Conservation properties for the Galerkin and stabilised forms of the advection-diffusion and incompressible Navier-Stokes equations

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A common criticism of continuous Galerkin finite element methods is their perceived lack of conservation. This may in fact be true for incompressible flows when advective, rather than conservative, weak forms are employed. However, advective forms are often preferred on grounds of accuracy despite violation of conservation. It is shown here that this deficiency can be easily remedied, and conservative procedures for advective forms can be developed from multiscale concepts. As a result, conservative stabilised finite element procedures are presented for the advection-diffusion and incompressible Navier-Stokes equations.

Keywords

Conservation, continuous Galerkin methods, stabilised methods, advection-diffusion equation, Navier-Stokes equations, multiscale methods.

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1 Introduction

Conservation is a highly sort after property in numerical simulations of transport phenomena. Global conservation is intuitively appealing and often used for validating a computational model. A common criticism of continuous Galerkin methods is that they are not globally or locally conservative. This misunderstanding arises from the problem that it is not possible in the general case to set the weight function equal to unity in a continuous Galerkin method as the weight function must vanish where Dirichlet boundary conditions are applied. Recently, this issue has been clarified by Hughes et al. [1], where it is was shown that continuous Galerkin methods for advection-diffusion equations are both globally and locally conservative.

For advection-diffusion problems, when the weak form is written in advective, rather than conservative, form and the flow field is divergence-free (at least in an appropriate weak sense), global conservation of the advected quantity is preserved. However, this condition is not often satisfied by stable finite element methods based on the Galerkin formulation and when the velocity field is computed from a pressure-stabilised finite element formulation. This leads to a lack of conservation of the transported quantity, despite the method being convergent.

It is shown here how global conservation can be preserved when using common stabilised finite element methods through a small residual-based modification of existing formulations. As a starting point, global conservation for stabilised methods applied to the advection-diffusion equation is proven for the case where the advective velocity is known to be solenoidal. The examination is then extended to the case where the velocity comes from the solution of a stabilised incompressible flow problem and the weak form is in the advective, rather than conservative, form. It is shown that satisfaction of the inf-sup condition will, in general, preclude conservation for classical Galerkin methods, and stabilisation of the continuity equation in common stabilised finite element formulations results in global conservation not being guaranteed. Considering the governing equations in a multiscale context [2, 3] points the way to a formulation which is globally conservative. The procedure is then generalised to the Navier-Stokes equations. It is again shown that the addition of particular residual-based terms to the stabilised formulation leads to correct conservation of momentum. Implementation of the procedures requires only minor additions to existing stabilised finite element codes.

2 Conservation for the scalar advection-diffusion equation

2.1 Strong and weak forms

Before examining conservation properties, it is useful to define carefully the problem (for background, see Hughes et al. [4]). Consider Ω, an open, bounded region in $\mathbb{R}^d$, where d is the spatial dimension. The boundary of Ω is denoted by $\Gamma = \partial\Omega$, and the outward unit normal to Γ is denoted $\boldsymbol{n} = (n_1, n_2, \ldots, n_d)$. Given a solenoidal advective velocity field $\boldsymbol{a}$ ($\nabla \cdot \boldsymbol{a} = 0$), consider the following definitions:

$$a_n = \boldsymbol{n} \cdot \boldsymbol{a} \tag{1}$$

$$a_n^+ = \frac{a_n + |a_n|}{2} \tag{2}$$

$$a_n^- = \frac{a_n - |a_n|}{2}. \tag{3}$$

Note that a_n^+ is equal to a_n at an outflow boundary, and is equal to zero at an inflow boundary. Conversely, a_n^- is equal to a_n at an inflow boundary, and is equal to zero at an

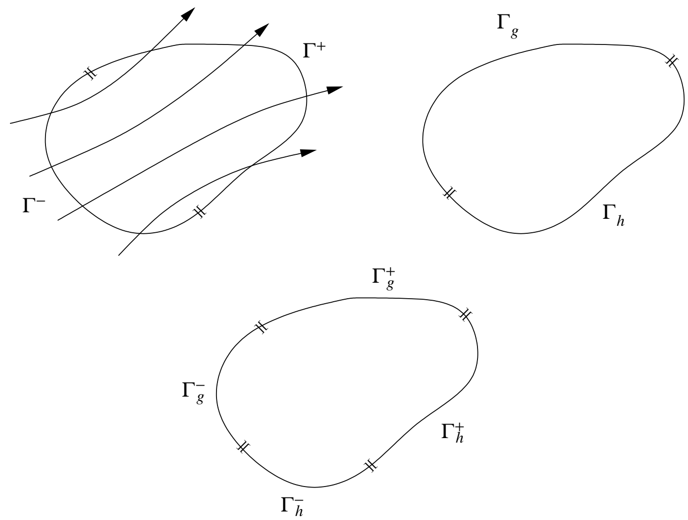

Figure 1: Definition of boundary partitions.

outflow boundary. Let $\{\Gamma^-, \Gamma^+\}$ and $\{\Gamma_g, \Gamma_h\}$ be partitions of Γ, defined by:

$$\Gamma^- = \{\boldsymbol{x} \in \Gamma \mid a_n(\boldsymbol{x}) < 0\} \qquad \text{inflow boundary} \tag{4}$$

$$\Gamma^+ = \Gamma - \Gamma^- \qquad \text{outflow boundary} \tag{5}$$

and

$$\Gamma_g^\pm = \Gamma_g \cap \Gamma^\pm \tag{6}$$

$$\Gamma_h^\pm = \Gamma_h \cap \Gamma^\pm. \tag{7}$$

The various partitions are illustrated in Figure 2.1.

Denoting the diffusivity as κ, which is assumed to be constant and positive ($\kappa > 0$), the following fluxes are defined:

$$\boldsymbol{\sigma}^a(\phi) = -\boldsymbol{a}\phi \tag{8}$$

$$\boldsymbol{\sigma}^d(\phi) = \kappa\nabla\phi \tag{9}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^a + \boldsymbol{\sigma}^d \tag{10}$$

$$\sigma_n^a = \boldsymbol{n}\cdot\boldsymbol{\sigma}^a \tag{11}$$

$$\sigma_n^d = \boldsymbol{n}\cdot\boldsymbol{\sigma}^d \tag{12}$$

$$\sigma_n = \boldsymbol{n}\cdot\boldsymbol{\sigma}. \tag{13}$$

Advective-diffusive transport of a scalar ϕ is assumed to be governed by:

$$-\nabla\cdot\boldsymbol{\sigma}(\phi) = f \qquad \text{in } \Omega \tag{14}$$

$$\phi = g \qquad \text{on } \Gamma_g \tag{15}$$

$$-a_n^-\phi + \sigma_n^d(\phi) = h \qquad \text{on } \Gamma_h \tag{16}$$

where $f:\Omega\to\mathbb{R}$, $g:\Gamma_g\to\mathbb{R}$ and $h:\Gamma_h\to\mathbb{R}$ are prescribed. The boundary condition on Γ_h can be interpreted as:

$$h = \begin{cases} h^- & \text{on } \Gamma_h^- \\ h^+ & \text{on } \Gamma_h^+ \end{cases} \tag{17}$$

where h^- is the total flux and h^+ is the diffusive flux,

$$\sigma_n(\phi) = h^- \qquad \text{total flux boundary condition} \tag{18}$$

$$\sigma_n^d(\phi) = h^+ \qquad \text{diffusive flux boundary condition.} \tag{19}$$

For the variational formulation of the advection-diffusion equation, the following function spaces are required:

$$\mathcal{S} = \left\{ \phi \in H^1(\Omega) \;|\; \phi = g \text{ on } \Gamma_g \right\} \tag{20}$$

$$\mathcal{V} = \left\{ w \in H^1(\Omega) \;|\; w = 0 \text{ on } \Gamma_g \right\}. \tag{21}$$

It is useful to define another function space $\mathcal{W}$, such that

$$H^1(\Omega) = \mathcal{V} \oplus \mathcal{W} \tag{22}$$

The standard variational problem for advection-diffusion consists of: find $\phi \in \mathcal{S}$ such that

$$B(w, \phi) = L(w) \qquad \forall\, w \in \mathcal{V} \tag{23}$$

where

$$B(w, \phi) = (\nabla w, \boldsymbol{\sigma}(\phi))_\Omega + \left(w, a_n^+ \phi\right)_{\Gamma_h} \tag{24}$$

and

$$L(w) = (w, f)_\Omega + (w, h)_{\Gamma_h}. \tag{25}$$

Consistency of the variational form is easily proven by integrating equation (23) by parts,

$$\begin{aligned} 0 &= B(w, u) - L(w) \\ &= -(w, \nabla \cdot \boldsymbol{\sigma}(\phi) + f)_\Omega + \left(w, -a_n^- + \sigma_n^d(\phi) - h\right)_{\Gamma_h} \end{aligned} \tag{26}$$

which, under suitable smoothness hypotheses, implies the original strong form of the governing boundary-value problem. Stability can be proven by showing that $B(w, \phi)$ is coercive,

$$\begin{aligned} B(w,w) &= (\nabla w, -\boldsymbol{a}w + \kappa\nabla w)_\Omega + \left(w, a_n^+ w\right)_{\Gamma_h} \\ &= \frac{1}{2}\left(w, (\nabla\cdot\boldsymbol{a})\, w\right)_\Omega - \frac{1}{2}\left(w, a_n w\right)_{\Gamma_h} + \kappa\|\nabla w\|_\Omega^2 + \left(w, a_n^+ w\right)_{\Gamma_h} \\ &= \frac{1}{2}\left(w, (\nabla\cdot\boldsymbol{a})\, w\right)_\Omega + \kappa\|\nabla w\|_\Omega^2 + \frac{1}{2}\left\||a_n|^{1/2} w\right\|_{\Gamma_h}^2 . \end{aligned} \tag{27}$$

If the advective field is divergence-free, equation (27) is equivalent to:

$$B(w,w) = \kappa\|\nabla w\|_\Omega^2 + \frac{1}{2}\left\||a_n|^{1/2} w\right\|_{\Gamma_h}^2 \tag{28}$$

which proves that the stability is assured. The boundary conditions considered here were described in Hughes et al. [4], and they result in a well-posed variational problem if the flow field is divergence-free.

