A Stabilized Mixed Discontinuous Galerkin Method
for Darcy Flow

by

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Abstract

A new mixed, stabilized, discontinuous Galerkin formulation for Darcy flow is presented. The formulation combines several attributes not simultaneously satisfied by other methods: it is convergent for any combination of velocity and pressure interpolation higher than first-order, it exactly satisfies a mass balance on each element, and it passes two- and three-dimensional constant-flow “patch tests” for distorted element geometries. The key ingredient in the formulation is a volumetric, residual-based, stabilization term that does not involve any mesh-dependent parameters.
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1. Introduction

In a previous work, Masud and Hughes [30] presented a mixed stabilized finite element formulation of Darcy flow. Pressure interpolation could be discontinuous or continuous, but velocity interpolation was required to satisfy normal continuity. Convergence and robustness studies were performed for a variety of elements for which both normal and tangential velocity were taken to be continuous. In all cases, good results were obtained but we realized that enforcing tangential continuity of velocity imposed a non-physical constraint across layers in which material parameters were discontinuous (Wan [37]). This provided motivation to pursue development of a formulation in which tangential velocity was free to be discontinuous. To this end we develop herein a mixed, stabilized, discontinuous Galerkin formulation of Darcy flow. Some recent works on discontinuous Galerkin methods are [6,13,16-18,29,33,34,38]. In some respects, this is a straightforward generalization of Masud and Hughes [30]. However, the present methodology results in a unique combination of attributes which has not been attained by any other Darcy flow formulation. In particular, the method is stable and convergent for any combination of velocity and pressure interpolation, first-order and higher. The method is locally conservative in that each element satisfies an exact mass balance. The method produces two- and three-dimensional elements which satisfy constant-flow patch tests in distorted configurations. As far as we are aware, there are no other stable, three-dimensional elements which pass constant-flow patch tests in distorted configurations. This topic has been recently investigated thoroughly by Tom Russell and his colleagues [27,32], wherein a number of formulations are studied.

An outline of the remainder of the paper follows: In Section 2 we present the discontinuous Galerkin finite element formulation. In Section 3 we present the corresponding weak formulation of the continuous problem. An error analysis is presented in Section 4. A shortcoming of the analysis is that it requires interface pressure jump terms. Conjecturing that these terms are unnecessary, a number of numerical calculations are performed in Section 5
in which good results are attained without them. Therein we study one-dimensional elements employing various combinations of Lagrange polynomial interpolations, and two-dimensional triangles and quadrilaterals employing various combinations of standard polynomial interpolations in parametric coordinates. Recently, a more penetrating mathematical error analysis has been performed in which it is proved that the pressure jump terms are indeed unnecessary (see Brezzi et al. [10]). Additional numerical results are reported upon in Section 5 involving three-dimensional constant-flow patch tests, and a problem in which the exact solution exhibits discontinuous tangential velocity. Good results are obtained for all cases with the new formulation. The key ingredient in the formulation is a volumetric, residual-dependent stabilization term. (Some recent works on stabilized methods are [3,7,8,10-15,19-23,26,28-31,36,37].) Without this term, the formulation is not stable. An interesting observation, made in Brezzi et al. [10], is that the present method may be viewed as a stable, linear combination of two unstable methods, namely, Bassi and Rebay [4] and Baumann and Oden [5]. An appealing feature is that no mesh-dependent parameters are required. In Section 6, we draw conclusions.

2. The Finite Element Method

Consider a decomposition of $\Omega$ into open, non-overlapping, shape-regular element subdomains, $\Omega^e$, $e = 1, 2, ..., n_{el}$, where $n_{el}$ is the total number of elements. The element interiors and interior boundary (or “skeleton”) are denoted, respectively,

$$
\tilde{\Omega} = \bigcup_{e=1}^{n_{el}} \Omega^e \tag{1}
$$

$$
\tilde{\Gamma} = \left( \bigcup_{e=1}^{n_{el}} \Gamma^e \right) \setminus \Gamma \tag{2}
$$

where $\Gamma^e = \partial \Omega^e$ denotes the boundary of element $e$. Let

$$
\mathcal{V}^h = \{ v^h | v^h|_{\Omega^e} \in (P^k(\Omega^e))^{n_{ed}}, e = 1, 2, ..., n_{el} \} \tag{3}
$$
\[
\mathcal{Q}^h = \left\{ q^h | q^h|_{\Omega^e} \in \mathcal{P}^l(\Omega^e), \; e = 1, 2, \ldots, n_{el}, \; \int_{\Omega} q^h \, d\Omega = 0 \right\}
\]

(4)

\[
\mathcal{Z}^h = \mathcal{V}^h \times \mathcal{Q}^h
\]

(5)

where \( \mathcal{P}^l(\Omega^e) \) is the space of complete polynomials in global coordinates of order \( l \) on \( \Omega^e \).

Note that no continuity across \( \tilde{\Gamma} \) is assumed. The set-up allows elements of various standard shapes, such as, triangles, quadrilaterals, tetrahedra, hexahedra, and also non-standard polygons and polyhedra.

The finite element method is then given by: Find \( \mathbf{V}^h = \{ \mathbf{v}^h, p^h \} \in \mathcal{Z}^h \) such that, for all \( \mathbf{W}^h = \{ \mathbf{w}^h, q^h \} \in \mathcal{Z}^h \),

\[
B_{\text{stab}}^{DG}(\mathbf{W}^h, \mathbf{V}^h) = L_{\text{stab}}^{DG}(\mathbf{W}^h)
\]

(6)

where

\[
B_{\text{stab}}^{DG}(\mathbf{W}^h, \mathbf{V}^h) = B^{DG}(\mathbf{W}^h, \mathbf{V}^h) + \frac{1}{2} \left( \left( -\frac{\mu}{\kappa} \mathbf{w}^h + \nabla q^h, \frac{\kappa}{\mu} \mathbf{v}^h + \nabla p^h \right) \right)_{\tilde{\Omega}}
\]

\[
+ \left( \frac{\beta}{h_\perp} < \frac{\kappa}{\mu} > [q^h], [p^h] \right)_{\tilde{\Gamma}}
\]

(7)

\[
B^{DG}(\mathbf{W}^h, \mathbf{V}^h) = (\mathbf{w}^h, \frac{\mu}{\kappa} \mathbf{v}^h)_{\tilde{\Omega}} - (\text{div} \; \mathbf{w}^h, p^h)_{\tilde{\Omega}} + (q^h, \text{div} \; \mathbf{v}^h)_{\tilde{\Omega}} + ([\mathbf{w}^h], < p^h >)_{\tilde{\Gamma}}
\]

\[
- (q^h, [\mathbf{v}^h])_{\tilde{\Gamma}} + (\mathbf{w}^h \cdot \mathbf{n}, p^h)_{\Gamma} - (q^h, \mathbf{v}^h \cdot \mathbf{n})_{\Gamma}
\]

