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A Residual Based Eddy Viscosity Model for the Large Eddy Simulation of Turbulent flows

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Abstract

A new eddy viscosity model for the large eddy simulation of turbulent flows is developed. The eddy viscosity is proportional to the product of the magnitude of the fine scale velocity and the grid size. The residual based variational multiscale formulation is used to obtain a computationally inexpensive approximation for the fine scale velocity field. It is noted that the fine scale velocity field, and hence the eddy viscosity, are proportional to the residual of the resolved scale solution. As a result the eddy viscosity automatically vanishes whenever or wherever the coarse scale solution is accurate. In that regard this model is inherently dynamic and does not require the application of a dynamic procedure. The performance of this model is tested in predicting the decay of incompressible and compressible homogeneous turbulence. In both cases it is found that it is at least as accurate as the dynamic version of the Smagorinsky-type eddy viscosity model. However it is easier to implement as it does not require a dynamic procedure in order to determine model parameters.

Keywords: Large eddy simulation, Turbulent eddy viscosity, Residual-based variational multiscale formulation

1 Introduction

Large eddy simulation (LES) has now emerged as a popular tool for simulating turbulent engineering flows. In LES the large scale fluctuations are resolved and the effect of the fine scale fluctuations on the large scales is modeled through terms that depend only on the large scales. Over the years several LES models have been developed, and a majority of these, to some extent the commonly used ones, are based on the concept of an eddy viscosity. This idea is motivated by direct analogy with a molecular viscosity. Just as the molecular viscosity represents momentum transfer through fluctuations of the atomistic particles, the turbulent eddy viscosity represents momentum transfer through fluctuating continuum fluid velocity.

The Smagorinsky model is the most popular eddy viscosity based LES model [1]. In this model the eddy viscosity is proportional to the magnitude of the local rate of strain tensor and a representative length scale, often chosen to be a measure of the grid size. Despite its popularity, this model has several drawbacks. In particular, it must be modified in space and time whenever the turbulence is decaying or damped in some spatial regions (close to a wall, for example). The most effective way to accomplish this is to employ the so-called dynamic approach based on the Germano identity [2, 3]. In this approach the difference between the subgrid stresses on the computational grid and on a test-filter scale is considered. This quantity, which can be explicitly determined once scale similarity is invoked, is used to estimate the magnitude of the Smagorinsky eddy viscosity. This approach has been successful in extending the application of the

Smagorinsky model to a wide range of flows. However, it is cumbersome in that it requires the use of at least one additional filter and it involves averaging, either in space, or time, or along material trajectories in order to in order to smooth a widely varying eddy viscosity field. In addition, the definition and the implementation of the test filter for complex, unstructured grids is a challenging task.

The dynamic approach may be thought of as a way of repairing an inconsistency within the Smagorinsky model in that it does not vanish when the flow field is devoid of any fluctuations. There have been several attempts to derive eddy viscosity models that are consistent and hence do not require a dynamic procedure for simulating most flows. In wall adapted LES (WALES) [4] an eddy viscosity based on the square of the velocity gradient tensor is proposed. This eddy viscosity includes straining and rotational effects through the inclusion of the symmetric and anti-symmetric components of the velocity gradient tensor, and is designed to achieve the appropriate y^3 behavior near solid walls. In [5] an eddy viscosity that depends on the invariants of product of the velocity gradient and its transpose is proposed. This eddy viscosity is constructed so as to vanish for a class of laminar and transitional flows. More recently in [6] an eddy viscosity that depends on the second and third invariants of the rate of strain tensor has been proposed. This viscosity is designed to suppress the dynamics of scales less than the filter scale and has been applied to a channel flow simulation with a constant parameter. However, in [7], a dynamic version of this viscosity is also developed. The dependence of this viscosity on the determinant of the rate of the strain is consistent with an earlier work of a different set of authors [8], who discovered a strong correlation between this quantity and the subgrid energy transfer rate.

In this manuscript we present a new eddy viscosity model that is also consistent and circumvents the use of a dynamic approach. Our model is based on ideas derived from the residual-based variational multiscale (VMS) formulation [9]. Within this approach an equation for the fine scales is derived from a variational formulation of the Navier Stokes equations. This equation is then approximated to obtain an explicit expression for the fine scales. In this expression it is observed that the fine scales are driven by the residual of the coarse scales. Thus when the coarse scales are accurate, the residual vanishes, as do the fine scales. We recognize that once an expression for the fine scales is obtained (albeit an approximate one), it may be used to estimate the viscosity induced by these scales on the coarse scales. In analogy to the molecular viscosity we may assume that the turbulent, or eddy, viscosity is proportional to the magnitude of the fine scale velocity times a length scale which plays the role of the mean free path. In the context of LES it makes sense to select this length scale to be proportional to the grid size. As a result we have $\nu_T \sim |\boldsymbol{u}'|h$. We dub this model the residual-based eddy viscosity model (RBEVM) and implement it in a Fourier-spectral code to predict the decay of incompressible and compressible turbulent flows. In doing so we do not evaluate a dynamic parameter for this model, instead we rely on its inherent dynamic nature to adjust to the flow. We note that this approach differs from the models described in [10, 11, 12] in several ways:

1. The fine scale velocity in these earlier works was constructed by solving the orig-

inal Navier-Stokes equations appended with an eddy viscosity. In the RBEVM the fine scale viscosity is constructed by simply scaling the residual of the coarse scale equations, a much simpler procedure.

2. The fine sale velocity in these works was used to construct a fine scale rate of strain tensor which was used in a Smagorinsky-type model. In the RBEVM the expression for ν_t is different from the Smagorinsky model.

3. In these earlier works the eddy viscosity was applied only to the equation for the fine-resolved scales, whereas in this manuscript it is applied to all the resolved scales.

It also differs from the approach developed in [14, 9], in which no eddy viscosity was employed.

The layout of the remainder of this paper is as follows. In Section 2, we review the VMS formulation and derive a residual-based eddy viscosity model from it. In Section 3, we derive an expression for the fine scales that appear within this viscosity. In Section 4, we apply this viscosity in simulating the decay of incompressible and compressible homogeneous isotropic turbulence. We compare the performance of the RBEVM with a static and a dynamic version of the Smagorinsky model and conclude that the former is at least as accurate as the dynamic model. However it is simpler in that it does not require a dynamic procedure. We end with conclusions in Section 5.

2 Residual-based eddy viscosity

We consider the incompressible and compressible Navier-Stokes equations, written together as:

$$\boldsymbol{\mathcal{L}U} = \boldsymbol{F}, \qquad \text{in } \Omega\times]0,T[, \tag{1}$$

where $\boldsymbol{U} = [\boldsymbol{u}, p]^T$ for the incompressible case and $\boldsymbol{U} = [\rho, \boldsymbol{m}, p]^T$ for the compressible case. Here ρ is the density, $\boldsymbol{u}$ is the velocity, $\boldsymbol{m} = \rho\boldsymbol{u}$ is the linear momentum (per unit volume) and p is the pressure. The operator $\boldsymbol{\mathcal{L}}$ is the differential operator associated with the Navier-Stokes equations, and the vector $\boldsymbol{F}$ contains volume sources of momentum and heat. They are defined as follows.

For the incompressible Navier Stokes equations

$$\boldsymbol{\mathcal{L}U} \equiv \begin{bmatrix} \nabla\cdot\boldsymbol{u} \\ \boldsymbol{u}_{,t} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p - \frac{1}{Re}\nabla^2\boldsymbol{u} \end{bmatrix}, \tag{2}$$

where Re is the Reynolds number, and

$$\boldsymbol{F} \equiv \begin{bmatrix} 0 \\ \boldsymbol{f} \end{bmatrix}, \tag{3}$$

where $\boldsymbol{f}$ represents the body force.

