Goal-Oriented Error Estimation and Adaptivity for Fluid–Structure Interaction using Exact Linearized Adjoint

by

K.G. van der Zee, E.H. van Brummelen, I. Akkerman, R. de Borst
Abstract

We develop duality-based a-posteriori error estimates for functional outputs of solutions of fluid–structure-interaction problems. The crucial complication in obtaining these estimates pertains to the derivation of the coupled dual (exact linearized-adjoint) problem owing to the free-boundary character of fluid–structure interaction. We present two approaches to derive the dual problem. In the domain-map linearization approach, the fluid subproblem is first transformed to a fixed reference domain, after which one essentially linearizes with respect to the domain transformation map. In the shape-linearization approach, fluid unknowns are fixed in the current configuration and a very weak formulation of the fluid subproblem is then linearized using shape-derivative techniques. We show that the dual problems correspond to coupled fluid–structure problems with nonstandard coupling conditions. Furthermore, we present numerical experiments that demonstrate the consistency of the dual-based error estimates and their usefulness in goal-oriented adaptive mesh-refinement.

Key words: fluid–structure interaction, goal-oriented error estimation, a posteriori error estimation, duality, exact linearized adjoint, domain-map linearization, shape linearization, shape derivative, adaptive discretization, adaptive mesh-refinement

1. Introduction

Fluid–structure-interaction problems constitute multiphysics, free-boundary problems. The interface at which the fluid and the structure interact acts as a free-boundary,
i.e., its position is part of the solution and therefore unknown a priori. This free-boundary character causes fluid–structure-interaction problems to be inherently nonlinear through the geometry of the fluid domain, and causes the numerical simulation of fluid–structure interaction phenomena to be a challenging endeavor.

Classically, the main motivation for the numerical simulation of fluid–structure interaction has been the prediction of aeroelastic phenomena in aerospace and civil engineering applications. Since the 1990s, many computational techniques have been developed and analyzed; see the works by Farhat et al. [35–37, 80], Le Tallec and Mouro [67], and the overviews in [33, 96, 97, 108]. As of the late 1990s, another important motivation for the numerical simulation of fluid–structure interaction is hemodynamics (vascular fluid dynamics or blood flow) in biomechanical applications; see the early works in [75, 86, 107], and recent applications in [2, 10, 47]. For these problems, a particular difficulty is the derivation of efficient iteration schemes to solve the nonlinear coupled system. The popular loosely-coupled schemes (also called explicit schemes) used in aeroelastic applications are generally unstable for biomechanical application owing to a significant so-called fluid added-mass effect; see Causin, Gerbeau and Nobile [17] and also [45, 98]. The design of efficient coupling schemes in this context is still ongoing research, see e.g. [9, 15, 39, 42, 62, 99, 100, 106]. Nevertheless, we are currently at a time where robust solvers can be devised for most fluid–structure-interaction problems allowing, in principle, the computation of solutions to many complex practical problems. These computations need, however, not be based on optimal discretizations.

1.1. Goal-oriented adaptive discretization

Numerical procedures for practical fluid–structure interaction problems require substantial computational effort. Typically, most of the computational resources are consumed by the fluid subsystem. Often, however, practical interest is restricted to a prescribed response quantity of the structure subsystem rather than full resolution of the complete coupled problem. In fact, one is often not so much interested in the solution itself, but uses the computed solution to quantify a certain goal. Example goals in aeroelastic computations are the structure displacement at specific points, global forces acting on the structure such as the lift or drag or the energy that is transferred from the fluid to the structure; see for instance [34, 37, 80]. Examples in biomechanical applications are the wall shear stress in the vicinity of aneurysms [88] or the volumetric flow rates at arterial cross sections [95].

Goal quantities of interest can usually be written in the form of a functional of the solution, the so-called goal functional, output functional or target functional. In computations, the accuracy of the goal quantity depends on both the accuracy of the fluid approximation and the accuracy of the structure approximation. In general, this dependence is non-obvious and any heuristic approach to constructing the meshes underlying the approximation spaces leads to an inefficient approximation of the goal quantity.

Finite-element discretization techniques employing goal-oriented adaptive strategies can offer a significant efficiency improvement in such simulations. Starting with a coarse discretization, only those refinements are made which substantially benefit to the accuracy of the goal quantity, in contrast to standard, norm-oriented, adaptive strategies (see [1, 6, 105]) which make refinements that benefit the accuracy of the solution in the full norm. Goal-oriented error estimation and goal-oriented adaptive methods have been developed in the late 1990s and have mostly been applied to linear and nonlinear problems in solid and fluid mechanics. Pioneering work in this field
has been performed by Eriksson, Estep, Hansbo and Johnson [28], Becker and Ran-nacher [11], Giles and Süli [49], Süli and Houston [94], Prudhomme and Oden [85], Patera and Peraire [79], Stein and Rüter [92], and Hoffman and Johnson [58]. An important recent development in this context is the combined discretization–modeling error estimation and adaptivity [14, 76–78, 92]. Very recent applications of goal-oriented adaptive methods include multiphysics problems involving multiple coupled boundary value problems; see Larson et al. [63–66], Estep et al. [16, 29–31], Fick [43] and Prudhomme et al. [84].

If applied appropriately to fluid–structure interaction problems, goal-oriented adaptive strategies result in an optimal discretization of both the fluid and structure variables for the goal functional under consideration, each with appropriate resolution. We would like to point out that from the viewpoint of the fluid (or structure) subsystem, a coupled adaptive strategy can be conceived of as a combined discretization–model adaptive procedure, where the structure (or fluid) model is refined in conjunction with the refinement of the fluid (or structure) discretization.

Goal-oriented adaptive strategies rely on local refinement indicators to guide the adaptive procedure. These refinement indicators are obtained from duality-based a posteriori error estimates for the goal functional of interest. To compute these goal-oriented error estimates, one requires the solution of a dual problem. For nonlinear problems, this dual problem is based on the linearized-adjoint operator. The corresponding dual solution, also called influence, extraction or generalized Green’s function, indicates the precise spatial influence of the residual functional on the error in the goal. That is, it provides a residual weighting that yields an estimate of the error in the goal.

Although, in principle, the goal-oriented-error estimation framework applies immediately to all (non)linear problems that can be cast in canonical variational form, fluid–structure-interaction problems elude this standard framework on account of their free-boundary character. Indeed, the usual variational form of fluid–structure-interaction problems is based on domain-dependent spaces, but the domain itself constitutes an unknown. It is exactly this unusual domain-dependent nonlinearity that renders the derivation of the linearized-adjoint operator of fluid–structure interaction problems highly nontrivial.

Recently, several approaches have been suggested to bypass the derivation of the linearized adjoint. The first approach is by reformulating the fluid–structure interaction problem a priori such that the domain-dependence vanishes. This has been pursued by Dunne [25, 26] who rewrites the structure to a Eulerian framework. The disadvantage being that one has to capture the interface afterwards. The second approach, by Grätsch and Bathe [54], is to replace the linearized adjoint with a finite difference approximation. In this approach one, of course, has the problem of choosing a sufficiently small difference, but not too small to affect conditioning. The third approach, which to the authors knowledge hasn’t been applied to fluid–structure interaction problems, is to compute the dual solution by a specific (infinite, defect-correction) iteration scheme between the subsystems, the limit of this iteration resulting in the solution of the (linearized) adjoint problem, cf. [31, 63, 64].

1.2. Exact linearized adjoint of fluid–structure interaction

In this paper, we do not bypass the derivation of the linearized-adjoint, but present rigorous derivations of exact linearized adjoints for the coupled fluid–structure problem. This allows the direct application of the established goal-oriented-error estimation framework. In particular, we obtain goal-oriented error estimates in the form of
a residual functional of the coupled problem weighted by the solution of suitable dual (linearized-adjoint) problems. We note that the current work is an extension of techniques introduced in our former work \[102, 103\], where we studied goal-oriented error estimation for a model free-boundary problem.

To derive the linearized-adjoint operator of fluid–structure interaction problems, we need to linearize with respect to the domain geometry. The scientific field that deals with this type of linearization is called \textit{shape} \textit{(differential) calculus}, and the corresponding derivatives are called \textit{shape derivatives}. Shape calculus has mostly been developed in the context of shape optimization by Simon \[89\], Pironneau \[81, 82\], and Zolésio \[111\]. Comprehensive treatments can be found in the books of Sokolowski and Zolésio \[90\], Haslinger and Neittaanmäki \[56\], Delfour and Zolésio \[22\], and Moubachir and Zolésio \[74\]. In the context of shape optimization, there are two notions of derivatives with respect to domain perturbations. The extension of these notions to a suitable linearization technique for fluid–structure interaction problems is not immediate.

One notion is the so-called \textit{material derivative}, and we refer to the associated linearization technique as \textit{domain-map linearization}. The idea is to use a suitable domain map (in discrete settings known as the ALE (arbitrary Langrangian–Eulerian) map \[9, 23\]) to transform the fluid–structure problem into an equivalent problem on a fixed reference domain. The variational formulation of the transformed problem is then in canonical form, although it contains intricate terms involving the domain map. One can then linearize the transformed problem with respect to the domain map. We shall show that the resulting domain-map linearized dual problem corresponds to a dual fluid subproblem coupled to a dual structure-displacement subproblem. These subproblems have a kinematically straightforward coupling, however, the complementary coupling condition depends in a nonstandard and \textit{nonlocal} manner on the dual fluid variables. This nonlocality is reminiscent of material derivatives in shape optimization \[56, 90\]. We note that the domain-map linearization technique has been introduced for goal-oriented error estimation of fluid–structure interaction in \[104\], in a formal setting though.

An elegant alternative notion in shape optimization is the so-called \textit{shape derivative}. We shall refer to the associated linearization technique as \textit{shape linearization}. Here the idea is to linearize with respect to the geometry by holding the fluid unknowns fixed in the current configuration assuming they have smooth extensions outside their domain. To perform this shape linearization, we need to remove the essential boundary conditions in the fluid velocity test and trial spaces. This is achieved by introducing a very weak form of the fluid subproblem. A linearization of this very weak form then allows the extraction of a suitable dual problem. We shall show that this dual problem corresponds to the same dual fluid subproblem coupled to the dual structure-displacement subproblem with the same kinematically straightforward coupling condition. The other coupling condition, however, although nonstandard, is \textit{local}.

Domain-map and shape-linearization approaches have recently been investigated for Newton-type iterative solution algorithms for free-boundary problems. In the context of fluid–structure-interaction problems, only the domain-map linearization approach has been used; see the works of Fernández et al. \[38, 42\] and Bazilevs et al. \[8, 9\] and also \[71\]. The shape-linearization approach, on the other hand, has only been investigated for simple free-boundary problems (in a formal sense); see Kärkkäinen and Tiihonen \[59–61\].

We note that the presented shape-linearization approach bears similarities to the so-called shape derivative used in shape optimization of fluid problems; see \[12, 13, 74\].
In dynamic settings, such derivatives are investigated in [27, 71, 74]. Furthermore, the shape-linearization approach is related to formal model linearizations of fluid–structure interaction giving rise to the so-called transpiration boundary condition; see [32, 40, 41].

The contents of this paper are arranged as follows. Sec. 2 introduces the fluid–structure-interaction model. We briefly review the goal-oriented error estimation framework in Sec. 3 and discuss the difficulties that are encountered when applying it to fluid–structure interaction. We then present the two approaches to deriving a suitable dual problem. In Sec. 4 we consider the domain-map linearization approach, and in Sec. 5 we consider the shape-linearization approach. Numerical experiments are presented in Sec. 6. Finally, Sec. 7 contains concluding remarks.

2. Problem statement

In principle, our interest concerns incompressible fluids interacting with deformable solids. However, since our focus will be on the free-boundary character of the fluid subproblem, we shall consider fluids governed by the Stokes equations and simple lower-dimensional structure models for which the extension to deformable solids is not too involved. To fix ideas, we shall describe a steady problem in two dimensions, although the extension to three dimensions is straightforward. As a one-dimensional structure model we consider the vector-string equation; see for example [109]. String models have the computational advantage of being low-order models. Moreover, we prefer a vector model over a scalar model, since in this case the traction coupling condition applies to the full fluid-traction vector. This will simplify the ensuing analysis considerably.

To describe the functional setting of our problem we shall employ standard notation for function spaces. In particular, for \( \omega \subseteq \mathbb{R}^2 \) we denote by \( L^2(\omega) \) and \( H^1(\omega) \) the (Hilbertian) Sobolev space of functions on \( \omega \) for which their values, and their values and derivatives, respectively, are square integrable. \( H^1_0(\omega) \subseteq H^1(\omega) \) is the subspace of functions with zero trace on \( \gamma = \partial \omega \). We set \( H^1_0(\omega) = H^1_{0,\partial \omega}(\omega) \). Furthermore, \( C^{0,1}(\overline{\omega}) \) denotes the (Banach) Hölder space of bounded Lipschitz continuous functions on \( \omega \).

