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by

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SUMMARY

We present a numerical formulation aimed at modeling the nonlinear response of elastic materials using large-deformation continuum mechanics in three dimensions. This finite element formulation is based on the Eulerian description of motion and the transport of the deformation gradient. When modeling a nearly incompressible solid, the transport of the deformation gradient is decomposed into its isochoric part and the Jacobian determinant as independent variables. A homogeneous isotropic hyper-elastic solid is assumed and a NURBS-based finite element method is used for the spatial discretization. A variational multiscale residual-based approach is employed to stabilize the transport equations. The performance of the scheme is explored for both compressible and nearly incompressible applications. The numerical results are in good agreement with theory illustrating the viability of the computational scheme. Copyright © 2010 John Wiley & Sons, Ltd.

KEY WORDS: Non-linear elasticity, Eulerian formulation, Large deformation, Variational multiscale methods; Isogeometric analysis, NURBS, Incompressibility

1. INTRODUCTION

Modeling geophysical problems such as lithospheric deformation [1] and ice sheet flow [2] involves complex phenomena such as multi-phase flow, localization of deformation, fracture of non-Newtonian visco-elasto-plastic media, and phase transformation. Further, evolution of rifted margins and development of crustal deformation involve large-scale plastic deformation over long time scales [3]. The computational challenges of such geophysical models are significant due to the presence of nonlinearities, moving boundaries and inhomogeneities. Therefore, specialized numerical methods are needed to produce accurate simulations. The development of the current Eulerian formulation is our first step toward accurately simulating complex geophysical processes.
The Lagrangian description is a popular choice for modeling the response of Earth materials [4–7]. In a Lagrangian formulation it is sufficient to find an admissible displacement field such that the momentum equation is satisfied in a weak sense. The deformation gradient can be evaluated from the computed displacement field. However, for finite-strain elastic deformation processes such an implementation often results in excessive mesh distortion because the mesh is attached to the material [3, 8]. This reduces the accuracy and ultimately terminates the computation due to an invalid mesh. In order to avoid this, remeshing is required at regular intervals and the model variables have to be mapped (projected or interpolated) onto the new mesh [3, 4, 8]. Therefore, such a scheme is computationally expensive and more importantly inaccurate and cumbersome.

On the other hand, an Eulerian description, in which the mesh is fixed and the material moves relative to the mesh, does not have the problem of mesh distortion. Several Eulerian formulations for modeling elastic and elastic-plastic response of solids can be found in the literature: Plohr & Sharp presented conservative Eulerian formulations for elastic flow and plasticity in [9, 10]; Dvorkin et al. implemented flow formulations using an Eulerian description of motion for analyzing transient and stationary metal forming processes in [11–15]; Trangenstein et al., Miller & Colella developed shock-capturing Eulerian numerical methods for general hyper-elasticity [16–19]. In addition to solving the momentum equations, Eulerian formulations require solving additional equations for the transport of either the stress or the deformation gradient. Therefore, they are computationally more expensive than Lagrangian schemes, and also need special treatment such as SUPG [20] and VMS stabilization [21], to handle the convective transport.

Deformations that involve very small volume changes occur for rubber-like materials and the elastic-plastic response of metals and undrained soils. Standard displacement-based finite elements have difficulty in modeling nearly incompressible deformations. Several numerical techniques such as mixed element formulations, reduced integration, stabilization, and projection techniques are available in the literature to deal with nearly incompressible elasticity using a Lagrangian framework. These techniques are useful for avoiding instabilities and volumetric locking in standard finite elements (see [22] for a literature review). The mixed Galerkin finite element formulation is one of the most common techniques used. In such a formulation, the order of approximation for pressure and displacement/velocity variables cannot be chosen arbitrarily. The method must satisfy the LBB condition to ensure stability and optimal convergence or must be used in conjunction with stabilization techniques [23, 24].

The development of isogeometric analysis, recently proposed by Hughes et al. [25], introduced a new version of the finite element method in an attempt to improve geometry, solution representation, and mesh refinement compared with the standard finite element analysis. The first implementation of isogeometric analysis was based on Non-Uniform Rational B-Splines (NURBS). A higher-order approach called the “k-refinement” has emerged from isogeometric analysis in which discretizations of order p achieve \( C^{(p-1)} \) continuity on “patches” (roughly speaking, subdomains). Recently Elguedj et al. [22] proposed \( \bar{B} \) and \( \bar{F} \) projection methods for nearly incompressible linear and non-linear elasticity and plasticity using higher-order NURBS elements. This study demonstrated that for the accuracy and stability of finite-strain elasticity formulations, the order of approximation used for representing the displacement/velocity fields needs to be of higher-order than the one used for the Jacobian determinant/deformation gradient.

The numerical scheme proposed herein involves solving the momentum equations and
the transport equations for the deformation gradient in a staggered fashion in time for a compressible material. Alternatively, in the incompressible limit, the scheme we propose solves the linear momentum balance and the transport of the isochoric and volumetric parts of the deformation gradient in a staggered manner in time. These staggered schemes yield a system of coupled linear problems that are advanced in time until the steady state is achieved. The remainder of this paper is organized as follows: Section 2 presents the equations of motion of a hyper-elastic, isotropic solid in an Eulerian framework and the variational formulation; Section 3 describes the NURBS based finite element discretization, scale separation, and the solution algorithm; Section 4 presents numerical examples demonstrating the robustness of the scheme and some concluding remarks are made in Section 5.

2. FORMULATION OF THE GOVERNING EQUATIONS

2.1. Governing equations

• The Eulerian description of motion: We use a Cartesian coordinate system \( \{ x = x_j \hat{e}_j \mid j = 1, 2, 3 \} \), where \( \hat{e}_j \) are the orthonormal basis vectors. The velocity of a particle at a given point at time \( t \) is given by,

\[
\mathbf{v} = \mathbf{v}(x,t).
\]

(1)

• Momentum balance: In an Eulerian framework, the equilibrium equations are

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \rho \mathbf{f} + \nabla \cdot \mathbf{\sigma}.
\]

(2)

In the above, \( \nabla = (\partial/\partial x_k \hat{e}_k) \) is the spatial gradient, the dot (\( \cdot \)) denotes dot product or contraction, where in components, \( \mathbf{v} \cdot \nabla \mathbf{v} = v_j \frac{\partial v_i}{\partial x_j} \) and \( \nabla \cdot \mathbf{\sigma} = \frac{\partial \sigma_{ij}}{\partial x_j} \), in components, \( \rho \) is the density, \( \mathbf{\sigma} \) the Cauchy stress tensor, and \( \mathbf{f} \) is the external load vector per unit mass. Following Einstein’s summation convention, repeated indices are considered to be summed over the number of spatial dimensions unless specified otherwise.

• Mass balance: The mass balance equation (also called the continuity equation) can be written as given in

\[
\frac{\partial J}{\partial t} + \mathbf{v} \cdot \nabla J - J \nabla \cdot \mathbf{v} = 0,
\]

(3)

where \( J = \rho^0 / \rho \) and \( \rho^0 \) is the density of the reference configuration at \( t = 0 \). Using Cartesian coordinates it can be shown that

\[
J = \det|\mathbf{F}|,
\]

(4)

where \( \mathbf{F} \) is the deformation gradient tensor. Thus, the instantaneous density at time \( t \) is given by,

\[
\rho(x,t) = \rho^0 / J(x,t).
\]

(5)
**Remark 1.** For a solid composed of one type of material, the instantaneous density, \( \rho(x,t) \), can be evaluated using a reference state density (a material parameter), \( \rho^0 \), and the value of the Jacobian determinant at the instant, \( J(x,t) \). The reference state density, \( \rho^0 \), may be taken as the density of the material in the unstressed state at a specific temperature. Therefore, the local density, \( \rho \), is not an independent variable in the current formulation. If the solid is composed of a mixture of materials then the local density cannot be defined just using the Jacobian determinant and therefore we need a new independent variable which could be either the density itself, or a phase variable which defines the local density at any time, \( t \). Recall, that the density at a point is required for evaluating the body force contribution to the equilibrium equation at that point.