For the compressible Navier Stokes equations

$$\boldsymbol{\mathcal{L}U} \equiv \begin{bmatrix} \rho_{,t} + \nabla\cdot\boldsymbol{m} \\ \boldsymbol{m}_{,t} + \nabla\cdot(\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho}) + \nabla p - \frac{1}{Re}\nabla\cdot\boldsymbol{\sigma} \\ p_{,t} + \nabla\cdot(\boldsymbol{u}p) + (\gamma-1)p\nabla\cdot\boldsymbol{u} - \frac{(\gamma-1)}{Re}\phi - \frac{1}{M_\infty^2 Pr Re}\nabla\cdot(\mu\nabla T) \end{bmatrix}, \tag{4}$$

where $\boldsymbol{u} = \tilde{\boldsymbol{u}}(\boldsymbol{U}) \equiv \frac{\boldsymbol{m}}{\rho}$ is the velocity field, $\boldsymbol{S} = \tilde{\boldsymbol{S}}(\boldsymbol{U}) \equiv \nabla^s \tilde{\boldsymbol{u}}(\boldsymbol{U})$ is the rate of strain tensor, the subscript *dev* denotes the deviatoric component of a tensor, the temperature $T = \tilde{T}(\boldsymbol{U}) \equiv \gamma M_\infty^2 \frac{p}{\rho}$ is given by the ideal gas law, the non-dimensional viscosity $\mu = \tilde{\mu}(T) \equiv (\frac{\tilde{T}(\boldsymbol{U})}{T_0})^{0.76}$ is given by Sutherland's law, the viscous stress tensor $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}(\boldsymbol{U}) \equiv 2\tilde{\mu}(\boldsymbol{U})\tilde{\boldsymbol{S}}_{dev}(\boldsymbol{U})$, the dissipation $\phi = \tilde{\phi}(\boldsymbol{U}) \equiv \tilde{\boldsymbol{\sigma}}(\boldsymbol{U}) : \tilde{\boldsymbol{S}}(\boldsymbol{U})$, Pr is the Prandtl number, M_∞ is the free-stream Mach number, and γ is the ratio of specific heats. The forcing vector

$$\boldsymbol{F} \equiv \begin{bmatrix} 0 \\ \boldsymbol{f} \\ f_p \end{bmatrix}, \tag{5}$$

where $\boldsymbol{f}$ represents the body force and f_p represents sources of heat.

The weak form of these equations is given by: find $\boldsymbol{U} \in \mathcal{V}$ such that $\forall \boldsymbol{W} \in \mathcal{V}$

$$B(\boldsymbol{W}, \boldsymbol{U}) = (\boldsymbol{W}, \boldsymbol{F}). \tag{6}$$

Here $\boldsymbol{U}$ is the desired solution, $\boldsymbol{W}$ is the weighting function and the semi-linear operator $B(\cdot,\cdot)$ is derived from the Navier-Stokes equations and is defined below. It is linear in the first slot and nonlinear in the second. The weighting function for the incompressible case is given by $\boldsymbol{W} = [\boldsymbol{v}, q]^T$ and for the compressible case it is given by $\boldsymbol{W} = [r, \boldsymbol{v}, q]^T$. Further $(\boldsymbol{W}, \boldsymbol{F}) = \int_\Omega \boldsymbol{W} \cdot \boldsymbol{F} d\boldsymbol{x}$ denotes the L_2 inner product.

For the incompressible Navier Stokes equations

$$\begin{aligned} B(\boldsymbol{W}, \boldsymbol{U}) &= (q, \nabla \cdot \boldsymbol{u}) \\ &\quad + (\boldsymbol{w}, \boldsymbol{u}_{,t}) - (\nabla \boldsymbol{w}, \boldsymbol{u} \otimes \boldsymbol{u}) - (\nabla \cdot \boldsymbol{w}, p) + \frac{2}{Re}(\nabla^s \boldsymbol{w}, \nabla^s \boldsymbol{u}). \end{aligned} \tag{7}$$

For the compressible Navier Stokes equations

$$
\begin{aligned}
B(\boldsymbol{W},\boldsymbol{U}) \equiv\ & (r,\rho_{,t}) - (\nabla r,\boldsymbol{m}) \\
& +(\boldsymbol{w},\boldsymbol{m}_{,t}) - (\nabla \boldsymbol{w},\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho}) - (\nabla\cdot\boldsymbol{w},p) + \frac{1}{Re}(\nabla\boldsymbol{w},\boldsymbol{\sigma}) \\
& +(q,p_{,t}) - (\nabla q,\boldsymbol{u}p) + (\gamma-1)(q,p\nabla\cdot\boldsymbol{u}) \\
& -\frac{(\gamma-1)}{Re}(q,\Phi) + \frac{1}{M_\infty^2 Pr Re}(\nabla q,\mu\nabla T)
\end{aligned}
\tag{8}
$$

Note that for ease of notation we have assumed that the boundary conditions are such that the space of weighting functions and trial solutions are the same. We also note that for sufficiently smooth functions $\boldsymbol{W}$ and $\boldsymbol{V}$, and for vanishing boundary terms,

$$
B(\boldsymbol{W},\boldsymbol{V}) = (\boldsymbol{W},\mathcal{L}\boldsymbol{V}), \qquad \forall \boldsymbol{W}\in\mathcal{V}. \tag{9}
$$

An approximate solution to the Navier Stokes equations is obtained by approximating the infinite-dimensional space $\mathcal{V}$ with a finite-dimensional $\mathcal{V}^h \subset \mathcal{V}$. The equation for the approximate solution $\boldsymbol{U}^h$ is given by: find $\boldsymbol{U}^h \in \mathcal{V}^h$ such that $\forall \boldsymbol{W}^h \in \mathcal{V}^h$

$$
B(\boldsymbol{W}^h,\boldsymbol{U}^h) = (\boldsymbol{W}^h,\boldsymbol{F}). \tag{10}
$$

When $\mathcal{V}^h$ is sufficiently refined so as to resolve all scales of motion down to the Kolmogorov length scale, $\boldsymbol{U}^h$ represents the direct numerical simulation (DNS) solution. However, when this is not the case, and the fine scales are not represented, $\boldsymbol{U}^h$ is inaccurate and represents the coarse DNS solution. The accuracy of this solution may be improved by adding to it terms that model the effect of the missing or unresolved scales on the resolved scales. In this case (10) is replaced by: find $\boldsymbol{U}^h \in \mathcal{V}^h$ such that

$\forall \boldsymbol{W}^h \in \mathcal{V}^h$

$$B(\boldsymbol{W}^h, \boldsymbol{U}^h) + M(\boldsymbol{W}^h, \boldsymbol{U}^h) = (\boldsymbol{W}^h, \boldsymbol{F}), \tag{11}$$

where $M(\boldsymbol{W}^h, \boldsymbol{U}^h)$ denotes the model term.

In the incompressible case the model term is often represented by an eddy viscosity in direct analogy with the viscous models for transfer of momentum through molecular motion. The assumption is that the subgrid turbulent eddies redistribute momentum among the coarse velocity scales just like the thermal fluctuations of particles redistribute momentum among the continuum velocity scales. It is therefore reasonable to assume, in direct analogy with molecular diffusion of momentum that the eddy viscosity $\nu_T = \bar{C} h |\boldsymbol{u}'|$, where the subgrid velocity fluctuations $\boldsymbol{u}'$, play the role of the thermal velocity fluctuations and the grid size h plays the role of mean free path of these eddies. As a result, for incompressible flows we have

$$M(\boldsymbol{W}^h, \boldsymbol{U}^h) = 2\bar{C}(\nabla^s \boldsymbol{w}^h, h|\boldsymbol{u}'|\boldsymbol{S}^h). \tag{12}$$

In the equation above $\boldsymbol{S}^h = \nabla^s \boldsymbol{u}^h$ and the constant $\bar{C}$ may be determined by equating the dissipation induced by the model term to the total dissipation and is derived below.

For turbulent flows where compressibility is important, following Yoshizawa [13], the fine scale fluctuations introduce two additional terms. As a result we propose

$$M(\boldsymbol{W}^h, \boldsymbol{U}^h) = 2\bar{C}(\nabla^s \boldsymbol{w}^h, \rho^h h|\boldsymbol{u}'|\boldsymbol{S}^h_{dev}) - \frac{1}{3}(\nabla \cdot \boldsymbol{w}^h, \rho^h |\boldsymbol{u}'|^2) + \frac{\bar{C}}{Pr_t \gamma M_\infty^2}(\nabla q^h, \rho^h h |\boldsymbol{u}'| \nabla T^h). \tag{13}$$

In the equation above $\boldsymbol{S}^h = \tilde{\boldsymbol{S}}(\boldsymbol{U}^h)$, $T^h = \tilde{T}(\boldsymbol{U}^h)$, and $Pr_t = 0.5$ is the turbulent Prandtl number. The first term on the right hand side, which is the Smagorinsky model, is a model for the deviatoric component of the subgrid stresses. The second term is a model for the dilatational component of the subgrid stresses and the third term is a model for the subgrid heat flux vector.