2.1. Domain maps

We consider a fluid–structure system such as graphically depicted in Fig. 1. Let \( \Gamma_0 \) denote the reference configuration of the string. This also corresponds to the reference fluid–structure interface. Consider a (vector) string displacement \( \theta : \Gamma_0 \to \mathbb{R}^2 \) that is sufficiently smooth, i.e., \( \theta \in C^{0,1}(\Gamma_0) \cap H^1(\Gamma_0) \). To each admissible \( \theta \), we can associate the current fluid–structure interface \( \Gamma_\theta \) and the open bounded fluid domain \( \Omega_\theta \subseteq \mathbb{R}^2 \). In general, the boundary of the fluid domain, \( \partial \Omega_\theta \), also consists of in- and outflow boundaries \( \Gamma_{\text{in/out}} \) and wall boundaries \( \Gamma_{\text{wall}} \).

We next introduce for each \( \theta \), a domain map \( T_\theta \) that shall act between reference and current fluid–structure configurations. For this, we note that there exists a Lipschitz extension of \( \theta \) from \( \Gamma_0 \) onto \( \Omega_0 \) such that it is zero on the other boundaries of \( \Omega_0 \). We shall denote this extension also by \( \theta \) and note that it is a member of the space

\[
\Theta := \{ \theta = \text{Ext}_{\Gamma_0} \eta \in C^{0,1}(\overline{\Omega_0}) : \eta \in C^{0,1}(\overline{\Gamma_0}) : \theta = 0 \text{ on } \partial \Omega_0 \setminus \Gamma_0 \},
\]

2For vector functions \( \phi : \omega \to \mathbb{R}^2 \), we write, for example, \( \phi \in L^2(\omega)^2 \) or simply \( \phi \in L^2(\omega) \) when no confusion is possible, meaning that each component \( \phi_j (j = 1, 2) \) is in \( L^2(\omega) \).
Figure 1: Geometric set-up of a fluid–structure-interaction problem: fluid domain $\Omega_0$ and current fluid–structure interface $\Gamma_\theta$ for structure displacement $\theta$, the reference interface $\Gamma_0$, and in- and outflow boundaries $\Gamma_\text{in}$ and $\Gamma_\text{out}$. The thick lines correspond to the wall boundaries $\Gamma_\text{wall}$.

with $\text{Ext}_{\Gamma_0}$ a given (sufficiently smooth) extension operator. Accordingly, we introduce the domain map

$$T_\theta = \text{Id} + \theta : \Omega_0 \to T_\theta(\Omega_0) = \Omega_\theta,$$

where $\text{Id} : \Omega_0 \to \Omega_0$ is an identity map. Notice that $T_\theta$ extends the (deformation) map $T_\theta|_{\Gamma_0}$ that takes the material points of the string in the reference configuration $\Gamma_0$ to the current configuration $\Gamma_\theta = T_\theta(\Gamma_0)$.

For later usage, we note that for admissible $\theta$, $T_\theta$ is a $C^{0,1}$-diffeomorphism and the function transportation map $H^1(\Omega_0) \ni \phi_0 \mapsto \phi_0 \circ T_\theta^{-1} \in H^1(\Omega_\theta)$ is a linear bijection; see [22, p. 406] or [55, p. 21].

2.2. Fluid problem statement

Within the domain $\Omega_\theta$, we consider the Stokes problem for the velocity $u : \Omega_\theta \to \mathbb{R}^2$ and pressure $p : \Omega_\theta \to \mathbb{R}$:

\[
\begin{align*}
-\nu \Delta u + \nabla p &= f & \text{in } \Omega_\theta, \\
-\text{div} u &= 0 & \text{in } \Omega_\theta, \\
u \frac{\partial u}{\partial n} + p n &= 0 & \text{on } \Gamma_\text{out},
\end{align*}
\]  

(1a)  

(1b)

where $\nu > 0$ is the fluid kinematic viscosity and $f$ is a sufficiently smooth body force. We shall require derivatives of $f$ later on, hence, for the sake of simplicity we shall assume $f \in H^1(\mathbb{R}^2)$. The first and second equation in (1a) are the momentum and continuity equation, respectively. Eq. (1b) is the kinematic coupling condition which in this steady case imposes a no-slip condition. On the other boundaries, suitable boundary conditions hold. A typical choice specifies no-slip on $\Gamma_\text{wall}$, nonzero inflow on $\Gamma_\text{in}$ and zero traction on $\Gamma_\text{out}$, i.e.,

\[
\begin{align*}
u \frac{\partial u}{\partial n} + p n &= 0 & \text{on } \Gamma_\text{out},
\end{align*}
\]

where $h_\text{in}$ is a given inflow velocity. However, for the sake of presentation, we shall assume that only homogeneous Dirichlet and Neumann (traction) boundary conditions are imposed on the other boundaries. Let us denote these other boundaries collectively as $\Gamma_\not\theta = (\Gamma_\text{D} \cup \Gamma_\text{N})$ if Dirichlet (Neumann) boundary conditions are imposed on them. Note that

$$\partial \Omega_\theta = \Gamma_\theta \cup \Gamma_\not\theta.$$

6
We stipulate that both $\Gamma_D$ and $\Gamma_N$ are nonempty.

2.3. Structure problem statement

The structure subproblem for a string model in a fluid–structure interaction problem is simply the traction coupling condition, where the fluid traction at the interface drives the string equation. That is, the string (vector) displacement $\theta : \Gamma_0 \rightarrow \mathbb{R}^2$ satisfies the (vector) equation

$$-E \partial_s^2 \theta = g + \left( -\nu \partial_n u + p n \right) \circ T_0 \partial_s T_0$$

in $\Gamma_0$, \hspace{1cm} (1c)

where the constant $E$ is the string’s Young modulus, $\partial_s (\cdot) := \partial(\cdot)/\partial s$ is the derivative along $\Gamma_0$, and $g : \Gamma_0 \rightarrow \mathbb{R}^2$ is a sufficiently smooth external (referential) force on the string. Note that in (1c), the fluid traction is mapped to the reference configuration by means of $T_0$. The function $|\partial_s T_0| : \Gamma_0 \rightarrow \mathbb{R}$ is referred to as a tangential Jacobian (or surface Jacobian) and corrects the fluid traction for its evaluation in the reference configuration. For a horizontal $\Gamma_0$ such as depicted in Fig. 1, we have

$$|\partial_s T_0| = \sqrt{(1 + \partial_s \theta_1)^2 + (\partial_s \theta_2)^2}.$$

Suitable boundary conditions need to be imposed on the string. We shall employ homogeneous Dirichlet boundary conditions, i.e.,

$$\theta = 0 \quad \text{on } \partial \Gamma_0.$$

The presented vector-string model corresponds to the constitutive behavior of a so-called perfectly elastic material, in which tension is equal to $E |\partial_s T_0|$ (rubber-like behavior); see [109] for a concise derivation.\footnote{We note that the derivation in [109] is not based on a small displacement assumption.}

2.4. Coupled weak formulation with weakly-enforced tractions

A preliminary weak formulation of the coupled problem can be given by employing standard weak formulations of the subproblems: Find $\theta \in H^1_0(\Gamma_0)$ and $(u, p) \in H^1_{0,\Gamma_D \cup \Gamma_N}(\Omega_\theta) \times L^2(\Omega_\theta)$ such that

$$\int_{\Omega_\theta} \left( \nu \nabla u \cdot \nabla v - p \ \text{div} \ v \right) = \int_{\Omega_\theta} f \cdot v \quad \forall v \in H^1_{0,\Gamma_D \cup \Gamma_N}(\Omega_\theta), \hspace{1cm} (2a)$$

$$\int_{\Omega_\theta} -q \ \text{div} \ u = 0 \quad \forall q \in L^2(\Omega_\theta), \hspace{1cm} (2b)$$

$$\int_{\Gamma_0} E \partial_n \theta \cdot \partial_n \eta = \int_{\Gamma_0} \left( g + ( -\nu \partial_n u + p n ) \circ T_0 \partial_s T_0 \right) \cdot \eta \quad \forall \eta \in H^1_0(\Gamma_0), \hspace{1cm} (2c)$$

where $\nabla u \cdot \nabla v := \sum_{i=1,2} \nabla u_i \cdot \nabla v_i$. The first two equations (2a)–(2b) have been obtained by employing the standard weak formulation for the fluid problem (1a)–(1b) with strongly enforced Dirichlet boundary conditions. The third equation (2c) has been obtained by employing a straightforward weak formulation for the elliptic problem in (1c), leaving the right-hand side as is.

A fundamental shortcoming of the coupled weak formulation (2) is that the fluid traction appearing in the last equation is not necessarily defined for arbitrary velocities...
and pressures in the a priori indicated spaces. To address this issue, the associated term is rewritten in a consistent manner. Assuming sufficient smoothness, note that we can transform the integral to $\Gamma_0$ by means of $T^{-1}_\theta$, i.e.,

$$\int_{\Gamma_0} \left( (-\nu \partial_n u + p) \cdot T \partial_s T \right) \cdot \eta = \int_{\Gamma_0} \left( (-\nu \partial_n u + p) \cdot (\eta \circ T^{-1}_\theta) \right).$$

The main idea is then to use a Green’s identity to replace this interface integral by domain integrals and subsequently invoke the momentum equation in (1a). To make sense of $\eta \circ T^{-1}_\theta$ in $\Omega_\theta$, we introduce the operator $E_{\theta}$:

$$\eta \mapsto E_{\theta} \eta := (\text{Ext}_{\Gamma_0} \eta) \circ T^{-1}_\theta : H^1_0(\Gamma_0) \rightarrow H^1_{0,\Gamma_\theta}(\Omega_\theta).$$

This final expression is bounded for $f, p \in L^2(\Omega_\theta)$ and $u, E_{\theta} \eta \in H^1(\Omega_\theta)$. Note that this expression is equal to the residual form of the momentum equation, see (2a), with $E_{\theta} \eta$ substituted for $v$. The residual evaluated at $E_{\theta} \eta$ is, in general, nonzero since $E_{\theta} \eta \in H^1_{0,\Gamma_\theta}(\Omega_\theta) \supset H^1_{0,\Gamma_\theta}(\Omega_\theta)$.

To summarize the final coupled weak formulation, let us introduce the following semilinear functionals. Let $R_m$ and $R_c$ denote the momentum- and continuity-equation residual functionals:

$$R_m(\Omega, (u, p); v) := \int_{\Omega} \left( f \cdot v - \nu \nabla u \cdot \nabla v + p \ \text{div} \ v \right),$$

$$R_c(\Omega, u; q) := \int_{\Omega} q \ \text{div} \ u.$$

We then define a semilinear form $N$ for the total fluid problem, i.e.,

$$N(\Omega, (u, p); (v, q)) := -R_m(\Omega, (u, p); v) - R_c(\Omega, u; q).$$

By $S$ and $G$, we denote the bilinear form associated with the structure operator and the linear functional associated with external structure forces:

$$S(\theta, \eta) := \int_{\Gamma_0} E \partial_s \theta \cdot \partial_s \eta,$$

$$G(\eta) := \int_{\Gamma_0} g \cdot \eta.$$

---

4The fluid traction term should actually be understood as a duality pairing. The consistent rewriting provides an interpretation of this pairing.
The coupled weak formulation can then be condensed into the following problem:

\[
\begin{align*}
\text{Find } & \theta \in H^1_0(\Gamma_0) \quad \text{and} \quad (u, p) \in H^1_0(\Gamma_0, \mathbb{F}) \times L^2(\Omega_0) : \\
& N(\Omega_0, (u, p); (v, q)) = 0 , \\
& S(\theta, \eta) - \mathcal{R}^m(\Omega_0, (u, p); E_0 \eta) = \mathcal{G}(\eta) , \\
& \forall (\eta, v, q) \in H^1_0(\Gamma_0) \times H^1_0(\Gamma_0, \mathbb{F}) \times L^2(\Omega_0) .
\end{align*}
\]

In principle, many other steady fluid–structure-interaction problems fit into the above abstract form (with suitable spaces substituted), such as three dimensional extensions as well as more complicated structure models.

The abstract form in (4) highlights the connection of the fluid subproblem with free-boundary problems. Indeed, the dependence of the fluid part on the structure displacement \( \theta \) is entirely via the underlying fluid domain \( \Omega_0 \). This dependence manifests itself through the first argument of \( N \) as well as the function spaces upon which \( N(\Omega_0, \cdot, \cdot); (\cdot, \cdot) \) acts. This corresponds exactly to free-boundary problems; see e.g. [103, 110].

