left( N^T - \frac{K_b}{K_s} N'' \right) \right|_{\Gamma_Q}
\]  

(53)

Special attention has to be given to the imposition of boundary conditions. Due to the fact that the bending displacement \(w_b\) is discretized instead of the total displacement \(w\) and the rotation \(\varphi\), boundary conditions on displacement and rotation cannot be imposed by directly assigning the respective values of the boundary degrees of freedom. Instead, such boundary conditions lead to constraints involving the degrees of freedom near the boundary. In the following, this is shown in detail for zero displacement and zero rotation at the left boundary \(x = 0\).

The total displacement and rotation of the discretized model are obtained as:

\[
w(0) = w_b(0) - \frac{K_b}{K_s} w''_b(0) = \sum_i \left( N_i(0) - \frac{K_b}{K_s} N''_i(0) \right) \hat{w}_{b,i}
\]

(54)

\[
\varphi(0) = - w'_b(0) = - \sum_i N'_i(0) \hat{w}_{b,i}
\]

(55)

Due to the open knot vector properties, only certain functions and derivatives are non-zero at the boundary, as described in Section 3.1. Using equations (44)-(48), equations (54) and (55) reduce to:

\[
w(0) = \hat{w}_{b,1} - \frac{K_b}{K_s} (N''_1(0) \hat{w}_{b,1} + N''_2(0) \hat{w}_{b,2} + N''_3(0) \hat{w}_{b,3})
\]

(56)

\[
\varphi(0) = - N'_1(0) (\hat{w}_{b,1} - \hat{w}_{b,2})
\]

(57)

For a zero displacement boundary condition, i.e., \(w(0) = 0\), equation (56) yields the following linear constraint among the first three degrees of freedom:

\[
\left( N''_1(0) - \frac{K_s}{K_b} \right) \hat{w}_{b,1} + N''_2(0) \hat{w}_{b,2} + N''_3(0) \hat{w}_{b,3} = 0
\]

(58)

For a zero rotation boundary condition, i.e., \(\varphi(0) = 0\), equation (57) simply yields:

\[
\hat{w}_{b,1} = \hat{w}_{b,2}
\]

(59)

In case of a clamped support, i.e., \(w(0) = \varphi(0) = 0\), equation (59) can be substituted into (58), which then reduces to:

\[
\left( N''_1(0) + N''_2(0) - \frac{K_s}{K_b} \right) \hat{w}_{b,1} + N''_3(0) \hat{w}_{b,3} = 0
\]

(60)

Equations (58)-(60) are linear constraints among the boundary degrees of freedom which can be easily enforced during the assembly of the stiffness matrix and load vector. The weighting functions are likewise constrained to satisfy the homogeneous forms of (54) and (55).
3.3. Isogeometric collocation

Besides the Galerkin formulation presented in the previous section, we also consider isogeometric collocation to solve the Timoshenko beam problem. Isogeometric collocation has been recently proposed as a high-order, low-cost alternative to Galerkin-based isogeometric analysis [1–4, 7, 9, 14, 17, 18]. The main idea is to discretize the strong form equations and to collocate them on a set of suitable points such that a square system of equations is obtained. The so-called Greville abscessae are chosen as collocation points, which are knot averages defined by the knot vector and the polynomial degree. The Greville abscessae related to a spline space of degree $p$ and a knot vector $\{\xi_1, \ldots, \xi_{n+p+1}\}$ are defined as:

$$
\overline{\xi}_i = \frac{\xi_{i+1} + \xi_{i+2} + \ldots + \xi_{i+p}}{p} \quad (61)
$$

Furthermore, the Greville abscessae related to the $k$-th derivative space are defined as:

$$
\overline{\xi}_i^{(k)} = \frac{\xi_{i+k} + \xi_{i+2+k} + \ldots + \xi_{i+p}}{p - k} \quad (62)
$$

From equation (62) one obtains $n - k$ Greville abscessae, where $n$ equals the number of degrees of freedom.

