Babuška ⇔ Brezzi ??

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Abstract

Equivalence of Babuška's and Brezzi's stability conditions for mixed variational formulations is discussed. In particular, the Brezzi's conditions are derived from the Babuška's condition.

1 Introduction

The following notes were prepared for a TICAM seminar, I gave almost five years ago, to mark the (then) 30th anniversary of the ground breaking result of Prof. Ivo Babuška [1], and the oncoming (then) 30th anniversary of the famous result of Prof. Franco Brezzi [2] for saddle point problems. After the over thirty years, the $\inf \sup$ condition, referred to as the *Babuška-Brezzi (BB) condition* (at least in the Finite Element community), remains to be a crucial tool in understanding and designing mixed discretizations.

The seminar was addressed to TICAM graduate students with a goal to explain the various versions of the $\inf \sup$ conditions, and discuss the equivalence of Babuška's and Brezzi's stability conditions for mixed problems. The "working horse" of the underlying algebra is the possibility of switching the order of variables in the $\inf \sup$ condition, and this was the main point, I tried to elucidate in my presentation.

Additionally, I had decided to include in the presentation the nice result of Xu and Zikatanov (reproducing the original result of Kato), showing how to eliminate constant "one" in Babuška's estimate (result valid for Hilbert spaces only). Contrary to presentation in [5], I had set up the whole framework in reflexive Banach spaces, and focused on showing how Babuška's condition implies Brezzi's conditions. In the presentation I use the language of bilinear forms but all results generalize in a straightforward way to sesquilinear forms as well.

The presented results are over 30 years old, and there is nothing new in the following notes written simply in response to several students and colleagues who have asked for a copy of the seminar notes. And to Ivo and Franco, let this be a part of a happy celebration of their famous contributions.

2 Bilinear forms

We begin by recalling fundamental facts considering bilinear and sesquilinear forms in reflexive Banach spaces. For detailed proofs consult, e.g. [3].

Bilinear (sesquilinear) form. Let U, V be two reflexive real (complex) Banach spaces with corresponding norms $\|u\| = \|u\|_U, \|v\| = \|v\|_V$. A function,

$$U \times V \ni (u,v) \to b(u,v) \in \mathbb{R}(\mathbb{C}) ,$$

is called a *bilinear (sesquilinear) form* if $b(u,v)$ is linear in u and linear (antilinear) in v.

Continuity. The following conditions are equivalent to each other.

$$b(u,v) \text{ is continuous,}$$

$$b(u,v) \text{ is continuous at } (0,0),$$

$$\exists M > 0 \ : \ |b(u,v) \leq M\|u\| \, \|v\| ,$$

$$|b(u,v)| \leq \|b\| \, \|u\| \, \|v\| .$$

Here $\|b\|$ denotes the *norm of the bilinear form* [1] defined as follows,

$$\begin{aligned} |b\| &= \inf \{M \ : \ |b(u,v)| \leq M\|u\| \, \|v\|\} \\ &= \sup_{u \neq 0, v \neq 0} \frac{|b(u,v)|}{\|u\| \, \|v\|} \\ &= \sup_{\|u\|=1, \, \|v\|=1} |b(u,v)| . \end{aligned}$$

The infimum and supremum above are actually attained, and can be replaced with minimum and maximum.

[1]The bilinear forms themselves form a reflexive Banach space

Operators associated with bilinear forms. Every continuous bilinear form generates two corresponding continuous operators,

$$\begin{aligned} B : U \to V' \qquad & < Bu, v >_{V' \times V} &= b(u,v) \quad \forall u \in U, v \in V \\ B' : V \to U' \qquad & < B'v, u >_{U' \times U} &= b(u,v) \quad \forall u \in U, v \in V \; . \end{aligned}$$

If we identify spaces U, V with their *biduals* U'', V'', operator B' is *conjugate* to operator B and, analogously, operator B is *conjugate* to operator B', e.g.,

$$< Bu, v >_{V' \times V} = b(u,v) = < B'v, u >_{U' \times U} = < u, B'v >_{U'' \times U'} \; .$$

Bounded below operators. The following conditions are equivalent to each other.

$$\exists \gamma > 0 \; : \; \|Bu\|_{V'} \geq \gamma \|u\|_U \; ,$$

$$\exists \gamma > 0 \; : \; \sup_{v \neq 0} \frac{| < Bu, v > |}{\|v\|} \geq \gamma \|u\| \; ,$$

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|b(u,v)|}{\|u\| \, \|v\|} =: \gamma > 0 \; .$$

The operator $B : U \to V'$ is said to be *bounded below*. Constant γ in the last condition is called the $\inf \sup$ constant. The $\inf$ and $\sup$ are actually attained and can be replaced with $\min$ and $\max$.