**Remark 2.1** The reformulation of the traction term is a well-known procedure applied in finite-element discretizations of fluid–structure interaction, see for instance [9, 39, 48, 67]. In such discrete settings, the fluid traction is essentially enforced in a weak sense. This procedure is also related to the post-processing (or extraction) approach of outputs (such as stresses and fluxes) in the finite-element method as investigated in the early works of Babuška and Miller [3–5]; see also [6, p. 726] and [49, 72].

**Remark 2.2** The existence theory for a coupled solution \((\theta, u, p)\) of the fluid–structure-interaction problem (4) is highly nontrivial, even with the introduced weak traction term. One of the issues of the current setting is that structure displacements can be rather nonsmooth, i.e., arbitrary in \( H^1_0(\Gamma_0) \). These displacements can give rise to kinked fluid domains or, even worse, cusped fluid domains, which are non-Lipschitz domains. In current existence theories, it is, however, preferred to work with Lipschitz domains. Therefore, structure displacements are a priori assumed to be smoother, for example, members of the non-integer Sobolev space \( H^{3/2+\epsilon}(\Gamma_0) \cap H^1_0(\Gamma_0) \) (with \( \epsilon > 0 \) small), see [7, 50, 51] or the non-Hilbertian Sobolev space \( W^{1,\infty}(\Gamma_0) \cap H^1_0(\Gamma_0) \), see [87]. Various extensions of these works have been studied. For example, an extension to three dimensions and (non)linear solids can be found in [52]. Extensions to unsteady fluid–structure interaction with various structural models are given in [18–20, 53] and references therein.

In the sequel, we assume the existence of an admissible displacement \( \theta \in C^{0,1}(\bar{\Gamma}_0) \cap H^1_0(\Gamma_0) \), leading to a Lipschitz fluid domain \( \Omega_0 \), and a corresponding velocity and pressure pair \((u, p)\) \( \in H^1_0(\Gamma_0, \mathbb{F}) \times L^2(\Omega_0) \) that solve (4).

### 2.5. Goal quantities and discretization errors

Our interest will be specific (bounded and continuous differentiable) goal functionals \( Q : (\theta, u, p) \in H^1_0(\Gamma_0) \times H^1_0(\Gamma_0, \mathbb{F}) \times L^2(\Omega_0) \rightarrow \mathbb{R} \) of the solution. For simplicity, we will restrict the ensuing analysis to functionals

\[
Q(\theta, u, p) = \int_{\Gamma_0} q^\text{disp} \cdot \theta + \int_{\Omega_0} (q^\text{vel} \cdot u + q^\text{press} p) . \tag{5}
\]
where \( q^{\text{disp}} \in L^2(\Gamma_0)^2 \) and \( q^{\text{pres}} \) are sufficiently smooth. As we shall require derivatives later on, we assume for simplicity \( q^{\text{disp}} \in H^1(\mathbb{R}^2)^2 \) and \( q^{\text{pres}} \in H^1(\mathbb{R}^2) \). We note that other bounded functionals are also possible.

Numerical approximations of the fluid–structure interaction problem can be obtained by employing a suitable discretization method. We shall focus on conforming finite-element discretizations that directly employ the weak formulation (4) using suitable discrete spaces. Let us assume that the approximate structure displacement \( \theta_h \) is in some approximation space \( V_h \subset H^1_0(\Gamma_0) \). To each \( \theta_h \) one can associate the approximate fluid domain \( \Omega_{\theta_h} = T_{\theta_h}(\Omega_0) \). Typically, the approximate velocity and pressure \((\mathbf{u}, p)\) are then members of the \( \theta^h \)-dependent space

\[
V^h_1(\theta^h) := \{ (v, q) = (v_0, q_0) \circ T_{\theta^h}^{-1} \ : \ (v_0, q_0) \in V^h_1 \} \subset H^1_0(\Gamma_0 \cup \Gamma_D(\Omega_{\theta_h})) \times L^2(\Omega_{\theta_h}),
\]

where \( V^h_1 := V^h_1(0) \) is some suitable approximate velocity and pressure space in the referential domain; see Fig. 2. Such an approximation is called conforming in the sense that the resulting velocity and pressure live on the approximate domain \( \Omega_{\theta_h} \). In practice, this is achieved by having matching fluid and structure meshes at the interface.

Given a conforming approximation \((\theta^h, \mathbf{u}^h, p^h)\), our objective is to provide a dual-based estimate of the goal-error \( Q(\theta, \mathbf{u}, p) - Q(\theta^h, \mathbf{u}^h, p^h) \). Such an estimate can then be employed by a goal-oriented adaptive strategy to refine the discretization and control the accuracy of the goal quantity.

### 3. The goal-oriented error estimation framework

A general paradigm for a-posteriori error estimation of quantities of interest has been established for canonical variational formulations; see in particular [1, 11, 49, 85]. We also refer to [101, Chapter 3] for an overview and analysis of various a priori and a posteriori error estimates of quantities of interest. Sections 3.1–3.3 give a brief summary of some of the essentials of the theory established in literature. In Sec. 3.4, we elucidate the complications that are encountered when applying the theory to fluid–structure interaction.

#### 3.1. Canonical setting

Let \( U \) and \( V \) denote reflexive Banach spaces. Consider the canonical variational problem, referred to as the primal problem,

\[
\text{Find } \mu \in U : \quad \mathcal{N}(\mu; \nu) = L(\nu) \quad \forall \nu \in V ,
\]
where \( N : U \times V \to \mathbb{R} \) is a semilinear form (nonlinear in the first entry) and \( L(\cdot) \) is a continuous linear functional on \( V \). The quantity of interest is the value of the (possibly nonlinear but continuous Fréchet differentiable) goal functional \( Q : U \to \mathbb{R} \) for the solution \( \mu \) of (6). Given any approximation \( \mu^h \in U \), the purpose of a posteriori error estimation is to obtain a computable estimate of the goal error, \( Q(\mu) - Q(\mu^h) \).

### 3.2. Dual-based error representation

In a dual-based approach, one solves the dual (or linearized-adjoint) problem

\[
N'(\mu^h; \zeta)(\delta \mu) = Q'(\mu^h)(\delta \mu) \quad \forall \delta \mu \in U ,
\]

where the prime indicates (Fréchet) differentiation with respect to the nonlinear arguments. That is, \( N'(\mu^h; \zeta) \) and \( Q'(\mu^h) \) are linear functionals on \( U \) such that

\[
N'(\mu^h; \zeta)(\delta \mu) = N(\mu^h + \delta \mu; \zeta) - N(\mu^h; \zeta) + o(\|\delta \mu\|_U),
\]

\[
Q'(\mu^h)(\delta \mu) = Q(\mu^h + \delta \mu) - Q(\mu^h) + o(\|\delta \mu\|_U),
\]

for all \( \delta \mu \in U \). Note that the dual problem (7) is a linear problem, obtained by linearization of \( N \) and \( Q \) at the approximate solution \( \mu^h \). The following theorem shows that the dual solution \( \zeta \) is the key element in relating the error in the quantity of interest to the residual at \( \mu^h \).

\[
\mathcal{R}(\mu^h; \cdot) := L(\cdot) - N(\mu^h; \cdot).
\]

**Theorem 3.A (Linearized-Adjoint Based Error Representation)** Assume that \( N \) and \( Q \) are continuous differentiable. Given any approximation \( \mu^h \in U \) of the solution \( \mu \) of (6), let \( \zeta \in V \) be the solution of the dual problem (7). It holds that

\[
Q(\mu) - Q(\mu^h) = \mathcal{R}(\mu^h; \zeta) + r ,
\]

with remainder \( r = o(\|\mu - \mu^h\|_U) \). If \( N \) and \( Q \) are twice continuous differentiable, then the remainder is given by \( r = R_Q - R_N = O(\|\mu - \mu^h\|_U^2) \), where

\[
R_Q := \int_0^1 (1 - t) Q'(\mu^h + te; e)(e) \, dt ,
\]

\[
R_N := \int_0^1 (1 - t) N''(\mu^h + te; \zeta)(e)(e) \, dt ,
\]

and \( e := \mu - \mu^h \) is the primal error.

**Proof** The proof uses the following standard Taylor-series formulae:

\[
Q(\mu) = Q(\mu^h) + Q'(\mu^h)(e) + r_Q ,
\]

\[
N(\mu; \zeta) = N(\mu^h; \zeta) + N'(\mu^h; \zeta)(e) + r_N ,
\]

where \( r_Q, r_N = o(\|\mu - \mu^h\|_U) \). If \( Q \) and \( N \) are twice continuously differentiable, then \( r_Q = R_Q \) and \( r_N = R_N \). The first Taylor series formula gives

\[
Q(\mu) - Q(\mu^h) = Q'(\mu^h)(e) + R_Q = N'(\mu^h; \zeta)(e) + R_Q ,
\]
where we used the dual problem (7) in the second step. It follows from the second Taylor-series formula that

\[ Q(\mu) - Q(\mu_h) = N(\mu; \zeta) - N(\mu_h; \zeta) + r_Q - r_N. \]

Finally, we obtain the proof by noting that \( N(\mu; \zeta) = L(\zeta) \) according to the primal problem (6), and by definition of \( R \). \( \Box \)

Note that the remainder term \( r \) in (8) is of higher order. Hence, the residual evaluated at the dual solution, \( R(\mu_h; \zeta) \), provides a higher-order accurate error estimate. This estimate is exact if \( N \) and \( Q \) are linear.

### 3.3. Approximate dual solution

The dual problem (7) cannot in general be solved exactly and one has to deal with approximations instead. Let \( \zeta^h \in V \) be an approximation to the solution \( \zeta \) of (7). Furthermore, setting \( e_\zeta := \zeta - \zeta^h \), we have the representation formula

\[ Q(\mu) - Q(\mu_h) = R(\mu_h; \zeta^h) + R(\mu_h; e_\zeta) + r. \]  

(9)

Accordingly, we can estimate the goal error by using the residual evaluated at the approximate dual solution giving the dual-based error estimate

\[ \text{Est}_Q := R(\mu_h; \zeta^h). \]

**Remark 3.1** Typically the primal problem is solved using a Galerkin approximation employing a discrete trial space \( \hat{U} \subset U \) and corresponding test space \( \hat{V} \subset V \). Owing to Galerkin orthogonality, \( R(\mu^h; \nu^h) = 0 \) for all \( \nu^h \in \hat{V} \). Hence, one can subtract an arbitrary element in \( \hat{V} \) from \( \zeta^h \) in the final estimate:

\[ \text{Est}_Q = R(\mu_h; \zeta^h - \nu^h) \quad \forall \nu^h \in \hat{V}. \]

**Remark 3.2** If one uses a test space \( \hat{V} \subset V \) for the Galerkin approximation of the primal problem and a trial space \( \bar{V} \subset V \) for the Galerkin approximation of the dual problem, then \( R(\mu^h; \zeta^h) = 0 \) if \( \bar{V} \subseteq \hat{V} \) on account of Galerkin orthogonality. The estimate is then useless, of course. Therefore, in practice, the dual problem is either solved using a larger space, \( \bar{V} \supset \hat{V} \), or it is solved on a dedicated dual-problem space such that \( \hat{V} \nsubseteq \bar{V} \).

**Remark 3.3** The above abstract framework is very powerful. Indeed, the main conclusion is that one can compute a goal-oriented error estimate by determining an approximate solution of the dual (linearized-adjoint) problem and subsequently evaluating the residual functional. To control the error, typically goal-oriented adaptive refinement is performed based on refinement indicators that are extracted from the error representation formula (9).

See the mentioned references at the beginning of Sec. 3 for more details.

### 3.4. Nontrivial nonlinearity in fluid–structure interaction

The major difficulty in applying the above framework to fluid–structure interaction pertains to the determination of a suitable dual problem: How do we linearize the operators in these problems?
Let us see why the nonlinearity is nontrivial for fluid–structure interaction by recalling the weak formulation (4): Find \( \theta \in H^1_0(\Gamma_0) \) and \((u, p) \in H^1_{0,\Gamma_{in} \cup \Gamma_{ff}}(\Omega_\theta) \times L^2(\Omega_\theta) \) such that

\[
S(\theta, \eta) + N(\Omega_\theta, (u, p); (v + E_\theta \eta, q)) = G(\eta) \quad \forall (\eta, v, q) \in H^1_0(\Gamma_0) \times H^1_{0,\Gamma_{in} \cup \Gamma_{ff}}(\Omega_\theta) \times L^2(\Omega_\theta).
\]

It should directly be noticed that this weak form does not fit the canonical abstract form (6). We have the following two complications.