(8)

\[
L_{\text{stab}}^{DG}(\mathbf{W}^h) = L^{DG}(\mathbf{W}^h) - \frac{1}{2} \left( \left( -\frac{\mu}{\kappa} \mathbf{w}^h + \nabla q^h, \frac{\kappa}{\mu} \left( \frac{\rho}{g_c} \mathbf{g} \right) \right)_{\tilde{\Omega}}
\]

(9)

\[
L^{DG}(\mathbf{W}^h) = - (\mathbf{w}^h, \frac{\rho}{g_c} \mathbf{g})_{\tilde{\Omega}} + (q^h, \varphi)_{\tilde{\Omega}} - (q^h, \psi)_{\Gamma}
\]

(10)

where \([\cdot]\) and \(< \cdot >\) are the jump and mean-value operators, respectively. For future reference we note the important identities

\[
[w^h, p^h] = [w^h] < p^h > + < w^h > [p^h]
\]

(11)

\[
[v^h, q^h] = [v^h] < q^h > + < v^h > [q^h]
\]

(12)
We employ the Brezzi convention in which the jump of a vector is scalar-valued, and the jump of a scalar is vector-valued, viz.,

\[
[w^h] = w^h_+ \cdot n_+ + w^h_- \cdot n_-
\]

\[
= (w^h_+ - w^h_-) \cdot n_+
\]

\[
= (w^h_- - w^h_+) \cdot n_-
\]

(13)

\[
[p^h] = p^h_+ n_+ + p^h_- n_-
\]

\[
= (p^h_+ - p^h_-) n_+
\]

\[
= (p^h_- - p^h_+) n_-
\]

(14)

The jump operator is invariant under reversal of the \pm designations. The averages are defined as

\[
<w^h> = \frac{(w^h_+ + w^h_-)}{2}
\]

(15)

\[
<p^h> = \frac{(p^h_+ + p^h_-)}{2}
\]

(16)

Remarks

1. The discontinuous Galerkin interface and boundary terms in (8) were selected in the “skew form” as first introduced by Baumann and Oden [5]. This choice makes the stability of the method transparent.

2. The stabilization term on \( \Omega \) (see (7) and (9)) was first introduced by Masud and Hughes [30]. Note that there are no element length-scale parameters present in this term. The weighting function operator is in the “adjoint”, or “multiscale”, form, as originally advocated by Franca and Russo [20] (see also Hughes [24]). This term should only be added if the order of pressure interpolation is linear or higher (i.e., if \( l \geq 1 \)). If the pressure interpolation is constant, its gradient drops out and the term represents an inconsistent modification of the variational equation.
3. The interface stabilization term in (7) is of the form proposed by Hughes and Franca [26]. The element length scale, $h_\perp$, appearing in the integrals should be thought of as being assigned edge- or face-wise, and, roughly speaking, should be taken as a measure of element size perpendicular to the edge or face in question. An acceptable definition for most purposes is

$$h_\perp = \frac{\text{meas}(\Omega_+) + \text{meas}(\Omega_-)}{2 \text{meas}(\Gamma_\pm)}$$

(17)

where $\Omega_+$ and $\Omega_-$ share interface $\Gamma_\pm$ (see Figure 1). In one dimension we interpret this formula as the average, namely, $h_\perp = \frac{h_+ + h_-}{2}$. We assume throughout that the smallest element dimension divided by the maximum element diameter, denoted by $h$, is bounded from below by a constant $\alpha$, and that $\alpha \leq \min_e \frac{h_e}{h} \leq 1$. $\beta$ is assumed to be a positive, non-dimensional, $O(1)$ constant. Note that we assume that $\frac{\kappa}{\mu}$ is possibly discontinuous across element interfaces and we have accounted for this by employing the average $\langle \frac{\kappa}{\mu} \rangle = \left( \frac{\kappa_+}{\mu_+} + \frac{\kappa_-}{\mu_-} \right) / 2$. It may be more appropriate to employ a harmonic average of $\frac{\kappa}{\mu}$, rather than the arithmetic average, but we have not studied this issue sufficiently to say definitively. The $\beta$-stabilization term is necessary for the general convergence theorem established in the sequel. However, there is now considerable computational and mathematical evidence that it is unnecessary in many cases of practical interest (see Brezzi et al. [10]).

4. It will be useful in the sequel to employ an integrated-by-parts form of the method. In this case (8) is written as

$$B^{DG}(W^h, V^h) = (w^h, \frac{\mu}{\kappa}v^h)_\Omega + (w^h, \nabla p^h)_\Omega - (\nabla q^h, v^h)_\Omega - \langle w^h, [p^h] \rangle _\Gamma$$

$$+ (\langle q^h \rangle, < v^h >) _\Gamma$$

(18)

We refer to (18) as the “gradient form” and (8) as the “divergence form.” They are, of course, mathematically equivalent.

5. Observe that, if we replace $q^h$ by $-q^h$, then the method becomes symmetric, that is,
3. A Weak Formulation of the Continuous Problem

Let

\( \tilde{V} = H(\text{div}, \tilde{\Omega}) \)

\[ \tilde{V} = \{ w | w|_{\Omega^e} \in (L_2(\Omega^e))^{n_{sd}}, \text{div} w|_{\Omega^e} \in L_2(\Omega^e), \ e = 1, 2, \ldots, n_{el} \} \] (20)

\( \tilde{Q} = H^1(\tilde{\Omega}) \backslash \mathbb{R} \)

\[ \tilde{Q} = \{ q | q|_{\Omega^e} \in L_2(\Omega^e), \nabla q|_{\Omega^e} \in (L_2(\Omega^e))^{n_{sd}}, \ e = 1, 2, \ldots, n_{el}, \int_{\Omega} q \, d\Omega = 0 \} \] (21)

\( \tilde{Z} = \tilde{V} \times \tilde{Q} \) (22)
For further elaboration, see Brezzi-Fortin [9]. Note that $V^h \subset \widetilde{V}, Q^h \subset \widetilde{Q}$ and $Z^h \subset \widetilde{Z}$, and no continuity is assumed across element interfaces. These spaces ensure that the same variational equation describing the discrete case, namely (6), is well defined. Furthermore, integration-by-parts, used to obtain (18), is also valid, and both the “divergence” and “gradient” forms of $B^{DG}(\cdot, \cdot)$, namely, (8) and (18), respectively, are again mathematically equivalent. Thus, we may write the continuous problem as: Find $V = \{v, p\} \in \widetilde{Z}$, such that, for all $W = \{w, q\} \in \widetilde{Z}$,

$$B^{DG}_{\text{stab}}(W, V) = L^{DG}_{\text{stab}}(W)$$

(23)

Remarks

1. It is interesting to note that the relevant form of the continuous problem is inspired by the numerical method. Generally, one assumes the numerical method is implied by the weak form of the continuous problem. It seems that discontinuous Galerkin methods represent an interplay of discrete and continuous idea without a clear starting point.