THEOREM 1

Let operator B be bounded below and its conjugate B' be injective, i.e.

$$b(u,v) = 0 \quad \forall u \quad \Rightarrow v = 0 \; .$$

Then operator B' is bounded below as well, and the two $\inf \sup$ *constants are equal,*

$$\inf_{u \neq 0} \sup_{v \neq 0} \frac{|b(u,v)|}{\|u\| \, \|v\|} = \inf_{v \neq 0} \sup_{u \neq 0} \frac{|b(u,v)|}{\|u\| \, \|v\|} =: \gamma > 0 \; .$$

$\blacksquare$

Proof: By *Banach Closed Range Theorem*, see e.g. [3], page 476, operator B has a continuous inverse $B^{-1} : V' \to U$, and

$$\|B^{-1}\| = \frac{1}{\gamma} \; .$$

By standard property for conjugate operators, see [3], page 472,

- the conjugate operator has a continuous inverse $(B')^{-1} \; : \; U' \to V$ as well,
- $(B')^{-1} = (B^{-1})'$, and

$$\|(B')^{-1}\| = \|(B^{-1})'\| = \|B^{-1}\| = \frac{1}{\gamma} \; .$$

$\blacksquare$

What if conjugate operator B' is not injective? In this case, we take the null space of operator B',

$$V_0 = \mathcal{N}(B') = \{v \in V \ : \ b(u,v) = 0 \quad \forall u \in U\} \ ,$$

and consider *quotient space* V/V_0 with the norm,

$$\|[v]\|_{V/V_0} = \inf_{v_0 \in V_0} \|v + v_0\|_V \ .$$

The quotient space is again a reflexive Banach space, see e.g. [3], page 478. In place of the original bilinear form, we introduce a bilinear form defined on the quotient space,

$$b(u,[v]) = b(u,w), \text{ where } w \in [v] \text{ is an arbitrary representative.}$$

The new bilinear form is well-defined and continuous. We repeat the whole reasoning to arrive at a more general result.

THEOREM 2
Let operator B be bounded below. Then

$$\begin{aligned} \inf_{[v]\neq 0} \sup_{u \neq 0} \frac{|b(u,[v])|}{\|u\| \, \|[v]\|} &= \inf_{u \neq 0} \sup_{[v]\neq 0} \frac{|b(u,[v])|}{\|u\| \, \|[v]\|} \\ &= \inf_{u \neq 0} \sup_{v \neq 0} \frac{|b(u,v)|}{\|u\| \, \|v\|} =: \gamma > 0 \ . \end{aligned}$$

∎

Thus, in the general case, we can still switch the order of arguments in the $\inf \sup$ condition, but at the expense of introducing the equivalence classes and the corresponding quotient space norm.

3 Babuška's Theorem

Let $U_h \subset U$ be a one-parameter family of *trial subspaces*, and let V_h be a corresponding family of *test spaces* of the same dimension, $\dim V_h = \dim U_h$, (not necessarily subspaces of V), with corresponding norms $\|u_h\| = \|u_h\|_U$ and $\|v_h\| = \|v_h\|_{V_h}$. Assume, we are given a family of continuous bilinear forms,

$$b_h(u, v_h), \quad u \in U, v_h \in V_h$$

$$|b_h(u, v_h)| \leq \|b_h\| \, \|u\| \, \|v_h\| \ ,$$

and we set to solve the (generalized) projection problem,

$$\begin{cases} u_h \in U_h \\ b_h(u_h, v_h) = b_h(u, v_h) \quad \forall v_h \in V_h \ . \end{cases} \tag{3.1}$$

THEOREM 3

Ivo Babuška, [1]

Suppose, the discrete $\inf \sup$ *condition holds,*

$$\inf_{u_h \in U_h} \sup_{v_h \in V_h} \frac{|b_h(u_h, v_h)|}{\|u_h\| \, \|v_h\|} =: \gamma_h > 0 \, .$$

Then

- *Projection problem (3.1) has a unique solution* u_h*, and*

$$\|u_h\| \leq \frac{\|b_h\|}{\gamma_h} \|u\| \, . \tag{3.2}$$

- *We have the following estimate,*

$$\|u - u_h\| \leq \left(1 + \frac{\|b_h\|}{\gamma_h}\right) \inf_{w_h \in U_h} \|u - w_h\| \, . \tag{3.3}$$

∎

Proof: The uniqueness (and, therefore, existence) of the solution, as well as stability estimate (3.2) follow immediately from the $\inf \sup$ condition.