In the incompressible case the average turbulent dissipation induced by the model is given by

$$
\begin{aligned}
\epsilon^h &= \langle 2\bar{C}h|\boldsymbol{u}'||\boldsymbol{S}^h|^2\rangle && (14)\\
&\approx 2\bar{C}h\langle|\boldsymbol{u}'|^2\rangle^{1/2}\langle|\boldsymbol{S}^h|^2\rangle && (15)\\
&= 2\bar{C}h\Big(2\int_{k^h}^{\alpha k^h} E(k)dk\Big)^{1/2} \times \int_0^{k^h} k^2 E(k)dk && (16)
\end{aligned}
$$

Note that the upper limit in the integral used to approximate $\langle|\boldsymbol{u}'|^2\rangle$ above is αk^h, with $\alpha > 1$. This acknowledges the fact that $\boldsymbol{u}'$ will be evaluated on a grid that is finer than the grid used to represent $\boldsymbol{u}^h$ by a factor equal to α. In the examples considered in this manuscript $\alpha = 3/2$. Using the Kolmogorov spectrum $E(k) = C_K\epsilon^{2/3}k^{-5/3}$ in the expression above, evaluating the integrals, setting $k^h = \frac{\pi}{h}$, and then equating the model dissipation to the molecular dissipation, that is $\epsilon = \epsilon^h$, we arrive at

$$
\bar{C} = \frac{2C_K^{-3/2}}{3\sqrt{3}\pi}\frac{1}{\sqrt{(1-\alpha^{-2/3})}}. \qquad (17)
$$

For $\alpha = 3/2$ (used in the examples in this work) and assuming that the Kolmogorov constant, $C_K = 1.4$, yields $\bar{C} = 0.15$. In the limit when the fine scale velocity is represented on a very fine grid, that is $\alpha \to \infty$, the value of $\bar{C}$ reduces to 0.074.

In the expressions above there is no undetermined parameter, however we have made use of $\boldsymbol{u}'$, the fine scale velocity field, which is obviously not known in a coarse scale simulation. It may be approximated using the VMS formulation as described below.

3 Expression for the fine scale velocity field

We rely on the variational multiscale (VMS) formulation in order to obtain an approximate expression for the fine scale velocity field. We note that the VMS formulation itself is an approach for developing LES methods [9, 14, 15, 16]. However, in this manuscript we do not use it in that context. Instead we utilize it to determine an approximation for the fine scale velocity field to be used in the eddy-viscosity models described above.

The weak form is the starting point for the variational multiscale (VMS) formulation [17, 18]. The goal is to derive a good numerical method whose solution, $\boldsymbol{U}^h$, is close to the exact solution, $\boldsymbol{U}$. The definition of the optimal solution within this space is made precise by a user-defined projection operator $\mathbb{P}^h : \mathcal{V} \rightarrow \mathcal{V}^h$. That is $\boldsymbol{U}^h = \mathbb{P}^h \boldsymbol{U}$ is the optimal finite-dimensional solution. Thus for any function $\boldsymbol{V} \in \mathcal{V}$ we have $\boldsymbol{V} = \boldsymbol{V}^h + \boldsymbol{V}'$, where $\boldsymbol{V}^h \in \mathcal{V}^h$, and $\boldsymbol{V}' \equiv \boldsymbol{V} - \boldsymbol{V}^h$. This implies a direct sum decomposition of $\mathcal{V} = \mathcal{V}^h \oplus \mathcal{V}'$. Applying this decomposition to the weighting functions and trial solutions in (6) we have,

$$B(\boldsymbol{W}^h, \boldsymbol{U}^h + \boldsymbol{U}') = (\boldsymbol{W}^h, \boldsymbol{F}), \qquad \forall \boldsymbol{W}^h \in \mathcal{V}^h \tag{18}$$

$$B(\boldsymbol{W}', \boldsymbol{U}^h + \boldsymbol{U}') = (\boldsymbol{W}', \boldsymbol{F}), \qquad \forall \boldsymbol{W}' \in \mathcal{V}'. \tag{19}$$

These equations may be interpreted as a set of coupled nonlinear equations for $\boldsymbol{U}^h$ and $\boldsymbol{U}'$. In particular (18) is the equation for the coarse scale solution $\boldsymbol{U}^h$, and (19) is the equation for the fine, or the subgrid scales $\boldsymbol{U}'$. In the VMS formulation the goal is to solve (19) for $\boldsymbol{U}'$ and substitute the result in (18) in order to obtain an equation for the coarse scales that includes the effect of the fine scales.

Solving (19) for $\boldsymbol{U}'$ is perhaps as challenging as solving the original Navier-Stokes equations for all the scales. However, once it is recognized that the $\boldsymbol{U}'$ obtained from this equation is to be used in the coarse scale equations, *where it is weighted by* $\boldsymbol{W}^h$*, a coarse-scale weighting function, it becomes clear that an approximate solution for* $\boldsymbol{U}'$ *may suffice.*

An approximation for $\boldsymbol{U}'$ is obtained by subtracting $B(\boldsymbol{W}', \boldsymbol{U}^h)$ from both sides of (19), and using (9) on the right hand side. This yields

$$B(\boldsymbol{W}', \boldsymbol{U}^h + \boldsymbol{U}') - B(\boldsymbol{W}', \boldsymbol{U}^h) \;=\; -(\boldsymbol{W}', \mathcal{L}\boldsymbol{U}^h - \boldsymbol{F}), \qquad \forall \boldsymbol{W}' \in \mathcal{V}'. \tag{20}$$

Assuming that the residual for the coarse scale solution is small, that is $\epsilon = \|\mathcal{L}\boldsymbol{U}^h - \boldsymbol{F}\| \ll 1$, where $\|\cdot\|$ is an appropriate norm, expanding $\boldsymbol{U}' = \boldsymbol{U}^{(0)} + \epsilon \boldsymbol{U}^{(1)} + O(\epsilon^2)$ in (20), we note that $\boldsymbol{U}^{(0)} = \boldsymbol{0}$ and that

$$B'(\boldsymbol{W}', \epsilon \boldsymbol{U}^{(1)}; \boldsymbol{U}^h) \;=\; -(\boldsymbol{W}', \mathcal{L}\boldsymbol{U}^h - \boldsymbol{F}), \qquad \forall \boldsymbol{W}' \in \mathcal{V}'. \tag{21}$$

where $B'(\cdot, \cdot; \boldsymbol{U}^h)$ is the linearization of $B(\cdot, \cdot)$ about $\boldsymbol{U}^h$. Note that the equation above is a time-depdendent linearized advective diffusive system of equations. The system may be solved in order to determine $\boldsymbol{U}'$. Instead, often an approximation is typically made in that

$$\boldsymbol{U}' \approx \epsilon \boldsymbol{U}^{(1)} \approx -\mathbb{P}'\boldsymbol{\tau}\mathbb{P}'^T(\mathcal{L}\boldsymbol{U}^h - \boldsymbol{F}) = -\mathbb{P}'\boldsymbol{\tau}\mathbb{P}'^T\mathcal{R}(\boldsymbol{U}^h). \tag{22}$$

where $\mathcal{R}(\boldsymbol{U}^h) = \mathcal{L}\boldsymbol{U}^h - \boldsymbol{F}$ is the residual of the coarse scale solution. This approximation replaces the inverse of the differential operator by an algebraic one denoted by the matrix $\boldsymbol{\tau}$. Further $\mathbb{P}' \equiv \mathbb{I} - \mathbb{P}^h$ is an operator that ensures $\boldsymbol{U}' \in \mathcal{V}'$.