2. The consistency of the weak formulation with the strong form of the problem can be investigated by performing integration-by-parts on (23) and obtaining the Euler-Lagrange form:

$$0 = B^{DG}_{\text{stab}}(W, V) - L^{DG}_{\text{stab}}(W)$$

$$= \left( \frac{\mu}{\kappa} w + \frac{1}{2} \left( - \frac{\mu}{\kappa} w + \nabla q \right), \kappa \left( \frac{\mu}{\kappa} v + \nabla p + \rho g_c \right) \right)_{\tilde{\Omega}}$$

Darcy’s law

$$- (\langle w \rangle - \frac{\beta}{h} \nabla q, \kappa [p])_{\Gamma}$$

(Continuity of $p$)

$$+ (q, \text{div} v - \varphi)_{\tilde{\Omega}}$$

(Balance of mass)

$$- (\langle q \rangle, [v])_{\Gamma}$$

(Continuity of $v_n$)

$$- (q, v_n - \psi)_{\Gamma}$$

(Boundary condition on $v_n$)
Note that each term represents a weighted residual of the strong solution (see Masud and Hughes [30]).

3. Note that $V^h, Q^h$ and $Z^h$ are closed linear subspaces of $\tilde{V}, \tilde{Q}$, and $\tilde{Z}$, respectively.

### 4. Error Analysis

Let $\| \cdot \|_D$ denote the $L_2(D)$ norm, where $D = \Omega, \tilde{\Omega}, \Gamma, \tilde{\Gamma}$, etc. We define “stability norms” $\forall W \in Z$ by

$$
\| W \|_{stab}^{DG} = \left( \| W \|_{stab}^{2} + \left( \frac{\beta}{\kappa} < \frac{\kappa}{\mu} > \right)^{\frac{1}{2}} \| q \|_{\Gamma}^{2} \right)^{\frac{1}{2}}
$$

$$
\| W \|_{stab} = \left( \frac{1}{2} \left( \frac{\mu}{\kappa} \| w \|_{\Omega}^{2} + \left( \frac{\kappa}{\mu} \| \nabla q \|_{\Omega}^{2} \right) \right) \right)^{\frac{1}{2}}
$$

**Lemma 1. (Stability).**

For all $W \in \tilde{Z}$,

$$
B_{stab}^{DG}(W, W) = (\| W \|_{stab}^{DG})^{2}
$$

**Proof.** The proof follows immediately from the definition of $B_{stab}^{DG}(\cdot, \cdot)$ (see (7) and (8)).

**Lemma 2. (Consistency).**

For all $W^h \in Z^h$,

$$
B_{stab}^{DG}(W^h, E) = 0
$$

where $E = V^h - V$ is the error in the finite element solution.

**Proof.** Substituting $W^h \in Z^h$ for $W$ in (23) and substracting from (6) yields the result.

**Interpolation Estimate**

Let

$$
V = H(div, \Omega)
$$
\[ Q = H^1(\Omega) \setminus \mathbb{R} \]  
\[ Z = \mathcal{V} \times Q \]  

These spaces are appropriate when the data \( \varphi \in L_2(\Omega), \frac{\rho}{y_c}g \in (L_2(\Omega))^{n_sd}, \psi \in H^{-\frac{1}{2}}(\Gamma) \), and the coefficient \( \frac{\kappa}{\mu} \in L^\infty(\Omega) \) is discontinuous but bounded above and below by positive constants. Note that \( \mathcal{V} \subset \tilde{\mathcal{V}}, Q \subset \tilde{Q} \) and \( Z \subset \tilde{Z} \). Let \( \mathbf{V}_I^h \in \mathcal{Z}^h \) be an “interpolate” of \( \mathbf{V} \in \mathcal{Z} \). We decompose the error in the usual way:

\[ E = \mathbf{V}_I^h - \mathbf{V} = (\mathbf{V}_I^h - \mathbf{V}_I^h) + (\mathbf{V}_I^h - \mathbf{V}) = \mathbf{E}_h + \mathbf{H} \]  

where \( \mathbf{E}_h = \{ e_v^h, e_p^h \} \in \mathcal{Z}^h \) and \( \mathbf{H} = \{ \eta_v, \eta_p \} \in \tilde{\mathcal{Z}} \). We have the following estimate for the interpolation error \( \mathbf{H} \):

\[ \| \mathbf{H} \|_{\text{stab}} \leq \tilde{C}_1 \left( \frac{h}{L} \right)^{\frac{k+1}{2}} |\mathbf{v}|_{k+1} + \tilde{C}_2 \left( \frac{h}{L} \right)^{l} |p|_{l+1} \]  

where \( h \) is the mesh parameter (i.e., the maximum element diameter), \( L \) is a characteristic dimension of the domain \( \Omega \), and \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are constants independent of \( h, \mathbf{v} \) and \( p \), but which depend on \( \frac{\kappa}{\mu} \) in the following way

\[ \tilde{C}_1 = C_1 \sup_{x \in \Omega} \left( \frac{\mu(x)}{\kappa(x)} \right)^{\frac{1}{2}} \]  
\[ \tilde{C}_2 = C_2 \sup_{x \in \Omega} \left( \frac{\kappa(x)}{\mu(x)} \right)^{\frac{1}{2}} L^{-1} \]  

in which \( C_1 \) and \( C_2 \) are non-dimensional constants, and \( |\cdot|_s \) is the \( s^\text{th} \) Sobolev seminorm defined by

\[ |\cdot|_s^2 = |\cdot|_{H^s(\Omega; L)}^2 = \int_{\Omega} (L^s \nabla \cdot \nabla (\cdot))^2 d\Omega \]
Trace Inequalities

Let $\Omega$ have a Lipschitz boundary and assume that the mesh is quasi-uniform. Then if $w \in (H^1(\tilde{\Omega}))^{n_{el}}$, 
\[
\sum_{e=1}^{n_{el}} \|w\|_{\tilde{\Gamma}^e}^2 \leq C_T \left\{ h^{-1} \|w\|_{\tilde{\Omega}}^2 + h \|\nabla w\|_{\tilde{\Omega}}^2 \right\}
\]  
where $C_T$ is a constant independent of $h$ and $w$. This is a standard result; see Arnold [2] and Agmon [1]. (The result of course also holds if we replace $w$ by a scalar-valued function $q \in H^1(\tilde{\Omega})$.) 