1. How to deal with the fact that the \((u, p)\)- and \((v, q)\)-space are defined on the a priori unknown domain \( \Omega_\theta \)?
2. How to linearize domain dependencies such as \( \Omega_\theta \mapsto N(\Omega_\theta, (u, p); (v, q)) \)?

A resolution of these issues should also deal with the following. Suppose that \( \theta^k \) is an approximate displacement with corresponding fluid domain \( \Omega_{\theta^k} \) and velocity and pressure approximation \((u^k, p^k) \in H^1_{0,\Gamma_{in} \cup \Gamma_{ff}}(\Omega_{\theta^k}) \times L^2(\Omega_{\theta^k}) \). In the abstract framework, we have continuously considered errors. However, how should the error between \((u, p)\) and \((u^k, p^k)\) be defined? Their difference, \((u - u^k, p - p^k)\), is meaningless, since we are comparing functions on different domains.

In this work, we present two approaches that deal with these complications.

**Domain-map linearization approach**

The first approach is referred to as the domain-map linearization approach. The basic idea is that the fluid subproblem is transformed to a fixed reference domain, say \( \hat{\Omega} \). This is accomplished by means of a suitable \( \theta \)-dependent transformation map

\[
\hat{T}_\theta : \hat{\Omega} \rightarrow \Omega_\theta.
\]

The transformed velocity and pressure trial and test functions, \((\hat{u}, \hat{p}) := (u, p) \circ \hat{T}_\theta \) and \((\hat{v}, \hat{q}) := (v, q) \circ \hat{T}_\theta \) are then members of the fixed space \( H^1_{0,\Gamma_{in} \cup \Gamma_{ff}}(\hat{\Omega}) \times L^2(\hat{\Omega}) \). Hence, the fluid subproblem can then be equivalently formulated on \( \hat{\Omega} \), thereby arriving in the canonical form (6). To obtain the dual problem, one essentially needs to linearize with respect to the domain map \( \hat{T}_\theta \). We consider the domain-map linearization approach in Sec. 4.

**Shape-linearization approach**

The second approach is referred to as the shape-linearization approach. Here the idea is to linearize with respect \( \Omega_\theta \) by holding the fluid unknowns fixed in the current configuration assuming they have smooth extensions outside their domain. To perform this shape linearization, we need to remove the essential boundary conditions in the fluid velocity space \( H^1_{0,\Gamma_{in} \cup \Gamma_{ff}}(\Omega_\theta) \). This is achieved by introducing a very weak form of the fluid subproblem. Since geometry maps such as \( \Omega \mapsto N(\Omega, (u, p); (v, q)) \) are so-called shape functionals, we can employ the techniques of shape differential calculus to linearize such dependencies. We consider the shape linearization approach in Sec. 5.

4. Goal-oriented error estimation by domain-map linearization

To enable the application of the standard goal-oriented error estimation framework, we shall cast the weak formulation in (4) in canonical form by reformulating the fluid
part on a fixed domain. We can then linearize the transformed problem with respect to
the domain map yielding the dual problem.

Essentially, it does not matter which fixed domain is chosen (cf. the analysis
in [102]). However, it is most natural to choose the fixed domain closest to the lin-
erization state, viz., the approximate domain corresponding to $\theta^h$. For convenience,
let us denote the approximate domain and interface throughout the rest of this paper by

$\hat{\Omega} := \Omega_{\theta^h}$ and $\hat{\Gamma} := \Gamma_{\theta^h}$.

Note that $\partial \hat{\Omega} = \hat{\Gamma} \cup \Gamma_D \cup \Gamma_N$.

4.1. Reformulation to the approximate domain

We transform the fluid problem to the approximate domain by means of the map
$\hat{T}_\theta : \hat{\Omega} \rightarrow \Omega_{\theta^h}$. This map is defined, using $T_\theta$ introduced in Sec. 2.1, as

$\hat{T}_\theta := T_\theta \circ T^{-1}_{\theta^h} = Id + (\theta - \theta^h) \circ T^{-1}_{\theta^h} \quad \forall \theta \in \Theta$ ;

see also Fig. 3. Similar to $T_\theta$, $\hat{T}_\theta$ constitutes a $C^{0,1}$-diffeomorphism for admissible $\theta$.
For such a diffeomorphism, it holds that

$H^1_{0,\Gamma_D,\Gamma_N}(\hat{\Omega}) = \left\{ v = \hat{v} \circ \hat{T}^{-1}_{\theta^h} : \hat{v} \in H^1_{0,\Gamma_D,\Gamma_N}(\hat{\Omega}) \right\},$

$L^2(\Omega_{\theta^h}) = \left\{ q = \hat{q} \circ \hat{T}^{-1}_{\theta^h} : \hat{q} \in L^2(\hat{\Omega}) \right\};$

see [55, p. 21] or [22, p. 406].

Let us now introduce the transformed semilinear form

$\hat{N} : \Theta \times (H^1(\hat{\Omega}) \times L^2(\hat{\Omega})) \times (H^1(\hat{\Omega}) \times L^2(\hat{\Omega})) \rightarrow \mathbb{R}$

defined as

$\hat{N}(\theta, (\hat{v}, \hat{\rho}); (\hat{v}, \hat{\rho})) := N(\Omega_{\theta^h}, (\hat{v}, \hat{\rho}) \circ \hat{T}^{-1}_{\theta^h}, (\hat{v}, \hat{\rho}) \circ \hat{T}^{-1}_{\theta^h})$

$\forall (\hat{v}, \hat{\rho}), (\hat{v}, \hat{\rho}) \in H^1(\hat{\Omega}) \times L^2(\hat{\Omega})$.

We can then formulate the fluid–structure interaction problem equivalently on fixed
domains as follows.
Proposition 4.1 Let \( \theta \) denote an admissible structure solution of (4) and let \((\hat{u}, \hat{p}) := (u, p) \circ \hat{T}_\theta \in H^1_{0, \Gamma, \Gamma_p} (\hat{\Omega}) \times L^2(\hat{\Omega}) \) denote the corresponding fluid solution of (4) transformed to \( \hat{\Omega} \). Then it holds that

\[
S(\theta, \eta) + \hat{N}(\theta, (\hat{u}, \hat{p}); (\hat{v} + \hat{E} \eta, \hat{q})) = \hat{G}(\eta)
\]

\[
\forall (\eta, \hat{v}, \hat{q}) \in H^1_0(\Gamma_u) \times H^1_{0, \Gamma, \Gamma_p} (\hat{\Omega}) \times L^2(\hat{\Omega}) ,
\]

(12)

where \( \hat{N} \) is defined in (11) and

\[
\hat{E} \eta := E_\theta \eta = (\text{Ext}_\theta \eta) \circ T^{-1}_\theta .
\]

Explicitly:

\[
\hat{N}(\theta, (\hat{u}, \hat{p}); (\hat{v}, \hat{q})) = -\hat{R}^m(\theta, (\hat{u}, \hat{p}); \hat{v}) - \hat{R}^e(\theta; \hat{u}, \hat{q}) ,
\]

(13a)

\[
\hat{R}^m(\theta, (\hat{u}, \hat{p}); \hat{v}) = \int_\Omega \left( f_\theta \cdot \hat{v} - \nu (A_\theta \nabla \hat{u}) \cdot \nabla \hat{v} + \hat{p} \left( B_\theta \nabla \right) \hat{v} \right)
\]

(13b)

\[
\hat{R}^e(\theta; \hat{u}, \hat{q}) = \int_\Omega \hat{q} \left( B_\theta \nabla \right) \hat{u} ,
\]

(13c)

with

\[
A_\theta := \hat{J}_\theta D^T_{\theta} D^T_{\theta} , \quad B_\theta := \hat{J}_\theta D^T_{\theta} \nabla , \quad f_\theta := \hat{J}_\theta (f \circ \hat{T}_\theta) ,
\]

(14)

\(D\hat{T}_\theta := \partial \hat{T}_\theta / \partial (x_1, x_2)\) is the Jacobian matrix, and \(\hat{J}_\theta := \det D\hat{T}_\theta\) is the Jacobian of \(\hat{T}_\theta\).

Proof To show that (12) holds, recall that \(E_\theta \eta = (\text{Ext}_\theta \eta) \circ T^{-1}_\theta\), see (3). Hence this extension transforms to \(\hat{\Omega}\) as

\[
(E_\theta \eta) \circ \hat{T}_\theta = (\text{Ext}_\theta \eta) \circ T^{-1}_\theta \circ \hat{T}_\theta = (\text{Ext}_\theta \eta) \circ T^{-1}_\theta = \hat{E} \eta ,
\]

where we used (10) in the second step. This result combined with (11) establishes (12).

The specification of \(\hat{N}\) given by (13b) and (13c) can be proven by transforming the integrals to \(\hat{\Omega}\). Consider any \(\nu \in H^1_{0, \Gamma, \Gamma_p} (\Omega_\theta)\). To transform \(\int_{\Omega_\theta} \nu \nabla u \cdot \nabla v = \int_{\Omega_\theta} \nu \nabla u \cdot \nabla v_i\), we use the identity

\[
(\nabla w_i) \circ \hat{T}_\theta = D\hat{T}_{\theta}^T \nabla (w_i \circ \hat{T}_\theta) \quad \forall w_i \in H^1(\Omega_\theta) ,
\]

to obtain

\[
\int_{\Omega_\theta} \nu \nabla u \cdot \nabla v = \int_{\Omega_\theta} \left( D\hat{T}_{\theta}^T \nabla (u_i \circ \hat{T}_\theta) \right) \cdot \left( D\hat{T}_{\theta}^T \nabla (v_i \circ \hat{T}_\theta) \right) \hat{J}_\theta
\]

\[
= \int_{\Omega} \left( A_\theta \nabla (u_i \circ \hat{T}_\theta) \right) \cdot \nabla (v_i \circ \hat{T}_\theta) .
\]

Replacing \(u \circ \hat{T}_\theta\) with \(\hat{u}\) and setting \(v \circ \hat{T}_\theta =: \hat{v} \in H^1_{0, \Gamma, \Gamma_p} (\hat{\Omega})\), we obtain the \(A_\theta\)-term in (13b). The other terms in (13b) and (13c) follow similarly. \(\square\)

---

5Expression involving matrices are meant to be evaluated as in ordinary matrix-matrix or matrix-vector products. For example, \(A_\theta \nabla u = \left( A_{\theta,j,k} (\nabla u)_{k,j} \right) = \left( A_{\theta,j,k} \partial_i u_j \right)\). Hence, \(A_\theta \nabla u \cdot \nabla v = \left( A_{\theta,j,k} \partial_i u_j \partial_h v_j \right) = \left( A_\theta \nabla u \right) \cdot \nabla v_i\).
The goal functional can be transformed to $\hat{\Omega}$ in a similar manner as $\mathcal{N}$, i.e.,

$$
\hat{Q}(\theta, \hat{u}, \hat{p}) = Q(\theta, \hat{u} \circ T_\theta^{-1}, \hat{p} \circ T_\theta^{-1}) = \int_{\Gamma_0} q^\text{disp} \cdot \theta + \int_{\Omega} (q^\text{vol}_\theta \cdot \hat{u} + q^\text{pres}_\theta \hat{p}) .
$$

with

$$
q^\text{vol}_\theta := J_\theta (q^\text{vol} \circ \hat{T}_\theta) \quad \text{and} \quad q^\text{pres}_\theta := J_\theta (q^\text{pres} \circ \hat{T}_\theta) .
$$

4.2. Dual problem by domain-map linearization

Having cast our problem in canonical form, we can derive the dual problem by linearization. Accordingly, we shall linearize the left-hand side of (12) and $\hat{\theta}$. Straightforward linearization of $\hat{\theta}$, gives

$$
\hat{\theta} = \hat{\theta} \circ T_\theta^{-1} \circ \delta \theta .
$$

The same linearization of $\hat{\theta}$ gives

$$
\delta_{(\hat{u}, \hat{p})} \hat{\mathcal{N}}(\theta^\phi, (u^h, p^h); (z, s)) (\delta \hat{u}, \delta \hat{p})
\quad = \int_{\Omega} \left( \nabla \delta \hat{u} \cdot \delta z + (u^h \cdot \nabla \delta \hat{u} + \delta \hat{u} \cdot \nabla u^h) \cdot \delta \hat{p} \right) - \int_{\partial \Omega} s \text{ div } \delta \hat{u} .
$$

Next, let us consider the linearization of $\hat{\mathcal{N}}$ and $\hat{\mathcal{Q}}$ with respect to $\theta$. Basically, this amounts to linearizing the $\theta$-dependent terms in (13b), (13c) and (15) and requires the derivatives of $A^\theta$, $B^\theta$, $f^\theta$, $q^\text{vol}_\theta$ and $q^\text{pres}_\theta$. Let us first state some elementary derivatives in the following lemmata. Generally, such derivatives are given for a linearization at $\theta = 0$, that is, at the unperturbed configuration; see [22, 90] and [101, Chapter 4], for example. However, linearizations about nonzero $\theta$ can simply be obtained by translation. In particular, note that $\hat{T}_\theta$ can be written as a perturbation of the identity starting from $\theta^0$:

$$
\hat{T}_{\theta^0 + t \delta \theta} = I_d + t (\delta \theta \circ T^{-1}_{\theta^0}) = I_d + t \delta \theta ;
$$

see (10), where

$$
\delta \theta := \delta \theta \circ T^{-1}_{\theta^0} \in \mathcal{H} := \{ \delta \theta = \delta \theta \circ T^{-1}_{\theta^0}, \forall \delta \theta \in \Theta \} .
$$

A proof of the following lemmata then follows from standard results in [22, 90, 101].
Lemma 4.2 For $\tilde{T}_v$ defined in (10) and its Jacobian $\tilde{J}_v$, we have

\[
\langle \partial_v \delta T_{v}, \delta \theta \rangle = D \delta \theta , \hspace{1cm} \langle \partial_v \tilde{J}_v, \delta \theta \rangle = \text{div} \, \delta \theta ,
\]

\[
\langle \partial_v \delta T_{v}^{-1}, \delta \theta \rangle = -D \delta \theta ,
\]

for all $\delta \theta \in \tilde{\Theta}$.