Note that the expression for $\boldsymbol{U}'$ above contains approximations for the fine scale pressure and density (in the compressible case) fields in addition to the fine scale velocity field. However, we are interested only in latter. From, (22), after assuming a diagonal form for $\boldsymbol{\tau}$ we arrive at:

$$\boldsymbol{u}' \approx -\frac{1}{\rho^h}\mathbb{P}'\tau_m\mathbb{P}'^T\mathcal{R}_m(\boldsymbol{U}^h). \tag{23}$$

where τ_m is a scalar and $\mathcal{R}_m(\boldsymbol{U}^h)$ is the residual of the coarse scale solution for the momentum equations. The expression above is valid for both the incompressible and compressible flows.

For incompressible flows $\rho^h \equiv 1$,and the scalar τ_m is given by (see [19] for example)

$$\tau_m = \left(\frac{4 < |\boldsymbol{u}^h|^2 >}{h^2} + C_I(\frac{4}{Reh^2})^2\right)^{-1/2}, \tag{24}$$

where h is the grid size, $< \cdot >$ denotes the spatial average of a quantity, and C_I is a constant obtained from inverse estimates. For the spectral method used in this manuscript $C_I = \pi$. Finally the residual of the coarse scales for the incompressible momentum equations is

$$\mathcal{R}_m(\boldsymbol{U}^h) = \boldsymbol{u}^h_{,t} + \nabla \cdot (\boldsymbol{u}^h \otimes \boldsymbol{u}^h) + \nabla p^h - \frac{1}{Re}\nabla^2\boldsymbol{u}^h - \boldsymbol{f}. \tag{25}$$

For compressible flows (see [20, 21] for example),

$$\tau_m = \left((\lambda^e)^2 + C_I(\frac{4}{h^2}\frac{< \mu^h >}{< \rho^h > Re})^2\right)^{-1/2} \tag{26}$$

with

$$\frac{1}{\lambda^e} = \frac{1-e^{-Ma}}{\lambda_1^e} + \frac{e^{-Ma}}{\lambda_2^e} \tag{27}$$

$$(\lambda_1^e)^2 = \frac{4}{h^2} < |\boldsymbol{u}^h|^2 > (1 + 2Ma^{-2} + Ma^{-1}\sqrt{4 + Ma^{-2}}) \tag{28}$$

$$(\lambda_2^e)^2 = \frac{4}{h^2} < |\boldsymbol{u}^h|^2 > \tag{29}$$

where $\boldsymbol{u}^h = \tilde{\boldsymbol{u}}(\boldsymbol{U}^h)$, $Ma = \sqrt{< |\boldsymbol{u}^h|^2 >} / < c^h >$ is the turbulent Mach number, and $c^h = \sqrt{T^h}/M_\infty$ is the local speed of sound. The coarse scale momentum residual is given by

$$\mathcal{R}_m(\boldsymbol{U}^h) = \boldsymbol{m}^h_{,t} + \nabla \cdot (\frac{\boldsymbol{m}^h \otimes \boldsymbol{m}^h}{\rho^h}) + \nabla p^h - \frac{1}{Re} \nabla \cdot \boldsymbol{\sigma}^h - \boldsymbol{f}, \tag{30}$$

where $\boldsymbol{\sigma}^h = \tilde{\boldsymbol{\sigma}}(\boldsymbol{U}^h)$.

There are several important observations to be made about the approximation described above:

1. Even though (22) represents a gross approximation for $\boldsymbol{U}'$, it has proven to be very useful one as evidenced by the success of the VMS formulation [9, 14, 15, 16] that makes use of this approximation. This is primarily because the $\boldsymbol{U}'$ it yields is weighted by the coarse scale weighting function in the coarse scale equation. As a result the details of $\boldsymbol{U}'$ are not important, rather its average effect is of consequence.

2. This approximation retains the consistency of the original equation for $\boldsymbol{U}'$. In particular from (20) we observe that when $\boldsymbol{U}^h$ is exact, that is $\mathcal{R}(\boldsymbol{U}^h) = \boldsymbol{0}$, then $\boldsymbol{U}' = \boldsymbol{0}$. This is a consistency condition that states that when the coarse scales are

accurate, in that coarse scale residual vanishes, the fine scales should also vanish. From (22) and (23) we observe that this consistency condition is inherited by the approximation. This is a critical feature that leads to the "dynamic" nature of the eddy viscosity proposed in this manuscript.

3. The operator B' represents a system of linear, time-dependent, advective-diffusive equations that is driven by the residual of the coarse scale solution. The matrix $\boldsymbol{\tau}$ is intended to approximate the inverse of this differential operator. To understand this, consider the following scalar, linear, advective-diffusive equation with frozen coefficients a and ν:

$$au'_{,x} - \nu u'_{,xx} = r'. \tag{31}$$

Further, let $r' = Ce^{ikx}$ be the right hand side. Then it is easy to show that

$$|u'| = \frac{|r'|}{\sqrt{k^2 a^2 + \nu^2 k^4}} \tag{32}$$

If we now substitute $|k| \approx 1/h$, $|a| \approx |\boldsymbol{u}^h|$, and $\nu \approx 1/Re$, an expression τ_m, similar to the one in (24), is obtained.

4. In the compressible case, when the local Mach number, Ma, is small, the largest advective speed is c, the local speed of sound. However, the turbulent fluctuations are not advected by acoustic waves. Instead they are advected by the coarse scale velocities, and as a result the appropriate advective time scale in this limit is $h/|\boldsymbol{u}^h|$. On the other hand, the in the high Mach number limit the fine scales are advected by the velocity $\boldsymbol{u}^h + \boldsymbol{c}$, which yields an advective time scale of $h/|\boldsymbol{u}^h + \boldsymbol{c}|$.

Note that for $Ma \gg 1$ this term is also approximately equal to $h/|\boldsymbol{u}^h|$. It is in the intermediate range, $Ma \approx 1$, when the fine scale velocity fluctuations are carried by waves that travel with velocity $\boldsymbol{u}^h + \boldsymbol{c}$, and when $|\boldsymbol{u}^h|$ is not so large that it completely overwhelms c, that the advective time scale departs significantly from $h/|\boldsymbol{u}^h|$. This is accounted for in the expression for the compressible τ_m.

Summary : The new residual-based eddy viscosity formulation is given by (11), where the model term is given by (12) for the incompressible case and by (13) for the compressible case. In each of these expressions the fine scale velocity $\boldsymbol{u}'$ is given by (23), where the parameter τ_m is given by (24) in the incompressible case and by (26) in the compressible case.

4 Numerical Results

In this section we apply the RBEVM to predict the decay of compressible and incompressible turbulence. We compare these results with DNS performed on much finer grids and with the dynamic version of popular eddy viscosity models. We emphasize that we do not use a dynamic procedure for the RBEVM. Our claim is that because the viscosity in this model is proportional to the residual of the coarse scale equations, it is inherently dynamic. We set $\Omega =]0, 2\pi[^3$ and require density, velocity and pressure fields to satisfy periodic boundary conditions. We simulate these problems using a Fourier basis, where modes with $|\boldsymbol{k}| < k^h$ are used to define $\mathcal{V}^h$. We note that these basis functions have the special property that they are orthogonal to each other in all H^m

inner-products. In addition, we define the projector $\mathbb{P}^h$ to be the H^1 projector and due to the orthogonality of the Fourier modes this is the low-pass sharp cut-off filter in wavenumber space. Then $\mathbb{P}'$ and $\mathbb{P}'^T$ are high-pass, sharp cutoff filters in wavenumber space. For the RBEVM we represent the fine scale velocities using Fourier modes with $|\boldsymbol{k}| \in (k^h, 3k^h/2)$, which implies $\alpha = 3/2$ and therefore from (17), $\bar{C} = 0.15$. For all models we perform spatial integration on a $(3N/2)^3$ grid, where $N = k^h$ is the number of modes used to represent the coarse scales in each direction. This choice exactly de-aliases all quadratic terms.