**Deformation gradient transport:** The material time derivative of the deformation gradient tensor, \( F \) is

\[
\frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F = lF, \tag{6}
\]

where the velocity gradient tensor \( l = \nabla \mathbf{v} \) and in components, \( l_{ik}F_{kj} = \frac{\partial v_i}{\partial x_k}F_{kj} \). Now, let us write the multiplicative split of the deformation gradient,

\[
F = J^{1/3} \hat{F}, \tag{7}
\]

where \( \hat{F} \) is the deviatoric (isochoric) part of the deformation gradient tensor \( F \). For Cartesian coordinates the transport equation for \( \hat{F} \) is given by,

\[
\frac{\partial \hat{F}}{\partial t} + \mathbf{v} \cdot \nabla \hat{F} = \left[ l - \frac{1}{3}(\nabla \cdot \mathbf{v}) \mathbf{1} \right] \hat{F}, \tag{8}
\]

where \( \mathbf{1} = \delta_{ij} \hat{e}_i \otimes \hat{e}_j \) is the identity tensor, \( \delta_{ij} \) is Kronecker’s delta defined as 1 if \( i = j \) and 0 otherwise. Note that \( \det(\hat{F}) = 1 \).

**Remark 2.** By multiplying (3) by \( \frac{1}{3}J^{-2/3} \hat{F} \) and (2) by \( J^{1/3} \) and then adding, we can recover (1). This means the fulfillment of (3) and (2) automatically satisfies (1) and vice-versa.

**Constitutive relations:** We characterize the stress response using the stored-energy function proposed by Simo et al. [26, 27],

\[
W = U(J) + \hat{W}(\hat{b}), \tag{9}
\]

where \( U(J) \) and \( \hat{W}(\hat{b}) \) are the volumetric and deviatoric parts of \( W \), respectively. In the above equation, \( \hat{b} \) is given by,

\[
\hat{b} = \hat{F} \hat{F}^T = J^{-2/3} FF^T. \tag{10}
\]

We consider the following explicit forms [27]

\[
U(J) = \frac{K}{2} (\ln(J))^2, \quad \hat{W}(\hat{b}) = \frac{\mu}{2} (\tr(\hat{b}) - 3). \tag{11}
\]
where \( \mu > 0 \) and \( \kappa > 0 \) are interpreted as the shear modulus and the bulk modulus, respectively, at zero deformation. Using the above, the Cauchy stress tensor, \( \sigma \), can be evaluated as \([26]\),

\[
\sigma = \frac{2}{J} \mathbf{F} \frac{\partial W}{\partial C} \mathbf{F}^T = J U'(J) \mathbf{1} + s,
\]

\[
U'(J) = \frac{dU}{dJ} = \kappa \frac{\ln(J)}{J},
\]

\[
s = 2 \text{ dev} \left[ \hat{\mathbf{F}} \frac{\partial W}{\partial C} \hat{\mathbf{F}}^T \right] = \mu \text{ dev} [\hat{\mathbf{b}}],
\]

where \( C = \mathbf{F}^T \mathbf{F} \) is the right Cauchy-Green tensor and \( \hat{C} = \hat{\mathbf{F}}^T \hat{\mathbf{F}} \). The final expression for \( \sigma \) is given by,

\[
\sigma = \frac{1}{J} \left[ \kappa \ln(J) \mathbf{1} + \mu \text{ dev} [\hat{\mathbf{b}}] \right].
\]

2.2. Strong form

We consider a space-time domain \( Q = \Omega \times [0,T] \) where \( \Omega \) is any regular region bounded by a smooth boundary \( \Gamma \). Two alternate initial/boundary-value problems describing the elastic deformation process are given below.

- **\( \mathbf{v} - \mathbf{F} \) formulation:** The strong form is given by: find \( \mathbf{v} \) and \( \mathbf{F} \) such that

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \rho \mathbf{f} + \nabla \cdot \mathbf{\sigma} \text{ in } Q,
\]

\[
\mathbf{v} = \mathbf{v}^g \text{ on } \Gamma_v^g \times ]0,T[,
\]

\[
\mathbf{t} = \mathbf{t}^h \text{ on } \Gamma_v^h \times ]0,T[,
\]

where \( \Gamma_v^g \) and \( \Gamma_v^h \) denote the Dirichlet and Neumann parts of the boundary for the momentum equation, respectively, and, using Cauchy’s principle, \( \mathbf{t} = \mathbf{n} \cdot \mathbf{\sigma} \) is the traction vector on a surface with outward normal \( \mathbf{n} \). The boundary of the computational domain is partitioned such that \( \Gamma_v^g \cap \Gamma_v^h = \emptyset \) and \( \Gamma_v^g \cup \Gamma_v^h = \Gamma \). In addition,

\[
\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{F} = \mathbf{lF} \text{ in } Q,
\]

\[
\mathbf{F} = \mathbf{F}^g \text{ on } \Gamma_f^g \times ]0,T[,
\]

\[
\mathbf{F}(t = 0) = \mathbf{1} \text{ in } \Omega,
\]

where \( \Gamma_f^g \) denotes the Dirichlet boundary for the transport equation for \( \mathbf{F} \).

- **\( \mathbf{v} - \hat{\mathbf{F}} - J \) formulation:** The strong form is given by: find \( \mathbf{v}, \hat{\mathbf{F}} \) and \( J \) such that,

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \rho \mathbf{f} + \nabla \cdot \mathbf{\sigma} \text{ in } Q,
\]

\[
\mathbf{v} = \mathbf{v}^g \text{ on } \Gamma_v^g \times ]0,T[,
\]

\[
\mathbf{t} = \mathbf{t}^h \text{ on } \Gamma_v^h \times ]0,T[,
\]

\[
\mathbf{F}(t = 0) = \mathbf{1} \text{ in } \Omega,
\]

where \( \Gamma_f^g \) denotes the Dirichlet boundary for the transport equation for \( \mathbf{F} \).
\[
\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{F} = \left[ \mathbf{I} - \frac{1}{3} (\nabla \cdot \mathbf{v}) \right] \mathbf{F} \quad \text{in} \ Q, \]
\[
\mathbf{F} = \mathbf{F}^g \quad \text{on} \ \Gamma^g_0 \times ]0, T[, \]
\[
\mathbf{F}(t = 0) = 1 \quad \text{in} \ \Omega, \]
\[
\frac{\partial J}{\partial t} + \mathbf{v} \cdot \nabla J = J \nabla \cdot \mathbf{v} \quad \text{in} \ Q \]
\[
J = J^g \quad \text{on} \ \Gamma^g_0 \times ]0, T[, \]
\[
J(t = 0) = 1 \quad \text{in} \ \Omega, \]
\]

**Remark 3.** The strong form for the momentum equation is the same for both the formulations, but, the expressions for the consistent tangents upon linearization differ due to the procedure described in Section 3.5.

2.3. Weak form

In this section we shall only derive the weak form of the governing equations for the “\(\mathbf{v} - \mathbf{F} - J\) formulation”. The procedure to derive the weak form for the “\(\mathbf{v} - \mathbf{F}\) formulation” is similar.