Lemma 4.3 Let $\phi \in H^1(\mathbb{R}^2)$. Then the map $\theta \mapsto \phi \circ \tilde{T}_v$ is differentiable at $\theta^0 \in \Theta$ in $L^2(\tilde{\Omega})$. The derivative is given by

\[
\langle \partial_\theta (\phi \circ \tilde{T}_v)|_{\theta^0}, \delta \theta \rangle = \nabla \phi \cdot \delta \theta .
\]

for all $\delta \theta \in \tilde{\Theta}$.

Using these results, we can easily obtain the derivatives of $A_{\theta}, B_{\theta}, f_{\theta}, q_{\theta}^{\text{vel}}$ and $q_{\theta}^{\text{pres}}$ from their definitions in (14) and (16). We collect these derivatives in the following proposition.

Proposition 4.4 Denoting $\delta \theta := \delta \theta \circ T_{v}^{-1}$, it holds that

\[
\langle \partial_\theta A_{\theta}, \delta \theta \rangle = (\text{div} \, \delta \theta) I - D \delta \theta - D \delta \tilde{\theta} , \hspace{1cm} \langle \partial_\theta q_{\theta}^{\text{pres}}, \delta \theta \rangle = \text{div}(q_{\theta}^{\text{pres}} \delta \theta) ,
\]

\[
\langle \partial_\theta B_{\theta}, \delta \theta \rangle = (\text{div} \, \delta \theta) I - D \delta \theta , \hspace{1cm} \langle \partial_\theta q_{\theta}^{\text{vel}}, \delta \theta \rangle = \text{div}(q_{\theta}^{\text{vel}} \delta \theta) ,
\]

\[
\langle \partial_\theta f_{\theta}, \delta \theta \rangle = \text{div}(f \, \delta \theta) ,
\]

with $I$ the identity matrix.

The linearizations lead to the following dual problem:

Find $\zeta \in H^1_0(\Gamma_o)$ and $(z, s) \in H^1(\tilde{\Omega}) \times L^2(\tilde{\Omega})$:

\[
S(\delta \theta, \delta z) + \tilde{N}'((\delta \theta, (u^\theta, p^\theta)); (z, s))(\delta \theta, \delta u, \delta p) = \tilde{Q}'((\delta \theta, (u^\theta, p^\theta)); (z, s))(\delta \theta, \delta u, \delta p)
\]

\[
\forall (\delta \theta, \delta u, \delta p) \in H^1_0(\Gamma_o) \times H^1_{\tilde{\Omega} \setminus \Gamma_o}(\tilde{\Omega}) \times L^2(\tilde{\Omega}) ,
\]

with the $\delta u$-, $\delta p$- and $\delta \theta$-equation specified by

\[
\int_{\tilde{\Omega}} \left( \nu \nabla \delta u \cdot \nabla z - s \, \text{div} \, \delta u \right) = \int_{\tilde{\Omega}} q_{\text{vel}} \cdot \delta u ,
\]

\[
- \int_{\tilde{\Omega}} \delta p \, \text{div} \, z = \int_{\tilde{\Omega}} q_{\text{pres}} \delta p ,
\]

\[
\int_{\Gamma_o} E \, \delta \theta \cdot \partial_\nu \zeta + \delta \theta \tilde{N}((\delta \theta, (u^\theta, p^\theta)); (z, s))(\delta \theta) = \partial_\nu \tilde{Q}(\delta \theta, (u^\theta, p^\theta))(\delta \theta) .
\]

We remark that the precise form of the derivatives $\partial_\theta \tilde{N}$ and $\partial_\theta \tilde{Q}$ in (19c) follow straightforwardly from their definitions in (13) and (15), and the derivatives in Prop. 4.4.

Remark 4.5 It should be noted that the dual fluid problem, (19a) and (19b), corresponds to a Stokes problem for $(z, s)$ on the fixed approximate domain $\tilde{\Omega}$. Furthermore, note that, in (18), the dual velocity satisfies a nonzero kinematic boundary condition at the interface $\Gamma$, coupling it to the dual structure displacement $\zeta$. The dual structure equation (19c) is essentially a string equation coupled to the dual fluid via nonlocal terms in $\partial_\theta \tilde{N}$. Note that this coupling condition is nonlocal.
4.3. Dual-based error representation

In the following theorem, we show that the dual problem is suitable to provide a goal-oriented error estimate.

**Theorem 4.A (Domain-Map Linearized Dual-Based Error Representation)**

Given any admissible structure approximation $\theta^h$ and corresponding fluid domain $\hat{\Omega} = \Omega_0$ and fluid approximation $(u^h, p^h) \in H^1_{\text{free}}(\hat{\Omega}) \times L^2(\hat{\Omega})$ of the fluid–structure solution $(\theta, u, p)$ of (4), assume the dual problem (18) has a solution $(\zeta, z, s)$. Then it holds that

$$Q(\theta, u, p) - Q(\theta^h, u^h, p^h) = \mathcal{R}(\theta^h, u^h, p^h; \zeta, z, s) + r,$$

where the coupled fluid–structure residual is given by

$$\mathcal{R}(\theta^h, u^h, p^h; \zeta, z, s) = \mathcal{G}(\zeta) - S(\theta^h, \zeta) - N(\hat{\Omega}, (u^h, p^h); (z, s)) .$$

The remainder $r = o(||e||_{H^2(\Gamma_0)}, ||\tilde{e}||_{H^2(\hat{\Omega})}, ||\tilde{e}||_{L^2(\hat{\Omega})})$ and the errors are defined as

$$e^h := \theta - \theta^h, \quad e^u := u \circ \hat{T}_h - u^h, \quad e^p := p \circ \hat{T}_h - p^h .$$

This error representation formula is a specification of the abstract formula in Theorem 3.A for our fluid–structure-interaction problem. The goal-quantity error can be estimated by evaluating the fluid–structure residual $\mathcal{R}$ with a discrete dual solution. In specific discretized settings, one can invoke Galerkin orthogonality to subtract arbitrary discrete test-functions, as usual; see the remark in Sec. 3.3.

Note that the theorem shows how fluid variables, that live on distinct domains, are compared. Indeed, the remainder forms a higher-order term of errors, $u \circ \hat{T}_h - u^h$ and $p \circ \hat{T}_h - p^h$, measured in the approximate domain.

**Proof** The proof is similar to the proof of Theorem 3.A. An essential element of the proof are the following Taylor-series-like formulæ:

$$Q(\theta, u, p) = Q(\theta^h, u^h, p^h) + \hat{Q}(\theta^h, u^h, p^h)(e^h, e^u, e^p) + r_\hat{Q} , \quad (20a)$$

$$N(\Omega_0, (u, p); (v, q) \circ \hat{T}_h^{-1}) = N(\hat{\Omega}, (u^h, p^h); (v, q))$$

$$+ \hat{N}'((\theta^h, (u^h, p^h); (v, q))(e^h, e^u, e^p) + r_{\hat{N}} . \quad (20b)$$

for any $(v, q) \in H^1(\hat{\Omega}) \times L^2(\hat{\Omega})$, with remainders $r_\hat{Q}, r_{\hat{N}}$ of the same order as $r$ as provided in the theorem. Let us show that (20b) holds. By definition of $\hat{N}$ in (11), we have the identity

$$N(\Omega_0, (u, p); (v, q) \circ \hat{T}_h^{-1}) - N(\hat{\Omega}, (u^h, p^h); (v, q))$$

$$= \hat{N}(\theta, (u, p) \circ \hat{T}_h; (v, q)) - \hat{N}(\theta^h, (u^h, p^h); (v, q)) .$$

Since $\hat{N}$ acts on fixed spaces, we can apply a standard Taylor-series formula to the right-hand side. This yields (20b). Eq. (20a) can be established analogously. Next, consider the goal error $E_Q := Q(\theta, u, p) - Q(\theta^h, u^h, p^h)$. Using (20a), and subsequently invoking the dual problem (18), we obtain

$$E_Q = \hat{Q}(\theta^h, u^h, p^h)(e^h, e^u, e^p) + r_\hat{Q}$$

$$= S(e^h, \zeta) + \hat{N}'((\theta^h, (u^h, p^h); (z, s))(e^h, e^u, e^p) + r_{\hat{N}} .$$
Next, applying (20b), it follows that
\[ E_Q = S(e^\theta, \zeta) + N(\Omega_\theta, (u, p); (z, s) \circ T_\theta^{-1}) - N(\hat{\Omega}, (u^b, p^b); (z, s)) + r_Q - r_N. \]

The proof follows by noticing that, since \( z|_{\hat{\Gamma}} = \zeta \), we have
\[ S(\theta, \zeta) + N(\Omega_\theta, (u, p); (z, s) \circ T_\theta^{-1}) = G(\zeta), \]
according to the primal weak formulation (4).

5. Goal-oriented error estimation by shape linearization

In the previous section, we considered goal-oriented error estimation using the domain-map linearization approach. The resulting dual problem, however, contains a nonstandard, nonlocal boundary condition. Motivated by the results in [103], where local boundary conditions were obtained by the shape linearization approach in case of the Bernoulli free-boundary problem, we consider in this section the shape-linearization approach to derive the linearized adjoint of our fluid–structure-interaction problem (4).

5.1. Very-weak fluid formulation

For a shape linearization of (a weak form of) the fluid–structure-interaction problem, it is important that the involved spaces do not depend on \( \theta \). Hence, we need to remove the essential boundary conditions at \( \Gamma_\theta \) from the fluid velocity test and trial spaces in (4). The constraint on test functions \( v \) can be removed by absorbing \( E_\theta \eta \) in \( v \), i.e., the structure test function \( \eta \) is eliminated in favor of a nonzero \( v \). The constraint on trial velocity functions can be removed by going to a very weak formulation.\(^6\)

Furthermore, we shall view fluid variables as living on a sufficiently-large hold-all domain \( D \subseteq \mathbb{R}^2 \). For this hold-all domain, it holds that \((\Gamma_D \cup \Gamma_N) \subset \partial D\) and \( \Omega_\theta \subset D \) for all admissible \( \theta \).

Let us define the very weak fluid form \( \mathcal{N}_{vw} \) as
\[ \mathcal{N}_{vw}(\Omega, (u, p); (v, q)) := \int_{\Omega} \left( -f \cdot v + u \cdot (-\nu \Delta v + \nabla q) - p \ \text{div} \ v \right) + \int_{\Gamma} u \cdot (\nu \partial_n v - q \ n). \]

We summarize the consistent reformulation in the following proposition.

**Proposition 5.1** Let \((\theta, u, p)\), viewed as a member of \(H_0^1(\Gamma_0) \times H_0^1(\Gamma_\theta \cap \Gamma_\theta^c) \times L^2(D)\), denote the solution of (4). It holds that
\[ -\mathcal{R}_{vw}(\theta, (u, p); (v, q)) := S(\theta, (v \circ T_\theta)|_{\Gamma_\theta}) + \mathcal{N}_{vw}(\Omega_\theta, (u, p); (v, q)) - \mathcal{G}((v \circ T_\theta)|_{\Gamma_D}) = 0, \]
for all \( v \in H^2(D) \cap H_0^1(\Gamma_\theta \cap \Gamma_\theta^c) \) and \( q \in H^1(D) \).