Proving conservation for the weak form of the advective-diffusion equation in both conservative and advective formats is straightforward for the case $\Gamma_g = \emptyset$. Considering equation (23) and setting $w = 1$,

$$\begin{aligned} 0 &= B(1,\phi) - L(1) \\ &= \int_\Gamma a_n^+ \phi \, d\Gamma - \int_\Omega f \, d\Omega - \int_\Gamma h \, d\Gamma, \end{aligned} \tag{29}$$

which can also be expressed as:

$$0 = -\left(\int_{\Gamma^+} \left(-a_n\phi + h^+\right) \, d\Gamma + \int_{\Gamma^-} h^- \, d\Gamma + \int_\Omega f \, d\Omega\right) \tag{30}$$

which proves that ϕ is conserved (recalling that h^+ is the diffusive flux, hence $(h^+ - a_n\phi)\,|_{\Gamma^+}$ is the total flux, and h^- is also the total flux). Establishing conservation for the case $\Gamma_g \neq \emptyset$ is more complicated as it is not possible to set $w = 1$ in the entire domain since w must be equal to zero on Γ_g. Proving conservation requires the definition of an auxiliary flux h_g on the boundary Γ_g [1]. Consider the problem: find $\phi \in \mathcal{S}$ and $h_g \in \mathcal{W}$ such that

$$B(w, \phi) = L(w) + (w, h_g)_{\Gamma_g} \quad \forall\, w \in H^1(\Omega)\,. \tag{31}$$

Recalling equation (22), this problem can be alternatively expressed as: find $\phi \in \mathcal{S}$ and $h_g \in \mathcal{W}$ such that

$$B(w, \phi) = L(w) \quad \forall\, w \in \mathcal{V} \tag{32}$$

$$B(q, \phi) = L(q) + (q, h_g)_{\Gamma_g} \quad \forall\, q \in \mathcal{W}. \tag{33}$$

Setting $w = 1$ in equation (31), which is possible since $1 \in H^1(\Omega)$,

$$\begin{aligned} 0 &= B(1, \phi) - L(1) - (1, h_g)_{\Gamma_g} \\ &= -\left(\int_{\Gamma_h^+} \left(-a_n \phi + h^+\right) \, d\Gamma + \int_{\Gamma_h^-} h^- \, d\Gamma + \int_{\Omega} f \, d\Omega + \int_{\Gamma_g} h_g \, d\Gamma \right) \end{aligned} \tag{34}$$

which proves that ϕ is conserved.

Conservation has been shown above for the weak form of the advection-diffusion equation in conservation form, with no reliance on the field $\boldsymbol{a}$ being divergence-free. From the identity:

$$\nabla \cdot (\boldsymbol{a}\phi) = (\nabla \cdot \boldsymbol{a})\, \phi + \boldsymbol{a} \cdot \nabla \phi \tag{35}$$

and assuming $\nabla \cdot \boldsymbol{a} = 0$, the weak advection-diffusion equation in advective format is expressed as:

$$B_a(w, \phi) = (w, \boldsymbol{a} \cdot \nabla \phi)_{\Omega} + (\nabla w, \kappa \nabla \phi)_{\Omega} - (w, a_n \phi)_{\Gamma_g} - (w, a_n^- \phi)_{\Gamma_h} \tag{36}$$

where the subscript 'a' has been added to denote the advective format. Considering the advective form and setting $w = 1$ for the case $\Gamma_g \neq \emptyset$,

$$
\begin{aligned}
0 &= B_a(1, \phi) - L(1) - \left(w, h_g\right)_{\Gamma_g} \\
&= -\left(1, (\nabla \cdot \boldsymbol{a})\, \phi\right)_{\Omega} + \left(1, a_n \phi\right)_{\Gamma} - \left(1, a_n \phi\right)_{\Gamma_g} - \left(1, a_n^{-} \phi\right)_{\Gamma_h} - (1, f)_{\Omega} - (1, h)_{\Gamma_h} - \left(1, h_g\right)_{\Gamma_g} \\
&= -\left(1, (\nabla \cdot \boldsymbol{a})\, \phi\right)_{\Omega} + \left(1, a_n^{+} \phi\right)_{\Gamma_h} - (1, f)_{\Omega} - (1, h)_{\Gamma_h} - \left(1, h_g\right)_{\Gamma_g} \\
&= -\left(\int_{\Omega} (\nabla \cdot \boldsymbol{a})\, \phi \, d\Omega + \int_{\Gamma_h^{+}} \left(-a_n \phi + h^{+}\right) \, d\Gamma + \int_{\Gamma_h^{-}} h^{-} \, d\Gamma + \int_{\Gamma_g} h_g \, d\Gamma + \int_{\Omega} f \, d\Omega \right),
\end{aligned}
\tag{37}
$$

hence ϕ is conserved if $\nabla \cdot \boldsymbol{a} = 0$.

2.2 Conservation in Galerkin and stabilised Galerkin formulations

Conservation for the case in which Dirichlet boundary conditions are applied ($\Gamma_g \neq \emptyset$) in a Galerkin formulation can be proven in a similar way to the infinite-dimensional case in the previous section. To prove conservation, again an auxiliary flux must be defined on Γ_g [1]. Consider now $\mathcal{S}^h \subset \mathcal{S}$ and $\mathcal{V}^h \subset \mathcal{V}$ to be finite element spaces. The space $\mathcal{S}^h$ contains basis functions, each associated with a node, which satisfy the Dirichlet boundary conditions (technically speaking, $\mathcal{S}^h$ is a linear manifold, not a linear space). The space $\mathcal{V}^h$ is the span of the basis functions associated with each node in $\bar{\Omega}$ *excluding* those lying on Γ_g. A complementary space $\mathcal{W}^h$ is defined which is the span of of the basis functions associated with all nodes lying on Γ_g. It is assumed that the support of the traces of functions in $\mathcal{W}^h$ is contained in Γ_g. A further space $\mathcal{X}^h$ is defined as:

$$
\mathcal{X}^h = \mathcal{V}^h \oplus \mathcal{W}^h. \tag{38}
$$

Hence, $\mathcal{X}^h$ is the span of the shape functions associated with all nodes. Note that $\mathcal{S}^h \subset \mathcal{X}^h$.

For generality, conservation will be shown for the case of stabilised methods, which are a superset of the Galerkin method. The domain Ω is subdivided into 'elements' Ω_e, and the domain of element interiors $\tilde{\Omega}$ is defined as:

$$\tilde{\Omega} = \bigcup_e \Omega_e \tag{39}$$

which does not include the element boundaries.

The standard Galerkin problem consists of: find $\phi^h \in \mathcal{S}^h$ such that

$$B\left(w^h, \phi^h\right) = L\left(w^h\right) \quad \forall\, w^h \in \mathcal{V}^h. \tag{40}$$

A class of stabilised Galerkin methods can be expressed as:

$$B\left(w^h, \phi^h\right) + \left(\mathcal{L}_a w^h, \tau_a r\right)_{\tilde{\Omega}} = L\left(w^h\right) \quad \forall\, w^h \in \mathcal{V}^h \tag{41}$$

in which r is the residual of the advection-diffusion equation,

$$r = \boldsymbol{a} \cdot \nabla \phi^h - \kappa \Delta \phi^h - f, \tag{42}$$

τ_a is a stabilisation parameter and $\mathcal{L}_a$ is an operator which defines the stabilisation method. For the streamlined-upwind Petrov-Galerkin method (SUPG) [5], $\mathcal{L}_a$ is given by:

$$\mathcal{L}_a w^h = \boldsymbol{a} \cdot \nabla w^h. \tag{43}$$

For the Galerkin/least-squares method (GLS) [4],

$$\mathcal{L}_a w^h = \boldsymbol{a} \cdot \nabla w^h - \kappa \Delta w^h \tag{44}$$

and for the multiscale method (MS) [2, 6]

$$\mathcal{L}_a w^h = \boldsymbol{a} \cdot \nabla w^h + \kappa \Delta w^h. \tag{45}$$

Expressions for τ_a may be found in References [7–11].

Proving conservation for the case $\Gamma_g = \emptyset$ for the presented stabilised Galerkin methods is trivial as the stabilisation term involves the gradient of w^h, which vanishes when setting $w^h = 1$ everywhere in Ω. For the non-trivial case of $\Gamma_g \neq \emptyset$, consider the following Galerkin problem: find $\phi^h \in \mathcal{S}^h$ and $h_g^h \in \mathcal{W}^h$ such that

$$B\left(w^h, \phi^h\right) + \left(\mathcal{L}_a w^h, \tau_a r\right)_{\tilde{\Omega}} = L\left(w^h\right) + \left(w^h, h_g^h\right)_{\Gamma_g} \quad \forall\, w^h \in \mathcal{X}^h \tag{46}$$

where h_g^h is the flux across Γ_g. This problem can be split, yielding: find $\phi^h \in \mathcal{S}^h$ and $h_g^h \in \mathcal{W}^h$ such that

$$B\left(w^h, \phi^h\right) + \left(\mathcal{L}_a w^h, \tau_a r\right)_{\tilde{\Omega}} - L\left(w^h\right) = 0 \quad \forall\, w^h \in \mathcal{V}^h \tag{47}$$

$$B\left(q^h, \phi^h\right) + \left(\mathcal{L}_a q^h, \tau_a r\right)_{\tilde{\Omega}} - L\left(q^h\right) = \left(q^h, h_g^h\right)_{\Gamma_g} \quad \forall\, q^h \in \mathcal{W}^h. \tag{48}$$

Equation (47) is the standard stabilised advection-diffusion problem, which can be solved without knowing h_g. Once ϕ^h has been computed from equation (47), it can be inserted into equation (48) to calculate the auxiliary flux, which is merely a post-processing procedure (see Hughes et al. [1] and the references therein).

Conservation for the case $\Gamma_g \neq \emptyset$ can be proven by setting $w^h = 1$ in equation (46) (note that $w^h = 1 \in \mathcal{X}^h$, irrespective of the boundary conditions),

$$
\begin{aligned}
0 &= B\left(1, \phi^h\right) + \left(\mathcal{L}_a 1, \tau_a r\right)_{\tilde{\Omega}} - L\left(1\right) - \left(1, h_g^h\right) \\
&= \int_{\Gamma_h} a_n^+ \phi^h \, d\Gamma - \int_{\Gamma_h} h \, d\Gamma - \int_{\Omega} f \, d\Omega - \int_{\Gamma_g} h_g^h \, d\Gamma \\
&= -\left(\int_{\Gamma_h^+} \left(-a_n \phi^h + h^+\right) \, d\Gamma + \int_{\Gamma_h^-} h^- \, d\Gamma + \int_{\Omega} f \, d\Omega + \int_{\Gamma_g} h_g^h \, d\Gamma\right)
\end{aligned}
\tag{49}
$$

which proves that Galerkin and the considered stabilised Galerkin finite element methods are globally conservative. While the present arguments are of a global nature, the same procedure can be extended to local subdomains consisting of individual elements or unions of elements, thus establishing that continuous Galerkin and stabilised Galerkin methods are also locally conservative [1].