- **Variational multiscale formulation of the transport process:**
  We consider the sum decomposition of \(\mathbf{F}\) into “coarse-scale” and “fine-scale” components, \(\mathbf{F}^c\) and \(\mathbf{F}'\), respectively \[21\],
  \[
  \mathbf{F} = \mathbf{F}^c + \mathbf{F}'. \]
  Let \((\cdot, \cdot)_{\Omega}\) denote the \(L^2\) inner product with respect to the domain, \(\Omega\). The weak form of the transport equation for \(\mathbf{F}\) is given by,
  \[
  0 = \left( W, \frac{\partial \mathbf{F}^c}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{F} - l \dot{\mathbf{F}} + \frac{1}{3} \nabla \cdot \mathbf{v} \mathbf{F} \right)_{\Omega}, \]
  where \(W\) denotes the weighting function for the isochoric part of the deformation gradient transport. Introducing scale separation and neglecting the contribution from \(\frac{\partial \mathbf{F}^c}{\partial t}\), we get,
  \[
  0 = \left( W, \frac{\partial \mathbf{F}'}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{F} - l \dot{\mathbf{F}} + \frac{1}{3} \nabla \cdot \mathbf{v} \mathbf{F} \right)_{\Omega} + \left( W, \mathbf{v} \cdot \nabla \dot{\mathbf{F}} - l \dot{\mathbf{F}} + \frac{1}{3} \nabla \cdot \mathbf{v} \mathbf{F} \right)_{\Omega'}, \]
  where \(\Omega'\) is the union of the element interiors. Assuming the fine scales of \(\mathbf{F}\) are zero at the element boundaries and using integration by parts for the term \(\left( W, \mathbf{v} \cdot \nabla \dot{\mathbf{F}}\right)_{\Omega}\), we have,
  \[
  0 = \left( W, \frac{\partial \mathbf{F}'}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{F} - l \dot{\mathbf{F}} + \frac{1}{3} \nabla \cdot \mathbf{v} \mathbf{F} \right)_{\Omega} - \left( \mathbf{v} \cdot \nabla W, \dot{\mathbf{F}}\right)_{\Omega} \]
  \[
  - \left( W, l \dot{\mathbf{F}}\right)_{\Omega} - \left( W, \frac{2}{3} \nabla \cdot \mathbf{v} \mathbf{F}'\right)_{\Omega}. \]

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Let us now consider the weak form of the continuity equation, that is, the transport equation for $J$,

$$0 = \left( w, \frac{\partial J}{\partial t} + \mathbf{v} \cdot \nabla J - J \nabla \cdot \mathbf{v} \right)_\Omega,$$

(23)

where $w$ denotes the weighting function for the Jacobian determinant transport. Introducing scale separation for $J$,

$$J = \bar{J} + J',$$

(24)

and neglecting the contribution from $\frac{\partial J'}{\partial t}$ we get,

$$0 = \left( w, \frac{\partial \bar{J}}{\partial t} + \mathbf{v} \cdot \nabla \bar{J} - \bar{J} \nabla \cdot \mathbf{v} \right)_\Omega + \left( w, \mathbf{v} \cdot \nabla J' - J' \nabla \cdot \mathbf{v} \right)_\Omega.$$

(25)

Again, assuming the fine scales of $J$ are zero at the element boundaries and using integration by parts for the term $\left( w, \mathbf{v} \cdot \nabla J' \right)_\Omega$ we have,

$$0 = \left( w, \frac{\partial \bar{J}}{\partial t} + \mathbf{v} \cdot \nabla \bar{J} - \bar{J} \nabla \cdot \mathbf{v} \right)_\Omega - \left( \mathbf{v} \cdot \nabla w, J' \right)_\Omega - \left( \mathbf{v}, 2 J' \nabla \cdot \mathbf{v} \right)_\Omega.$$

(26)

**Variational formulation of the momentum equation:**

For “slow” deformation processes, it is reasonable to consider just the mechanical equilibrium equation by neglecting the contribution of the inertial terms. The weak form of the equilibrium equation is given by,

$$0 = \left( \mathbf{w}, \nabla \cdot \mathbf{\sigma} + \rho \ddot{\mathbf{F}} \right)_\Omega,$$

(27)

where $\mathbf{w}$ denotes the weighting function for the linear momentum balance. Using the divergence theorem and Cauchy’s principle we get,

$$0 = - \left( \nabla \mathbf{w}, \mathbf{\sigma} \right)_\Omega + \left( \mathbf{w}, \rho \ddot{\mathbf{F}} \right)_\Omega - \left( \mathbf{w}, \mathbf{t} \right)_{\Gamma_h},$$

(28)

where $\Gamma_h$ denotes the Neumann boundary and $\mathbf{t} = \mathbf{n} \cdot \mathbf{\sigma}$ is the traction vector on a surface with outward normal $\mathbf{n}$.

The weak form of the “$\mathbf{v} - \ddot{\mathbf{F}} - J$ formulation” in the domain $\Omega$ is: find $\mathbf{v}$, $\ddot{\mathbf{F}}$, and $J$, such that, for all test functions $\mathbf{w}$, $\mathbf{W}$, and $w$,

$$- \left( \nabla \mathbf{w}, \mathbf{\sigma} \right)_\Omega + \left( \mathbf{w}, \rho \ddot{\mathbf{F}} \right)_\Omega - \left( \mathbf{w}, \mathbf{t} \right)_{\Gamma_h} = 0,$$

$$\left( \mathbf{W}, \frac{\partial \ddot{\mathbf{F}}}{\partial t} + \mathbf{v} \cdot \nabla \ddot{\mathbf{F}} - \mathbf{l} \cdot \ddot{\mathbf{F}} + \frac{1}{3} \nabla \cdot \mathbf{v} \ddot{\mathbf{F}} \right)_\Omega - \left( \mathbf{v} \cdot \nabla \mathbf{W}, \ddot{\mathbf{F}} \right)_\Omega = 0,$$

$$\left( w, \frac{\partial \bar{J}}{\partial t} + \mathbf{v} \cdot \nabla \bar{J} - \bar{J} \nabla \cdot \mathbf{v} \right)_\Omega - \left( \mathbf{v} \cdot \nabla w, J' \right)_\Omega - \left( \mathbf{v}, 2 J' \nabla \cdot \mathbf{v} \right)_\Omega = 0.$$

(29)

The trial functions, $\mathbf{v}$, $\ddot{\mathbf{F}}$, and $J$, are to be chosen such that they satisfy all the Dirichlet boundary conditions and the corresponding test functions $\mathbf{w}$, $\mathbf{W}$, and $w$ are chosen to be zero on the appropriate Dirichlet boundaries.


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3. SOLUTION STRATEGY

3.1. NURBS-based finite element approximation

Let Ω\(^h\) represent the discretization of the domain, Ω, where the \(h\)-superscript denotes the mesh parameter. Let the Ω\(^h\) be partitioned into a set of subdomains using NURBS elements. NURBS functions are built from B-Splines that are piecewise polynomial curves composed of a linear combination of B-Spline basis functions. A more detailed description of NURBS functions and NURBS based isogeometric analysis can be found in [25, 28]. The discretization of a field, generically named \(u\), is given by,

\[
u^h(x, t) = \sum_{A=1}^{n} N^A(x) u^A(t), \tag{30}\]

where \(N^A(x)\) are the NURBS basis functions and \(u^A\) are the associated control variables. This naturally reduces to classical finite element analysis when polynomial shape functions are used.

In the current problem there are three fields to be described on Ω\(^h\), the spatial velocity field, \(v^h\), the isochoric part of the deformation gradient tensor, \(\tilde{F}^h\), and the Jacobian determinant, \(J^h\).