\(^6\)It is also possible to introduce Lagrange multipliers to remove the constraints. This is, however, more involved.
Proof} Any \( v \in H^2(D) \cap H^1_{0\Gamma_D}(D) \) can be split as \( v = v_0 + v_1 \), with \( v_0 = 0 \) on \( \Gamma_D \) and \( v_1 \) is an extension of \( v_{\|\Gamma_D} \). First consider \( v_0 \). Then (22) implies that
\[
N_{vw}(\Omega_D, (u, p); (v_0, q)) = 0.
\]
Since \( N_{vw} \) is a very weak form of \( N \), a simple integration by parts of (21) and using \( u = 0 \) on \( \Gamma_D \) yields
\[
N(\Omega_D, (u, p); (v_0, q)) = 0,
\]
which is known to hold: the fluid subproblem in (4). Next consider \( v_1 \). Note that there exists an \( \eta \in H^1(\Omega_0) \) such that \( v_1 = \eta \circ T^{-1}_{\theta} \) on \( \Gamma_D \). Hence,
\[
S(\theta, \eta) + N_{vw}(\Omega_D, (u, p); (v_1, q)) - G(\eta) = 0.
\]
Again, apply an integration by parts on \( N_{vw} \) to obtain the structure subproblem in (4). \( \square \)

In view of the reformulation, we can linearize \( R_{vw} \) in (22) as well as \( Q \) (defined in (5)) with respect to \( (\theta, u, p) \in H^1_0(\Gamma_0) \times H^1_{\Gamma_D}(D) \times L^2(D) \). Let \( (\theta^h, u^h, p^h) \in H^1_0(\Gamma_0) \times H^1_{\Gamma_D}(\Omega_{\theta^h}) \times L^2(\Omega_{\theta^h}) \) denote the approximation at which the linearization takes place. We shall extract the dual problem for \( (z, s) \) from the linearized-adjoint equation:

\[
-\mathcal{R}_{vw}'(\theta^h, (u^h, p^h); (z, s))(\delta \theta, \delta u, \delta p) = Q'(\theta^h, u^h, p^h)(\delta \theta, \delta u, \delta p)
\]
\[
\forall (\delta \theta, \delta u, \delta p) \in H^1_0(\Gamma_0) \times H^1_{\Gamma_D}(D) \times L^2(D).
\]

We first consider the case of smooth interfaces. The extension to the more general case of nonsmooth interfaces is discussed in Sec. 5.5.

For convenience, we shall also uphold the following shorthand notation throughout this section:
\[
\hat{\Omega} := \Omega_{\theta^h} \quad \text{and} \quad \hat{\Gamma} := \Gamma_{\theta^h}.
\]

5.2. Linearization with respect to fluid variables

Let us consider the linearization of \( (u, p) \mapsto -R_{vw}(\theta^h, (u, p); (z, s)) \). This is straightforward as only \( N_{vw} \) depends on it and the involved terms are linear. Denoting the derivative by \( \partial_{(u, p)}(\cdot)\), we obtain

\[
-\partial_{(u, p)}R_{vw}(\theta^h, (u^h, p^h); (z, s))(\delta u, \delta p) = \int_{\Omega} \left( \delta u \cdot (-\nu \Delta z + \nabla s) - \delta p \ \text{div} \ z \right)
\]
\[
+ \int_{\Gamma_D} \delta u \cdot (\nu \delta_{\nu}z - s n).
\]

Similarly, for the goal functional in (5), we obtain

\[
\partial_{(u, p)}Q(\theta^h, u^h, p^h)(\delta u, \delta p) = \int_{\Omega} q^\text{vel} \cdot \delta u + \int_{\hat{\Omega}} q^\text{press} \delta p.
\]
These results already show that the fluid part of the dual problem corresponds simply to a Stokes problem:

\[
\begin{align*}
-\nu \Delta z + \nabla s &= q^{\text{vol}} \quad \text{in } \Omega, \\
- \text{div } z &= q^{\text{pres}}
\end{align*}
\]

(24)

with a homogeneous traction boundary condition: \(-\nu \partial_n z + s n = 0\) on \(\Gamma_s\). More involved are the boundary conditions at the interface, which follow from the linearization with respect to \(\theta\).

5.3. Shape linearization at smooth interfaces

Next we consider the linearization with respect to \(\theta\) in the direction \(\delta \theta\). Let us assume that the approximate displacement \(\theta_h\) is smooth, in particular, \(\theta_h \in C^1(\Gamma_0) \cap H^1_0(\Gamma_0)\). The approximate interface \(\hat{\Gamma}\) is then a \(C^1\)-boundary. We shall first take up the \(\theta\)-linearization of \(N_{vw}\) and subsequently consider \(Q, S\) and \(G\).

It is of key importance to note that the linearization of the very weak fluid form, i.e., of the map

\[
\theta \mapsto N_{vw}(\Omega; (u^h, p^h); (z, s)) = \int_{\Omega_h} \left( - f \cdot z + u^h \cdot ( -\nu \Delta z + \nabla s) - p^h \text{div } z \right),
\]

(25)
corresponds to the so-called shape derivative of \(\Omega \mapsto N_{vw}(\Omega; (u^h, p^h); (z, s))\) at \(\hat{\Omega}\) in the direction

\[
\delta \theta = \delta \theta \circ T^{-1}_{\theta_h}.
\]

In the following lemma, we recall the shape derivative of domain integrals; see [22, 90] or [101, Chapter 4].

**Lemma 5.2** Let \(J\) be the shape functional defined as the domain integral of \(\phi \in W^{1,1}(D)\), i.e.,

\[
J(\Omega) = \int_{\Omega} \phi \, d\Omega.
\]

The shape derivative of \(J\) at \(\hat{\Omega}\) in the direction \(\delta \theta \in C^{0,1}(\hat{\Omega})\) is given by

\[
J'(\hat{\Omega})(\delta \theta) = \lim_{t \to 0} \frac{J(\hat{\Omega} \circ T(\delta \theta t)) - J(\hat{\Omega})}{t} = \int_{\partial \hat{\Omega}} \phi \delta \theta \cdot n.
\]

It is clear that the shape derivative of \(J\) vanishes if \(\phi = 0\) at \(\partial \hat{\Omega}\). This holds also for less regular \(\phi\) defined as the product \(\phi = \phi_1 \phi_2\) where \(\phi_1 \in L^2(D)\) and \(\phi_2 \in H^1(D)\) with \(\phi_2 = 0\) on \(\partial \hat{\Omega}\); see [102, Proposition 3.3].

Next, we use these results to linearize (25). First note that \(u^h\) is zero at \(\hat{\Gamma}\). Furthermore, we already know from (24) that \(- \text{div } z = q^{\text{pres}} \in H^1(D)\). Hence we obtain\(^7\)

\[
\partial_\Omega N_{vw}(\hat{\Omega}; (u^h, p^h); (z, s))(\delta \theta \circ T^{-1}_{\theta_h}) = \int_\Gamma \left( - f \cdot z + q^{\text{pres}} p^h \right) \delta \theta \cdot n.
\]

(26)

\(^7\) Actually, we should momentarily require more regularity for \(p^h\) than \(p^h \in L^2(D)\). However as will soon become clear, the corresponding term cancels.
Similarly, the velocity contribution to the derivative of the goal functional vanishes:

\[ \partial_\theta Q(\theta^h, u^h, p^h)(\delta \theta) = \int_{\Gamma_0} q_{\text{disp}} \cdot \delta \theta + \int_{\Gamma} q_{\text{prec}} \ p^h \delta \theta \cdot n \ . \]

Next, consider the structure related maps:

\[ \theta \mapsto S(\theta, (z \circ T_\theta)|_{\Gamma_0}) = \int_{\Gamma_0} E \partial_i \theta \cdot \partial_j (z \circ T_\theta) \]

\[ \theta \mapsto G((z \circ T_\theta)|_{\Gamma_0}) = \int_{\Gamma_0} g \cdot (z \circ T_\theta) \]

We collect their derivatives in the following lemma.

**Lemma 5.3** Denoting the total derivative by \( d_\theta(\cdot)|_{\theta}(\delta \theta) \), it holds that\(^8\)

\[ d_\theta S(\theta, (z \circ T_\theta)|_{\Gamma_0})|_{\theta}(\delta \theta) = \int_{\Gamma_0} \left( E \partial_i \delta \theta \cdot \partial_j (z \circ T_\theta) - E \partial_i^2 \delta \theta \cdot (\nabla_z^T \circ T_\theta) \cdot \delta \theta \right) , \]

\[ d_\theta G((z \circ T_\theta)|_{\Gamma_0})|_{\theta}(\delta \theta) = \int_{\Gamma_0} g \cdot (\nabla_z^T \circ T_\theta) \cdot \delta \theta . \]

**Proof** To differentiate the \((z \circ T_\theta)\)-terms at \( \theta^h \), we recall that

\[ \tilde{\Gamma}_\theta = T_\theta \circ T_\theta^1 : \tilde{\Omega} \rightarrow \Omega_\theta ; \]

see also Fig. 3. We then have from Lemma 4.3:

\[ \langle \partial_\theta(z \circ T_\theta), \delta \theta \rangle = \langle \partial_\theta(z \circ \tilde{\Gamma}_\theta \circ T_\theta^1), \delta \theta \rangle = (\nabla z \circ T_\theta) \cdot \delta \theta , \]

for \( i = 1, 2 \). Hence,

\[ \langle \partial_\theta(z \circ T_\theta), \delta \theta \rangle = (\nabla z^T \circ T_\theta) \cdot \delta \theta . \]

Note that to arrive at the second term in the derivative of \( S \), we integrated by parts and used \( \delta \theta = 0 \) at \( \partial \Gamma_0 \). \( \square \)

Collecting the \( \theta \)-derivatives, and noting that the \( p^h \)-term cancels, we can write the \( \delta \theta \)-equation of the linearized-adjoint equation (23) as:

\[
\int_{\Gamma_0} \left( E \partial_i \delta \theta \cdot \partial_j (z \circ T_\theta) - (g + E \partial_i^2 \delta \theta) \cdot (\nabla_z^T \circ T_\theta) \cdot \delta \theta \right) \\
- \int_{\Gamma} f \cdot z \delta \theta \cdot n = \int_{\Gamma_0} q_{\text{disp}} \cdot \delta \theta .
\]

Note that we can write the integral at \( \Gamma \) as an integral on \( \Gamma_0 \):

\[ \int_{\Gamma} f \cdot z \delta \theta \cdot n = \int_{\Gamma_0} (f \cdot z) \circ T_\theta (n \circ T_\theta) \cdot \delta \theta \omega_\theta = \int_{\Gamma_0} (f \cdot z) \circ T_\theta J_{\theta} DT_\theta^T n \cdot \delta \theta , \]

\(^8\)The innerproduct of a matrix with a vector is defined as \( A \cdot v = A_{ij} v_j \) and \( v \cdot A = v_i A_{ij} \). Hence 
\( \nabla z^T \cdot v = \partial_{z_j} z_i v_j \).
where $\omega_{\theta^h} := J_{\theta^h}|DT_{\theta^h}^T n|$ is the surface Jacobian. $J_{\theta^h} := \det DT_{\theta^h}$ the Jacobian and where we used $n \circ T_{\theta^h} = DT_{\theta^h}^T n/[DT_{\theta^h}^T n]$; see [90, p. 79]. Hence, we finally arrive at the $\delta\theta$-equation:

$$\int_{\Gamma_0} E \partial_s \delta\theta \cdot \partial_s (z \circ T_{\theta^h}) - \int_{\Gamma_0} \left((g + E \partial_s^2 \theta^h) \cdot (\nabla z^T \circ T_{\theta^h}) + (f \cdot z) \circ T_{\theta^h} J_{\theta^h} DT_{\theta^h}^T n\right) \cdot \delta\theta = \int_{\Gamma_0} q_{\text{disp}} \cdot \delta\theta.$$  \tag{27}

### 5.4. Shape-linearized dual problem and dual-based error estimate

We are now ready to define the dual problem. Note that the $\delta\theta$-equation (27) can be interpreted as a dual structure problem. This motivates the introduction of the dual structure displacement $\zeta = (z \circ T_{\theta^h})|_{\hat{\Gamma}_0}$. The aggregated dual problem is then defined as

$$-\nu \Delta z + \nabla q_{\text{vol}} = 0 \quad \text{in} \quad \hat{\Omega},$$

$$-\nu \partial_n z + s n = 0 \quad \text{on} \quad \Gamma_{\partial},$$

$$\zeta = 0 \quad \text{on} \quad \partial\Gamma_{\partial^h},$$

$$-E \partial_s^2 \zeta - (\nabla z \circ T_{\theta^h}) \cdot (g + E \partial_s^2 \theta^h)$$

$$-\left((f \cdot z) \circ T_{\theta^h} J_{\theta^h} DT_{\theta^h}^T n = q_{\text{disp}} \quad \text{in} \quad \Gamma_0,$$

$$\zeta = 0 \quad \text{on} \quad \partial\Gamma_{\theta^h}.$$  \tag{28a}

The dual fluid and dual structure subproblems can be identified as (28a–28d) and (28e–28f), respectively. These subproblems are coupled kinematically by (28d). This coupling condition is the same as obtained by the domain-map linearization in the previous section. The complementary coupling mechanism, see (28e), although nonstandard, is local. This result is contrary to what was obtained before.