For the isogeometric collocation scheme, we discretize the bending displacement as in (50) and collocate the strong form equation (24) on the Greville abscessae of the fourth derivative space, corresponding to the order of the differential equation:

$$
K_b N'''''(\overline{\xi}_i^{(4)} ) \hat{w}_b = f(\overline{\xi}_i^{(4)}) \quad \text{for } i = 1, \ldots, n - 4 \quad (63)
$$

Equation (63) yields $n - 4$ equations. A square equation system is obtained by adding four boundary conditions, which are obtained as the discrete version of equations (25)-(28):

$$
(N - \frac{K_b}{K_s} N'') \hat{w}_b = \hat{w} \quad \text{on } \Gamma_w \quad (64)
$$

$$
-N' \hat{w}_b = \hat{\varphi} \quad \text{on } \Gamma_\varphi \quad (65)
$$

$$
K_b N'' \hat{w}_b = \pm \hat{M} \quad \text{on } \Gamma_M \quad (66)
$$

$$
K_b N''' \hat{w}_b = \pm \hat{Q} \quad \text{on } \Gamma_Q \quad (67)
$$

As can be seen, in the collocation approach, all boundary conditions can be implemented in a straightforward way. The fact that the bending displacement $w_b$ is discretized instead of the total displacement and rotation does not pose any difficulty for collocation.
4. Numerical tests

In this section, we study several examples to investigate various aspects of the presented methods. First, we test the imposition of different boundary conditions. After that, we study convergence behavior. Finally, we perform a shear-locking study varying the beam slenderness, and compare the results to those of standard Timoshenko and Bernoulli-Euler beam formulations.

![Displacement plots for different boundary conditions](image)

Figure 4: Beam with constant load and different boundary conditions. Displacement plots for the following cases: (a) simply supported; (b) clamped; (c) left end clamped, right end free (cantilever); (d) left end simply supported with boundary moment $\bar{M}_l$, right end with zero-rotation slider support and a boundary force $\bar{Q}_r$. The dashed lines show the undeformed beam

4.1. Beam with uniform load and different boundary conditions

To test the imposition of different boundary conditions, we consider the following cases:

(a) simply supported;
(b) clamped;

(c) left end clamped, right end free (cantilever);

(d) left end simply supported with boundary moment $\bar{M}_l$, right end with zero-rotation slider support and a boundary force $\bar{Q}_r$;

Assuming the parameters $f = \bar{Q}_r = \bar{M}_l = l = 1$ the analytical solution for the total displacement $w$ is given for the four cases as follows:

(a) $w(x) = \frac{1}{24K_b} \left(x^4 - 2x^3 + x\right) + \frac{1}{2K_s} \left(-x^2 + x\right)$

(b) $w(x) = \frac{1}{24K_b} \left(x^4 - 2x^3 + x^2\right) + \frac{1}{2K_s} \left(-x^2 + x\right)$

(c) $w(x) = \frac{1}{24K_b} \left(x^4 - 4x^3 + 6x^2\right) + \frac{1}{2K_s} \left(-x^2 + 2x\right)$

(d) $w(x) = \frac{1}{24K_b} \left(x^4 - 8x^3 + 12x^2 - 4x\right) + \frac{1}{2K_s} \left(-x^2 + 4x\right)$

Since $f$ is constant, the analytical solution is of fourth order in all cases. Therefore, we model the beam by one quartic element, which should be capable of representing the solution exactly, using the Galerkin formulation and applying the boundary conditions as described in Section 3.2. For the collocation formulation, strictly speaking, quintic basis functions are necessary since this is the minimum polynomial degree for which the fourth derivatives are continuous and thus for computing the Greville abscissae corresponding to the fourth derivative space, see (62). However, we can develop a successful formulation for quartic basis functions too, if standard Greville abscissae are used. This is the analog of what has been done in the second order case, for which quadratic basis functions have been shown effective [4].

For the computations, we consider a beam with a rectangular cross-section and the following geometrical and material parameters: width 0.1, thickness 0.01, Young’s modulus $10^7$, Poisson’s ratio 0.2 and a shear correction factor of $5/6$. Figure 4 shows the displacement results for the four cases, which are obtained by both the Galerkin and collocation methods and exactly represent the analytical solutions reported above.

4.2. Convergence studies

We apply a sinusoidal load function $f(x) = 16\pi^4 \cos(2\pi x)$ and consider a clamped boundary condition on the left end and a slider support on the right end. The problem setup is shown in Figure 5, the beam length is $l = 1$. The analytical solution for the total displacement is given by:

$$w(x) = \frac{1}{K_b} \left(\cos(2\pi x) - 1\right) + \frac{1}{K_s} \left(4\pi^2 \cos(2\pi x) - 4\pi^2\right)$$

(68)
From equation (68), the analytical solution for the variables $\varphi$, $M$, and $Q$ can easily be derived. A rectangular cross-section is considered with the same geometrical and material parameters as in the previous case, with the exception of the beam thickness, for which two cases are considered, a thin beam ($t = 10^{-4}$) and a thick beam ($t = 10^{-1}$). We perform refinement studies for different polynomial degrees and evaluate the error in the $L^2$-norms for the total displacement $w$, the rotation $\varphi$, the bending moment $M$, and the shear force $Q$.