4.1 Incompressible Taylor Green Vortex Flow

The equations for the evolution of the Fourier coefficients (denoted by a hat) for the incompressible case, after eliminating pressure, are given by

$$\frac{\partial \widehat{\boldsymbol{u}}^h}{\partial t} + \boldsymbol{R}(\boldsymbol{k}) \cdot \widehat{\boldsymbol{u}^h \otimes \boldsymbol{u}^h} = -\frac{|\boldsymbol{k}|^2}{Re} \widehat{\boldsymbol{u}}^h + i\boldsymbol{k} \cdot \widehat{2\nu_t \boldsymbol{S}^h_{dev}} \tag{33}$$

where $R(\boldsymbol{k})_{nmj} = i(\delta_{nm} - \frac{k_n k_m}{k^2})k_j$. All the models considered in this manuscript can be obtained by defining the turbulent viscosity ν_t as follows:

1. For no model (DNS), $\nu_t = 0$.

2. For the static Smagorinsky eddy viscosity model (SSEVM), $\nu_t = C_0^2 h^2 |\boldsymbol{S}^h|$, and we consider $C_0 = 0.1$ & 0.16.

3. For the dynamic Smagorinsky eddy viscosity model (DSEVM), $\nu_t = C_0^2 h^2 |\boldsymbol{S}^h|$, and C_0 is determined dynamically.

4. For the residual based eddy viscosity model (RBEVM) $\nu_t = \bar{C}h|\boldsymbol{u}'|$, where $\boldsymbol{u}'$ is given by (23), and $\bar{C} = 0.15$.

The DNS runs were performed using 512^3 modes while the LES runs were performed with 32^3 modes.

The test problem corresponds to the turbulent decay of Taylor-Green vortices with the Reynolds number $Re = 6,369$ and an initial velocity field given by

$$\boldsymbol{u}^h(\boldsymbol{x}, 0) = \begin{bmatrix} \sin(x)\cos(y)\cos(z) \\ -\cos(x)\sin(y)\cos(z) \\ 0 \end{bmatrix}. \tag{34}$$

The velocity field evolves from this smooth initial condition to a field with more energy in the higher wavenumber components, and finally transitions to a turbulent state.

This problem presents an interesting test case for a dynamic eddy viscosity model. Initially the velocity field is smooth and well captured by the resolved scales, and as a result the eddy viscosity ought to be small. However as the energy cascades to finer sales, the coarse LES grid is unable to represent the exact solution, and a model becomes necessary to capture the effects of the fine scales. At this instant one would expect a dynamic model to switch itself on. In Figure 1 we have plotted the spatially-averaged viscosity for different models as a function of time. As expected we observe that the viscosity of the two static Smagorinsky models starts from a non-zero value. The viscosity of the dynamic Smagorinsky model is zero initially and increases as the simulation progresses. This clearly indicates the "dynamic" behavior of this model. The viscosity for the RBEVM also starts out very close to zero and then increases to a finite value, much like the dynamic model, even though the parameter in the RBEVM is kept fixed. The apparent dynamic behavior of this model may be attributed to the fact that the eddy viscosity in this model is proportional to the residual of the coarse scales which becomes large as the solution develops fine scale features. Note that, initially, the dynamic model and the RBEVM are almost identical, but they begin to differ at about $t \cong 5$, and become quite different thereafter with RBEVM much less oscillatory.

The energy spectra for all the models is compared with the DNS in Figures 2 & 3 at $t = 8$ & 12, respectively. At $t = 8$ we observe that the RBEVM, the DSYEVM and the SSYEVM(0.1) are all accurate, while the SSEVM(0.16) is too dissipative. At $t = 12$ the SSEVM(0.1) appears to be not dissipative enough while the SSEVM(0.16) model is overly dissipative. The RBEVM and the DSEVM are the most accurate and their results are comparable. This is explained by recognizing that the turbulent eddy viscosity for these models is very similar (see Figure 1).

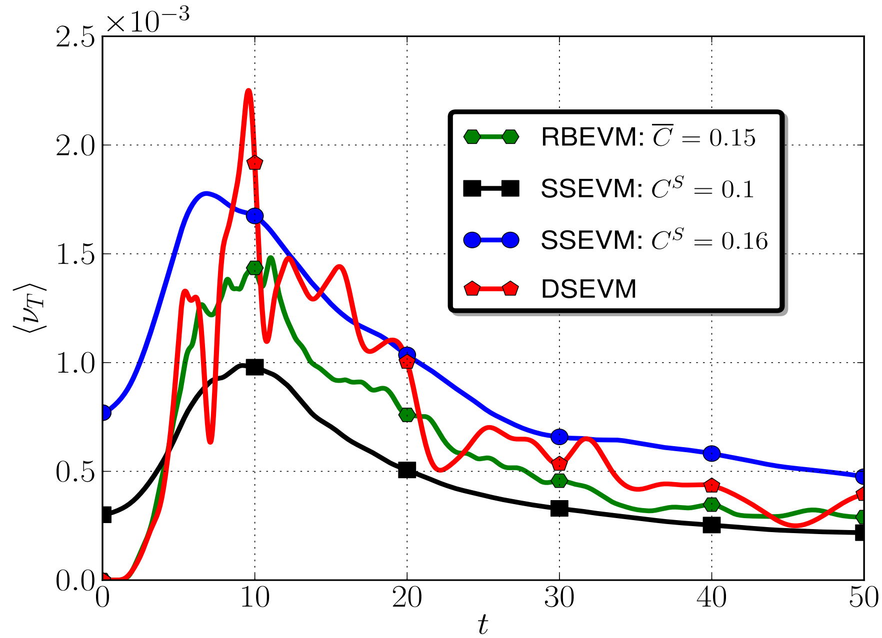

Figure 1: Evolution of the average turbulent viscosity in the incompressible case.

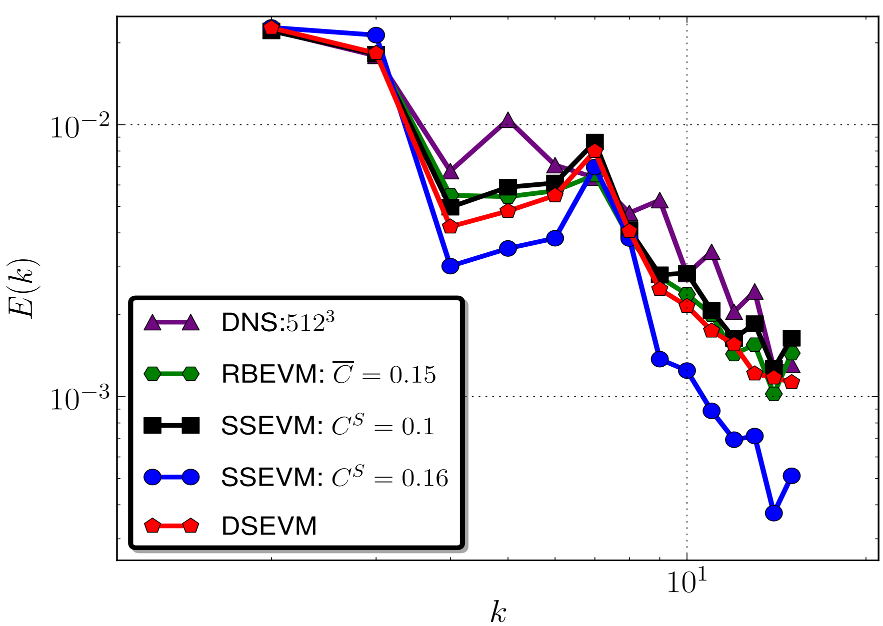

Figure 2: Energy spectra at $t = 8$ for the incompressible case.

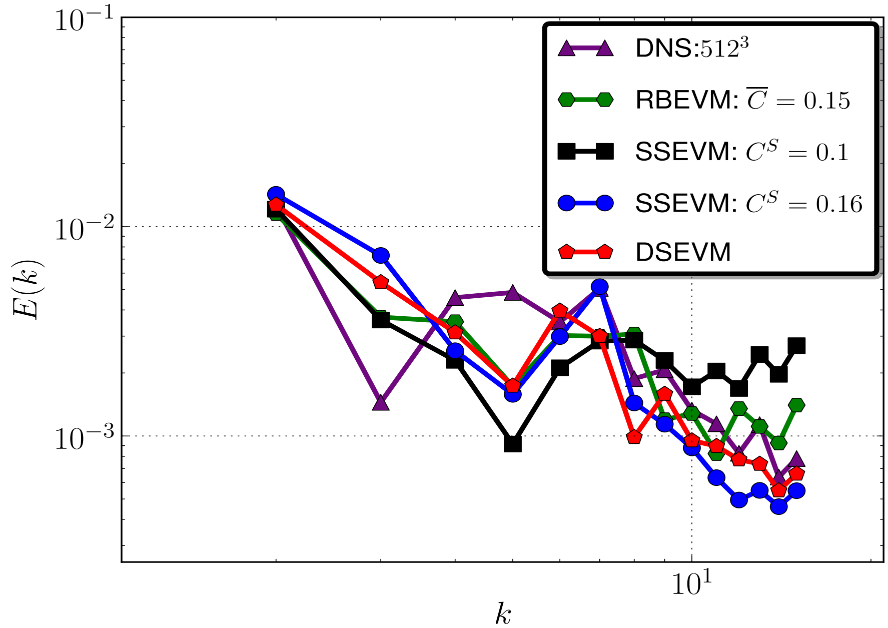

Figure 3: Energy spectra at $t = 12$ for the incompressible case.