If we assume that $w^h \in V$, we have the stronger result 
\[
\sum_{e=1}^{n_{el}} \|w^h\|_{\tilde{\Gamma}^e}^2 \leq C_k h^{-1} \|w^h\|_{\tilde{\Omega}}^2
\]  
where $C_k$ depends on $k$, the polynomial order of functions in $V$. Note that $C_k$ is an increasing function of $k$. This result can be proved by consideration of the properties of the element eigenproblem 
\[
\int_{\Gamma^e} r^h \cdot w^h \, d\Gamma = \lambda^h \int_{\Omega^e} r^h \cdot w^h \, d\Omega \quad \forall r^h \in V
\]  

Lemma 3

Assume the trace inequality, (37), holds where $C_T$ is the constant in (37) and define $\tilde{C}_1$ by (34). Then if $\eta_v \in (H^1(\tilde{\Omega}))^{n_{el}}$, 
\[
\left\| \left( \frac{h^+}{\beta} \frac{\mu}{\kappa} > \right)^{1/2} \eta_v \right\|_{\tilde{\Gamma}} \leq \left( \frac{C_T}{2\beta} \right)^{1/2} \tilde{C}_1 \left( \frac{h}{L} \right)^{k+1} |v|_{k+1}
\]  
Proof

\[
\left\| \left( \frac{h^+}{\beta} \frac{\mu}{\kappa} > \right)^{1/2} \eta_v \right\|_{\tilde{\Gamma}}^2 = \int_{\tilde{\Gamma}} \left( \frac{h^+}{\beta} \frac{\mu}{\kappa} > \right)^{1/2} \eta_v \, d\Gamma \leq \frac{h}{2\beta} \sup_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) \left\{ \int_{\tilde{\Gamma}} (|\eta_v|^2 + |\eta_v|^2) \, d\Gamma + \int_{\tilde{\Gamma}} |\eta_v|^2 \, d\Gamma \right\}
\]
\[
= \frac{h}{2\beta} \sup_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) \sum_{e=1}^{n_{el}} \int_{\Gamma^e} |\eta_v|^2 \, d\Gamma
\]
\[ \leq \frac{C_T}{2\beta} \sup_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) \left( \int_{\Omega} |\eta_v|^2 \, d\Omega + h^2 \int_{\Omega} |\nabla \eta_v|^2 \, d\Omega \right) \]
\[ \leq \frac{C_T}{2\beta} \sup_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) C_1^2 \left( \frac{h}{L} \right)^{2(k+1)} |\mathbf{v}|_{k+1}^2 \]
\[ = \frac{C_T}{2\beta} \tilde{C}_1^2 \left( \frac{h}{L} \right)^{2(k+1)} |\mathbf{v}|_{k+1}^2 \quad (41) \]

**Lemma 4**

With the same assumptions as Lemma 3, and if \( \eta_p \in \left( H^1(\tilde{\Omega}) \right)^{n_{ad}} \), then

\[ \| \left( \frac{\beta}{h_{\perp}} < \frac{\kappa}{\mu} > \right)^{1/2} [\eta_p] \|_{\Gamma} \leq \left( \frac{2\beta}{\alpha} C_T \right) \tilde{C}_2 \left( \frac{h}{L} \right)^{1/2} |p|_{l+1} \quad (42) \]

**Proof**

\[ \| \left( \frac{\beta}{h_{\perp}} < \frac{\kappa}{\mu} > \right)^{1/2} [\eta_p] \|_{\Gamma}^2 = \int_{\Gamma} \left( \frac{\beta}{h_{\perp}} < \frac{\kappa}{\mu} > \right)^{1/2} [\eta_p]^2 \, d\Gamma \]
\[ \leq \frac{2\beta}{\alpha h} \sup_{x \in \Omega} \left( \frac{\kappa}{\mu} \right) \left\{ \int_{\Gamma} |\eta_p^+|^2 + |\eta_p^-|^2 \, d\Gamma + \int_{\Gamma} |\eta_p|^2 \, d\Gamma \right\} \]
\[ = \frac{2\beta}{\alpha h} \sup_{x \in \Omega} \left( \frac{\kappa}{\mu} \right) \left( \sum_{e=1}^{n_{ed}} \int_{\Gamma_e} |\eta_p|^2 \, d\Gamma \right) \]
\[ \leq \frac{2\beta}{\alpha} C_T \sup_{x \in \Omega} \left( \frac{\kappa}{\mu} \right) \left( h^{-2} \int_{\Omega} |\eta_p|^2 \, d\Omega + \int_{\Omega} |\nabla \eta_p|^2 \, d\Omega \right) \]
\[ \leq \frac{2\beta}{\alpha} C_T \sup_{x \in \Omega} \left( \frac{\kappa}{\mu} \right) C_2^2 L^{-2} \left( \frac{h}{L} \right)^{2l} |p|_{l+1}^2 \]
\[ = \frac{2\beta}{\alpha} C_T \tilde{C}_2^2 \left( \frac{h}{L} \right)^{2l} |p|_{l+1}^2 \quad (43) \]

**Lemma 5**

Let \( C_k \) denote the constant in the trace inequality (38). Then

\[ \| \left( \frac{h_{\perp}}{\beta} < \frac{\mu}{\kappa} > \right)^{1/2} < \mathbf{e}_v^h > \|_{\Gamma} \leq \gamma^{1/2} \left( \frac{\mu}{\kappa} \right)^{1/2} \mathbf{e}_v^h \|_{\Omega} \quad (44) \]
where

\[
\gamma = \frac{C_k}{2\beta} \sup_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) / \inf_{x \in \Omega} \left( \frac{\mu}{\kappa} \right)
\]  

(45)

Proof

\[
\left\| \left( \frac{\beta}{\mu} \right) > \frac{1}{2} \right\| < e_v^h > \right\|_\Gamma^2 = \int_\Gamma \frac{h}{2\beta} \sup_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) \left\{ \int_\Gamma (|e_v^h|^2 + |e_v^h|_\perp^2) d\Gamma + \int_\Gamma |e_v^h|^2 d\Gamma \right\}
\]

\[
= \frac{h}{2\beta} \sup_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) \sum_{e=1}^{n_{el}} \int_{\Gamma_e} |e_v^h|^2 d\Gamma
\]

\[
\leq \frac{C_k}{2\beta} \sup_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) \int_\Omega |e_v^h|^2 d\Omega
\]

\[
= \frac{C_k}{2\beta} \inf_{x \in \Omega} \left( \frac{\mu}{\kappa} \right) \int_\Omega |e_v^h|^2 d\Omega
\]

\[
= \gamma \left\| < \frac{\mu}{\kappa} > \right\|_\Omega^2
\]  

(46)

Theorem

\[
\| E^h \|_{DG}^{stab} \leq C \| H \|_{stab}
\]  

(47)

where \( C \) is a non-dimensional constant.