Using the discrete $\inf \sup$ condition, we have,

$$\begin{aligned} \gamma_h \|u_h - w_h\| \; &\leq \sup_{v_h \neq 0} \frac{|b(u_h - w_h, v_h)|}{\|v_h\|} \\ &= \sup_{v_h \neq 0} \frac{|b(u - w_h, v_h)|}{\|v_h\|} \leq \|b_h\| \|u - w_h\| \, , \end{aligned}$$

and, by triangle inequality,

$$\begin{aligned} \|u - u_h\| \; &\leq \|u - w_h\| + \|w_h - u_h\| \\ &\leq \left(1 + \frac{\|b_h\|}{\gamma_h}\right) \|u - w_h\| \, . \end{aligned}$$

∎

The quantity on the right-hand side in estimate (3.3) is known as the *best approximation error*.

In the case of a Hilbert space, estimate (3.3) can be improved [5].

THEOREM 4

Let U *be a Hilbert space with inner product* $(\cdot, \cdot)_U$*, and let all the assumptions of the previous theorem hold. Then*

$$\|u - u_h\| \leq \frac{\|b_h\|}{\gamma_h} \inf_{w_h \in U_h} \|u - w_h\| \, . \tag{3.4}$$

∎

Lemma 1
Tosio Kato, [4]
Let $X \subset U$ be a subspace of Hilbert space U, and let P denote a non-trivial linear projection onto X, i.e.

$$P \,:\, U \to X, \quad P^2 = P \quad P \neq 0, I \quad .$$

Then

$$\|I - P\| = \|P\| \, .$$

∎

We present two alternative proofs of the result following [4] and [5].

Proof: (Kato)

Step 1: Since $I - P$ is also a projection, it is sufficient to prove that,

$$\|P\| \leq \|I - P\| \, .$$

Step 2: Since $I - P \neq 0$, $\|I - P\| \geq 1$. If $\|P\| = 1$, the inequality above holds. It is sufficient, therefore, to consider the case when $\|P\| > 1$.

Step 3: Let $1 < a < \|P\|$ be an arbitrary constant. Definition of the operator norm implies that there exists $u \neq 0$ such that

$$\|Pu\| \geq a\|u\| \, .$$

Let X be the range of P. Project u onto the range and consider,

$$v = (u, \frac{Pu}{|Pu\|})\frac{Pu}{|Pu\|} \, ,$$

(comp. Fig. 1). The Pythagoras theorem implies that,

$$\|v\|^2 = \|u\|^2 - \frac{|(u, Pu)|^2}{\|Pu\|^2} \, .$$

Also, $(I - P)v = (I - P)u$. We claim that,

$$\frac{\|(I - P)v\|^2}{\|v\|^2} = \frac{\|(I - P)u\|^2 \|Pu\|^2}{\|u\|^2 \|Pu\|^2 - |(u, Pu)|^2} \geq \frac{\|Pu\|^2}{\|u\|^2} \geq a^2 \, .$$

The middle inequality is equivalent to,

$$\|(I - P)u\|^2 \|u\|^2 \geq \|u\|^2 \|Pu\|^2 - |(u, Pu)|^2 \, ,$$

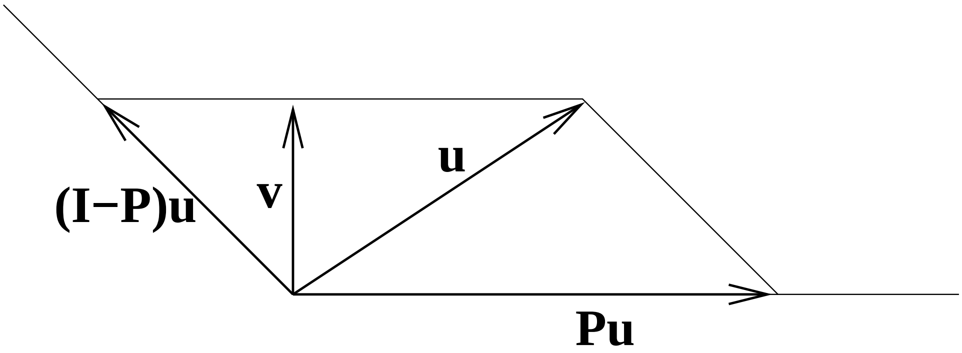

Figure 1: Kato's idea: choice of vector v

which, in turn is equivalent to,

$$(\|u\|^2 + \Re(u, Pu))^2 - \Re(u, Pu)^2 \geq |(u, Pu)|^2 \, .$$

Step 4: Taking supremum with respect constant a, we finish the proof.