Thus far, it has been assumed that $\boldsymbol{a}$ is solenoidal. Consequently, the conservative and advective forms of the weak advection-diffusion equations are interchangeable. Often, the advective form is preferred in practice. However, a problem arises in the advective form when $\boldsymbol{a}$ is computed numerically from another equation, such as the incompressible Navier-Stokes equations, and fails to satisfy the divergence-free condition. In this case,

$$
\left(w^h, \boldsymbol{a} \cdot \nabla \phi^h\right)_{\Omega} = -\left(\nabla w^h, \boldsymbol{a} \phi^h\right)_{\Omega} - \left(w^h, \left(\nabla \cdot \boldsymbol{a}\right) \phi^h\right)_{\Omega} + \left(w^h, a_n \phi^h\right)_{\Gamma} \tag{50}
$$

in which the term involving $\nabla \cdot \boldsymbol{a}$ is not identically zero. For conservation to hold, the term involving $\nabla \cdot \boldsymbol{a}$ in equation (50) must vanish when $w^h = 1$. That is,

$$
\int_{\Omega} \left(\nabla \cdot \boldsymbol{a}\right) \phi^h \, d\Omega = 0. \tag{51}
$$

In a Galerkin finite element method for the incompressible Navier-Stokes equations, this condition will be satisfied when the space of pressure interpolations $\mathcal{P}^h$ contains $\mathcal{X}^h$, because, in this case, the discrete approximation of incompressibility is given by:

$$\int_{\Omega} q^h \left(\nabla \cdot \boldsymbol{a} \right) \, d\Omega = 0 \quad \forall \, q^h \in \mathcal{P}^h \tag{52}$$

which implies that equation (51) holds as $\mathcal{P}^h \supseteq \mathcal{X}^h$. Unfortunately, $\mathcal{P}^h \not\supseteq \mathcal{X}^h$ is often the case due to restrictions imposed by the inf-sup condition (see Brezzi and Fortin [12]). Furthermore, it is a common practice to use some form of pressure stabilisation in which case equation (52) definitely will not hold. In order to construct conservative methods for these cases, an approach based on multiscale considerations proves fruitful, and is pursued herein.

2.3 Conservation when using a computed flow field from a stabilised Galerkin method

To construct a conservative advection-diffusion scheme for cases when the advective flow field comes from a numerical simulation, the starting point is a multiscale decomposition of the advective flow field. Following the multiscale concept [2, 3], the velocity field is decomposed additively,

$$\boldsymbol{a} = \bar{\boldsymbol{a}} + \boldsymbol{a}' \tag{53}$$

where $\bar{\boldsymbol{a}}$ is the 'coarse' scale component of the velocity field and $\boldsymbol{a}'$ is the 'fine' scale component of the velocity field. It is assumed that $\boldsymbol{a} = \bar{\boldsymbol{a}}$ on Γ. Returning to equation (50) and inserting the multiscale decomposition,

$$\begin{aligned} \left(w, \boldsymbol{a} \cdot \nabla \phi \right)_{\Omega} &= \left(w, \left(\bar{\boldsymbol{a}} + \boldsymbol{a}' \right) \cdot \nabla \phi \right)_{\Omega} \\ &= \left(w, \nabla \cdot \left(\bar{\boldsymbol{a}} \phi \right) \right)_{\Omega} - \left(w, \left(\nabla \cdot \bar{\boldsymbol{a}} \right) \phi \right)_{\Omega} + \left(w, \boldsymbol{a}' \cdot \nabla \phi \right)_{\Omega} . \end{aligned} \tag{54}$$

Integrating the first term by parts on the right-hand side of equation (54), and considering that $\boldsymbol{a} = \bar{\boldsymbol{a}}$ on Γ,

$$(w, \boldsymbol{a} \cdot \nabla \phi)_\Omega = -\left(\nabla w, \bar{\boldsymbol{a}}\phi\right)_\Omega + (w, a_n \phi)_\Gamma - (w, (\nabla \cdot \bar{\boldsymbol{a}})\, \phi)_\Omega + \left(w, \boldsymbol{a}' \cdot \nabla \phi\right)_\Omega . \tag{55}$$

Inserting this expression into the advective form of the advection-diffusion equation (36) and including the boundary conservation term,

$$\begin{aligned}
0 =& B_a\,(w, \phi) - L\,(w) - \left(w, h_g\right)_{\Gamma_g} \\
=& -\left(\nabla w, \bar{\boldsymbol{a}}\phi\right)_\Omega + (w, a_n \phi)_\Gamma - (w, (\nabla \cdot \bar{\boldsymbol{a}})\, \phi)_\Omega + \left(w, \boldsymbol{a}' \cdot \nabla \phi\right)_\Omega \\
&+ (\nabla w, \kappa \nabla \phi)_\Omega - (w, a_n \phi)_{\Gamma_g} - \left(w, a_n^- \phi\right)_{\Gamma_h} - (w, f)_\Omega - (w, h)_{\Gamma_h} - \left(w, h_g\right)_{\Gamma_g} .
\end{aligned} \tag{56}$$

Setting $w = 1$,

$$0 = \left(1, a_n^+ \phi\right)_{\Gamma_h} - (1, (\nabla \cdot \bar{\boldsymbol{a}})\, \phi)_\Omega + \left(1, \boldsymbol{a}' \cdot \nabla \phi\right)_\Omega - (1, f)_\Omega - (1, h)_{\Gamma_h} - \left(1, h_g\right)_{\Gamma_g} \tag{57}$$

which can be equivalently expressed as:

$$\begin{aligned}
0 = -\Bigg(\int_{\Gamma_h^+} \left(-a_n \phi + h^+\right)\, d\Gamma + \int_\Omega f \, d\Omega + \int_{\Gamma_h^-} h^- \, d\Gamma + \int_{\Gamma_g} h_g \, d\Gamma \\
+ \int_\Omega (\nabla \cdot \bar{\boldsymbol{a}})\, \phi \, d\Omega - \int_\Omega \boldsymbol{a}' \cdot \nabla \phi \, d\Omega \Bigg) .
\end{aligned} \tag{58}$$

From the above equation, it is clear that (58) reduces to (34), and ϕ is conserved, if the last two terms cancel. That is, if:

$$\int_\Omega \left((\nabla \cdot \bar{\boldsymbol{a}})\, \phi - \nabla \phi \cdot \boldsymbol{a}'\right)\, d\Omega = 0. \tag{59}$$

Requiring that:

$$\int_{\Omega} \left(q \left(\nabla \cdot \bar{\boldsymbol{a}} \right) - \nabla q \cdot \boldsymbol{a}' \right) \, d\Omega = 0 \qquad \forall q \in H^1 \left(\Omega \right) \tag{60}$$

ensures that equation (59) holds, and conservation is guaranteed. From multiscale considerations and in light of the connection to stabilised finite element methods [2, 3], an approximation of the fine scale velocity on element interiors is given by [2]:

$$\boldsymbol{a}' \approx -\tau_p \boldsymbol{r} \tag{61}$$

where τ_p is a stabilisation parameter (pressure stabilisation) and $\boldsymbol{r}$ is the residual from the equation governing the advective flow field (e.g., the incompressible Navier-Stokes equations). Considering now that $\bar{\boldsymbol{a}}$ is represented by the finite-dimensional solution $\boldsymbol{a}^h$, inserting the model for $\boldsymbol{a}'$ into equation (60) yields:

$$\left(q^h, \nabla \cdot \boldsymbol{a}^h \right)_{\Omega} + \left(\nabla q^h, \tau_p \boldsymbol{r} \right)_{\tilde{\Omega}} = 0 \quad \forall \, q^h \in \mathcal{P}^h \tag{62}$$

which is the pressure-stabilised continuity equation for the SUPG, GLS and MS methods. Hence, a conservative form for advection-diffusion, in advection format, for problems involving an advective velocity field which satisfies equation (62) and $\mathcal{P}^h \supseteq \mathcal{S}^h$, is given by:

$$\begin{aligned} \left(w^h, \bar{\boldsymbol{a}} \cdot \nabla \phi^h \right)_{\Omega} + \left(\nabla w^h, \kappa \nabla \phi^h \right)_{\Omega} - \left(w^h, a_n \phi^h \right)_{\Gamma_g} - \left(w^h, a_n^- \phi^h \right)_{\Gamma_h} \\ - \left(w^h, \tau_p \boldsymbol{r} \cdot \nabla \phi^h \right)_{\tilde{\Omega}} = \left(w^h, f \right)_{\Omega} + \left(w^h, h \right)_{\Gamma_h} + \left(w^h, h_g^h \right)_{\Gamma_g} \qquad \forall w^h \in \mathcal{X}^h. \end{aligned} \tag{63}$$

The last term on the right-hand side of equation (63) has the form of an advection term, hence, in addition to the usual advective term, it too needs to be stabilised. A stabilised, conservative formulation is given by:

$$
\begin{aligned}
&\left(w^h, \bar{\boldsymbol{a}} \cdot \nabla \phi^h\right)_\Omega + \left(\nabla w^h, \kappa \nabla \phi^h\right)_\Omega - \left(w^h, a_n \phi^h\right)_{\Gamma_g} - \left(w^h, a_n^- \phi^h\right)_{\Gamma_h} \\
&\quad + \left(\mathcal{L}_{\bar{a}} w^h, \tau_a\, r\right)_{\tilde{\Omega}} \boxed{+ \left(w^h, \boldsymbol{a}' \cdot \nabla \phi^h\right)_{\tilde{\Omega}} + \left(\mathcal{L}'_a w^h, \tau' \left(\boldsymbol{a}' \cdot \nabla \phi^h\right)\right)_{\tilde{\Omega}}} \\
&\qquad = \left(w^h, f\right)_\Omega + \left(w^h, h\right)_{\Gamma_h} + \left(w^h, h_g^h\right)_{\Gamma_g} \quad \forall w^h \in \mathcal{X}^h
\end{aligned}
\tag{64}
$$

where $\boldsymbol{a}' = -\tau_p \boldsymbol{r}$, τ' is a stabilisation parameter for the fine-scale term, and

$$
\mathcal{L}'_a(\cdot) = \boldsymbol{a}' \cdot \nabla(\cdot) = -\tau_p \boldsymbol{r} \cdot \nabla(\cdot) \tag{65}
$$

$$
\mathcal{L}_{\bar{a}} w^h = \begin{cases} \bar{\boldsymbol{a}} \cdot \nabla w^h & \text{(SUPG)} \\ \bar{\boldsymbol{a}} \cdot \nabla w^h - \kappa \Delta w^h & \text{(GLS)} \\ \bar{\boldsymbol{a}} \cdot \nabla w^h + \kappa \Delta w^h & \text{(MS).} \end{cases} \tag{66}
$$

The first term in the box is the key to conservation, while the second term in the box stabilises the first and does not affect conservation. The second term in the box can be equivalently expressed as:

$$
\left(\mathcal{L}'_a w^h, \tau' \left(\boldsymbol{a}' \cdot \nabla \phi^h\right)\right)_{\tilde{\Omega}} = \left(\mathcal{L}'_a w^h, \tau' \mathcal{L}'_a \phi^h\right)_{\tilde{\Omega}} \tag{67}
$$

which elucidates its symmetric, positive-definite nature. Note that the required modifications to an existing computer code for stabilised Galerkin methods to ensure conservation are minor.