3.2. Fine-scale approximation

We assimilate the coarse scale part of the solution to the numerical approximation, thus, \(\bar{J} \equiv J^h\) and \(\bar{F} \equiv \tilde{F}^h\). Let us approximate the fine scales by an algebraic scaling of the residual of the coarse scales as described in [21],

\[
\begin{align*}
\tilde{F}' &= -\tau_F r^F, \\
J' &= -\tau_J r^J,
\end{align*}
\tag{31}
\]

where

\[
\begin{align*}
r^F &= \frac{\partial \tilde{F}^h}{\partial t} + v^h \cdot \nabla \tilde{F}^h - l^h \tilde{F}^h + \frac{1}{3} \nabla \cdot v^h \tilde{F}^h, \\
r^J &= \frac{\partial J^h}{\partial t} + v^h \cdot \nabla J^h - J^h \nabla \cdot v^h, \\
\tau_F &= \left( \frac{4}{\Delta t^2} + v^h \cdot G v^h + l^h : l^h \right)^{-1/2}, \\
\tau_J &= \left( \frac{4}{\Delta t^2} + v^h \cdot G v^h + (\nabla \cdot v^h)^2 \right)^{-1/2}, \\
G_{ij} &= \sum_{k=1}^{3} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_k}{\partial x_j}
\end{align*}
\tag{32}
\]

In the above equations \(x = \{x_i\}_{i=1}^{3}\) denote the coordinates of the NURBS element in physical space, \(\xi = \{\xi_i\}_{i=1}^{3}\) denote the coordinates of the element in parametric space, and : denotes double contraction, thus, \(l^h : l^h = l_{ij} l_{ij}\) in components.
3.3. Spatial discretization of the transport equations

The spatially discretized transport equations for \( \hat{F} \) and \( J \) are obtained by substituting equations (31) and (32) in (22) and (20), resulting in:

\[
0 = \left( W^h, \frac{\partial \hat{F}^h}{\partial t} + v^h \cdot \nabla \hat{F}^h - t^h \hat{F}^h + \frac{1}{3} \nabla \cdot v^h \hat{F}^h \right)_\Omega \\
+ \left( v^h \cdot \nabla W^h, \tau_F r^F \right)_\Omega + \left( W^h, \frac{2}{3} \nabla \cdot v^h \tau_F r^F \right)_\Omega, \tag{33}
\]

\[
0 = \left( \frac{\partial J^h}{\partial t} + v^h \cdot \nabla J^h - J^h \nabla \cdot v^h \right) + \left( v^h \cdot \nabla w^h, \tau_F r^F \right)_\Omega + \left( \frac{1}{3} \nabla \cdot v^h \right)_\Omega. \tag{34}
\]

The corresponding residuals, \( R^F = [R^F_{A,ij}] \) and \( R^J = [R^J_A] \) are:

\[
R^F_{A,ij} = \left( N^A, r^F_{ij} \right)_\Omega + \left( v^h_p, N^A_{ij}, \tau_F r^F_{ij} \right)_\Omega + \left( N^A, \tau_F v^h_{i,m} r^F_{mj} \right)_\Omega + \left( N^A, \frac{2}{3} \tau_F v^h_{k,k} r^F_{ij} \right)_\Omega, \tag{35}
\]

\[
R^J_A = \left( N^A, r^J \right)_\Omega + \left( v^h_p, N^A_{i,j}, \tau_J r^J \right)_\Omega + \left( N^A, 2 \tau_J v^h_{k,k} r^J \right)_\Omega,
\]

where the “,” preceding a subscript denotes partial spatial differentiation, \( \frac{\partial \hat{F}^h}{\partial t} \) + \( v^h_p \hat{F}^h_{i,j} - v^h_{i,m} \hat{F}^h_{mj} + \frac{1}{3} v^h_{k,k} \hat{F}^h_{ij} \) and \( \frac{\partial J^h}{\partial t} + v^h_p J^h_p - v^h_{i,m} J^h_m \).

3.4. Numerical implementation of the transport equations

Let \( \hat{F} \) and \( \hat{\hat{F}} \) denote the degrees of freedom of the deformation gradient and its partial time derivative. At time \( t \), let us assume we know the solution, \( \hat{F}(t) \) and \( \hat{\hat{F}}(t) \). The solution at \( t + \Delta t \) is given by,

\[
\begin{align*}
\hat{F}(t + \Delta t) &= \hat{F}(t) + \Delta \hat{F}, \\
\hat{\hat{F}}(t + \Delta t) &= \hat{F}(t) + \Delta \hat{F} \Delta t,
\end{align*}
\tag{36}
\]

where \( \Delta \hat{F} \) is the increment in \( \hat{F} \). Thus, finding the solution at \( t + \Delta t \) involves evaluating \( \Delta \hat{F} \).

We start with the expression for the residual, \( R^F \) at \( (t + \Delta t) \) given by (32),

\[
R^F_{A,ij}(t + \Delta t) = \left( N^A, r^F_{ij}(t + \Delta t) \right)_\Omega + \left( v^h_p N^A_{ij}, \tau_F r^F_{ij}(t + \Delta t) \right)_\Omega \\
+ \left( N^A, \tau_F v^h_{i,m} r^F_{mj}(t + \Delta t) \right)_\Omega + \left( N^A, \frac{2}{3} \tau_F v^h_{k,k} r^F_{ij}(t + \Delta t) \right)_\Omega, \tag{37}
\]

where \( r^F_{ij}(t + \Delta t) = \hat{F}_{ij}(t + \Delta t) + v^h_p \hat{F}_{ij,p}(t + \Delta t) - v^h_{i,m} \hat{F}_{mj}(t + \Delta t) + \frac{1}{3} v^h_{k,k} \hat{F}_{ij}(t + \Delta t) \). We substitute the expressions for \( \hat{F}(t + \Delta t) \) and \( \hat{\hat{F}}(t + \Delta t) \) given by (30) in the above equation (37). Since \( R^F(t + \Delta t) \) is a linear function of \( \hat{F}(t + \Delta t) \) and \( \hat{\hat{F}}(t + \Delta t) \), the resulting equation upon substitution can be simplified and written as,

\[
R^F(t + \Delta t) = R^F(t) + T(t) \Delta \hat{F}. \tag{38}
\]
where $\mathbf{T} = \left[T_{ijrs}^{AB}\right]$ is the approximation of the consistent tangent. We seek a solution such that $\mathbf{R}^F(t + \Delta t) = 0$, therefore, the linear system for $\mathbf{F}$ is given by,

$$
\mathbf{T}(t) \Delta \mathbf{F} = -\mathbf{R}^F(t). \tag{39}
$$

where

$$
T_{ijrs}^{AB} = \int_\Omega N^A N^B \delta_{ir} \delta_{js} d\Omega + \int_\Omega N^A v_k N^B_k \Delta t \delta_{ir} \delta_{js} d\Omega \\
- \int_\Omega N^A v_i, r N^B \Delta t \delta_{js} d\Omega + \int_\Omega \frac{1}{3} \tau_{p,p} N^A N^B \Delta t \delta_{ir} \delta_{js} d\Omega \\
+ \int_\Omega \tau_{p,v} N^A_m N^B \delta_{ir} \delta_{js} d\Omega + \int_\Omega \tau_{p,v} N^A_m v_k N^B_k \Delta t \delta_{ir} \delta_{js} d\Omega \\
- \int_\Omega \tau_{p,i} N^A v_i, r N^B \Delta t \delta_{js} d\Omega + \int_\Omega \frac{1}{3} \tau_{p,v} p N^A_i v_k N^B_k \Delta t \delta_{ir} \delta_{js} d\Omega \\
+ \int_\Omega \tau_{p,v} N^A_i v_{ir} N^B \Delta t \delta_{js} d\Omega + \int_\Omega \frac{1}{3} \tau_{p,v} p N^A_i v_{ir} N^B_k \Delta t \delta_{ir} \delta_{js} d\Omega \\
- \int_\Omega \tau_{p,v} N^A_i v_{ir} N^B \Delta t \delta_{js} d\Omega + \int_\Omega \frac{2}{3} \tau_{p,v} p N^A_i v_{ir} N^B_k \Delta t \delta_{ir} \delta_{js} d\Omega \tag{40}
$$