**Remark 5.4** The condition in (28e) appears to be similar to the curvature-dependent boundary condition obtained in the shape-linearized dual problem of the Bernoulli free-boundary problem; see [103]. Indeed, the second derivative $\partial_s^2 \theta^h$ is a measure of the curvature of $\hat{\Gamma}$.

A weak formulation of the dual problem (28) can be derived by employing standard formulations. Let us recall the space for the dual velocity $z$ introduced in (17):

$$H^1_t(\hat{\Omega}) = \{ v \in H^1(\hat{\Omega}) : v|_\Gamma = \zeta \circ T_{\theta^h}^{-1}, \ v|_{\Gamma_{\partial^h}} = 0 \}.$$
A weak formulation of (28) then reads:

\[
\begin{aligned}
\text{Find } \zeta & \in H^1_0(\Gamma_0) \text{ and } (z, s) \in H^1_0(\hat{\Omega}) \times L^2(\hat{\Omega}) : \\
\int_{\Omega} (\nu \nabla u \cdot \nabla z - s \text{ div } \delta u) & = \int_{\Omega} q_{\text{vel}} \cdot \delta u , \\
\int_{\Omega} -\delta p \text{ div } z & = \int_{\Omega} q_{\text{press}} \delta p , \\
\int_{\Gamma_0} (E \partial_z \delta \theta \cdot \delta \zeta - \delta \theta \cdot (\nabla z \circ T_{\theta}) \cdot (g + E \partial_z^2 \theta)) & - \int_{\Gamma} (f \cdot z) \delta \theta \cdot n = \int_{\Gamma_0} q_{\text{disp}} \cdot \delta \theta , \\
\forall (\delta \theta, \delta u, \delta p) & \in H^1_0(\Gamma_0) \times H^1_0(\hat{\Omega}) \times L^2(\hat{\Omega}).
\end{aligned}
\]  

(29)

Unfortunately, this weak formulation, as it stands, contains unbounded functionals due to the \(V_z\)-term at \(\Gamma_0\) in the \(\delta \theta\)-equation. Furthermore, it is not clear under which conditions (29) admits a solution. We shall however proceed under the assumption that (29) admits a solution. We note that boundary conditions involving the gradient are so-called oblique boundary conditions. Problems with such boundary conditions are studied in [55, p. 167] and [21, p. 398], for instance.

Since we have derived the dual problem from the linearization of \(R_{vw}\) and \(Q\), any sufficiently smooth solution \((z, s)\) of (29) satisfies the linearized adjoint equation (23). Accordingly, we can present the following error representation formula.

**Theorem 5.A (Shape-Linearized Dual-Based Error Representation)** Given any admissible structure approximation \(\theta^h \in C^{1,1}(\overline{\Gamma_0}) \cap H^1_0(\Gamma_0)\) and corresponding fluid domain \(\hat{\Omega} = \Omega_{\theta^h}\) and fluid approximation \((u^h, p^h) \in H^1_0(\hat{\Omega}) \times L^2(\hat{\Omega})\) of the fluid–structure solution \((\theta, u, p)\) of (4), assume that the dual problem (29) has a sufficiently smooth solution \((\zeta, z, s)\). Then it holds that

\[
Q(\theta, u, p) - Q(\theta^h, u^h, p^h) = R(\theta^h, u^h, p^h; \zeta, z, s) + r ,
\]

where the coupled fluid–structure residual is given by

\[
R(\theta^h, u^h, p^h; \zeta, z, s) = G(\zeta) - S(\theta^h, \zeta) - N(\hat{\Omega}, (u^h, p^h); (z, s)).
\]

The remainder \(r = o(\|e^\theta\|_{H^1(\Gamma_0)}\), \|e^z\|_{H^1(\hat{\Omega})}, \|e^p\|_{L^2(\hat{\Omega})})\) and the errors are defined as

\[
e^\theta := \theta - \theta^h , \quad e^z := u - u^h , \quad e^p := p - p^h .
\]

Note that this error representation is very similar to the one obtained in the previous section using domain-map linearization. Indeed, both representations consist of the fluid–structure residual evaluated at a dual solution and a higher-order remainder.

**Proof** The proof follows closely the proof of Theorem 3.A. Consider the goal error \(E_Q = Q(\theta, u) - Q(\theta^h, u^h)\). Using a Taylor-series formula for \(Q\) and invoking (23), we obtain

\[
E_Q = Q(\theta^h, u^h)(e^\theta, e^z, e^p) + r_Q = -R_{vw}(\theta^h, (u^h, p^h); (z, s))(e^\theta, e^z, e^p) + r_Q ,
\]

24
with \( r_Q = o(\|e^\theta\|_{H^1(\Gamma_0)}), \|e^\theta\|_{H^1_{\text{inter}}(\Omega)}, \|e^\theta\|_{L^2(\Omega)}). \) Applying a Taylor-series formula for \( R_{vw} \), we obtain

\[
E_Q = -R_{vw}(\theta, (u, p); (z, s)) + R_{vw}(\theta^h, (u^h, p^h); (z, s)) + r_Q + r_{R_{vw}}.
\]

with \( r_{R_{vw}} = o(\|e^\theta\|_{H^1(\Gamma_0)}, \|e^\theta\|_{H^1_{\text{inter}}(\Omega)}, \|e^\theta\|_{L^2(\Omega)}). \) The first term on the right-hand side vanishes on account of consistency of the solution with \( R_{vw} \); see Prop. 5.1. Expanding the second term in accordance with (22), it follows that

\[
E_Q = G((z \circ T_{\theta^h})|_{\Gamma_0}) - S(\theta^h, (z \circ T_{\theta^h})|_{\Gamma_0}) - N_{vw}(\hat{\Omega}, (u^h, p^h); (z, s)) + r.
\]

Applying an integration by parts on \( N_{vw} \) thereby using \( u^h = 0 \) on \( \partial \Omega \), we can replace \( N_{vw} \) by \( N \). Finally, note that \((z \circ T_{\theta^h})|_{\Gamma_0} = \xi\).

\[\blacksquare\]

5.5. Extension to nonsmooth interfaces

In finite-element discretizations, the structure approximation \( \theta^h \) is generally only continuous and not continuously differentiable. The interface \( \tilde{\Gamma} \) is then Lipschitz continuous. In this case, the dual problem in (28) and its weak formulation (29) are not differentiable. In this case, the dual problem in (28) and its weak formulation (29) are not differentiable. The interface \( \tilde{\Gamma} \) is then Lipschitz continuous. This complication can, however, be resolved by taking the singular points into account during the derivation of the dual problem.

Let us consider, for the sake of simplicity, a piecewise-linear structure approximation, i.e.,

\[
\theta^h \in S^1(\Gamma_0) := \left\{ \eta \in H^1_0(\Gamma_0) : \eta|_K \in \mathbb{P}^1(K), \forall K \in \tau(\Gamma_0) \right\},
\]

where \( \tau \) is a partition of \( \Gamma_0 \) into finite elements \( K \) and \( \mathbb{P}^1(K) \) is the space of polynomials up to first order on \( K \). Let us denote the set of inter-element edges (points) \( \hat{x} \) by \( \chi \).

The complication can be traced back to the linearization of the map

\[
\theta \mapsto S(\theta^h, (z \circ T_{\theta^h})|_{\Gamma_0}) = \int_{\Gamma_0} E \partial_\tau \theta^h : \partial_\tau (z \circ T_{\theta^h}),
\]

in Sec. 5.3. Prior to the linearization, we invoked a global integration by parts on this integral and obtained \( \hat{\partial}_\tau^2 \theta^h \). For a nonsmooth \( \theta^h \), we have to integrate by parts element-wise prior to linearization, i.e.,

\[
\int_{\Gamma_0} E \partial_\tau \theta^h : \partial_\tau (z \circ T_{\theta^h}) = \sum_{K \in \tau} \left[ E \partial_\tau \theta^h : (z \circ T_{\theta^h}) \right]_{\partial K}.
\]

If we now linearize with respect to \( \theta \), we obtain

\[
\sum_{K \in \tau} \left[ E \partial_\tau \theta^h : (\nabla_{\hat{x}} \circ T_{\theta^h}) \cdot \delta \theta \right]_{\partial K} = \sum_{K \in \tau} \left[ E \partial_\tau \theta^h : (\nabla_{\hat{x}} \circ T_{\theta^h}) \right]_{\hat{x}} \cdot \delta \theta,
\]

where \( \llbracket \cdot \rrbracket_{\hat{x}} \) denotes the jump of \((\cdot \circ T_{\theta^h})\) at \( \hat{x} \). Hence, the weak form of the dual problem which is suitable at nonsmooth interfaces is essentially given by (29) with the replacement:

\[
\int_{\Gamma_0} -\delta \theta \cdot (\nabla_{\hat{x}} \circ T_{\theta^h}) \cdot E \hat{\partial}_\tau^2 \theta^h \leftarrow \sum_{K \in \tau} \delta \theta \cdot \left[ (\nabla_{\hat{x}} \circ T_{\theta^h}) \cdot E \partial_\tau \theta^h \right]_{\hat{x}}.
\]
6. Numerical experiments

In this section, we present numerical experiments based on Galerkin finite element discretizations. The fluid subproblem is discretized using standard \((p^2-p^1)\) Taylor-Hood finite elements on triangles. The string equation is discretized using linear finite elements. These string elements coincide with the adjacent fluid-element edges ensuring that the coupled problem is conforming; see Fig. 2. To solve the nonlinear coupled problem, we use a standard subiteration (fixed-point iteration) scheme, where at each iteration the fluid mesh is deformed to accommodate the new displacement.

To obtain dual-based estimates of the error in goal quantities, we compute the estimate

\[ \text{Est}_Q = R(\theta^h, u^h, p^h; \zeta^h, z^h, s^h), \]

where \(R\) is the coupled fluid–structure residual as given in Theorem 4.A and 5.A. The discrete dual solution \((\zeta^h, z^h, s^h)\) is obtained by discretizing the domain-map or shape-linearized dual problem, (18) or (29), respectively, using the same mesh as the primal problem, but with shapefunctions of one order higher.

To drive adaptive mesh refinement, element refinement indicators are extracted from the dual-based estimate as usual; see [11, 94]. In particular, we subtract a lower-order interpolant from the dual (using Galerkin orthogonality), integrate by parts element-wise and assign weighted interior and edge residuals to fluid and structure elements to obtain element contributions. At the interface, we combine the structure contribution of a particular structure element with the fluid contribution of the adjacent fluid element into one fluid–structure residual contribution for that fluid element. The absolute values of the fluid element contributions are then identified as the element refinement indicators.

Based on these indicators, we mark a set of fluid elements for refinement. This set is the minimal set for which the sum is a fraction of the total sum of indicators (a so-called Dörfler-type marking; see [24, 73]). We take this fraction as 0.4. The fluid elements are refined using newest-vertex bisection [70, 93]. Structure elements are bisected if the adjacent fluid edge has been bisected. This preserves conformity of the resulting discretization.

6.1. Parabolic interface testcase

First, we consider a manufactured fluid–structure interaction problem. It corresponds to flow in a channel with a flexible top. The momentum and string equation have a forcing such that the interface solution is parabolically shaped; see Fig. 4 [left]. The parameters in this problem are \(\nu = 1\) and \(E = 10\). The right boundary of the domain is a Neumann boundary.

Let us briefly describe, how we compute the forcings. We start with the fluid solution \((u_0, p_0)\) in a \(2 \times 1\) rectangular channel (parabolic velocity, linear pressure). The velocity is then mapped to the domain with the parabolic interface by means of the Piola transform:

\[ u = (J_0^{-1} DT_\theta u_0) \circ T_\theta^{-1}, \]

where \(T_\theta\) is the map corresponding to a parabolic \(\theta_2\). The Piola transform has the nice property that the velocity remains divergence-free; see [13, 74]. The pressure is simply extended onto \(\Omega_0\). The so-defined solution \((\theta, u, p)\) is then substituted in the primal problem to yield the forcings.
Figure 4: Parabolic interface testcase. Primal solution on a fine mesh [left] and on the coarsest 8-element mesh [right]. The illustrations show velocity vectors over the pressure.