Remark

The boxed terms are *consistent* modifications of the method due to the fact that $\boldsymbol{r}$ is a residual vanishing for the exact solution. The stabilisation parameter, τ', is $O\left(h/|\boldsymbol{a}'|\right) =$

$O\left(h/\left(\tau_p|\boldsymbol{r}|\right)\right)$. Consequently, the additional term vanishes when $\boldsymbol{r} = \boldsymbol{0}$. A more precise expression for τ' is presented in Taylor et al. [13]. □

2.4 Time-dependent case

For generality, the preceding developments are extended to the unsteady case. Consider the following space-time domains:

$$\mathsf{Q} = \Omega\times]0,T[\tag{68}$$

$$\tilde{\mathsf{Q}} = \tilde{\Omega}\times]0,T[\tag{69}$$

$$\mathsf{P} = \Gamma\times]0,T[\tag{70}$$

$$\mathsf{P_g} = \Gamma_g\times]0,T[\tag{71}$$

$$\mathsf{P_h} = \Gamma_h\times]0,T[\tag{72}$$

$$\mathsf{P_h^+} = \Gamma_h^+\times]0,T[\tag{73}$$

$$\mathsf{P_h^-} = \Gamma_h^-\times]0,T[. \tag{74}$$

Solution of the unsteady advection-diffusion equation involves: find $\phi = \phi\left(\boldsymbol{x},t\right)\ \forall\boldsymbol{x}\in\Omega,\ \forall t\in[0,T]$ such that:

$$\frac{\partial\phi}{\partial t} - \nabla\cdot\boldsymbol{\sigma}\left(\phi\right) = f \qquad \text{in } \mathsf{Q} \tag{75}$$

$$\phi\left(\boldsymbol{x},0\right) = \phi_0 \qquad \text{in } \Omega \tag{76}$$

$$\phi = g \qquad \text{on } \mathsf{P_g} \tag{77}$$

$$-a_n^-\phi + \sigma_n^d\left(\phi\right) = h \qquad \text{on } \mathsf{P_h} \tag{78}$$

where $f:\mathsf{Q}\to\mathbb{R}$, $g:\mathsf{P_g}\to\mathbb{R}$ and $h:\mathsf{P_h}\to\mathbb{R}$ are prescribed.

For the discrete problem, the space-time domain Q is divided into 'time slabs'. Consider the following space-time slabs,

$$\mathsf{Q}^n = \Omega \times]t_n, t_{n+1}[\tag{79}$$

$$\tilde{\mathsf{Q}}^n = \tilde{\Omega} \times]t_n, t_{n+1}[\tag{80}$$

$$\mathsf{P}^n = \Gamma \times]t_n, t_{n+1}[\tag{81}$$

$$\mathsf{P}^n_{\mathsf{g}} = \Gamma_g \times]t_n, t_{n+1}[\tag{82}$$

$$\mathsf{P}^n_{\mathsf{h}} = \Gamma_h \times]t_n, t_{n+1}[\tag{83}$$

$$\mathsf{P}^{+n}_{\mathsf{h}} = \Gamma^+_h \times]t_n, t_{n+1}[\tag{84}$$

$$\mathsf{P}^{-n}_{\mathsf{h}} = \Gamma^-_h \times]t_n, t_{n+1}[\tag{85}$$

where $0 = t_0 < t_1 < \ldots < t_N = T$. The relevant function spaces on each slab are similar to those for the steady case, with infinite dimensional spaces defined by:

$$\mathcal{S}^n = \left\{ \phi \in H^1\left(\mathsf{Q}^n\right) \mid \phi = g \text{ on } \mathsf{P}^n_{\mathsf{g}} \right\} \tag{86}$$

$$\mathcal{V}^n = \left\{ w \in H^1\left(\mathsf{Q}^n\right) \mid w = 0 \text{ on } \mathsf{P}^n_{\mathsf{g}} \right\} \tag{87}$$

$$H^1\left(\mathsf{Q}^n\right) = \mathcal{V}^n \oplus \mathcal{W}^n \tag{88}$$

and relevant finite-dimensional spaces defined by:

$$\left(\mathcal{S}^n\right)^h \subset \mathcal{S}^n \tag{89}$$

$$\left(\mathcal{V}^n\right)^h \subset \mathcal{V}^n \tag{90}$$

$$\left(\mathcal{S}^n\right)^h \subset \left(\mathcal{X}^n\right)^h = \left(\mathcal{V}^n\right)^h \oplus \left(\mathcal{W}^n\right)^h. \tag{91}$$

Solution of the unsteady problem requires: find $\phi^h \in (\mathcal{S}^n)^h$ such that,

$$
\begin{aligned}
&\left(w^h\left(t_{n+1}^-\right),\phi^h\left(t_{n+1}^-\right)\right)_\Omega - \left(\frac{\partial w^h}{\partial t},\phi^h\right)_{\mathsf{Q}^n} + \left(w^h,\bar{\boldsymbol{a}}\cdot\nabla\phi^h\right)_{\mathsf{Q}^n} + \left(\nabla w^h,\kappa\nabla\phi^h\right)_{\mathsf{Q}^n} \\
&\quad - \left(w^h,a_n\phi^h\right)_{\mathsf{P}_\mathsf{g}^n} - \left(w^h,a_n^-\phi^h\right)_{\mathsf{P}_\mathsf{h}^n} + \left(\mathcal{L}_{\bar{a}}w^h,\tau_a r\right)_{\tilde{\mathsf{Q}}^n} \\
&\quad \boxed{+\left(w^h,\boldsymbol{a}'\cdot\nabla\phi^h\right)_{\mathsf{Q}^n} + \left(\mathcal{L}_a' w^h,\tau'\left(\boldsymbol{a}'\cdot\nabla\phi^h\right)\right)_{\tilde{\mathsf{Q}}^n}} \\
&= \left(w^h,f\right)_{\mathsf{Q}^n} + \left(w^h,h\right)_{\mathsf{P}_\mathsf{h}^n} + \left(w^h,h_g^h\right)_{\mathsf{P}_\mathsf{g}^n} + \left(w^h\left(t_n^+\right),\phi^h\left(t_n^-\right)\right)_\Omega \\
&\qquad \forall \boldsymbol{w}^h \in \left(\mathcal{X}^n\right)^h,\ n = 0,1,\ldots,N-1 \qquad (92)
\end{aligned}
$$

where $\boldsymbol{a}' = -\tau_p \boldsymbol{r}$, the operator $\mathcal{L}_a'$ is defined by equation (65), and

$$
\mathcal{L}_{\bar{a}}w^h = \begin{cases} \dfrac{\partial w^h}{\partial t} + \bar{\boldsymbol{a}}\cdot\nabla w^h & \text{(SUPG)} \\ \dfrac{\partial w^h}{\partial t} + \bar{\boldsymbol{a}}\cdot\nabla w^h - \kappa\Delta w^h & \text{(GLS)} \\ \dfrac{\partial w^h}{\partial t} + \bar{\boldsymbol{a}}\cdot\nabla w^h + \kappa\Delta w^h & \text{(MS).} \end{cases} \qquad (93)
$$

The boxed terms are the same as for the steady-state case (see equation (64)). Considering

$$
\int_{\mathsf{Q}^n} \left(\nabla\cdot\bar{\boldsymbol{a}}\right)\phi^h \, d\mathsf{Q} + \int_{\tilde{\mathsf{Q}}^n} \tau_p \boldsymbol{r}\cdot\nabla\phi^h \, d\mathsf{Q} = 0, \qquad (94)
$$

which holds when $\bar{\boldsymbol{a}}$ is computed from a pressure-stabilised finite element method and components of the advective velocity field $\bar{a}_i$ come from the same space as ϕ^h, and setting $w^h = 1$ in equation (92),

$$
\begin{aligned}
\int_\Omega \phi^h\left(t_{n+1}^-\right)\, d\Omega &= \int_\Omega \phi^h\left(t_n^-\right)\, d\Omega + \int_{\mathsf{P}_\mathsf{g}^n} a_n\phi^h \, d\mathsf{P} + \int_{\mathsf{P}_\mathsf{h}^n} a_n^-\phi^h \, d\mathsf{P} + \int_{\mathsf{Q}^n} f \, d\mathsf{Q} \\
&\quad + \int_{\mathsf{P}_\mathsf{h}^n} h \, d\mathsf{P} + \int_{\mathsf{P}_\mathsf{g}^n} h_g^h \, d\mathsf{P} - \int_{\mathsf{Q}^n} \bar{\boldsymbol{a}}\cdot\nabla\phi^h \, d\mathsf{Q} + \int_{\mathsf{Q}^n} \tau_p\boldsymbol{r}\cdot\nabla\phi^h \, d\mathsf{Q}. \qquad (95)
\end{aligned}
$$

Expanding the term involving $\bar{\boldsymbol{a}}$ and considering equation (94),

$$\int_{\Omega} \phi^h \left(t_{n+1}^-\right) \, d\Omega = \int_{\Omega} \phi^h \left(t_n^-\right) \, d\Omega + \int_{\mathsf{P}_h^{+n}} \left(-a_n \phi^h + h^+\right) \, d\mathsf{P} + \int_{\mathsf{P}_h^{-n}} h^- \, d\mathsf{P} + \int_{\mathsf{Q}^n} f \, d\mathsf{Q} + \int_{\mathsf{P}_g^n} h_g^h \, d\mathsf{P} \quad (96)$$

which shows that ϕ^h is conserved on space-time slabs.