Following the above procedure, the linear system for $J$ can be written as,

$$
\mathbf{D}(t) \Delta J = -\mathbf{R}^J(t). \tag{41}
$$

where $\mathbf{D} = [D^{AB}]$ is the approximation of the consistent tangent given by,

$$
D^{AB} = \int_\Omega N^A N^B d\Omega + \int_\Omega N^A v_k N^B_k \Delta t d\Omega - \int_\Omega N^A v_{p,p} N^B \Delta t d\Omega \\
+ \int_\Omega \tau_{v,v} N^A_m N^B \Delta t d\Omega + \int_\Omega \tau_{v,v} N^A_m v_k N^B_k \Delta t d\Omega - \int_\Omega \tau_{v,v} N^A_m v_{p,p} N^B \Delta t d\Omega \\
+ \int_\Omega \tau_{v,v} N^A \Delta t d\Omega + \int_\Omega \tau_{v,v} N^A v_k N^B_k \Delta t d\Omega \\
- \int_\Omega \tau_{v,v} N^A v_{p,p} N^B \Delta t d\Omega \tag{42}
$$

The $h$-superscript on $v$ has been removed in the above expressions to improve readability. The linear systems (39) and (41) are solved for the increments $\Delta \mathbf{F}$ and $\Delta J$, respectively, at each time step using a preconditioned GMRES algorithm to a specified tolerance, $TOLF = TOLF = 10^{-12}$. We used two different preconditioners: (1) block diagonal (2) element-by-element (described in detail in (31)). We found the convergence to be faster with the element-by-element preconditioner. Also, the linear solve is more efficient when we include...
the variational multiscale stabilization terms. Next, we update the solution as follows:

\[
\begin{align*}
\dot{F}(t + \Delta t) &= \dot{F}(t) + \Delta \dot{F}, \\
\ddot{F}(t + \Delta t) &= \ddot{F}(t) + \Delta \ddot{F}, \\
\dot{j}(t + \Delta t) &= \dot{j}(t) + \Delta \dot{j}, \\
J(t + \Delta t) &= J(t) + \Delta J.
\end{align*}
\] (43)

3.5. Linearization of the momentum equation

The discretized form of the equilibrium equation can be written using (28) as,

\[
0 = -\left(\nabla w^h, \sigma(F^h)\right)_\Omega + (w^h, \rho f)_\Omega - (w^h, t)_{\Gamma_h},
\] (44)

The residual vector when no body force, \(f\), is present, \(R^M = [R^M_{A,i}]\), is given by,

\[
R^M_{A,i} = -(N^A_j, \sigma_{ij}(F^h))_\Omega - (N_{A,i})_{\Gamma_h}.
\] (45)

Using Taylor’s expansion, we linearize the above equation at time \(t + \Delta t\) [13],

\[
0 = R^M(t + \Delta t) = R^M(t) + \left[\frac{\partial R^M}{\partial t}\right]_t \Delta t.
\] (46)

Rewriting the above we get,

\[
\left[\frac{\partial R^M}{\partial t}\right]_t = -\frac{1}{\Delta t} R^M(t),
\] (47)

where

\[
\frac{\partial R^M_{A,i}}{\partial t} = -(N^A_j, \frac{\partial \sigma_{ij}(F^h)}{\partial t})_\Omega.
\] (48)

Now using the chain rule we can write,

\[
\frac{\partial \sigma_{ij}(F^h)}{\partial t} = \frac{\partial \sigma_{ij}}{\partial \dot{F}^h_{lm}} \frac{\partial \dot{F}^h_{lm}}{\partial t} + \frac{\partial \sigma_{ij}}{\partial J^h} \frac{\partial J^h}{\partial t}.
\] (49)

From the transport equation for \(\dot{F}\), given in (3), we have,

\[
\frac{\partial \dot{F}_{l,m}^h}{\partial t} = v_{l,r}^h \dot{F}_{r,m}^h - v_r^h \dot{F}_{l,m,r}^h - \frac{1}{3} v_{k,k}^h \dot{F}_{l,m}^h,
\] (50)

and from the transport equation for \(J\), given in (3), we have,

\[
\frac{\partial J^h}{\partial t} = v_{r,r}^h J^h - v_r^h J^h.
\] (51)

From the constitutive relation (13) we get the following relations:

\[
\frac{\partial \sigma_{ij}}{\partial \dot{F}_{l,m}^h} = \frac{\mu}{J^h} \left( \delta_{il} \dot{F}_{j,m}^h + \delta_{ij} \dot{F}_{l,m}^h - \frac{2}{3} \delta_{ij} \dot{F}_{l,m}^h \right),
\] (52)
\[
\frac{\partial \sigma_{ij}}{\partial J^h} = \frac{1}{J^h} \left( \frac{\kappa}{J^h} \delta_{ij} - \sigma_{ij} \right).
\] (53)

Note that by substituting (50) and (51) in (49), we can express \( \frac{\partial \sigma_{ij}}{\partial t} \) in terms of the velocity field, \( v \). Thus, using the equations (48)–(53), the linear system given by (47) can be written as,

\[
K(t)v(t + \Delta t) = -\frac{1}{\Delta t} R^M(t),
\] (54)

where \( K = [K^{AB}_{ir}] \) is the approximation of the consistent tangent given by,

\[
K^{AB}_{ir} = \int_\Omega N^A_j \frac{\mu}{J^h} \left( \delta_{ij} \tilde{F}^h_{jm} + \delta_{ij} \tilde{F}^h_{im} - \frac{2}{3} \delta_{ij} \tilde{F}^h_{im} \right) \left( N^B_p \hat{F}^h_{pm} \delta_{lr} - N^B_{pr} \hat{F}^h_{lm} \right) d\Omega \\
+ \int_\Omega N^A_j \frac{1}{J^h} \left( \frac{\kappa}{J^h} \delta_{ij} - \sigma_{ij} \right) \left( N^B_r J^h - N^B_{jr} J^h \right) d\Omega.
\] (55)

The above equation is subject to the boundary conditions described in the Section 4. The linear system is then solved using an element-by-element preconditioned GMRES algorithm to a specified tolerance, \( TOLV = 10^{-12} \).

3.6. Algorithm

The solution strategy is designed to attain a steady state by solving a sequence of fictitious equilibrium steps. The parameter controlling this evolution is a pseudo time. The transient dynamics are irrelevant since the initial distributions of stress and deformation gradient may not be in equilibrium. The partitioning scheme adopted here helps express the original coupled nonlinear problem as a set of three linear coupled problems in \( v, \hat{F}, \) and \( J \), which can then be solved in a staggered fashion as we march forward in time. The algorithm is as follows:

1. At time \( t = 0 \) set \( \hat{F}(0) = 1 \) and \( J(0) = 1 \) everywhere (no deformation to start with);
2. Given \( \hat{F}(t) \) and \( J(t) \) compute \( v(t + \Delta t) \) by solving the linear system given by (54) subject to the boundary conditions in (16);
3. Given \( v(t + \Delta t) \) calculate the stabilization parameters \( \tau_\nu \) and \( \tau_J \) using (32);
4. Given \( v(t + \Delta t) \) compute \( \hat{F}(t + \Delta t) \) and \( J(t + \Delta t) \) by solving the linear systems given by (39) and (41), respectively, subject to the boundary conditions in (17) and (18);
5. If \( \| \hat{F}(t + \Delta t) - \hat{F}(t) \| < TOLF \) & \( \| J(t + \Delta t) - J(t) \| < TOLJ \) & \( \| v(t + \Delta t) - v(t) \| < TOLV \) then the stationary regime has been reached (end of computation); Else set \( t = t + \Delta t \) and go to step 2.

4. NUMERICAL EXAMPLES

We investigate the performance of the NURBS-based finite element formulation by analyzing the following benchmark examples presented in [14].