Figure 5: Parabolic interface testcase. Domain-map linearized dual solution [top] and shape-linearized dual solution [bottom] on a fine mesh [left] and on the coarsest 8-element mesh [right]. The dual solutions correspond to a goal quantity of interest of the vertical displacement of the string at a point $\sqrt{2}/4$ from the left side (note that the width of the domain is 2). The illustrations show dual velocity vectors over the dual pressure.

Our goal quantity of interest is the vertical displacement of the string at a point $s_0 = \sqrt{2}/4 \approx 0.3536$ from the left-side, i.e.,

$$Q(\theta, u, p) = Q^{\text{disp}}(\theta) = \theta(s_0).$$

The exact value of this quantity is $(4\sqrt{2} - 1)/16$. Fig 5 displays several corresponding dual solutions. In this figure, the domain-map linearized and shape-linearized dual solutions are visible obtained on a fine mesh and the coarsest 8-element mesh (the coarsest 8-element primal solution is visible in Fig 4 [right]). Note that there is a small difference between domain-map linearization and shape linearization.

The convergence of the dual-based estimate $\text{Est}_{Q^{\text{disp}}}$ and exact error $E_{Q^{\text{disp}}} = Q^{\text{disp}}(\theta) - Q^{\text{disp}}(\theta^h)$ on uniformly-refined meshes is reported in Table 1. Note that the effectivity $\text{Est}_{Q^{\text{disp}}}/E_{Q^{\text{disp}}}$ is close to 1. This clearly demonstrates the consistency of both the error estimates.

Next, we investigate the applicability of the error estimates to drive adaptive mesh refinement. In Fig. 6, we plot the convergence of the exact error and the dual-based error estimate on uniform and adaptive meshes versus the total number of degrees.
of freedom, which is denoted by \( n \). Both the domain-map linearization and shape-linearization approach yield adaptive meshes with an asymptotic convergence rate of \( O(n^{-2}) \). This corresponds to optimal behavior for Taylor-Hood elements with quadratic velocities and a linear pressure, and linear structure displacements. Note that for uniform refinements, the rate of approximately \( O(n^{-1}) \) is suboptimal. Fig. 7 displays several adaptively-refined meshes. The refinement near the interface as well as the point of interest are noteworthy.

6.2. Driven cavity with flexible bottom

Next, we consider a lid-driven cavity problem with a flexible bottom; see Fig. 8. This testcase has been studied in [107] and is similar to the test-case in [25]. The undeformed domain corresponds to a unit square. The parameters in this problem are \( \nu = 1 \) and \( E = 5 \). We impose a quadratic inflow with maximum 1 at the top quarter of the left side, and a unit horizontal velocity at the top of the domain. A homogeneous Neumann condition is imposed at the top quarter of the right side. This is also referred to as a leaky lid. Figure 9 shows the solution on a fine mesh.

Our goal quantity of interest is now the total integral of the vertical displacement of the string, i.e.,

\[
Q(\theta, u, p) = Q^{\text{disp}}(\theta) = \int_{\Gamma_0} \theta^2 \, ds.
\]

Since the domain-map linearization approach and shape linearization approach have similar results, we shall only present results from domain-map linearization.

In Fig. 10, we display the domain-map linearized dual solution on the coarsest mesh. Note the dual pressure singularity at the connection point of the Neumann and Dirichlet boundary. In Fig. 11, we plot the convergence of the dual-based error estimate on uniform and adaptive meshes. We also plot the error with respect to a reference value \( Q^{\text{disp}}(\theta) \approx 0.163332 \) obtained on a uniform mesh with 16,384 elements and 74,501 degrees of freedom. It should be noted that an optimal convergence rate is obtained on the adaptive meshes. On uniform meshes the convergence is suboptimal and displays

<table>
<thead>
<tr>
<th>Elements</th>
<th>Dofs</th>
<th>( Q^{\text{disp}} )</th>
<th>( \mathcal{E}_{\hat{n}^{\text{disp}}} )</th>
<th>Domain-map lin. Est(_{\text{disp}}^{\text{Eff}}) Effect.</th>
<th>Shape linearization Est(_{\text{disp}}^{\text{Eff}}) Effect.</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>60</td>
<td>0.1843</td>
<td>0.10674</td>
<td>0.10750 1.007</td>
<td>0.10116 0.948</td>
</tr>
<tr>
<td>16</td>
<td>115</td>
<td>0.2683</td>
<td>0.02280</td>
<td>0.02272 0.996</td>
<td>0.02319 1.017</td>
</tr>
<tr>
<td>32</td>
<td>187</td>
<td>0.2677</td>
<td>0.02331</td>
<td>0.02396 1.028</td>
<td>0.02432 1.044</td>
</tr>
<tr>
<td>64</td>
<td>369</td>
<td>0.2842</td>
<td>0.00683</td>
<td>0.00682 0.999</td>
<td>0.00701 1.026</td>
</tr>
<tr>
<td>128</td>
<td>657</td>
<td>0.2842</td>
<td>0.00689</td>
<td>0.00708 1.027</td>
<td>0.00702 1.020</td>
</tr>
<tr>
<td>256</td>
<td>1,309</td>
<td>0.2901</td>
<td>0.00092</td>
<td>0.00093 1.005</td>
<td>0.00063 0.683</td>
</tr>
<tr>
<td>512</td>
<td>2,461</td>
<td>0.2901</td>
<td>0.00093</td>
<td>0.00098 1.053</td>
<td>0.00072 0.772</td>
</tr>
<tr>
<td>1,024</td>
<td>4,917</td>
<td>0.2907</td>
<td>0.00039</td>
<td>0.00040 1.006</td>
<td>0.00041 1.042</td>
</tr>
<tr>
<td>2,048</td>
<td>9,525</td>
<td>0.2907</td>
<td>0.00039</td>
<td>0.00041 1.032</td>
<td>0.00037 0.953</td>
</tr>
<tr>
<td>4,096</td>
<td>19,045</td>
<td>0.2910</td>
<td>0.00009</td>
<td>0.00009 1.008</td>
<td>0.00009 0.963</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \infty )</td>
<td>0.2911</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 6: Parabolic interface testcase. Convergence of the error and error estimate under uniform and adaptive mesh refinement versus the total number of degrees of freedom \( n \), using domain-map linearization [top] and shape linearization [bottom].

Figure 7: Parabolic interface testcase. The initial mesh and adaptively-refined meshes after 8, 16 and 24 iterations, controlling the error in the vertical structure displacement at a point, using domain-map linearization [top] and shape linearization [bottom].

wiggly odd-even behavior. This behavior is most likely caused by the fact that a uniform mesh refinement by bisection may or may not bisect a structure element. Fig. 12 displays several adaptively-refined meshes. These meshes have particularly been refined at the Dirichlet–Neumann boundary transition point and near the fluid–structure interface.
6.3. Backward-step with flexible bottom

The last application we consider is the test problem introduced in [39]. It involves a backward step and a flexible bottom; see Fig. 13. Similar flexible channels without a step have been investigated in [57, 69], for example. The parameters in our problem are \( \nu = 1 \) and \( E = 15 \). A quadratic inflow with maximum 3 is imposed at the left inflow boundary. The Neumann boundary condition \( \nu \partial_n u - p n = (12, 0) \) is imposed at the boundary \( \Gamma_N \). Fig. 14 displays the solution on a fine mesh.

Our goal quantity of interest is the integral of the vorticity in the quarter circular domain \( \Omega^\nu \) indicated in Fig. 13:

\[
Q(\theta, u, p) = Q^\nu(u) = \int_{\Omega^\nu} \nabla \times u ,
\]

where \( \nabla \times u = \partial u_2 / \partial x_1 - \partial u_1 / \partial x_2 \). Such a goal is of interest when computing the circum-
Figure 10: Driven cavity with flexible bottom testcase. Domain-map linearized dual solution on the coarsest mesh corresponding to the goal quantity of interest of the integral of the vertical displacement: dual velocity vectors [top-left], dual pressure [top-right], horizontal dual velocity [bottom-left] and vertical dual velocity [bottom-right].

Figure 11: Driven cavity with flexible bottom testcase. Convergence of the error and error estimate under uniform and adaptive mesh refinement versus the total number of degrees of freedom $n$, using domain-map linearization.

lation at local regions; see [83, p. 129]. Since the domain-map linearization approach and shape linearization approach have similar results, we shall now only present results from shape linearization. For the other results, we refer to [101].

In Fig. 15, we depict the approximate shape-linearized dual solution for the coarsest mesh. Note that the dual solution is local to $\Omega^\omega$. In fact, the dual velocity changes strongly at the circular boundary of $\Omega^\omega$ while the dual pressure displays singularities at the endpoints of the quarter circle.

In Fig. 16, we plot the convergence of the dual-based error estimate on uniform and adaptive meshes. For uniformly refined meshes, we also plot the error with respect to a
reference value $Q^\omega(\mu) = -1.154$ obtained on a uniform mesh with 15,360 elements and 70,085 degrees of freedom. Owing to the observed convergence rate of approximately $O(n^{-1/2})$, the reference value is very inaccurate to be compared with the adaptive results. We do however observe an optimal convergence rate of $O(n^{-2})$ for the dual-based error estimate on adaptive meshes. Fig. 17 shows several adaptively-refined meshes. Note the refinement at the boundary of the circular domain $\Omega^\omega$.

7. Concluding remarks

We explained that the major difficulty in applying the goal-oriented-error estimation framework to fluid–structure interaction pertains to the determination of a suitable dual (linearized-adjoint) problem. That is, the domain-dependent nonlinearity inherent in fluid–structure interaction renders the derivation of the linearized adjoint highly non-trivial. We introduced two approaches to derive exact linearized operators and obtain the dual problem.

In the domain-map linearization approach, the fluid subproblem is first transformed to a fixed reference domain. The dual problem is then obtained by essentially linearizing with respect to the domain transformation map. We showed that the dual problem corresponds to a dual fluid subproblem coupled to a dual structure displacement subproblem. These subproblems are coupled kinematically in a simple manner: the dual structure displacement acts as a Dirichlet boundary condition for the dual fluid velocity. The complementary coupling condition, however, involves nonstandard, nonlocal fluid terms.

In the shape-linearization approach, fluid unknowns are fixed in the current configuration. To perform the linearization a very weak formulation of the fluid subproblem was introduced. This linearization can be carried out using shape-derivative techniques.
Figure 14: Backward step with flexible bottom testcase. Solution on a fine mesh, from top to bottom: velocity vectors, pressure, horizontal velocity and vertical velocity.

from shape calculus. We showed that the dual problem corresponds to the same fluid and structure subproblem with the same kinematic coupling as obtain by the domain-map linearization approach. The complementary coupling is again nonstandard, but local, contrary to what was obtained before.

We presented numerical experiments that showed the consistency of the goal-oriented error estimates based on both linearization approaches. Furthermore, we demonstrated the applicability of the estimates in a goal-oriented adaptive refinement strategy.

Extensions of the current work. The presented work can be extended in various directions. For example, one could consider more complex fluid–structure interaction problems, such as three dimensional extensions or more realistic structural models, e.g., deformable solids. In such extensions, the main challenge is the derivation of dual coupling conditions. The extension to unsteady problems is also very important. We believe that the derivation of unsteady dual problems proceeds similarly as presented.

A different extension of our work is to apply the derived dual-based error estimates in more sophisticated adaptive strategies, such as hp-adaptive strategies [91, 94] and anisotropic remeint strategies [46, 68].

Finally, we note that the presented linearization approaches may be useful for many other applications involving fluid–structure interaction or free-boundary problems in
Figure 15: Backward step with a flexible bottom testcase. Shape-linearized dual solution on the coarsest mesh for a local vorticity goal functional. From top to bottom: dual velocity vectors, dual pressure, horizontal dual velocity and vertical dual velocity.

Figure 16: Backward step with a flexible bottom testcase. Convergence of the error and error estimate under uniform and adaptive mesh refinement versus the total number of degrees of freedom $n$, using shape linearization.

general. We mention the derivations of Newton-type algorithms such as in [42, 44] and the application of linearized adjoints in modeling-error estimation [77] and optimal control [74].
Acknowledgements

This research was supported by the Dutch Technology Foundation STW, applied science division of NWO and the Technology Program of the Ministry of Economic Affairs.

References


[47] J.-F. Gerbeau, M. Vidrascu, P. Frey, Fluid–structure interaction in blood flows on geome-


[86] A. Quarteroni, M. Tuveri, A. Veneziani, Computational vascular fluid dynamics: prob-