3 Conservation for the incompressible Navier-Stokes equations

3.1 Incompressible Navier-Stokes equations

The developments in this section for the incompressible Navier-Stokes equations mirror the developments for the advection diffusion equation. Before proceeding to the Navier-Stokes equations, it is useful to establish some definitions. Denoting the velocity field $\boldsymbol{u}$,

$$u_n = \boldsymbol{n} \cdot \boldsymbol{u} \quad (97)$$

$$u_n^+ = \frac{u_n + |u_n|}{2} \quad (98)$$

$$u_n^- = \frac{u_n - |u_n|}{2}, \quad (99)$$

where, similar to before, u_n^+ is equal to u_n at an outflow boundary and zero elsewhere, and u_n^- is equal to u_n at inflow boundary and zero elsewhere. Let $\{\Gamma^-, \Gamma^+\}$ and $\{\Gamma_g, \Gamma_h\}$ be partitions of Γ, defined by:

$$\Gamma^- = \{\boldsymbol{x} \in \Gamma \mid u_n(\boldsymbol{x}) < 0\} \qquad \text{inflow boundary} \quad (100)$$

$$\Gamma^+ = \Gamma - \Gamma^- \qquad \text{outflow boundary} \quad (101)$$

and, as before,

$$\Gamma_g^{\pm} = \Gamma_g \cap \Gamma^{\pm} \tag{102}$$

$$\Gamma_h^{\pm} = \Gamma_h \cap \Gamma^{\pm}. \tag{103}$$

The partitions of Γ are illustrated in Figure 2.1.

Assuming the kinematic viscosity to be constant and positive ($\nu > 0$), the following fluxes are defined:

$$\boldsymbol{\sigma}^a(\boldsymbol{u}, \boldsymbol{a}, p) = -\boldsymbol{u} \otimes \boldsymbol{a} - p\boldsymbol{I} \qquad \text{‘advective’ flux} \tag{104}$$

$$\boldsymbol{\sigma}^d(\boldsymbol{u}) = 2\nu\nabla^s\boldsymbol{u} \qquad \text{diffusive flux} \tag{105}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{a}, p) = \boldsymbol{\sigma}^a + \boldsymbol{\sigma}^d \qquad \text{total flux} \tag{106}$$

$$\boldsymbol{\sigma}_n^a = \boldsymbol{n} \cdot \boldsymbol{\sigma}^a \tag{107}$$

$$\boldsymbol{\sigma}_n^d = \boldsymbol{n} \cdot \boldsymbol{\sigma}^d \tag{108}$$

$$\boldsymbol{\sigma}_n = \boldsymbol{n} \cdot \boldsymbol{\sigma} \tag{109}$$

where p is the pressure (divided by the density) and $\boldsymbol{a}$ is the advective velocity.

Solution of the steady Navier-Stokes equations involves: find $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, $p = p(\boldsymbol{x})$ $\forall \boldsymbol{x} \in \Omega$ such that:

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{u}, p) = \boldsymbol{f} \qquad \text{in } \Omega \tag{110}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega \tag{111}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_g \tag{112}$$

$$-u_n^- \boldsymbol{u} - p\boldsymbol{n} + \boldsymbol{\sigma}_n^d(\boldsymbol{u}) = \boldsymbol{h} \qquad \text{on } \Gamma_h \tag{113}$$

where $\boldsymbol{f} : \Omega \to \mathbb{R}^d$, $\boldsymbol{g} : \Gamma_g \to \mathbb{R}^d$ and $\boldsymbol{h} : \Gamma_h \to \mathbb{R}^d$ are prescribed. The boundary condition on Γ_h can be interpreted as:

$$\boldsymbol{h} = \begin{cases} \boldsymbol{h}^- & \text{on } \Gamma_h^- \\ \boldsymbol{h}^+ & \text{on } \Gamma_h^+ \end{cases} \tag{114}$$

where

$$\boldsymbol{\sigma}_n(\boldsymbol{u}, p) = \boldsymbol{h}^- \qquad \text{momentum flux boundary condition} \tag{115}$$

$$-p\boldsymbol{n} + \boldsymbol{\sigma}_n^d(\boldsymbol{u}, p) = \boldsymbol{h}^+ \qquad \text{traction boundary condition.} \tag{116}$$

For the variational formulation of the Navier-Stokes equations, the following function spaces are required:

$$\mathcal{S} = \left\{ \boldsymbol{u} \in \left(H^1(\Omega) \right)^d \mid \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma_g \right\} \tag{117}$$

$$\mathcal{V} = \left\{ \boldsymbol{w} \in \left(H^1(\Omega) \right)^d \mid \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_g \right\} \tag{118}$$

$$\mathcal{P} = \left\{ p \in L^2(\Omega) \right\}. \tag{119}$$

The variational problem consists of: find $\boldsymbol{u} \in \mathcal{S}$, $p \in \mathcal{P}$ such that:

$$B(\{\boldsymbol{w}, q\}; \{\boldsymbol{u}, p\}, \boldsymbol{u}) = L(\boldsymbol{w}) \qquad \forall \boldsymbol{w} \in \mathcal{V},\ \forall q \in \mathcal{P} \tag{120}$$

where

$$B(\{\boldsymbol{w}, q\}; \{\boldsymbol{u}, p\}, \boldsymbol{a}) = (\nabla \boldsymbol{w}, \boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{a}, p))_\Omega + \left(\boldsymbol{w}, a_n^+ \boldsymbol{u} \right)_{\Gamma_h} - (\nabla q, \boldsymbol{u})_\Omega + (q, u_n)_\Gamma \tag{121}$$

and

$$L(\boldsymbol{w}) = (\boldsymbol{w}, \boldsymbol{f})_\Omega + (\boldsymbol{w}, \boldsymbol{h})_{\Gamma_h}. \tag{122}$$

Remark

If q and $\boldsymbol{u}$ are sufficiently smooth, $-(\nabla q, \boldsymbol{u})_\Omega + (q, u_n)_\Gamma = (q, \nabla \cdot \boldsymbol{u})_\Omega$. It is assumed that this is the case. For the finite element case, differentiability and continuity will be addressed more precisely. □

Consistency of the weak formulation with the strong form, namely equations (110) to (113), can be proven by integrating equation (120) by parts, leading to:

$$\begin{aligned} 0 &= B\left(\{\boldsymbol{w}, q\}; \{\boldsymbol{u}, p\}, \boldsymbol{u}\right) - L\left(\boldsymbol{w}\right) \\ &= -\left(\boldsymbol{w}, \nabla \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}, \boldsymbol{u}, p\right) + \boldsymbol{f}\right)_\Omega + \left(\boldsymbol{w}, -u_n^- \boldsymbol{u} - p\boldsymbol{n} + \boldsymbol{\sigma}_n^d\left(\boldsymbol{u}\right) - \boldsymbol{h}\right)_{\Gamma_h} + \left(q, \nabla \cdot \boldsymbol{u}\right)_\Omega . \end{aligned} \tag{123}$$

Stability requires that the form in equation (121) be coercive in the following sense. Setting $\{\boldsymbol{u}, p\} = \{\boldsymbol{w}, q\}$ and $\boldsymbol{a} = \boldsymbol{u}$ in equation (121),

$$\begin{aligned} B\left(\{\boldsymbol{w}, q\}; \{\boldsymbol{w}, q\}, \boldsymbol{u}\right) &= \left(\nabla \boldsymbol{w}, -\boldsymbol{w} \otimes \boldsymbol{u} - q\boldsymbol{I} + 2\nu\nabla^{\mathrm{s}}\boldsymbol{w}\right)_\Omega + \left(\boldsymbol{w}, u_n^+ \boldsymbol{w}\right)_{\Gamma_h} - \left(\nabla q, \boldsymbol{w}\right)_\Omega \\ &= \frac{1}{2}\left(\boldsymbol{w}, \left(\nabla \cdot \boldsymbol{u}\right)\boldsymbol{w}\right)_\Omega - \frac{1}{2}\left(\boldsymbol{w}, u_n \boldsymbol{w}\right)_{\Gamma_h} + 2\nu\|\nabla^{\mathrm{s}}\boldsymbol{w}\|_\Omega^2 + \left(\boldsymbol{w}, u_n^+ \boldsymbol{w}\right)_{\Gamma_h} \\ &= \frac{1}{2}\left(\boldsymbol{w}, \left(\nabla \cdot \boldsymbol{u}\right)\boldsymbol{w}\right)_\Omega + \frac{1}{2}\left(\boldsymbol{w}, |u_n|\boldsymbol{w}\right)_{\Gamma_h} + 2\nu\|\nabla^{\mathrm{s}}\boldsymbol{w}\|_\Omega^2. \end{aligned} \tag{124}$$

This can be restated as:

$$B\left(\{\boldsymbol{w}, q\}; \{\boldsymbol{w}, q\}, \boldsymbol{u}\right) = \frac{1}{2}\left(\boldsymbol{w}, \left(\nabla \cdot \boldsymbol{u}\right)\boldsymbol{w}\right)_\Omega + \frac{1}{2}\||u_n|^{1/2}\boldsymbol{w}\|_{\Gamma_h}^2 + 2\nu\|\nabla^{\mathrm{s}}\boldsymbol{w}\|_\Omega^2 \tag{125}$$

which, for the case of a divergence-free flow field ($\nabla \cdot \boldsymbol{u} = 0$), guarantees stability of the field $\boldsymbol{w}$. However, analogous to the advection-diffusion equation, the problem is not necessarily stable if the flow field is not divergence-free.