Proof of Theorem

\[
\left( \| E^h \|_{\text{stab}}^{DG} \right)^2 = B_{\text{stab}}^{DG} (E^h, E^h)
\]  

(Stability)

\[
= B_{\text{stab}}^{DG} (E^h, E - H)
\]  

(Definition of \( E^h \))

\[
= B_{\text{stab}}^{DG} (E^h, E) - B_{\text{stab}}^{DG} (E^h, H)
\]  

(Bilinearity)

\[
= -B_{\text{stab}}^{DG} (E^h, H)
\]  

(Consistency)

\[
= |B_{\text{stab}}^{DG} (E^h, H)|
\]  

(48)
where

\[
B_{\text{stab}}^{DG}(E^h, H) = B_{\text{stab}}^{DG}(e^h_v, e^h_p, \eta_v, \eta_p)
\]

\[
= (e^h_v, \frac{\mu}{\kappa} \eta_v)_{\Omega} - (\text{div} e^h_v, \eta_p)_{\Omega} + (e^h_p, \text{div} \eta_v)_{\Omega} + (\|e^h_v\|, \eta_p)_{\Gamma}
\]

\[
- \epsilon (e^h_v, [\eta_v])_{\Omega} + (e^h_v \cdot n, \eta_p)_{\Gamma} - (e^h_v, \eta_v \cdot n)_{\Gamma} - \frac{1}{2} (e^h_v, \frac{\mu}{\kappa} \eta_v)_{\Omega}
\]

\[
- \frac{1}{2} (e^h_v, \nabla \eta_p)_{\Omega} + \frac{1}{2} (\nabla e^h_v, \eta_v)_{\Omega} + \frac{1}{2} \left( \nabla e^h_v, \frac{\kappa}{\mu} \nabla \eta_p \right)_{\Omega} + \left( \frac{\beta}{h_{\perp}} < \frac{\kappa}{\mu} \right) [e^h_p, [\eta_p]]_{\Gamma}
\]

\[
= \frac{1}{2} (e^h_v, \frac{\mu}{\kappa} \eta_v)_{\Omega} + \frac{1}{2} (e^h_v, \nabla \eta_p)_{\Omega} - \frac{1}{2} (\nabla e^h_v, \eta_v)_{\Omega} - (e^h_v, [\eta_v])_{\Gamma}
\]

\[
+ (e^h_p, [\eta_v])_{\Gamma} + \frac{1}{2} \left( \nabla e^h_v, \frac{\kappa}{\mu} \nabla \eta_p \right)_{\Omega} + \left( \frac{\beta}{h_{\perp}} < \frac{\kappa}{\mu} \right) [e^h_p, [\eta_p]]_{\Gamma}
\]

\[
\leq \frac{1}{2} \left\{ \frac{\mu}{\kappa} \frac{1}{2} \|e^h_v\|^2_{\Omega} + \frac{1}{2} \frac{\mu}{\kappa} \|\eta_v\|^2_{\Omega} + \frac{\mu}{\kappa} \|\eta_p\|^2_{\Gamma} \right\}
\]

\[
+ \frac{\kappa}{\mu} \frac{1}{2} \left\{ \frac{1}{2} \|e^h_p\|^2_{\Omega} + \frac{1}{2} \frac{\kappa}{\mu} \|\eta_p\|^2_{\Gamma} \right\}
\]

\[
(49)
\]
We can hide the first term in curly brackets on the left-hand side by selecting

\[ \varepsilon_1 + \varepsilon_2 + 2 \varepsilon_4 \gamma = 1 \]  
(50)

\[ \varepsilon_3 + \varepsilon_6 = 1 \]  
(51)

\[ \varepsilon_5 + \varepsilon_7 = 1 \]  
(52)

To this end we choose

\[ \varepsilon_1 = \varepsilon_2 = \frac{1}{4}, \quad \varepsilon_4 = \frac{1}{4 \gamma} \]  
(53)

\[ \varepsilon_3 = \varepsilon_6 = \varepsilon_5 = \varepsilon_7 = \frac{1}{2} \]  
(54)

Then,

\[
\left( \| E^h \|_{\text{stab}}^{DG} \right)^2 \leq 6 \| H \|_{\text{stab}}^2 + 4 \left( 1 + 2 \gamma \right) \frac{\beta}{\alpha} C_T \tilde{C}_2 \left( \frac{h}{L} \right)^l |p|_{l+1} + \frac{C_T}{\beta} \tilde{C}_1^2 \left( \frac{h}{L} \right)^{k+1} |v|_{k+1}
\]

\[
\leq C^2 \| H \|_{\text{stab}}^2
\]  
(55)

**Corollary**

\[
\| E \|_{\text{stab}}^{DG} \leq C \left\{ \tilde{C}_1 \left( \frac{h}{L} \right)^{k+1} |v|_{k+1} + \tilde{C}_2 \left( \frac{h}{L} \right)^l |p|_{l+1} \right\}
\]  
(56)

**Remarks**

1. Note that under the hypothesis stated, we expect the exact pressure to be continuous almost everywhere. In this case, if the discontinuous pressure interpolation is capable of continuously interpolating the exact solution, then we may take \( \[ \eta_p \] = 0 \) almost everywhere. This means that all terms multiplied by \( \varepsilon_4, \varepsilon_7, \varepsilon_4^{-1}, \) and \( \varepsilon_7^{-1} \) in (49) are zero. Consequently, the term

\[
4 \left( 1 + 2 \gamma \right) \frac{2 \beta}{\alpha} C_T \tilde{C}_2 \left( \frac{h}{L} \right)^2 |p|_{l+1}
\]  
(57)
is absent from the right-hand side of (55). (57) is the only term in the proof which is potentially troublesome because it contains $\gamma$ in the numerator and $\alpha$ in the denominator. From Lemma 5, (45), we see that $\gamma$ scales with the range of material parameters, which may be large in engineering applications, and from Lemma 4, (42), that $\alpha$ scales with the ratio of the $\min_e h_e^e \perp$, to maximum element diameter, $h$. Both $\gamma$ and $\alpha$ have an adverse effect on the constant $C$ in the estimate as can be seen from (55) and (57). It is not known at this juncture whether or not this result is an artifact of the proof, or is fundamental to the method. It is gratifying to note that a sufficiently rich pressure interpolation can completely remove this difficulty. This observation may have important practical implications. An advantage of the stabilized formulation in this instance is that it accommodates high-order pressure interpolations, unlike the classical mixed Galerkin formulation of Darcy flow.

2. The constant $\beta$, which may be selected to optimize results, seems to be more benign. It appears in both the numerator of (57) and in the denominator of the last term in (55), namely,

$$\frac{CT}{\beta} \tilde{C}_1^2 \left( \frac{h}{L} \right)^{k+1} |v|_{l+1}$$

(58)

$\gamma$ scales with $\beta^{-1}$ so there is no problem in (57) if $\beta$ is small. On the other hand, when $\beta$ is small, (58) is a problem, and if $\beta$ is large, (57) is a problem. But there seems no need for $\beta$ to be either large or small. It should be thought of as an $O(1)$ constant for all elements and orders of interpolation.