$\blacksquare$

Proof: (Xu,Zikatanov)

Case 1: dim$U = 2, P \neq 0$. See Fig. 2 for notation.

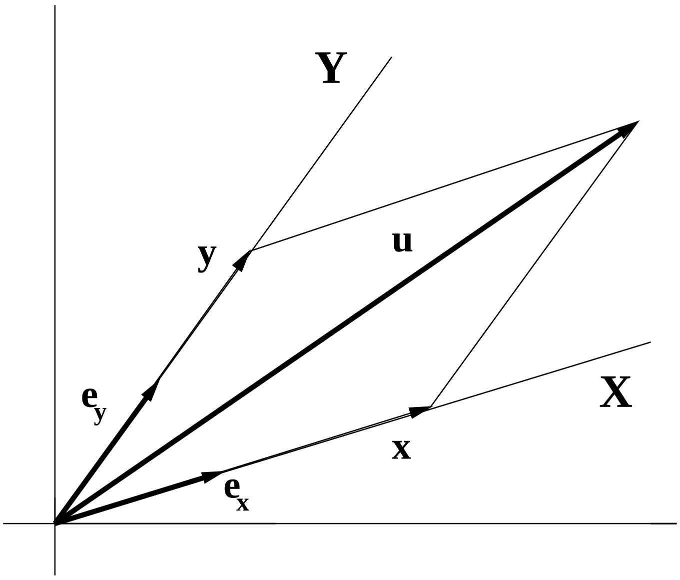

Figure 2: Linear projection in a two-dimensional space

Let $X = \mathcal{R}(P) = \mathcal{N}(I - P)$ and $Y = \mathcal{N}(P) = \mathcal{R}(I - P)$, and let $\boldsymbol{u} \in U$. Decomposing vector $\boldsymbol{u}$ into its X and Y components, we have

$$\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{y} = x\boldsymbol{e}_x + y\boldsymbol{e}_y, \quad P\boldsymbol{u} = \boldsymbol{x} \ .$$

Here $\boldsymbol{e}_x, \boldsymbol{e}_y$ are unit vectors of components $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$, respectively. Denote the corresponding co-basis by $\boldsymbol{e}^x, \boldsymbol{e}^y$. Expressing co-basis vector $\boldsymbol{e}^x$ in the original basis, we write

$$\boldsymbol{e}^x = \alpha\boldsymbol{e}_x + \beta\boldsymbol{e}_y \ .$$

Upon multiplying both sides with $\boldsymbol{e}^x$ and $\boldsymbol{e}^y$, we learn that $\alpha = \|\boldsymbol{e}^x\|^2, \quad \beta = (\boldsymbol{e}^x, \boldsymbol{e}^y)$ and, consequently,

$$\boldsymbol{e}^x = \|\boldsymbol{e}^x\|^2\boldsymbol{e}_x + (\boldsymbol{e}^x, \boldsymbol{e}^y)\boldsymbol{e}_y \ .$$

Multiplying, in turn, both sides with $\boldsymbol{e}_x$, we get

$$1 = \|\boldsymbol{e}^x\|^2\|\boldsymbol{e}_x\|^2 + (\boldsymbol{e}^x, \boldsymbol{e}^y)(\boldsymbol{e}_x, \boldsymbol{e}_y) \ , \tag{3.5}$$

or,

$$\|\boldsymbol{e}_x\|\|\boldsymbol{e}^x\| = [1 - (\boldsymbol{e}_x, \boldsymbol{e}_y)(\boldsymbol{e}^x, \boldsymbol{e}^y)]^{\frac{1}{2}} \ .$$

An analogous result holds for vector $\boldsymbol{e}_y$.