3.2 Conservation of mass and momentum

Conservation of momentum is examined for the general case $\Gamma_g \neq \emptyset$. Analogous to the advection-diffusion case, this requires the definition of a space $\mathcal{W}$ such that

$$\left(H^1\left(\Omega\right)\right)^d = \mathcal{V} \oplus \mathcal{W}. \tag{126}$$

The problem then involves: find $\boldsymbol{u} \in \mathcal{S}$, $p \in \mathcal{P}$ and $\boldsymbol{h}_g \in \mathcal{W}$, where $\boldsymbol{h}_g$ is the momentum flux across Γ_g, such that

$$B\left(\{\boldsymbol{w}, q\}; \{\boldsymbol{u}, p\}, \boldsymbol{u}\right) = L\left(\boldsymbol{w}\right) + \left(\boldsymbol{w}, \boldsymbol{h}_g\right)_{\Gamma_g} \quad \forall\, \boldsymbol{w} \in \left(H^1\left(\Omega\right)\right)^d, \ \forall q \in \mathcal{P}. \tag{127}$$

Setting $\{\boldsymbol{w}, q\} = \{\boldsymbol{e}_i, 0\}$, where $\boldsymbol{e}_i$ is the canonical unit basis vector, in the above equation,

$$\begin{aligned} 0 &= B\left(\{\boldsymbol{e}_i, 0\}, \{\boldsymbol{u}, p\}, \boldsymbol{u}\right) - \left(\boldsymbol{e}_i, \boldsymbol{f}\right)_\Omega - \left(\boldsymbol{e}_i, \boldsymbol{h}\right)_{\Gamma_h} - \left(\boldsymbol{e}_i, \boldsymbol{h}_g\right)_{\Gamma_g} \\ &= \int_{\Gamma_h} u_n^+ \boldsymbol{e}_i \cdot \boldsymbol{u}\, d\Gamma - \int_\Omega \boldsymbol{e}_i \cdot \boldsymbol{f}\, d\Omega - \int_{\Gamma_h} \boldsymbol{e}_i \cdot \boldsymbol{h}\, d\Gamma - \int_{\Gamma_g} \boldsymbol{e}_i \cdot \boldsymbol{h}_g\, d\Gamma. \end{aligned} \tag{128}$$

This can be rewritten as

$$0 = \int_\Omega \boldsymbol{e}_i \cdot \boldsymbol{f}\, d\Omega + \int_{\Gamma_h^-} \boldsymbol{e}_i \cdot \boldsymbol{h}^-\, d\Gamma + \int_{\Gamma_h^+} \boldsymbol{e}_i \cdot \left(-u_n \boldsymbol{u} + \boldsymbol{h}^+\right)\, d\Gamma + \int_{\Gamma_g} \boldsymbol{e}_i \cdot \boldsymbol{h}_g\, d\Gamma \tag{129}$$

which proves that momentum is conserved. Setting $\{\boldsymbol{w}, q\} = \{\boldsymbol{0}, 1\}$,

$$\begin{aligned} 0 &= B\left(\{\boldsymbol{0}, 1\}, \{\boldsymbol{u}, p\}, \boldsymbol{u}\right) - \left(\boldsymbol{0}, \boldsymbol{f}\right)_\Omega - \left(\boldsymbol{0}, \boldsymbol{h}_g\right)_{\Gamma_g} - \left(\boldsymbol{0}, \boldsymbol{h}\right)_{\Gamma_h} \\ &= \int_\Gamma u_n\, d\Gamma \end{aligned} \tag{130}$$

which shows that mass is also conserved. Note that in proving conservation, no reliance has been placed on the flow field being strictly divergence-free.

3.3 Conservation for the advective form of the Navier-Stokes equations

In practice, the conservation form of the Navier-Stokes equations is rarely used in Galerkin computations. Application of the conservation form results in spurious oscillations due to the $\nabla \cdot \boldsymbol{u}$ term in equation (124), which acts as a distribution of sinks and sources. To achieve accuracy, the advective form of the Navier-Stokes equations has been found much superior and is usually adopted. Using the identity:

$$\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{a}) = \boldsymbol{a} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \left(\nabla \cdot \boldsymbol{a}\right) \tag{131}$$

the conservation and advective forms of the Navier-Stokes equations are interchangeable for an advective field $\boldsymbol{a}$ which is divergence-free everywhere in Ω.

Considering the advective term in the Navier-Stokes equations in isolation,

$$-\left(\nabla \boldsymbol{w}, \boldsymbol{u} \otimes \boldsymbol{a}\right)_{\Omega} = \left(\boldsymbol{w}, \boldsymbol{a} \cdot \nabla \boldsymbol{u}\right)_{\Omega} + \left(\boldsymbol{w}, \left(\nabla \cdot \boldsymbol{a}\right) \boldsymbol{u}\right)_{\Omega} - \left(\boldsymbol{w}, a_n \boldsymbol{u}\right)_{\Gamma} . \tag{132}$$

Inserting equation (132) into equation (121), and assuming that $\boldsymbol{a}$ is divergence-free,

$$\begin{aligned} B_a \left(\{\boldsymbol{w}, q\}; \{\boldsymbol{u}, p\}, \boldsymbol{a}\right) = \left(\boldsymbol{w}, \boldsymbol{a} \cdot \nabla \boldsymbol{u}\right)_{\Omega} + \left(\nabla \boldsymbol{w}, -p\boldsymbol{I} + 2\nu \nabla^{s} \boldsymbol{u}\right)_{\Omega} - \left(\boldsymbol{w}, a_n^{-} \boldsymbol{u}\right)_{\Gamma_h} \\ - \left(\boldsymbol{w}, a_n \boldsymbol{u}\right)_{\Gamma_g} - \left(\nabla q, \boldsymbol{u}\right)_{\Omega} + \left(q, u_n\right)_{\Gamma} \end{aligned} \tag{133}$$

which is the advective weak form of the Navier-Stokes equations. It is this form which is usually utilised in finite element analysis. Setting $\{\boldsymbol{w}, q\} = \{\boldsymbol{e}_i, 0\}$,

$$
\begin{aligned}
0 &= B_a\left(\{\boldsymbol{e}_i, 0\}; \{\boldsymbol{u}, p\}, \boldsymbol{u}\right) - L\left(\boldsymbol{e}_i\right) - \left(\boldsymbol{e}_i, \boldsymbol{h}_g\right)_{\Gamma_g} \\
&= \left(\boldsymbol{e}_i, \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right)_\Omega - \left(\boldsymbol{e}_i, u_n^- \boldsymbol{u}\right)_{\Gamma_h} - \left(\boldsymbol{w}, u_n \boldsymbol{u}\right)_{\Gamma_g} - \left(\boldsymbol{e}_i, \boldsymbol{f}\right)_\Omega - \left(\boldsymbol{e}_i, \boldsymbol{h}\right)_{\Gamma_h} - \left(\boldsymbol{e}_i, \boldsymbol{h}_g\right)_{\Gamma_g} \\
&= -\left(\boldsymbol{e}_i, \left(\nabla \cdot \boldsymbol{u}\right) \boldsymbol{u}\right)_\Omega + \left(\boldsymbol{e}_i, u_n \boldsymbol{u}\right)_\Gamma - \left(\boldsymbol{e}_i, u_n^- \boldsymbol{u}\right)_{\Gamma_h} - \left(\boldsymbol{e}_i, u_n \boldsymbol{u}\right)_{\Gamma_g} - \left(\boldsymbol{e}_i, \boldsymbol{f}\right)_\Omega - \left(\boldsymbol{e}_i, \boldsymbol{h}\right)_{\Gamma_h} - \left(\boldsymbol{e}_i, \boldsymbol{h}_g\right)_{\Gamma_g} \\
&= -\left(\boldsymbol{e}_i, \left(\nabla \cdot \boldsymbol{u}\right) \boldsymbol{u}\right)_\Omega + \left(\boldsymbol{e}_i, u_n^+ \boldsymbol{u}\right)_{\Gamma_h} - \left(\boldsymbol{e}_i, \boldsymbol{f}\right)_\Omega - \left(\boldsymbol{e}_i, \boldsymbol{h}\right)_{\Gamma_h} - \left(\boldsymbol{e}_i, \boldsymbol{h}_g\right)_{\Gamma_g}.
\end{aligned}
\tag{134}
$$

Equation (134) implies that for momentum to be conserved when using the advective form of the Navier-Stokes equations, it is necessary that:

$$\int_\Omega \boldsymbol{e}_i \cdot \left(\nabla \cdot \boldsymbol{u}\right) \boldsymbol{u} \, d\Omega = 0 \tag{135}$$

which resembles the weak form of the continuity equation, which requires that:

$$\int_\Omega q \cdot \left(\nabla \cdot \boldsymbol{u}\right) \; d\Omega = 0 \quad \forall \, q \in \mathcal{P}. \tag{136}$$

Hence, satisfaction of weak continuity and requiring that:

$$\mathcal{P} \supseteq H^1\left(\Omega\right) \tag{137}$$

guarantees that momentum will be conserved in the advection form of the Navier-Stokes equations.

The difficulty that arises when adopting a Galerkin approach by replacing the relevant infinite-dimensional function spaces with finite dimensional counterparts is that satisfaction of the inf-sup stability condition will, in general, require that [12]:

$$\mathcal{P}^h \not\supseteq \mathcal{Y}^h \tag{138}$$

where in a finite element context $\mathcal{Y}^h$ is the span of the individual velocity shape function components associated with all nodes. Satisfaction of the inf-sup condition will in general preclude momentum conservation. This is the case for Galerkin methods. For stabilised Galerkin methods, equation (138) is often satisfied, but the continuity equation is stabilised, hence a form other than (136) is employed. This in fact is a key advantage, as will become clear.