(1) an elastic material compressed in a converging channel
(2) an elastic material expanded in a diverging channel
For both channels we assume the walls are frictionless and the material fills the entire channel. We take the Young’s modulus of the material to be $E = 2.1 \times 10^6$ kg/cm$^2 \approx 206$ GPa and examine two cases of material behavior:

(A) a compressible material with a Poisson’s ratio, $\nu = 0.1$
(B) a nearly incompressible material with a Poisson’s ratio, $\nu = 0.49$

We also examine the results for two different boundary conditions:

(I) Pull-out experiment - velocity, $\mathbf{v}(x = 1m) = (1 \text{ m/s}, 0, 0)$, is prescribed at the channel outlet and the surface is unloaded at the inlet, that is, $\mathbf{F}(x = 0) = 1$, $J = 1$ and $\mathbf{\sigma}(x = 0) = 0$ (see Figures 1(a) & 1(c)).

(II) Push-in experiment - velocity, $\mathbf{v}(x = 0) = (1 \text{ m/s}, 0, 0)$, is prescribed at the channel inlet and the material is entering the channel undeformed, that is, $\mathbf{F}(x = 0) = 1$, $J = 1$ and $\mathbf{\sigma}(x = 0) = 0$ (see Figures 1(b) & 1(d)).

Figure 1: A schematic showing the boundary conditions implemented for pull out and push-in experiments for the converging and diverging channels. The material is entering the channel at the left end ($x = 0$) and exiting the channel at the right end ($x = 1$ m).
Figure 1: A schematic showing the boundary conditions implemented for pull out and push-in experiments for the converging and diverging channels. The material is entering the channel at the left end \((x = 0)\) and exiting the channel at the right end \((x = 1 \text{ m})\).

The material is entering the channel at the left end \((x = 0)\) and exiting the channel at the right end \((x = 1 \text{ m})\). Since the channel is uniform in \(z\)-direction we only show the 2D view in the \(x - y\) plane in Figure 1. The initial condition is taken to be \(F = 1\) in the entire domain, that is, no deformation at \(t = 0\). The velocity boundary condition on the side walls (\(i.e.,\) the surfaces other than the inlet and outlet) is,

\[
\int_S v \cdot n \, dS = 0
\]  

where \(S\) represents the surface described by the side walls. Now, let \(S\) be divided into \(N_s\) 2D NURBS elements and let \(N_g\) denote the number of Gauss points per element. The velocity boundary condition on the side walls is given by,

\[
\sum_{i=1}^{N_s} \int_{S_i} v \cdot n \, dS_i = \sum_{i=1}^{N_s} \sum_{j=1}^{N_g} W_j v(x_j) \cdot n(x_j) = \sum_{i=1}^{N_s} \sum_{j=1}^{N_g} \sum_{A=1}^{N_n} W_j N^A(x_j) n_p(x_j)v_p^A = 0,
\]
where $S_i$ is the surface of the $i^{th}$ element, $W_j$ is the quadrature weight and $x_j$ is the location of the $j^{th}$ Gauss point, respectively, and $N_n$ is the number of nodes per element. We replace the linear equation for the equilibrium in $y$ direction at node $A$ with the above equation to impose this boundary condition. We use the above implementation rather than directly imposing $\mathbf{v}_A \cdot \mathbf{n}_A = 0$ at the control points since the normal may not be well defined at the nodes (or control points) at the kinks in the geometry, if any.

**Remark 4.** Note that this boundary condition imposition only enforces the correct velocity and equilibrium constraints in the steady state limit, which is achieved when the solutions for the velocity and the deformation gradient stop varying in time. Alternately, one may use a penalty method or a weak form implementation for the boundary conditions on the side walls.

We now explore the performance of the scheme in simulating elastic behavior of compressible and nearly incompressible materials in a converging channel and a diverging channel using higher-order NURBS elements. In the following sections we present the results from the above computational experiments and the conclusions from the each study are italicized.

### 4.1. Higher-order NURBS elements

Let us explore the performance of linear and quadratic NURBS elements for a compressible elastic material with Poisson’s ratio, $\nu = 0.1$ in the converging channel. The boundary conditions are set for the pullout experiment (see Figure 1(a)). We examine the results from the analysis using four meshes (control nets) with hexahedral elements:

- **Mesh(i)** consists $C^0$ linear NURBS elements (same as linear Lagrange finite elements),
- **Mesh(ii)** consists $C^0$ quadratic NURBS elements (span the same discrete space as conventional quadratic Lagrange finite elements),
- **Mesh(iii) & Mesh(iv)**: $C^1$ quadratic NURBS elements.

The grid plots of the meshes in physical spaces for the converging channel are given in Figure 2. The intersection points of vertical and horizontal lines mark the physical location of the nodes or control points or DOFs (degrees of freedom). Mesh(i) has $49 \times 5 \times 2$ control points. Mesh(ii) is obtained by $p$-refining mesh(i) and has $97 \times 9 \times 3$ control points. Eventhough mesh(ii) has more control points than mesh(i), the number of integration elements ($48 \times 4 \times 1 = 192$) is the same for both. The nodes of mesh(i) and mesh(ii) in knot space are uniformly distributed in the $x$, $y$ and $z$ directions except near the kinks ($x = 0.375$ and $x = 0.625$) where mid knots are introduced to increase mesh resolution. Mesh(iii) and mesh(iv) have $C^1$ continuity in the domain interior. Mesh(iii) has $42 \times 6 \times 3$ control points and in knot space are uniformly distributed in the $x$, $y$ and $z$ directions. Mesh(iv) is obtained by $h$-refining mesh(iii) by inserting mid knots in the $x$-direction in the transition zone (between $x = 0.4$ and $x = 0.7$) of the channel. Thus, mesh(iv) has $54 \times 6 \times 3$ control points. The converging channel length is 1 m, its width in $z$-direction is 0.1 m, and its height is 0.1 m at the inlet and 0.05 m at the outlet. The diverging channel is just a vertical mirror image of the corresponding converging channel. The channel is a single patch.

**Remark 5.** The results shown herein are obtained using meshes with one quadratic element or one linear element in the $z$-direction. The boundary condition imposed is $v_z = 0$ on the planes defined by $z = 0$ and $z = 0.1$. This means that the material entering the channel inlet
cannot leave the channel from the sides and can only exit through the channel outlet. At the mid-node of the quadratic element in z-direction (i.e. nodes at $z=0.5$) we do not impose $v_z = 0$, although it turns out to be that $v_z \approx 0$ in the whole domain. Therefore, the simulations results herein essentially describe the two dimensional behavior.

![Mesh plots](image)

Figure 2: Meshes (control nets) with hexahedral NURBS-based finite elements for the converging channel. The grid plots of the meshes in physical spaces are given here. The intersection points of vertical and horizontal lines mark the physical location of the nodes or control points or DOFs (degrees of freedom). Mesh(i) has $49 \times 5 \times 2$ control points. Mesh(ii) is obtained by $p$-refining mesh(i) and has $97 \times 9 \times 3$ control points. Mesh(iii) and mesh(iv) have $C^1$ continuity in the domain interior. Mesh(iii) has $42 \times 6 \times 3$ control points and mesh(iv) has $54 \times 6 \times 3$ control points.

Recall that our aim here is to identify an accurate and stable numerical strategy for simulating the elastic behavior of a compressible material with $\nu = 0.1$. We examine the following strategies:

**$C^0$ linears for $v$, $\hat{F}$ & $J$:** This is same as using trilinear Lagrange finite elements. We examine the performance of the scheme with and without the variational multiscale (VMS) stabilization. The analysis results using meshes(i) are given in Figure 3. We see that oscillations in $F_{xx}$ between $x = 0.0$ to $x = 0.4$ (Figure 3(a)) are significantly reduced when the VMS stabilization terms are included (see Figure 3(b)). Also, the sparse GMRES solver used to solve the linear system converges in about 30 iterations whereas it converges in about 60 iterations without the VMS stabilization terms. Thus,
using VMS stabilization makes the scheme more efficient. Some oscillations still remain in the solution between $x = 0.7$ to $x = 1.0$ (see Figure 3(b)).