3. The theorem establishes convergence in the stability norm as long as $\min\{k + 1, l\} \geq 1$. If we set $\beta = 0$, the theorem does not guarantee convergence of the constant part of the pressure in the pressure-discontinuous case. Furthermore, to control the constant part of the pressure we need to rely on “inf-sup” stability. Our numerical experience indicates that if velocity interpolation is of sufficiently high-order, and pressure is at least first-
order, then the constant part of the pressure is under control. For discontinuous pressure elements, discontinuous linear velocity is sufficient. If the velocity is continuous, it is sufficient to employ quadratic velocity (see Masud and Hughes [30]). Constant velocities seem unable to control the constant part of discontinuous pressure. In more recent work with F. Brezzi and L.D. Marini [10], we have mathematically established that, if $k$ and $l$ are both $\geq 1$, the $\beta$-term may be taken to be 0 and the convergence result still holds. Other interesting properties of the method and its relation to other discontinuous Galerkin methods are also described in [10].

4. In this paper we are primarily interested in elements which behave well when we omit the $\beta$-term. These elements have the appealing feature that there are no mesh-dependent terms. In the numerical studies herein, we focus on these cases.

5. If the velocity interpolation possesses continuous normal derivatives across element interfaces, and the $\beta$-term is omitted, the present formulation reduces to that of our prior study (Masud and Hughes [30]). This reference may be consulted for numerical tests involving continuous velocity, combined with continuous or discontinuous pressure.
5. Numerical Examples

In all cases, the pressure jump term is omitted (i.e., $\beta \equiv 0$).

5.1 Convergence study for one-dimensional elements

The domain under consideration is $\Omega = [0, 1]$. The exact pressure solution is given by

$$p = \sin 2\pi x$$

(59)

The velocity field is computed from Darcy’s law, in which $\rho g/g_c$ is taken to be zero and $\kappa/\mu = 1$; $\varphi$ is calculated by taking the derivative of the velocity field, and $\psi$ is calculated by taking its normal components. In specifying the boundary-value problem, $\varphi$ is prescribed over $\Omega$ while $\psi$ is prescribed weakly at the boundary. Uniform meshes were employed in obtaining the results presented in this section.

![Discontinuous elements in one dimension](image)

*Fig. 2.* Discontinuous elements in one dimension.
Figure 2 depicts the discontinuous velocity-pressure elements studied. Dots correspond to the velocity nodes and circles correspond to the pressure nodes. Note that, due to the omission of the pressure jump term, we have no guaranty of convergence. However, we observe that for all cases in which $k \geq 1$ and $l \geq 1$, convergence was obtained and observed to be at approximately optimal rates in the sense that the $L_2$–rate of velocity convergence was approximately the minimum of $k + 1$ and $l$, and the $L_2$–rate of pressure convergence was approximately the minimum of $k + 1$ and $l + 1$. It would appear that the most sensible elements would be on or near the equal-order diagonal of Figure 2. The most interesting cases thus seem to be ones for which $k = l + 1, k = l$, and $k + 1 = l$. The $k = l + 1$ cases might be viewed as fully discontinuous analogues of the Raviart-Thomas elements [35]. The $k + 1 = l$ cases are the most balanced in the spirit of the convergence theorem because the stability norm involves an $L_2$–velocity term and an $H^1$–pressure term. Sample convergence rate results for various combinations of the velocity and pressure interpolations are presented in Figures 3-9.
**Fig. 3.** Convergence rates for linear equal-order one-dimensional elements.

**Fig. 4.** Convergence rates for quadratic equal-order one-dimensional elements.
Fig. 5. Convergence rates for cubic equal-order one-dimensional elements.

Fig. 6. Convergence rates for quadratic-velocity linear-pressure one-dimensional elements.
Fig. 7. Convergence rates for cubic-velocity quadratic-pressure one-dimensional elements.
Fig. 8. Convergence rates for linear-velocity quadratic-pressure one-dimensional elements.

Fig. 9. Convergence rates for quadratic-velocity cubic-pressure one-dimensional elements.
5.2 Convergence study for one-dimensional non-uniform meshes

This section presents the convergence rates and line plots for one-dimensional non-uniform meshes.

![Convergence plot](image)

**Fig. 10.** Configuration of the coarsest non-uniform mesh for linear equal-order one-dimensional elements.

Figure 10 shows the configuration of the non-uniform mesh that is composed of coarse and fine zones. Nested meshes were constructed by bisecting the elements. Figure 11 presents convergence results for linear equal-order elements. Figures 12a and 12b depict the velocity and the pressure fields, respectively.

![Convergence rates plot](image)

**Fig. 11.** Convergence rates for linear equal-order one-dimensional elements. Non-uniform meshes.
Fig. 12a. Comparison of velocity profiles. Linear equal-order one-dimensional elements. Non-uniform meshes.

Fig. 12b. Comparison of pressure profiles. Linear equal-order one-dimensional elements. Non-uniform meshes.
The configuration of the coarsest non-uniform mesh for quadratic equal-order one-dimensional elements is the same as shown in Figure 11 except in this case there are five elements instead of ten. Once again nested meshes were constructed by bisecting the elements. Figure 13 presents the convergence rates and Figures 14a and 14b present the velocity and pressure fields, respectively.

Fig. 13. Convergence rates for quadratic equal-order one-dimensional elements. Non-uniform meshes.
Fig. 14a. Comparison of velocity profiles. Quadratic equal-order one-dimensional elements. Non-uniform meshes.

Fig. 14b. Comparison of pressure profiles. Quadratic equal-order one-dimensional elements. Non-uniform meshes.
Results for unequal-order elements are presented in Figures 15-18. The non-uniform meshes are constructed in the same way as for the quadratic equal-order case considered previously. The convergence results are similar to those obtained for the same elements on uniform meshes.

*Fig. 15.* Convergence rates for quadratic-velocity linear-pressure one-dimensional elements. Non-uniform meshes.
Fig. 16a. Comparison of velocity profiles. Quadratic-velocity linear-pressure one-dimensional elements. Non-uniform meshes.

Fig. 16b. Comparison of pressure profiles. Quadratic-velocity linear-pressure one-dimensional elements. Non-uniform meshes.
This test case is also an unequal order discontinuous velocity-pressure element with linear velocity and quadratic pressure fields. Nearly optimal $L_2$- and $H^1$-rates of convergence for the pressure field are attained.

![Graph showing convergence rates for linear-velocity quadratic-pressure one-dimensional elements. Non-uniform meshes.](image)

*Fig. 17.* Convergence rates for linear-velocity quadratic-pressure one-dimensional elements. Non-uniform meshes.
Fig. 18a. Comparison of velocity profiles. Linear-velocity quadratic-pressure one-dimensional elements. Non-uniform meshes.