3.4 Conservation in Galerkin and stabilised Galerkin formulations of the Navier-Stokes equations – multiscale approach

Conservation can be restored to stabilised Galerkin methods in a relatively simple manner. Consider again a multiscale decomposition of the advective flow field, $\boldsymbol{a} = \bar{\boldsymbol{a}} + \boldsymbol{a}'$, in which $\bar{\boldsymbol{a}}$ is the coarse scale component and $\boldsymbol{a}'$ is the fine scale component. As before, $\boldsymbol{a}' = \boldsymbol{0}$ on Γ. Inserting the multiscale decomposition into equation (133),

$$B_{\mathrm{a,ms}}\left(\{\boldsymbol{w}, q\}; \{\boldsymbol{u}, p\}, \boldsymbol{a}\right) = B_a\left(\{\boldsymbol{w}, q\}; \{\boldsymbol{u}, p\}, \bar{\boldsymbol{a}}\right) + \left(\boldsymbol{w}, \boldsymbol{a}' \cdot \nabla \boldsymbol{u}\right)_\Omega \tag{139}$$

A Galerkin procedure is now followed in which $\mathcal{S}^h \subset \mathcal{S}$, $\mathcal{V}^h \subset \mathcal{V}$ and $\mathcal{P}^h \subset \mathcal{P}$. It is assumed that $\bar{\boldsymbol{a}}$ is $\boldsymbol{u}^h \in \mathcal{S}^h$, and $\boldsymbol{a}'$ is $\boldsymbol{u}'$, which will be defined shortly. Similar to before, a space $\mathcal{W}^h$ is defined, such that

$$\mathcal{Z}^h = \mathcal{V}^h \oplus \mathcal{W}^h \tag{140}$$

where, by assumption, $\mathcal{Z}^h = \left(\mathcal{Y}^h\right)^d$. A Galerkin procedure, including a multiscale decomposition of the advective velocity, involves: find $\boldsymbol{u}^h \in \mathcal{S}^h$, $p^h \in \mathcal{P}^h$ and $\boldsymbol{h}_g^h \in \mathcal{W}^h$ such that:

$$B_a\left(\{\boldsymbol{w}^h, q^h\}; \{\boldsymbol{u}^h, p^h\}, \boldsymbol{u}^h\right) + \left(\boldsymbol{w}^h, \boldsymbol{u}' \cdot \nabla \boldsymbol{u}^h\right)_\Omega$$
$$= L\left(\boldsymbol{w}^h\right) + \left(\boldsymbol{w}^h, \boldsymbol{h}_g^h\right)_{\Gamma_g} \quad \forall\, \boldsymbol{w}^h \in \mathcal{Z}^h,\ q^h \in \mathcal{P}^h. \quad (141)$$

Inserting $\{\boldsymbol{w}^h, q^h\} = \{\boldsymbol{e}_i, 0\}$ into equation (141) leads to:

$$\begin{aligned} 0 =& B_a\left(\{\boldsymbol{e}_i, 0\}; \{\boldsymbol{u}^h, p^h\}, \boldsymbol{u}\right) + \left(\boldsymbol{e}_i, \boldsymbol{u}' \cdot \nabla \boldsymbol{u}^h\right)_\Omega - L\left(\boldsymbol{e}_i\right) - \left(\boldsymbol{e}_i, \boldsymbol{h}_g^h\right)_{\Gamma_g} \\ =& -\left(\boldsymbol{e}_i, \left(\nabla \cdot \boldsymbol{u}^h\right)\boldsymbol{u}^h\right)_\Omega + \left(\boldsymbol{e}_i, \boldsymbol{u}' \cdot \nabla \boldsymbol{u}^h\right)_\Omega - \left(\boldsymbol{e}_i, \boldsymbol{h}^-\right)_{\Gamma_h^-} \\ & - \left(\boldsymbol{e}_i, -u_n^h \boldsymbol{u}^h + \boldsymbol{h}^+\right)_{\Gamma_h^+} - \left(\boldsymbol{e}_i, \boldsymbol{h}_g^h\right)_{\Gamma_g} - \left(\boldsymbol{e}_i, \boldsymbol{f}\right)_\Omega . \end{aligned} \quad (142)$$

Hence, momentum is conserved if:

$$\int_\Omega \boldsymbol{e}_i \cdot \left(\left(\nabla \cdot \boldsymbol{u}^h\right)\boldsymbol{u}^h - \boldsymbol{u}' \cdot \nabla \boldsymbol{u}^h\right)\, d\Omega = 0, \quad (143)$$

and this requirement is satisfied if:

$$\int_\Omega \left(q^h\left(\nabla \cdot \boldsymbol{u}^h\right) - \boldsymbol{u}' \cdot \nabla q^h\right)\, d\Omega = 0 \quad \forall\, q^h \in \mathcal{P}^h \quad (144)$$

subject to the condition that $\mathcal{P}^h \supseteq \mathcal{Y}^h$. Hence, if the modified continuity equation in (144) is satisfied and $\mathcal{P}^h \supseteq \mathcal{Y}^h$, the formulation in equation (141) conserves momentum.

The missing component to this point is the definition of the fine scale velocity. This requires the supposition of a model. In the same vein as for the advection-diffusion problem, an approximation of the fine scale velocity on element interiors is introduced [2]:

$$\boldsymbol{u}' = -\tau_p \boldsymbol{r} \quad (145)$$

where $\boldsymbol{r}$ is the residual of the incompressible Navier-Stokes equation:

$$\boldsymbol{r} = \boldsymbol{u}^h \cdot \nabla \boldsymbol{u}^h + \nabla p^h - 2\nu\Delta \boldsymbol{u}^h - \boldsymbol{f}. \tag{146}$$

Inserting the expression for the model for the fine scales (145) into the modified continuity equation (144) yields:

$$\int_{\Omega} \left(q^h \left(\nabla \cdot \boldsymbol{u}^h \right) + \nabla q^h \cdot \tau_p \boldsymbol{r} \right) \, d\Omega = 0 \quad \forall \, q^h \in \mathcal{P}^h \tag{147}$$

which is precisely the pressure-stabilised continuity equation. Therefore, a momentum conserving, pressure-stabilised Galerkin formulation in advection format involves: find $\boldsymbol{u}^h \in \mathcal{S}^h$, $p^h \in \mathcal{P}^h$ and $\boldsymbol{h}_g^h \in \mathcal{W}^h$ such that:

$$\begin{aligned} B_a \left(\{\boldsymbol{w}^h, q^h\}; \{\boldsymbol{u}^h, p^h\}, \boldsymbol{u}^h \right) + \left(\nabla q^h, \tau_p \boldsymbol{r} \right)_{\tilde{\Omega}} - \left(\boldsymbol{w}^h, \tau_p \boldsymbol{r} \cdot \nabla \boldsymbol{u}^h \right)_{\tilde{\Omega}} \\ = L \left(\boldsymbol{w}^h \right) + \left(\boldsymbol{w}^h, \boldsymbol{h}_g^h \right)_{\Gamma_g} \qquad \forall \boldsymbol{w}^h \in \mathcal{Z}^h, \; \forall q^h \in \mathcal{P}^h. \end{aligned} \tag{148}$$

While the above formulation is conservative, it is not stabilised for advection dominated flows. The following stabilised Galerkin problem is both stable and momentum conserving: find $\boldsymbol{u}^h \in \mathcal{S}^h$, $p^h \in \mathcal{P}^h$ and $\boldsymbol{h}_g^h \in \mathcal{W}^h$

$$\begin{aligned} B_a \left(\{\boldsymbol{w}^h, q^h\}; \{\boldsymbol{u}^h, p^h\}, \boldsymbol{u}^h \right) + \left(\nabla q^h, \tau_p \boldsymbol{r} \right)_{\tilde{\Omega}} + \left(\mathcal{L}_m \boldsymbol{w}^h, \tau_m \boldsymbol{r} \right)_{\tilde{\Omega}} \\ \boxed{+ \left(\boldsymbol{w}^h, \boldsymbol{u}' \cdot \nabla \boldsymbol{u}^h \right)_{\tilde{\Omega}} + \left(\mathcal{L}'_m \boldsymbol{w}^h, \tau' \left(\boldsymbol{u}' \cdot \nabla \boldsymbol{u}^h \right) \right)_{\tilde{\Omega}}} \\ - \left(\boldsymbol{w}^h, \boldsymbol{h}_g^h \right)_{\Gamma_g} = L \left(\boldsymbol{w}^h \right) \qquad \forall \boldsymbol{w}^h \in \mathcal{Z}^h, \; \forall q^h \in \mathcal{P}^h \end{aligned} \tag{149}$$

in which $\boldsymbol{u}' = -\tau_p \boldsymbol{r}$, and

$$\mathcal{L}'_m (\cdot) = \boldsymbol{u}' \cdot \nabla (\cdot) = -\tau_p \boldsymbol{r} \cdot \nabla (\cdot) \, . \tag{150}$$

The operator $\mathcal{L}_m$ defines the particular stabilisation method,

$$\mathcal{L}_m \boldsymbol{w}^h = \begin{cases} \boldsymbol{u}^h \cdot \nabla \boldsymbol{w}^h & \text{(SUPG)} \\ \boldsymbol{u}^h \cdot \nabla \boldsymbol{w}^h - 2\nu \Delta \boldsymbol{w}^h & \text{(GLS)} \\ \boldsymbol{u}^h \cdot \nabla \boldsymbol{w}^h + 2\nu \Delta \boldsymbol{w}^h & \text{(MS)}. \end{cases} \tag{151}$$

As for the advection-diffusion case, the term introduced to provide conservation (the first term in the box) is an advection term and is hence stabilised (second term in the box). The second term in the box can be interpreted in the same fashion as equation (67). Appropriate expressions for τ_p, τ_m and τ' can be found in Taylor et al. [13], in which it is assumed that $\tau_m = \tau_p$.

3.5 Time dependent case

Considering the space-time domains defined in equations (68)–(74), solution of the unsteady Navier-Stokes equations involves: find $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, $p = p(\boldsymbol{x}, t)$ $\forall \boldsymbol{x} \in \Omega$, $\forall t \in [0, T]$ such that:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{u}, p) = \boldsymbol{f} \qquad \text{in } \mathsf{Q} \tag{152}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0 \qquad \text{in } \Omega \tag{153}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \mathsf{Q} \tag{154}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \mathsf{P_g} \tag{155}$$

$$-u_n^- \boldsymbol{u} - p\boldsymbol{n} + \boldsymbol{\sigma}_n^d(\boldsymbol{u}) = \boldsymbol{h} \qquad \text{on } \mathsf{P_h} \tag{156}$$

where $\boldsymbol{f} : \mathsf{Q} \to \mathbb{R}^d$, $\boldsymbol{g} : \mathsf{P_g} \to \mathbb{R}^d$ and $\boldsymbol{h} : \mathsf{P_h} \to \mathbb{R}^d$ are prescribed.