Now, $P\boldsymbol{u} = \boldsymbol{x} = x\boldsymbol{e}_x = (\boldsymbol{u}, \boldsymbol{e}^x)\boldsymbol{e}_x$. So,

$$\|P\boldsymbol{u}\| = |(\boldsymbol{u}, \boldsymbol{e}^x)|\|\boldsymbol{e}_x\| \leq \|\boldsymbol{e}_x\|\|\boldsymbol{e}^x\| \ \|\boldsymbol{u}\| \ .$$

Set $\boldsymbol{u} = \boldsymbol{e}^x$ to learn that, actually, $\|P\| = \|\boldsymbol{e}_x\|\|\boldsymbol{e}^x\|$.

Similarly, $\|I - P\| = \|\boldsymbol{e}_y\|\|\boldsymbol{e}^y\|$ and , by (3.5),

$$\|P\| = \|I - P\| \ .$$

Case 2: Space U arbitrary. Let $u \in U$, $\|u\| = 1$, be an arbitrary unit vector. Consider the at most two-dimensional space $U_0 = \text{span}\{u, Pu\}$. If dim$U_0 = 2$, we have by the two-dimensional result,

$$\begin{aligned} \|(I-P)u\|_U &= \|(I-P)u\|_{U_0} \\ &\leq \|I-P\|_{\mathcal{L}(U_0,U_0)} \\ &= \|P\|_{\mathcal{L}(U_0,U_0)} = \sup_{u_0 \in U_0, \|u_0\|=1} \|Pu_0\| \\ &\leq \sup_{u \in U, \|u\|=1} \|Pu\| = \|P\|_{\mathcal{L}(U,U)} \ . \end{aligned}$$

If dim$U_0 = 1$ then it must be $Pu = \alpha u$, for some α and, consequently, $Pu = P^2u = \alpha Pu$. Thus, either $(I - P)u = u$ or $(I - P)u = 0$. In both cases, $\|(I - P)u\|$ is bounded by $\|P\|$ which is always ≥ 1. We have proved, therefore, that $\|I - P\| \leq \|P\|$. Reversing the argument,

$$\|P\| = \|I - (I - P)\| \leq \|I - P\| \ ,$$

and, therefore, the two norms must be equal.

∎

REMARK 1 It is easy to construct a 2D counterexample for $\|I-P\| \neq \|P\|$ in a non-Hilbert Banach space. ∎

Proof of Theorem 4. Let $P_h \ : \ U \to U_h, \quad u \to u_h$ be the projection operator defined by (3.1). Then

$$\begin{aligned} \|u-u_h\| &= \|u-P_h u\| \\ &= \|u-w_h-P_h(u-w_h)\| \\ &= \|(I-P_h)(u-w_h)\| \\ &\leq \|I-P_h\| \, \|u-w_h\| = \|P_h\| \, \|u-w_h\| \\ &= \frac{\|b_h\|}{\gamma_h}\|u-w_h\| \, . \end{aligned}$$

∎

4 Brezzi's theorem

We turn now to Hilbert spaces only. Let V, Q be two Hilbert spaces, and let

$$\begin{aligned} &a(u,v) \qquad u,v \in V \, , \\ &b(p,v) \qquad p \in Q, v \in V \, , \end{aligned}$$

be two continuous bilinear forms. Consider a *mixed variational problem*,

$$\left\{ \begin{aligned} &a(u,v)+b(p,v) &&= f(v) \qquad \forall v \in V \\ &b(q,u) &&= g(q) \qquad \forall q \in Q \, , \end{aligned} \right. \tag{4.6}$$

where $f \in V', g \in Q'$. Typical examples of the mixed formulation include *saddle point problems* where $a(u,v)$ represents an energy and $b(q,u)$ is a constraint.

Assume, we are given a family of finite-dimensional subspaces $V_h \subset V$, $Q_h \subset Q$, and consider the corresponding Galerkin approximation,

$$\left\{ \begin{aligned} &a(u_h,v_h)+b(p_h,v_h) &&= f(v_h) \qquad \forall v_h \in V_h \\ &b(q_h,u_h) &&= g(q_h) \qquad \forall q_h \in Q_h \, . \end{aligned} \right. \tag{4.7}$$

If linear forms $f(v), g(q)$ are defined by left-hand sides of (4.6), the problem above is a special case of (3.3).