4.2 Compressible Flow with Shocklets

The equations for the evolution of the Fourier coefficients (denoted by a hat) for the compressible case are given by

$$\frac{\partial \widehat{\rho}^h}{\partial t} + i\boldsymbol{k} \cdot \widehat{\boldsymbol{m}}^h = 0 \quad (35)$$

$$\frac{\partial \widehat{\boldsymbol{m}}^h}{\partial t} + i\boldsymbol{k} \cdot \widehat{\frac{\boldsymbol{m}^h \otimes \boldsymbol{m}^h}{\rho^h}} + i\boldsymbol{k}\widehat{p}^h = \frac{i\boldsymbol{k} \cdot \widehat{\boldsymbol{\sigma}^h}}{Re} + i\boldsymbol{k} \cdot 2\widehat{\rho^h \nu_t \boldsymbol{S}^h_{dev}} - \frac{i\boldsymbol{k}}{3}\widehat{\rho^h u^2_{rms}} \quad (36)$$

$$\frac{\partial \widehat{p}^h}{\partial t} + i\boldsymbol{k} \cdot \widehat{\boldsymbol{u}^h p^h} + (\gamma - 1)\widehat{p^h \nabla \cdot \boldsymbol{u}^h} = \frac{(\gamma - 1)}{Re}\widehat{\phi^h} + \frac{i\boldsymbol{k}}{M^2_\infty} \cdot (\widehat{\frac{\mu^h \nabla T^h}{PrRe}} + \widehat{\frac{\rho^h \nu_t \nabla T^h}{\gamma Pr_t}}) \quad (37)$$

where $\boldsymbol{u}^h = \tilde{\boldsymbol{u}}(\boldsymbol{U}^h)$, $\boldsymbol{S}^h = \tilde{\boldsymbol{S}}(\boldsymbol{U}^h)$, $\boldsymbol{\sigma}^h = \tilde{\boldsymbol{\sigma}}(\boldsymbol{U}^h)$, $\phi^h = \tilde{\phi}(\boldsymbol{U}^h)$, $T^h = \tilde{T}(\boldsymbol{U}^h)$, and $\mu^h = \tilde{\mu}(\boldsymbol{U}^h)$. All the models considered in this manuscript can be obtained by defining ν_t and u_{rms} as follows:

1. For no model (DNS), $\nu_t = u_{rms} = 0$.

2. For the static Smagorinsky-Yoshizawa eddy viscosity model (SSYEVM), $\nu_t = C_0^2 h^2 |\boldsymbol{S}^h|$ and $u_{rms}^2 = 2C_1 h^2 |\boldsymbol{S}^h|^2$, where $C_0 = 0.1, 0.16$, $C_1 = 0.020$ and $Pr_t = 0.5$.

3. For the dynamic Smagorinsky-Yoshizawa eddy viscosity model (DSYEVM), $\nu_t = C_0^2 h^2 |\boldsymbol{S}^h|$ and $u_{rms}^2 = 2C_1 h^2 |\boldsymbol{S}^h|^2$, where C_0, C_1 and Pr_t are determined dynamically.

4. For the residual based eddy viscosity model (RBEVM) $\nu_t = \bar{C} h |\boldsymbol{u}'|$ and $u_{rms} = |\boldsymbol{u}'|$, where $\boldsymbol{u}'$ is given by (23), $\bar{C} = 0.15$ and $Pr_t = 0.5$.

The test problem corresponds to a DNS simulation of the decay of homogeneous turbulence for a compressible fluid (similar to the case D9 in [22]). The physical parameters are listed in Table 1 and include the initial spectrum for the turbulent kinetic energy $E(k, 0)$, the initial total kinetic energy $\frac{q^2}{2}$, the free stream Mach number M_∞, the turbulent Mach number, Ma, the Reynolds number Re, and the ratio of initial compressible to total kinetic energy χ.

For compressible flows the velocity field is comprised of solenoidal (incompressible) and dilatational (compressible) components, $\boldsymbol{u}^s$ and $\boldsymbol{u}^c$ respectively. For the case of isotropic turbulence in Fourier space the Helmholtz decomposition is unique and is given by

$$\widehat{\boldsymbol{u}}^c = [(\boldsymbol{k} \cdot \widehat{\boldsymbol{u}})/k^2]\boldsymbol{k} \tag{38}$$

$$\widehat{\boldsymbol{u}}^s = \widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}^c \, . \tag{39}$$

Table 1: Parameters for the decay of compressible homogeneous turbulence.

$E(k,0)$	$\frac{q^2}{2}$	M_∞	Ma	Re	χ
$0.011k^4e^{-2\left(\frac{k}{4}\right)^2}$	1.3235	0.300	0.488	843.8	0.4

Let K^s and K^c denote the turbulent kinetic energy from the solenoidal and dilatational velocity components, respectively. We define $\chi = K^c/(K^c + K^s)$ as the ratio of compressible kinetic energy to the total kinetic energy. The turbulent Mach number $Ma = \sqrt{<|\boldsymbol{u}^h|^2>}/<c^h>$, where $<\cdot>$ represents the spatial average and c^h is the local speed of sound. We approximate the turbulent Mach number with the root-mean-square Mach number $Ma \approx \sqrt{<|\boldsymbol{u}^h|^2>/<c^{h2}>}$, which is easier to evaluate. We assume that there are no fluctuations in the initial values of the thermodynamic quantities (pressure, density and the temperature). We choose Prandtl number $Pr = 0.7$ and the adiabatic index for air to be $\gamma = 1.4$.

These parameters lead to the development of local, weak shocks, referred to as shocklets close to the regions where the local Mach number exceeds unity. As pointed out in [22] this happens when $\nabla \cdot \boldsymbol{u}$ attains a large negative value corresponding to the deceleration of a supersonic flow to a subsonic flow. During the decay of turbulence the Taylor microscale Reynolds number, $Re_\lambda \sim 396$ to 41.

For the DNS simulation we use 512^3 modes and for the LES simulations we use 32^3 modes. For the LES calculations we use the truncated velocity field obtained from the DNS at $t/T_e = 1.05$ ($T_e = 0.667$ is the eddy turn-over time [22]) as the initial condition. We compare the performance of the models at $t/T_e = 3$ and 6, which corresponds to a

Taylor micro-scale Reynolds number of $Re_\lambda = 69$ and 45, respectively.

From the plots of the spectra of the incompressible component of the velocity (Figures 4 and 5) we observe that the SSYEVM (0.1) is the most accurate, though it does display a small pile-up of energy at the large wavenumbers. The other models are too dissipative, but of these the RBEVM is the most accurate.

The results for the compressible component of the velocity are shown in Figures 6 and 7. In this case the SSYEVM (0.1) appears to be not dissipative enough, while the DSYEVM and SSYEVM (0.16) results appear to be overly dissipative. The RBEVM results are the most accurate.

The results for the density, pressure and temperature spectra (Figures 8, 9 and 10, respectively) at $t/T_e = 6$ follow the same trend. The RBEVM is the most accurate. The SSYEVM (0.1) appears to be not dissipative enough, while the DSYEVM and SSYEVM (0.16) appear to be overly dissipative.

The average turbulent viscosity, ν_t, induced by these models as a function of time is shown in Figure 11. We observe that the viscosity for the RBEVM is very close to that of DSYEVM and they are in between that of the two SSYEVMs.