Fig. 18b. Comparison of pressure profiles. Linear-velocity quadratic-pressure one-dimensional elements. Non-uniform meshes.
5.3 Convergence study for two-dimensional elements

The domain under consideration is $\Omega = [0, 1] \times [0, 1]$, and the exact pressure solution is given by

$$p = \sin 2\pi x \sin 2\pi y$$

(60)

As in the one-dimensional case, the velocity field is computed from Darcy’s law in which $\rho g / g_c$ is taken to be zero and $\kappa / \mu = 1$; $\varphi$ is computed from the divergence of the velocity field, and $\psi$ is determined by its normal component. In specifying the boundary-value problem, $\varphi$ is prescribed over $\Omega$ while $\psi$ is prescribed weakly at the boundary. A contour plot and elevation plot of the exact pressure are shown in Figure 19. Figures 20a,b present contours of the components of the exact velocity field.

The elements employed in the study are shown in Figure 21 and representative meshes used in the studies are shown in Figure 22. The element mesh parameter, $h$, is taken to be the edge length of the elements for quadrilaterals, and the short-edge length for triangles. For all cases considered, convergence results consistent with the one-dimensional cases were obtained.

Figures 23a and 23b show the pressure field for bilinear equal-order elements. Likewise, Figures 24a,b and Figures 25a,b present the components of the velocity field.
Fig. 19a. Contour plot of the exact pressure field.

Fig. 19b. Elevation plot of the exact pressure field.
Fig. 20a. Contour plot of the $v_x$ component of the exact velocity field.

Fig. 20b. Contour plot of the $v_y$ component of the exact velocity field.
Fig. 21a. Discontinuous equal-order elements in two dimensions.

Fig. 21b. Discontinuous unequal-order elements in two dimensions.
Figs. 22a. Meshes of 200 triangles and 100 quadrilaterals.

Figs. 22b. Meshes of 800 triangles and 400 quadrilaterals.
Fig. 23a. Contour plot of the pressure field for bilinear equal-order quadrilaterals. 100-element mesh.

Fig. 23b. Contour plot of the pressure field for bilinear equal-order quadrilaterals. 400-element mesh.
Fig. 24a. Contour plot of the $v_x$ component of the velocity field for bilinear equal-order quadrilaterals. 100-element mesh.

Fig. 24b. Contour plot of the $v_x$ component of the velocity field for bilinear equal-order quadrilaterals. 400-element mesh.
Fig. 25a. Contour plot of the $v_y$ component of the velocity field for bilinear equal-order quadrilaterals. 100-element mesh.

Fig. 25b. Contour plot of the $v_y$ component of the velocity field for bilinear equal-order quadrilaterals. 400-element mesh.
Convergence rates are presented in Figures 26-33. The results are consistent with the corresponding one-dimensional cases.
Fig. 26. Convergence rates for equal-order bilinear quadrilaterals.

Fig. 27. Convergence rates for equal-order linear triangles.
Fig. 28. Convergence rates for equal-order biquadratic quadrilaterals.

Fig. 29. Convergence rates for equal-order quadratic triangles.
Fig. 30. Convergence rates for biquadratic-velocity bilinear-pressure quadrilaterals.

Fig. 31. Convergence rates for bilinear-velocity biquadratic-pressure quadrilaterals.
**Fig. 32.** Convergence rates for quadratic-velocity linear-pressure triangles.

**Fig. 33.** Convergence rates for linear-velocity quadratic-pressure triangles.
5.4 Convergence study for two-dimensional distorted meshes

We performed a convergence study over structured, distorted meshes. Figure 34 presents the convergence rates for meshes composed of bilinear equal-order quadrilaterals. Similar trends were observed for the other element types and are therefore not shown. Figures 35-37 present profiles of the pressure and velocity.

![Graph showing convergence rates for equal-order bilinear quadrilaterals (distorted meshes).](image)

*Fig. 34.* Convergence rates for equal-order bilinear quadrilaterals (distorted meshes).
Fig. 35a. Contour plot of the pressure field for bilinear equal-order quadrilaterals. 100-element mesh.

Fig. 35b. Contour plot of the pressure field for bilinear equal-order quadrilaterals. 400-element mesh.
Fig. 36a. Contour plot of the $v_x$ component of the velocity field for bilinear equal-order quadrilaterals. 100-element mesh.

Fig. 36b. Contour plot of the $v_x$ component of the velocity field for bilinear equal-order quadrilaterals. 400-element mesh.
Fig. 37a. Contour plot of the $v_y$ component of the velocity field for bilinear equal-order quadrilaterals. 100-element mesh.

Fig. 37b. Contour plot of the $v_y$ component of the velocity field for bilinear equal-order quadrilaterals. 400-element mesh.
5.5 The five-spot problem

This section presents numerical results for the five-spot problem. The square domain shown in Figure 38 has prescribed velocity at the source and the sink. Due to symmetry of the problem, zero normal flow is prescribed along the boundaries. We assumed the divergence of the velocity field, $\varphi$, consists of Dirac delta functions acting at source and sink locations, with strength $+\frac{1}{4}$ and $-\frac{1}{4}$, respectively. We calculated an equivalent distribution of normal velocity, $\psi$, and drove the problem with $\psi$, setting $\varphi = 0$. In the case of linear velocity elements, we assumed a linear distribution of $\psi$ along the external edges of the corner elements, which is zero at the nodes adjacent to the corner nodes. This uniquely determines the distribution of $\psi$ on the edge (see Fig. 39a). In the case of quadratic velocity elements, we assumed a parabolic distribution along the external edges of the corner elements, which is zero, and has zero derivative, at the element vertex nodes away from the corner. Again, this uniquely defines the distribution of $\psi$ along the edge (see Fig. 39b).

The lower left-hand corner represents the source, or injection well, while the upper right
hand corner represents the sink, or production well. The exact pressure is shown in an elevation plot in Figure 40.

Pressure contours for bilinear equal-order quadrilaterals are shown in Figure 41. The circular nature of the exact solution about the source and sink are apparent in the numerical results. Similar trends were observed for the other element types and are therefore not shown here.

Fig. 39. Five-spot problem. Distribution of $\psi$ along the corner elements at the production well. The distribution of $\psi$ at the injection well is the same with opposite sign.
Fig. 40. Elevation plot of the exact pressure field for the five-spot problem.
Fig. 41a. Five spot problem. Pressure field for bilinear equal-order quadrilaterals. 100-element mesh.

Fig. 41b. Five spot problem. Pressure field for bilinear equal-order quadrilaterals. 400-element mesh.
5.6 The checkerboard domain

This simulation tests the formulation for cases in which there are abrupt changes in the permeability parameter. We consider the five-spot problem described earlier, now zoned as shown in Figure 42. Figures 43-a and 43-b show pressures for $\frac{\kappa}{\mu} = 1$ in zones I and IV, and $\frac{\kappa}{\mu} = 0.01$ in zones II and III, for bilinear equal-order quadrilaterals. Similarly, Figure 44-a and 44-b show pressures for $\frac{\kappa}{\mu} = 1$ in zones I and IV, and $\frac{\kappa}{\mu} = 0.001$ in zones II and III. In both the cases there are no oscillations in the pressure field, an indication of robustness.