Introduce the subspace of finite element functions $v_h \in V_h$ that satisfy the discrete constraint,

$$V_{h,0} = \{v_h \in V_h \ : \ b(q_h, v_h) = 0 \quad \forall q_h \in Q_h\} \ . \tag{4.8}$$

THEOREM 5

Franco Brezzi, [2]

Assume the following discrete stability conditions are satisfied:

- *An* $\inf\sup$ *condition relating spaces* Q_h *and* V_h*,*

$$\inf_{q_h \neq 0} \sup_{v_h \neq 0} \frac{|b(q_h, v_h)|}{\|q_h\| \, \|v_h\|} =: \beta_h > 0 \tag{4.9}$$

- $\inf\sup$ *in the kernel condition,*

$$\inf_{0 \neq u_{h,0} \in V_{h,0}} \sup_{0 \neq v_{h,0} \in V_{h,0}} \frac{|a(u_{h,0}, v_{h,0})|}{\|u_{h,0}\| \, \|v_{h,0}\|} =: \alpha_h > 0 \ . \tag{4.10}$$

Then, there exist a constant C_h*, depending solely upon stability constants* β_h*,*α_h *and continuity constant* $\|a\|$ *such that*

$$\|u - u_h\|_U + \|p - p_h\|_Q \leq C_h \inf_{w_h \in V_h, r_h \in Q_h} \left(\|u - w_h\|_U + \|p - r_h\|_Q \right) \ . \tag{4.11}$$

∎

The proof of the theorem will be given in the next section.

5 Is Babuška equivalent to Brezzi ?

Introducing group variables $(u, p), (v, q)$ and a *big bilinear form,*

$$\mathcal{B}((u, p), (v, q)) = a(u, v) + b(p, v) + b(q, u) \ ,$$

we can recast Brezzi's mixed problem into Babuska's formulation,

$$\mathcal{B}((u_h, p_h), (v_h, q_h)) = \mathcal{B}((u, p), (v_h, q_h)) \quad \forall (v_h, q_h) \in V_h \times Q_h \ .$$

The actual goal of this note is to investigate the relation between the two sets of stability conditions.

Brezzi $\Rightarrow$ Babuška

We shall proceed in three steps. For simplicity, we shall drop index h in the notation.

Step1: Changing order of variables in (4.9), see section 2, we get

$$\inf_{[v]\neq 0} \sup_{q\neq 0} \frac{|b(q,v)|}{\|q\|\|[v]\|} = \beta \, .$$

Let u_0 be the orthogonal projection of u onto V_0. Then

$$\|u-u_0\| = \inf_{w_0\in V_0} \|u-w_0\| = \|[u]\| \leq \frac{1}{\beta}\|g\| \, ,$$

where f, and later g are defined by (4.6).

Step 2: Restricting ourselves to test functions $v = v_0 \in V_0$ in $(4.6)_1$, we get

$$a(u_0,v_0) = f(v_0) \quad \forall v_0 \in V_0 \, .$$

Thus, the $\inf\sup$ in the kernel condition implies that

$$\alpha\|u_0\| \leq \sup_{v_0\in V_0} \frac{|a(u_0,v_0)|}{\|v_0\|} = \sup_{v_0\in V_0} \frac{|f(v_0)|}{\|v_0\|} \leq \|f\| \, ,$$

and, consequently, $\|u_0\| \leq \frac{1}{\alpha}\|f\|$. By triangle inequality we get,

$$\|u\| \leq \|u-u_0\| + \|u_0\| \leq \frac{1}{\alpha}\|f\| + \frac{1}{\beta}\|g\| \, .$$

Step 3: Rewrite $(4.6)_1$ as

$$b(p,v) = f(v) - a(u,v) \, .$$

Then

$$\begin{aligned} \beta\|p\| \quad &\leq \sup_{v\neq 0} \frac{|b(p,v)|}{\|v\|} = \sup_{v\neq 0} \frac{|f(v)-a(u,v)|}{\|v\|} \\ &\leq \|f\| + \|a\| \, \|u\| \\ &\leq \|f\| + \|a\| \left(\frac{1}{\alpha}\|f\| + \frac{1}{\beta}\|g\| \right) \, . \end{aligned}$$

Thus,

$$\|p\| \leq \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)\|f\| + \frac{\|a\|}{\beta^2}\|g\| \, .$$

Summing all up, we get the final stability result,

$$\|u\| + \|p\| \leq \left\{ \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right) + \frac{1}{\alpha} \right\} \|f\| + \left\{ \frac{1}{\beta}\left(1+\frac{\|a\|}{\beta}\right) \right\} \|g\| \, .$$

In the end, we can conclude that Brezzi's stability conditions (plus continuity of form a[2]) imply Babuška's stability condition and, with the l^1-norm used for the group variable, we have the following estimate for the Babuška's $\inf\sup$ constant in terms of the Brezzi's constants,

$$\frac{1}{\gamma} \leq \max\left\{\frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)+\frac{1}{\alpha}, \frac{1}{\beta}\left(1+\frac{\|a\|}{\beta}\right)\right\} .$$

For a more elegant estimate in terms of the l^2-norm used for the group variable, see [5].