$C^0$ quadratics for $v$ and $C^0$ linears for $\hat{F}$ & $J$: We use mesh(ii) for interpolating $v$ and mesh(i) for interpolating $\hat{F}$ & $J$. The analysis results are given in Figure 3(c). The solution looks smooth without any observable oscillations or noise. However, the solutions from both this strategy and the previous one (see Figures 3(a)–(c)) have kinks because the geometry defined by the $C^0$ elements has kinks in the vicinity of the points $x = 0.4$ and $x = 0.7$. It is possible to avoid these kinks in the geometry using higher-order NURBS elements.

$C^1$ quadratics for $v$, $\hat{F}$ & $J$: We use the coarse mesh(iii) for interpolating $v$, $\hat{F}$ & $J$. The solution is smooth in this case without any kinks since the geometry defined by the $C^1$ elements is smooth (see Figure 3(d)). The solution from the refined mesh(iv) is also plotted in Figure 3(d). The results for $F_{xx}, F_{yy}$ and $J$ obtained from mesh(iii) and mesh(iv) are practically indistinguishable, but, the results for the Cauchy stress and pressure obtained from mesh(iv) are superior to mesh(iii), implying that the solution improves upon local refinement (not shown).

$C^1$ quadratics for $v$ and $C^0$ linears for $\hat{F}$ & $J$: We use the coarse mesh(iii) for interpolating $v$ and mesh(i) for interpolating $\hat{F}$ & $J$. The results from this case are identical to those from the above where we use $C^1$ quadratic NURBS for $v$, $\hat{F}$ & $J$. This suggests that for compressible materials an equal-order interpolation scheme $v$, $\hat{F}$ & $J$ works well.

The above study suggests that using higher-order NURBS-based finite element mesh with VMS stabilization is desirable. In the next section we explore the strategies for simulating elastic deformation for nearly incompressible materials.

![Figure 3: $F_{xx}$ component of the deformation gradient tensor for an elastic material with $\nu = 0.1$ in a converging channel.](image-url)
4.2. Nearly incompressible hyper-elasticity

Let us consider the converging channel filled with an elastic material with $\nu = 0.49$. For a nearly incompressible material the Jacobian determinant should only change slightly as the material is pulled out of the channel, that is, $J = \det[F] \approx 1.0$. We present the results for Cauchy stress component, $\sigma_{yy}$ and Cauchy pressure, $p$, using the different interpolation strategies given below:

$C^1$ quadratics for $\mathbf{v}$, $\mathbf{F}$ & $J$: In this case we implement the “$\mathbf{v} - \mathbf{F} - J$ formulation”. We use mesh(iv) (see Figure 2) for interpolating $\mathbf{v}$, $J$ & $\mathbf{F}$. The results for $J$ and $\sigma_{yy}$ for this case are given in Figure 4(a) & (b). The results show oscillations indicating the existence of an instability.

$C^1$ quadratics for $\mathbf{v}$ and $C^0$ linears for $\mathbf{F}$: In this case we implement the “$\mathbf{v} - \mathbf{F}$ formulation”. Here, we solve for the transport of $\mathbf{F}$ using (1) and then evaluate $J$ and $\mathbf{F}$ using $\mathbf{F}$. This is acceptable because the fulfillment of (1) implies fulfillment of (3) and (5). We use mesh(iv) (see Figure 2) for interpolating $\mathbf{v}$. The corresponding $C^0$ linear NURBS mesh for interpolating $\mathbf{F}$ has the same number of elements and internal knots as mesh(iv). Note that while evaluating the shape Jacobian we always use $C^1$ quadratic NURBS mesh(iv) to ensure proper mapping from the parametric space to the physical space. The results for $J$ and $\sigma_{yy}$ are given in Figure 4(c) & (d). The results still show oscillations, however, the oscillations have reduced. This is because $J$ is a nonlinear function of $\mathbf{F}$ and, using linear interpolation for $\mathbf{F}$ is not the same as using linear interpolation for $J$.

$C^1$ quadratics for $\mathbf{v}$ and $C^0$ linears for $\mathbf{\hat{F}}$ & $J$: In this case we implement the “$\mathbf{v} - \mathbf{\hat{F}} - J$ formulation”. We use mesh(iv) (see Figure 2) for interpolating $\mathbf{v}$. The corresponding $C^0$ linear NURBS mesh for interpolating $J$ & $\mathbf{\hat{F}}$ has the same number of elements and internal knots as mesh(iv). As before, the shape Jacobian is evaluated using the $C^1$
quadratic NURBS mesh(iv) to ensure proper mapping from the parametric space to the physical space. The results for $J$ and $\sigma_{yy}$ are given in Figures 2(e) & (f). The results are free of oscillations suggesting that using $C^0$ linears for $J$ ameliorates the problem of instability for near incompressible elasticity [22].

It is interesting to note that for the diverging channel, shown in Figures 2(c) and (d), all the above three computational schemes work equally well. This study suggests that restricting the interpolation space for $^\mathbf{F}$ and $J$ to be lower order than that of $v$ is critical only for the converging channel where the material is in compression, whereas not so much for the diverging channel where the material is in tension. Also, note that for many applications the material behavior is assumed compressible (i.e., $\nu = 0.1$ to 0.4) and thus, one may choose to implement the “$v - F$” formulation using $C^1$ quadratics to interpolate both the variables. However, we used the “$v - x - F - J$” formulation to compute the benchmark results for the converging and diverging channels in the next section for uniformity.

We now examine the performance of the numerical method for $\nu = 0.4999$ using both the converging and diverging channel meshes in Figure 4 for the pull-out experiment. The results for $v$, $J$ and $^\mathbf{F}$ are in general agreement with theory (see Figures 4(a) & (b)). In the limiting case of $v \rightarrow 0.5$ we can set $J = 1$ and evaluate the deviatoric part of the stress tensor as, $\sigma^{\text{dev}} = \mu \text{ dev}[^\mathbf{b}]$. Taking $F_{xx} = 2.0, F_{yy} = 0.5, F_{zz} = 1.0$ and $F_{xy} = F_{yx} = F_{xz} = F_{yz} = F_{zx} = F_{zy} = 0$, we get the theoretical value of $\sigma_{yy}^{\text{dev}} = -103000\text{MPa}$. From the numerical scheme we get $\sigma_{yy}^{\text{dev}} = -103100\text{MPa}$ (see Figure 4(c)). Let us now examine the results for $J$ for Poisson’s ratio, $\nu = 0.49, 0.499, 0.4999$. Recalling that the bulk modulus, $\kappa = \frac{E}{(1-2\nu)}$, the material should therefore be compressible on the order of $(1 - 2\nu)$. This fact is evident from Figure 5 where $|J(x = 0) - J(x = 1)| \sim (1 - 2\nu).$ This study suggests that the mixed interpolation formulation is able to model near incompressibility.

Remark 6. For $\nu = 0.4999$ the tolerances for the linear solve for $v$, $^\mathbf{F}$, and $J$ need to be much smaller ($\text{TOLV} = \text{TOLF} = \text{TOLJ} = 10^{-16}$) in order to obtain accurate results. In this case, a large number of GMRES iterations is required and a better preconditioner would decrease the computational expense. Alternatively, one may use a direct linear solver and iterate the residual to the convergence tolerance with direct linear solves at each iteration. For difficult linear problems one direct solve may not be enough and so even linear problems may need to be treated as if they are nonlinear.

4.3. Converging channel

Here we shall summarize the results for the converging channel for the pull-out experiment for the two extreme values of Poisson’s ratio. We use mesh(iv) with hexahedral elements using $C^1$ quadratic NURBS basis functions. All the results presented here are for the stationary regime, that is, the system has reached equilibrium.