![Checkerboard domain with sharp change in permeability value.](image)

*Fig. 42.* Checkerboard domain with sharp change in permeability value.
Fig. 43a. Five-spot problem for the checkerboard domain. Pressure field for bilinear equal-order quadrilaterals. 100-element mesh. Zones I and IV: \( \frac{\kappa}{\mu} = 1 \). Zones II and III: \( \frac{\kappa}{\mu} = 0.01 \).

Fig. 43b. Five-spot problem for the checkerboard domain. Pressure field for bilinear equal-order quadrilaterals. 400-element mesh. Zones I and IV: \( \frac{\kappa}{\mu} = 1 \). Zones II and III: \( \frac{\kappa}{\mu} = 0.01 \).
Fig. 44a. Five-spot problem for the checkerboard domain. Pressure field for bilinear equal-order quadrilaterals. 100-element mesh. Zones I and IV: $\frac{\kappa}{\mu} = 1$. Zones II and III: $\frac{\kappa}{\mu} = 0.001$.

Fig. 44b. Five-spot problem for the checkerboard domain. Pressure field for bilinear equal-order quadrilaterals. 400-element mesh. Zones I and IV: $\frac{\kappa}{\mu} = 1$. Zones II and III: $\frac{\kappa}{\mu} = 0.001$. 
5.7 Three-dimensional patch tests

A number of constant-flow “patch tests” have been performed to assess the behavior of the formulation in three dimensions when elements have non-constant jacobian determinants. Patch tests have been used by engineers to assess convergence behavior of elements and to determine if elements are programmed correctly. A discussion of patch tests in the context of diffusion and elasticity problems is presented in Hughes [25], Chapter 4. In the present circumstances, boundary conditions are set to be consistent with a constant velocity vector and linear pressure variation. The requirement of the patch test is that the computed solution be exact pointwise. Interest in patch tests for Darcy flow has been stimulated by Naff et al. [32] wherein it has been shown that three-dimensional Raviart-Thomas elements fail constant-flow patch tests for elements with non-constant jacobians. Efforts to develop improved three-dimensional elements have naturally ensued. As far as we are aware, no previous formulation is entirely satisfactory in three dimensions.

Sample domains and meshes are shown in Figures 45 and 46. A mesh of eight equal-order trilinear hexahedral elements is employed in each test. Element boundaries are not parallel, resulting in non-constant element jacobian determinants. In each test, the boundary velocities are imposed strongly, consistent with a constant velocity vector, \( v = c \). The magnitude of \( c \) is \( O(1) \). The pressure at one corner node is also imposed strongly to render the solution of the boundary-value problem unique. The material parameters are: \( \frac{\kappa}{\mu} = 1 \) and \( \frac{\rho g}{g_c} = 0 \). In all cases, the solution was found to be exact.

A more involved patch test has been considered by Naff et al. [32] in which a cube of edge length 5 is decomposed into five layers of 25 non-uniform elements each. The elements have planar faces, with “quasi-random” orientation as shown in Figures 47 and 48. The constant-flow patch test is specified as in the previous case. None of the elements considered by Naff et al. [32] were able to exactly solve this problem. The present method again provides exact solutions.
Fig. 45. Three dimensional constant flow patch test. Configuration 1 mesh and pressure contours.

Fig. 46. Three dimensional constant flow patch test. Configuration 2 mesh and pressure contours.
Fig. 47. Three dimensional constant flow patch test. Configuration with quasi-random orientation of planar faces.

Fig. 48. Three dimensional constant flow patch test. Configuration with quasi-random orientation of planar faces. Views of interior mesh.
5.8 A comparison of continuous and discontinuous elements

This problem illustrates the importance of a Darcy flow formulation to allow discontinuous tangential velocity across element boundaries. The present discontinuous stabilized formulation is compared with the continuous version presented earlier (see Masud and Hughes [30]). The domain, problem specification, and mesh are shown in Figure 49. There are five horizontal, two-dimensional layers. In each layer $\frac{k}{\mu}$ is constant but takes on different values in each layer. The elements are equal-order, bilinear and square, with edge length 1. Boundary conditions are set to be consistent with a constant velocity and linear pressure variation in the $x$-direction in each layer. While zero normal-flow condition is set on the top and bottom boundaries, the normal velocity on the left-hand and right-hand boundaries is set to $\frac{k}{\mu}$ in each layer. The exact solution of the problem is zero vertical velocity everywhere, constant horizontal velocity equal to $\frac{k}{\mu}$ in each layer, and the same horizontal linear pressure variation in all layers. A reference pressure in the corner of the domain is set strongly to ensure a unique pressure solution. Results for the present, fully-discontinuous formulation using both bilinear pressure and velocity interpolations are presented in Figure 50. Note that the solution is exact pointwise.

In contrast, elements in which the velocity is taken to be continuous, as studied in Masud and Hughes [30], do not allow tangential velocity to be discontinuous. The configuration in Figure 51 resembles that in Figure 49, but releases the need to set discontinuous velocity boundary conditions using continuous velocities. A volumetric source and sink, $\phi$, is specified in each element adjacent to the left-hand and right-hand boundaries. The value of $\phi$ is $\frac{k}{\mu}$ in each layer. The zero normal-flow condition is set on all boundaries. The exact solution of the problem is zero vertical velocity everywhere, and constant horizontal velocity equal to $\frac{k}{\mu}$ in each layer between the elements where there are sources and sinks. In those elements, the horizontal velocity varies linearly in $x$ between the boundary value of zero and $\frac{k}{\mu}$. Figure 52 shows the velocity profile in the middle of all layers. The non-physical enforcement of
tangential velocity continuity creates overshoots and undershoots about the interfaces between layers. (In fact, it was these results which first motivated us to pursue development of a fully-discontinuous formulation; see Wan [37].)
Fig. 49. The layered system. Configuration 1.

Fig. 50. Velocity profile for the fully-discontinuous stabilized method.
Fig. 51. The layered system. Configuration 2.

Fig. 52. Velocity profile for two variants of the stabilized continuous-velocity method.
6. Conclusions

We have developed a new, mixed, stabilized, discontinuous Galerkin, finite element formulation for Darcy flow. The method is a generalization of previous work in which conforming velocity fields were required. The new approach possesses a unique combination of attributes: (1) it is stable and convergent for all velocity and pressure interpolations, first-order and higher, (2) it is locally strongly mass conservative; and (3) it passes constant-flow “patch tests” in two and three dimensions when elements are distorted. This last attribute is apparently not shared by existing formulations, including the popular, conforming Raviart-Thomas elements (see Naff et al. [32]). The key ingredient in the new approach is a volumetric stabilization term, first introduced in Masud and Hughes [30], which does not involve mesh-dependent parameters. A number of numerical calculations are presented verifying the good behavior of the method, including one exhibiting discontinuous behavior across material layers.

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