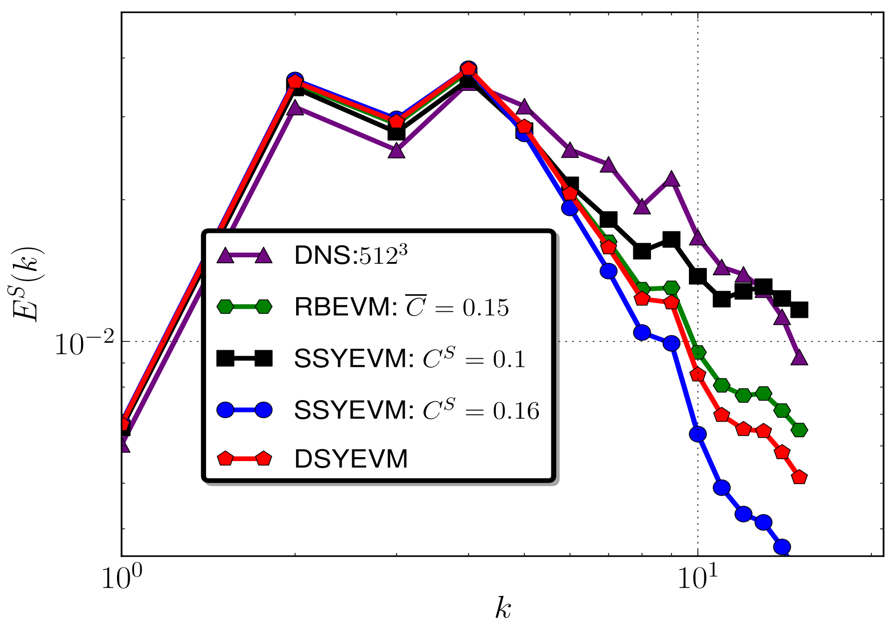

Figure 4: Incompressible velocity energy spectra at $t/T_e = 3$ for the compressible case.

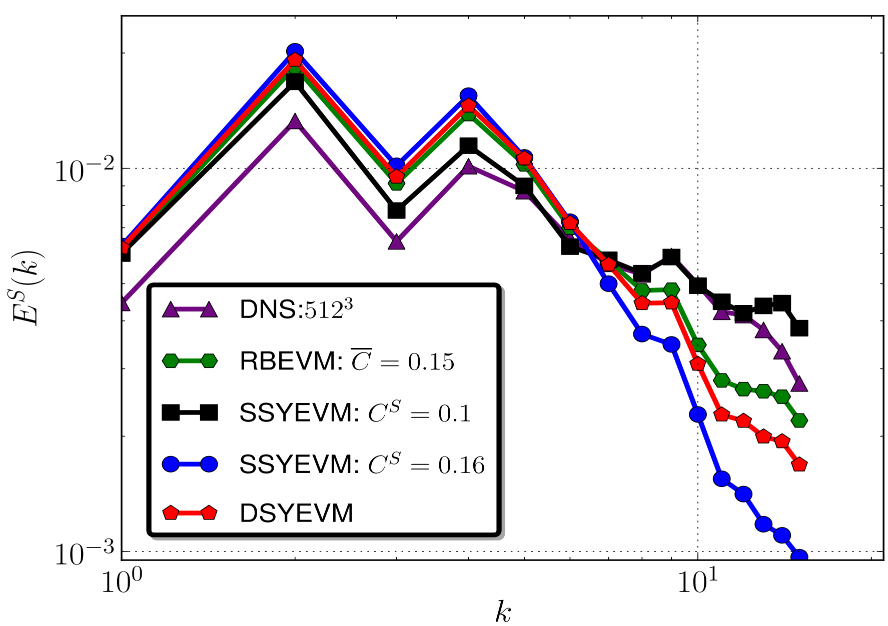

Figure 5: Incompressible velocity energy spectra at $t/T_e = 6$ for the compressible case.

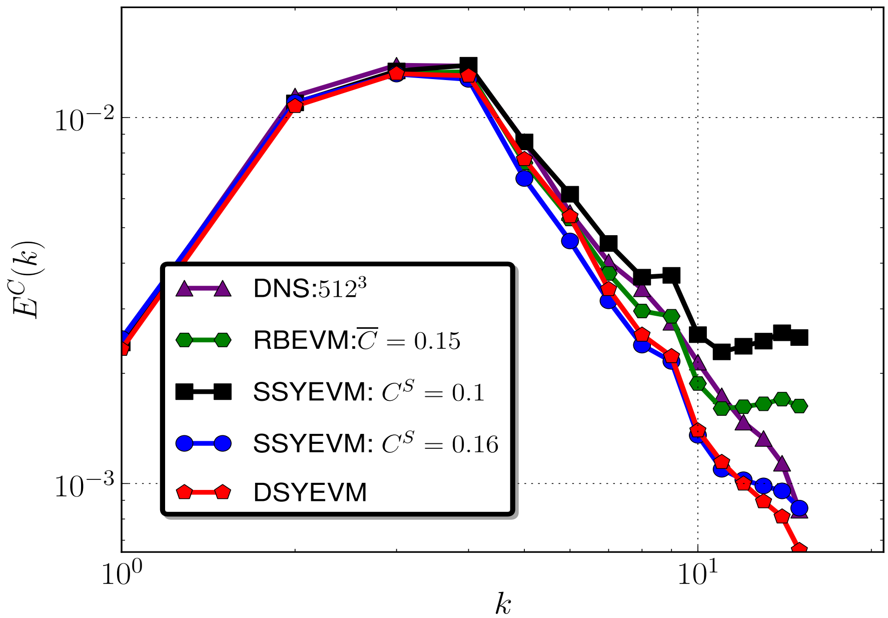

Figure 6: Compressible velocity energy spectra at $t/T_e = 3$ for the compressible case.

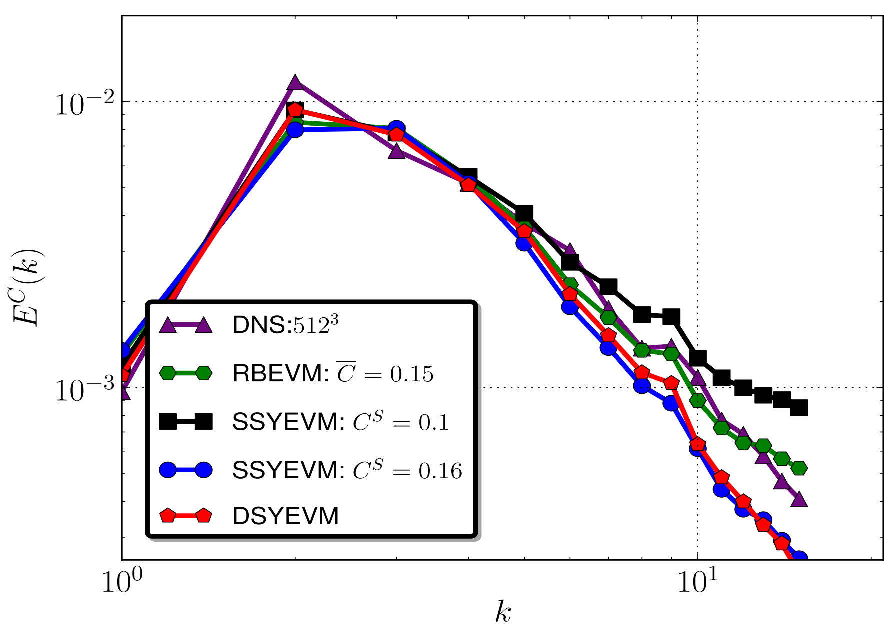

Figure 7: Compressible velocity energy spectra at $t/T_e = 6$ for the compressible case.