- **Material with $\nu = 0.1$:** The results from our simulations are given in Figure 6.
- **Material with $\nu = 0.49$:** The results from our simulations are given in Figure 7.

We see that for both values of Poisson’s ratio:

- $F_{yy} \rightarrow 0.5$ at the channel outlet (theoretical result from geometrical constraint)
Figure 4: Nearly incompressible elastic material in a converging channel, $\nu = 0.49$. The units for stress component $\sigma_{yy}$ are MPa.
Figure 4: Nearly incompressible elastic material in a converging channel, $\nu = 0.49$. The units for stress component $\sigma_{yy}$ are MPa.
Figure 5: Near incompressibility, $\nu = 0.4999$, using $C^1$ quadratics for $v$ and $C^0$ linears for $J$ & $\mathbf{F}$.
Figure 5: Near incompressibility, $\nu = 0.4999$, using $C^1$ quadratics for $\mathbf{v}$ and $C^0$ linears for $J$ and $\dot{F}$.

Figure 6: Jacobian determinant, $J$, along the line joining (0,0,0) and (1,0,0). Note that $|J(x = 0) - J(x = 1)| \sim (1 - 2\nu)$. 

(c) Deviatoric part of the Cauchy stress component, $\sigma^{\text{dev}}_{yy}$ in MPa
(a) Data along the line joining (0,0,0) and (1,0,0)

(b) Cauchy stress component, $\sigma_{yy}$ in MPa

Figure 7: Converging Channel, $\nu = 0.1$, using $C^1$ quadratics for $\mathbf{v}$ and $C^0$ linears for $J$ & $\tilde{F}$
(c) Cauchy pressure, \( p = \frac{\sigma_{kk}}{3} \) in MPa

(d) Horizontal component of velocity, \( v_x \), in m/s

Figure 7: Converging Channel, \( \nu = 0.1 \), using \( C^1 \) quadratics for \( \mathbf{v} \) and \( C^0 \) linears for \( J \) & \( \mathbf{\hat{F}} \). It is evident from sub-figure (d) that there are zones of extension at points \((x, y) = (0.4, 1.0) \) & \((0.7, 0.0) \) and compression at points \((x, y) = (0.4, 0.0) \) & \((0.7, 0.5) \). We believe a bending-like behavior of the material in the zones of higher curvature of the geometry explains the bumps in the data plots of \( F_{xx} \) and \( F_{yy} \).

- The stresses \( \sigma_{xx} = 0 \) and \( \sigma_{yy} = 0 \) at the channel inlet

Thus, our results are in good agreement with theory. Also, our results for the deformation gradient, \( \mathbf{F} \), are identical for the pull-out and push-in experiments, but, the results for the velocity, \( \mathbf{v} \), differ. Hence, only the plots for the pull-out experiment are presented here. In Figure 7(a) we find extrema in the plots of \( F_{xx} \) and \( F_{yy} \) at \( x = 0.4 \) and \( x = 0.7 \), that are the points of high geometric curvature. The value of \( F_{xx} = 1.14 \) at \( x = 0.7 \) and then drops to 1.09 at \( x = 0.8 \). It is evident from Figure 7(d) that there are zones of extension in the vicinity of the points \((x, y) = (0.4, 1.0) \) & \((0.7, 0.0) \) and zones of compression in the vicinity of the points \((x, y) = (0.4, 0.0) \) & \((0.7, 0.5) \). These zones of extension and compression and the extrema in \( F_{xx} \) and \( F_{yy} \) suggest a bending-like behavior of the material in the zones of higher curvature.
(a) Data along the line joining (0,0,0) and (1,0,0)

(b) Cauchy stress component, $\sigma_{yy}$ in MPa

Figure 8: Converging Channel, $\nu = 0.49$, using $C^1$ quadratics for $\mathbf{v}$ and $C^0$ linears for $J$ & $\tilde{F}$
4.4. Diverging channel

In this section we shall present the results for the diverging channel for the pull-out experiment for the two extreme values of Poisson’s ratio. We use mesh(iv) with hexahedral elements using $C^1$ quadratic NURBS basis functions. Again, all the results presented here are for the stationary regime.

- **Material with** $\nu = 0.1$: The results from our simulations are given in Figure 9.
- **Material with** $\nu = 0.49$: The results from our simulations are given in Figure 10.

We see that for both values of Poisson’s ratio:

- $F_{yy} \to 2.0$ at the channel outlet (theoretical result from geometrical constraint)
- The stresses $\sigma_{xx} = 0$ and $\sigma_{yy} = 0$ at the channel inlet

*Thus, our results are in good agreement with theory.* Again, our results for the deformation gradient, $F$, are identical for the pull-out and push-in experiments, but, the results for the velocity, $v$, differ. Hence, only the plots for the pull-out experiment are presented here.

4.5. Convergence and stability

The results obtained from the mixed interpolation strategy (see Sections 4.1 and 4.2) in conjunction with the VMS stabilization are free of oscillations and thus, illustrate the stability of this discretization scheme. The scheme is also able to deal well with near incompressibility for both the converging and the diverging channels. The pressure and stress distributions are free of oscillations for $\nu = 0.1$ and 0.49. For the limiting case of $\nu = 0.4999$ the solutions for $v$ and $\tilde{F}$ are free of oscillations and the numerical result for the deviatoric part of the Cauchy stress, $\sigma^{\text{dev}}$, is in good agreement with the theoretical result.
(a) Data along the line joining \((0,0,0)\) and \((1,0,0)\)

(b) Cauchy stress component, \(\sigma_{yy}\) in MPa

Figure 9: Diverging Channel, \(\nu = 0.1\), using \(C^1\) quadratics for \(v\) and \(C^0\) linears for \(J\) & \(\hat{F}\)
The results from the coarse mesh(iii) and the refined mesh(iv) (see Section 4.1) show converging behavior. Next, we tried different time step sizes, \( \Delta t = 10^{-3}, 10^{-2}, 10^{-1} \) seconds, and the differences in the results (compared with one another) are less than 0.5 %. The scheme does not seem to be sensitive to time step size and gives consistent results at steady state for any finite value of \( \Delta t \).

5. CONCLUSIONS

We have developed an Eulerian formulation for large deformation hyper-elasticity to address the shortcomings of the standard Lagrangian formulations in the presence of very large deformations. This approach allows us to model large deformations on a fixed mesh with the material moving relative to the mesh. In this Eulerian description, the deformation gradient is transported to recover the deformation history at each point of the domain. For near incompressible applications (\( \nu \rightarrow 0.5 \)), the deformation gradient is decomposed into its isochoric part, \( \hat{F} \), and the Jacobian determinant, \( J \), and then transported using a different order of interpolation than that used for the velocity field. This strategy yields a very robust and accurate method. The use of NURBS based isogeometric analysis allows us to avoid kinks in the geometry and helps us model smoother geometries. The variational multiscale formulation for the transport equations improves accuracy and stability, and accelerates the convergence of the computational scheme. The partitioning scheme adopted enables the original coupled fully nonlinear problem to be written as a set of linear coupled problems in \( \mathbf{v} \& \hat{F} \), or, \( \mathbf{v}, \hat{F} \& \mathbf{J} \), which can then be solved in a staggered fashion as we march forward in time. The formulation is accurate, stable and converges for compressible and nearly incompressible cases, and for different channel geometries and boundary conditions.
Figure 10: Diverging Channel, $\nu = 0.49$, using $C^1$ quadratics for $\mathbf{v}$ and $C^0$ linears for $J$ & $\mathbf{F}$.
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(c) Cauchy pressure, \( p = \frac{\sigma_{kk}}{3} \) in MPa

Figure 10: Diverging, \( \nu = 0.49 \), using \( C^1 \) quadratics for \( \mathbf{v} \) and \( C^0 \) linears for \( J \) & \( \mathbf{F} \)

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