For the Galerkin formulation, polynomial degrees $p = 3, 4, 5$ are used. The results for the thin beam are depicted in Figure 6, those for the thick beam in Figure 7. It can be observed that in the case of the thin beam, the convergence rates are, at least, $p + 1$ for $w$, $p$ for $\varphi$, $p - 1$ for $M$, and $p - 2$ for $Q$. These are the same convergence rates as obtained by the corresponding Bernoulli-Euler beam formulation (with the Bernoulli-Euler solution as reference solution).

This example is also solved by the proposed isogeometric collocation approach. In this case, polynomial degrees $p = 5, 6, 7$ are used. Figure 8 shows the results obtained with collocation for the thin beam and Figure 9 for the thick beam. As can be seen, the errors of all variables are practically identical for the thin and the thick case, with convergence rates of, at least, $p - 2$ for even and $p - 3$ for odd degrees. The same behavior has been observed for collocation methods of Bernoulli-Euler beams [17].

4.3. Locking study and comparison with standard Timoshenko and Bernoulli-Euler beam formulations

In this section, we perform a locking study by varying the slenderness, defined as length over thickness $l/t$, keeping the polynomial degree and the number of degrees of freedom fixed. Moreover, we compare the results obtained with the new Galerkin formulation to those obtained by “standard” Timoshenko and Bernoulli-Euler beam formulations. In the “standard” Timoshenko formulation displacement and rotation are discretized independently with equal order approximations. Clearly, such a formulation has more degrees of freedom for the same number of elements. For a fair
comparison, all methods use B-splines basis functions with the same polynomial degree and the same number of degrees of freedom. We consider a simply supported beam with the sinusoidal load described in Section 4.2. The beam is discretized with cubic B-splines and there are 16 degrees of freedom. Different slenderness ratios, ranging from 5 up to 10,000, are considered.

Figure 10 shows the normalized strain energy as a function of the slenderness for the different formulations. It can be seen that the Timoshenko formulation with standard discretization exhibits initial locking for a slenderness of 50, which dramatically increases for slenderness values higher than 100. It should also be noted that due to the higher polynomial degree employed here, the locking effects are less pronounced than for a standard linear Timoshenko finite element. The Bernoulli-Euler beam, on the other hand, yields correct results in the thin regime but underestimates the strain energy for slenderness values lower than 50, for which the shear deformation is not negligible. As can be seen, the presented single-variable Timoshenko formulation yields good results for the entire range of slenderness considered.
Figure 7: Convergence studies for a thick beam, $t = 10^{-1}$, solved with the Galerkin formulation. Error in the $L^2$-norm for (a) total displacement $w$, (b) rotation $\varphi$, (c) bending moment $M$, and (d) shear force $Q$. 
Figure 8: Convergence studies for a thin beam, $t = 10^{-4}$, solved with the collocation formulation. Error in the $L^2$-norm for (a) total displacement $w$, (b) rotation $\phi$, (c) bending moment $M$, and (d) shear force $Q$. 

\( \log_{10}(\frac{||w_{ex} - w_h||_{L^2}}{||w_{ex}||_{L^2}}) \)

\( \log_{10}(\frac{||\phi_{ex} - \phi_h||_{L^2}}{||\phi_{ex}||_{L^2}}) \)

\( \log_{10}(\frac{||M_{ex} - M_h||_{L^2}}{||M_{ex}||_{L^2}}) \)

\( \log_{10}(\frac{||Q_{ex} - Q_h||_{L^2}}{||Q_{ex}||_{L^2}}) \)
Figure 9: Convergence studies for a thick beam, $t = 10^{-1}$, solved with the collocation formulation. Error in the $L^2$-norm for (a) total displacement $w$, (b) rotation $\varphi$, (c) bending moment $M$, and (d) shear force $Q$. 
Figure 10: Locking study and comparison of the present single-variable formulation with standard Timoshenko and Bernoulli-Euler beam formulations: normalized strain energy as a function of the slenderness. Cubic B-splines are used for all formulations, “standard” Timoshenko indicates that $w$ and $\varphi$ are discretized independently with equal order approximation.
5. Conclusions

We have presented single-variable formulations for the Timoshenko beam problem with the bending displacement as the only unknown variable. The strong form differential equation is of fourth order, while the symmetric weak form is of third order. We use B-Splines of suitable continuity for discretization, and establish isogeometric Galerkin and collocation approaches based on the weak and the strong form, respectively. Since there is only one variable, these formulations involve only half of the degrees of freedom compared to standard Timoshenko formulations. At the same time shear locking is precluded ab initio. Several numerical examples demonstrate the validity and efficiency of the presented methods. In future research, we hope to extend this approach to shear-deformable plates and shells.

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