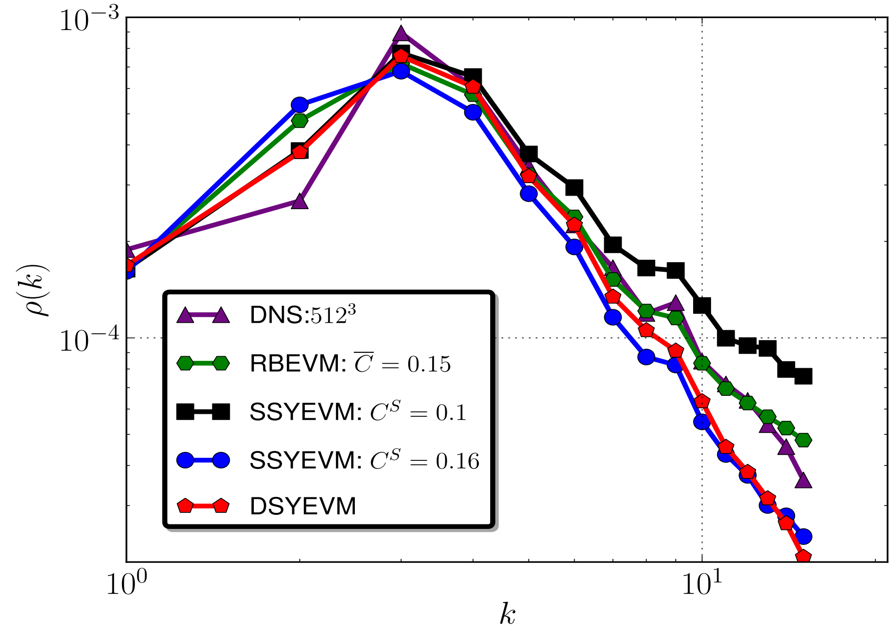

Figure 8: Density spectra at $t/T_e = 6$ for the compressible case.

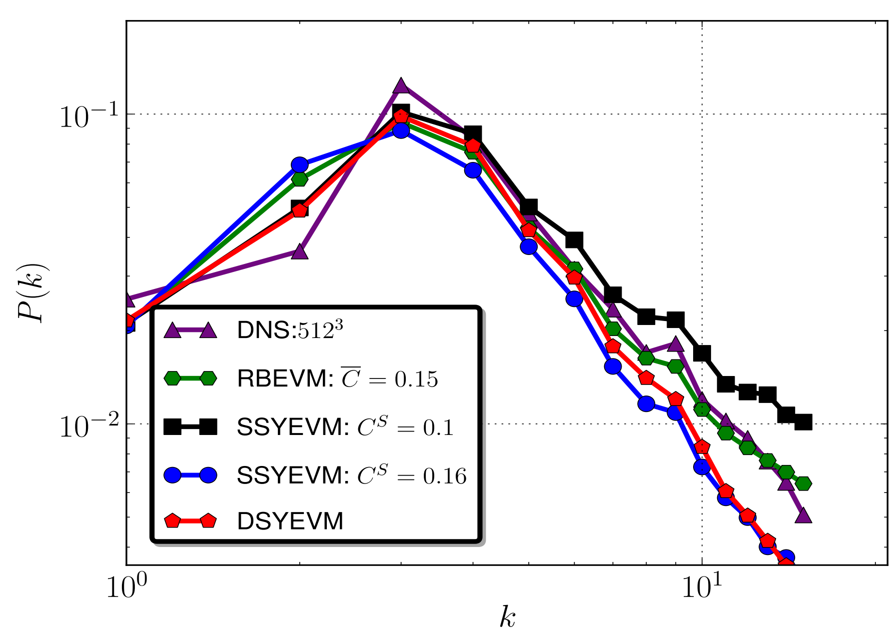

Figure 9: Pressure spectra at $t/T_e = 6$ for the compressible case.

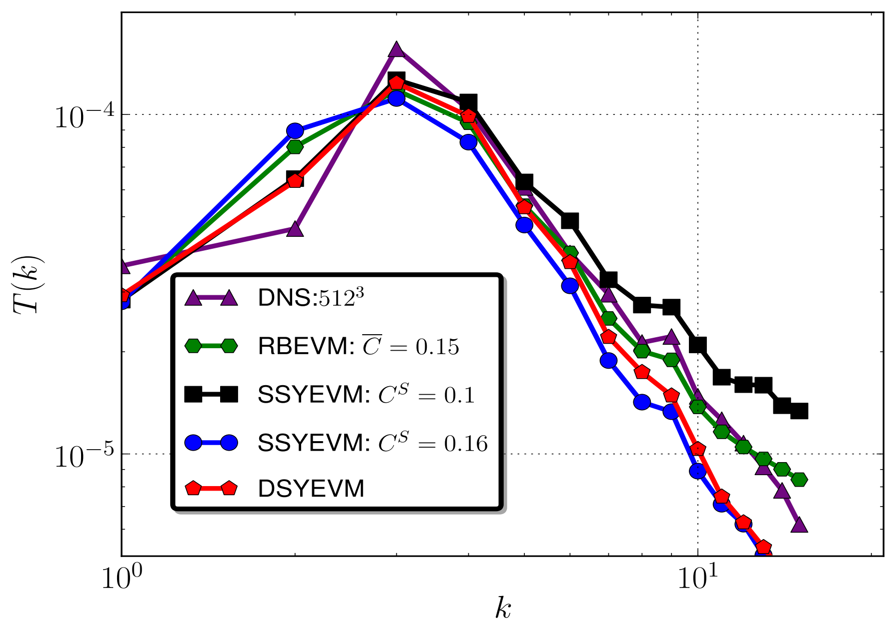

Figure 10: Temperature spectra at $t/T_e = 6$ for the compressible case.

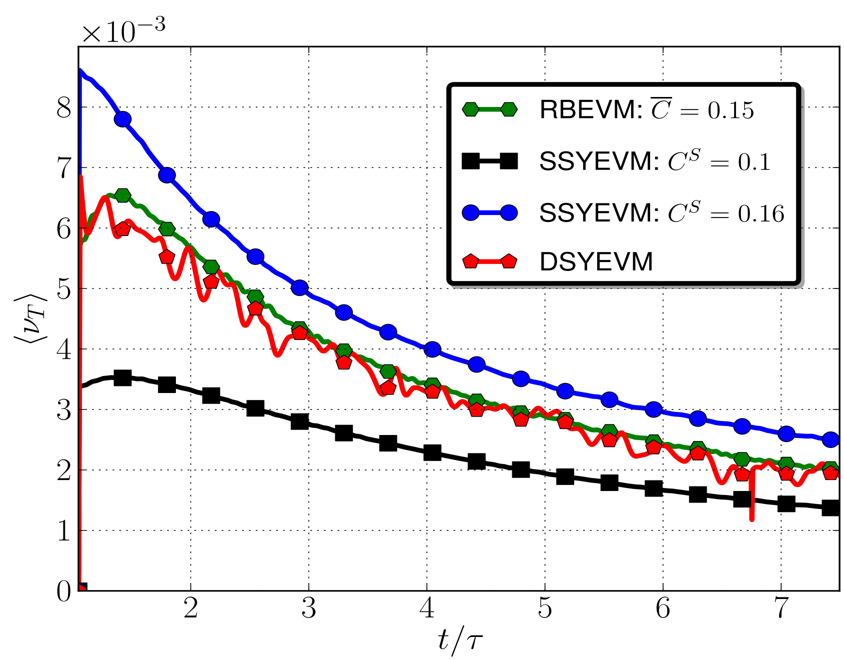

Figure 11: Evolution of average turbulent viscosity in the compressible case.

5 Conclusions

In this manuscript we have proposed and tested the performance of a new residual-based eddy viscosity model (RBEVM) for compressible and incompressible flows. In this model the eddy viscosity is proportional to $h|\boldsymbol{u}'|$, where h represents mesh size and $\boldsymbol{u}'$ is an estimate of the subgrid velocity that it proportional to the residual of the resolved scale solution. When the coarse scales fully resolve the solution of the Navier Stokes equations, that is when the residual of the resolved scales vanishes, the fine scale velocity also vanishes, and as a result the RBEVM also vanishes. This endows the RBEVM with an inherent dynamic behavior even though it does not require the dynamic calculation of any parameters.

The RBEVM is applied to study the decay of incompressible Taylor-Green vortices and that of a compressible flow with shocklets. In both cases it is found that it produces results that are at least as accurate as that of a more involved dynamic Smagorinsky-type model. However, the RBEVM is much easier to implement especially when working in complex geometries. Its performance for these types of flows will be considered in the future.

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