Babuška ⇒ Brezzi

This was the actual question that had intrigued me. Can we, in particular, *derive* Brezzi's conditions from the Babuška's condition, without guessing them?

So, let us assume the $\inf\sup$ condition for the *big bilinear form*,

$$\sup_{v\neq 0, q\neq 0} \frac{|a(u,v)+b(p,v)+b(q,u)|}{\|v\|+\|q\|} \geq \gamma\left(\|u\|+\|p\|\right) . \tag{5.12}$$

The first observation is easy. We set $u = 0$ in (5.12), and observe that the resulting supremum is attained for $q = 0$, to get the first Brezzi's condition,

$$\sup_{v\neq 0} \frac{|b(p,v)|}{\|v\|} \geq \gamma\|p\| ,$$

getting a bound $\beta \geq \gamma$.

Now, condition (5.12) implies that system (4.6) has a unique solution for any right-hand side f, g. Set $g = 0$ and restrict test functions in the first equation to $v = v_0 \in V_0$, comp. (4.8). We can conclude that equation,

$$\left\{\begin{array}{l} u_0 \in V_0 \\ a(u_0, v_0) = f(v_0) \quad \forall v_0 \in V_0 , \end{array}\right.$$

has a unique solution. The corresponding operator (matrix in the finite-dimensional setting) and its transpose are non-singular.

Now, restricting ourselves in (5.12) to $u = u_0 \in V_0$, we get,

$$\sup_{v\neq 0, q\neq 0} \frac{|a(u_0,v)+b(p,v)|}{\|v\|+\|q\|} \geq \gamma\left(\|u_0\|+\|p\|\right) .$$

Observing that the supremum is attained for $q = 0$, we conclude that,

$$\inf_{u_0\neq 0, p\neq 0} \sup_{v\neq 0} \frac{|a(u_0,v)+b(p,v)|}{(\|u_0\|+\|p\|)\|v\|} \geq \gamma .$$

[2] I believe, one cannot eliminate this assumption

We restrict now our big bilinear form to the Cartesian product $(V_0 \times Q) \times (V \times \{0\})$. Reversing the order of variables in the $\inf \sup$ condition above, we get

$$\inf_{[v]\neq 0} \sup_{u_0\neq 0, p\neq 0} \frac{|a(u_0,v)+b(p,v)|}{(\|u_0\|+\|p\|)\|[v]\|} \geq \gamma \, . \tag{5.13}$$

Here, the equivalence class $[v]$ is defined using the subspace

$$\begin{aligned} V_{00} &= \{v \in V \; : \; a(u_0,v)+b(p,v)=0 \quad \forall u_0 \in V_0, \forall p \in Q\} \\ &= \{v_0 \in V_0 \; : \; a(u_0,v_0)=0 \quad \forall u_0 \in V_0\} \, . \end{aligned}$$

But, by the argument above, space V_{00} is trivial and, therefore, condition (5.13) reduces to

$$\inf_{v\neq 0} \sup_{u_0\neq 0, p\neq 0} \frac{|a(u_0,v)+b(p,v)|}{(\|u_0\|+\|p\|)\|v\|} \geq \gamma \, .$$

Taking $v = v_0 \in V_0$, we conclude that

$$\inf_{v_0\neq 0} \sup_{u_0\neq 0} \frac{|a(u_0,v_0)|}{\|u_0\|\|v_0\|} \geq \gamma \, ,$$

or, equivalently,

$$\inf_{u_0\neq 0} \sup_{v_0\neq 0} \frac{|a(u_0,v_0)|}{\|u_0\|\|v_0\|} \geq \gamma \, .$$

Thus, Babuška's stability condition implies also Brezzi's $\inf \sup$ in the kernel condition, with a bound for constant $\alpha \; : \; \alpha \geq \gamma$.

References

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