Considering the space-time 'slabs' defined in equations (79)–(85), the pertinent function spaces are:

$$\mathcal{S}^n = \left\{ \boldsymbol{u} \in \left(H^1 \left(\mathsf{Q}^n \right) \right)^d \mid \boldsymbol{u} = \boldsymbol{g} \text{ on } \mathsf{P}^n_{\mathsf{g}} \right\} \tag{157}$$

$$\mathcal{V}^n = \left\{ \boldsymbol{w} \in \left(H^1 \left(\mathsf{Q}^n \right) \right)^d \mid \boldsymbol{w} = \boldsymbol{0} \text{ on } \mathsf{P}^n_{\mathsf{g}} \right\} \tag{158}$$

$$\mathcal{P}^n = \left\{ p \in L^2 \left(\mathsf{Q}^n \right) \right\} \tag{159}$$

$$\left(H^1 \left(\mathsf{Q}^n \right) \right)^d = \mathcal{V}^n \oplus \mathcal{W}^n \tag{160}$$

and relevant subspaces are:

$$(\mathcal{S}^n)^h \subset \mathcal{S}^n \tag{161}$$

$$(\mathcal{V}^n)^h \subset \mathcal{V}^n \tag{162}$$

$$(\mathcal{P}^n)^h \subset \mathcal{P}^n \tag{163}$$

$$(\mathcal{S}^n)^h \subset (\mathcal{Z}^n)^h = (\mathcal{V}^n)^h \oplus (\mathcal{W}^n)^h . \tag{164}$$

Solution of the unsteady problem requires: find $\boldsymbol{u}^h \in (\mathcal{S}^n)^h$, $p^h \in (\mathcal{P}^n)^h$ such that

$$\begin{aligned}
&\left(\boldsymbol{w}^h \left(t^-_{n+1} \right), \boldsymbol{u}^h \left(t^-_{n+1} \right) \right)_\Omega - \left(\frac{\partial \boldsymbol{w}^h}{\partial t}, \boldsymbol{u}^h \right)_{\mathsf{Q}^n} + \left(\boldsymbol{w}^h, \boldsymbol{u}^h \cdot \nabla \boldsymbol{u}^h \right)_{\mathsf{Q}^n} \\
&\quad + \left(\nabla \boldsymbol{w}^h, -p\boldsymbol{I} + 2\nu \nabla \boldsymbol{u}^h \right)_{\mathsf{Q}^n} - \left(\boldsymbol{w}^h, u^{h-}_n \boldsymbol{u} \right)_{\mathsf{P}^n_{\mathsf{h}}} - \left(\boldsymbol{w}^h, u^h_n \boldsymbol{u}^h \right)_{\mathsf{P}^n_{\mathsf{g}}} - \left(\nabla q^h, \boldsymbol{u}^h \right)_{\mathsf{Q}^n} \\
&\quad + \left(q^h, u^h_n \right)_{\mathsf{P}^n} - \left(\boldsymbol{w}^h, \boldsymbol{h}^h_g \right)_{\mathsf{P}^n_{\mathsf{g}}} \boxed{+ \left(\boldsymbol{w}^h, \boldsymbol{u}' \cdot \nabla \boldsymbol{u}^h \right)_{\mathsf{Q}^n} + \left(\mathcal{L}'_m \boldsymbol{w}^h, \tau' \left(\boldsymbol{u}' \cdot \nabla \boldsymbol{u}^h \right) \right)_{\tilde{\mathsf{Q}}^n}} \\
&\quad - \left(\nabla q^h, \boldsymbol{u}' \right)_{\mathsf{Q}^n} + \left(\mathcal{L}_m \boldsymbol{w}^h, \tau_m \boldsymbol{r} \right)_{\tilde{\mathsf{Q}}^n} = L \left(\boldsymbol{w}^h \right)_n \\
&\quad \forall \boldsymbol{w}^h \in (\mathcal{Z}^n)^h , \ \forall q^h \in (\mathcal{P}^n)^h , \ n = 0, 1, \ldots, N-1
\end{aligned} \tag{165}$$

where

$$L \left(\boldsymbol{w}^h \right)_n = \left(\boldsymbol{w}^h, \boldsymbol{f} \right)_{\mathsf{Q}^n} + \left(\boldsymbol{w}^h, \boldsymbol{h} \right)_{\mathsf{P}^n_{\mathsf{h}}} + \left(\boldsymbol{w}^h \left(t^+_n \right), \boldsymbol{u}^h \left(t^-_n \right) \right)_\Omega \tag{166}$$

and the fine scale velocity is given by $\boldsymbol{u}' = -\tau_p \boldsymbol{r}$ and $\mathcal{L}'_m$ is defined by equation (150). The operator $\mathcal{L}_m$ for the unsteady case is defined by:

$$\mathcal{L}_m \boldsymbol{w}^h = \begin{cases} \dfrac{\partial \boldsymbol{w}^h}{\partial t} + \boldsymbol{u}^h \cdot \nabla \boldsymbol{w}^h & \text{(SUPG)} \\ \dfrac{\partial \boldsymbol{w}^h}{\partial t} + \boldsymbol{u}^h \cdot \nabla \boldsymbol{w}^h - \nu \Delta \boldsymbol{w}^h & \text{(GLS)} \\ \dfrac{\partial \boldsymbol{w}^h}{\partial t} + \boldsymbol{u}^h \cdot \nabla \boldsymbol{w}^h + \nu \Delta \boldsymbol{w}^h & \text{(MS)}. \end{cases} \tag{167}$$

The boxed terms are the same as for the steady-state case (see equation (149)). Considering first $\{\boldsymbol{w}^h, q^h\} = \{\boldsymbol{0}, q^h\}$ and inserting the model $\boldsymbol{u}' = -\tau_p \boldsymbol{r}$,

$$\begin{aligned} 0 &= -\left(\nabla q^h, \boldsymbol{u}^h\right)_{\mathsf{Q}^n} + \left(q^h, u_n^h\right)_{\mathsf{P}^n} + \left(\nabla q^h, \tau_p \boldsymbol{r}\right)_{\tilde{\mathsf{Q}}^n} \\ &= \left(q^h, \nabla \cdot \boldsymbol{u}^h\right)_{\mathsf{Q}^n} + \left(\nabla q^h, \tau_p \boldsymbol{r}\right)_{\tilde{\mathsf{Q}}^n} \qquad \forall q^h \in \left(\mathcal{P}^n\right)^h \end{aligned} \tag{168}$$

which is equation (147), the weak form of the stabilised continuity equation.

Remark

Conservation of mass may be obtained from the first line of equation (168) by selecting $q^h = 1$. The integration-by-parts utilised in the second line of equation (168) assumes that q^h is continuous across element interfaces and exact quadrature is performed ($\boldsymbol{u}^h$ is always assumed to be continuous). If q^h is discontinuous across element interfaces, the first line of equation (168) requires additional pressure jump terms on element interfaces, whereas the second line is still valid. This follows by performing integration-by-parts element-wise:

$$\begin{aligned} \left(q^h, \nabla \cdot \boldsymbol{u}^h\right)_{\mathsf{Q}^n} &= -\left(\nabla q^h, \boldsymbol{u}^h\right)_{\tilde{\mathsf{Q}}^n} + \left(q^h, u_n^h\right)_{\mathsf{P}^n} + \sum_e \left(q^h, u_n^h\right)_{\mathsf{P}_e^n} \\ &= -\left(\nabla q^h, \boldsymbol{u}^h\right)_{\tilde{\mathsf{Q}}^n} + \left(q^h, u_n^h\right)_{\mathsf{P}^n} + \left(\left[\!\left[q^h \boldsymbol{n} \right]\!\right], \boldsymbol{u}^h\right)_{\tilde{\mathsf{P}}^n} \end{aligned} \tag{169}$$

where

$$\begin{aligned} \llbracket q^h \boldsymbol{n} \rrbracket &= q^h_+ \boldsymbol{n}_+ + q^h_- \boldsymbol{n}_- \\ &= \left(q^h_+ - q^h_- \right) \boldsymbol{n}_+ \\ &= - \left(q^h_+ - q^h_- \right) \boldsymbol{n}_- \end{aligned} \tag{170}$$

in which $\llbracket \cdot \rrbracket$ is the jump operator across element interfaces $\tilde{\Gamma}$,

$$\tilde{\Gamma} = \bigcup_e \Gamma_e \setminus \Gamma, \qquad \mathsf{P}^n_e = \Gamma_e \times]t_n, t_{n+1}[\,, \qquad \tilde{\mathsf{P}}^n = \tilde{\Gamma} \times]t_n, t_{n+1}[\tag{171}$$

and the $\pm$ designations are arbitrary labels of quantities associated with two different elements sharing an interface. Note that equation (170) is invariant under reversal of the $\pm$ labels. When inexact element quadrature is used for the continuous q^h case, the form in the first line of equation (168) is preferable because conservation is unaffected. This is not necessarily the case with the form on the second line of (168). Perhaps because most code developers are unaware of this fact, the second-line form is often employed in practice. □

Following the now familiar procedure of setting $\{\boldsymbol{w}^h, q^h\} = \{\boldsymbol{e}_i, 0\}$, and inserting this into the unsteady Navier-Stokes equation,

$$\begin{aligned} 0 =& B_a \left(\{\boldsymbol{e}_i, 0\}; \{\boldsymbol{u}^h, p^h\}, \boldsymbol{u}^h \right)_n - L\left(\boldsymbol{e}_i\right)_n - \left(\boldsymbol{e}_i, \boldsymbol{h}^h_g \right)_{\mathsf{P}^n_g} \\ =& \left(\boldsymbol{e}_i, \boldsymbol{u}^h\left(t^-_{n+1}\right) \right)_\Omega + \left(\boldsymbol{e}_i, u^{h+}_n \boldsymbol{u}^h \right)_{\mathsf{P}^n_h} - \left(\boldsymbol{e}_i, \left(\nabla \cdot \boldsymbol{u}^h \right) \boldsymbol{u}^h \right)_{\mathsf{Q}^n} - \left(\boldsymbol{e}_i, \tau_p \boldsymbol{r} \cdot \nabla \boldsymbol{u}^h \right)_{\tilde{\mathsf{Q}}^n} \\ & - \left(\boldsymbol{e}_i, \boldsymbol{h}^h_g \right)_{\mathsf{P}^n_g} - \left(\boldsymbol{e}_i, \boldsymbol{f} \right)_{\mathsf{Q}^n} - \left(\boldsymbol{e}_i, \boldsymbol{h} \right)_{\mathsf{P}^n_h} - \left(\boldsymbol{e}_i, \boldsymbol{u}^h\left(t^-_n\right) \right)_\Omega . \end{aligned} \tag{172}$$

Using the result in equation (168), the above equation is equivalent to:

$$\int_{\Omega} \boldsymbol{e}_i \cdot \boldsymbol{u}^h \left(t_{n+1}^-\right) \, d\Omega = \int_{\Omega} \boldsymbol{e}_i \cdot \boldsymbol{u}^h \left(t_n^-\right) \, d\Omega + \int_{\mathsf{P}_{\mathsf{h}}^{+n}} \boldsymbol{e}_i \cdot \left(-u_n^h \boldsymbol{u}^h + \boldsymbol{h}^+\right) \, d\mathsf{P}$$
$$+ \int_{\mathsf{P}_{\mathsf{h}}^{-n}} \boldsymbol{e}_i \cdot \boldsymbol{h}^- \, d\mathsf{P} + \int_{\mathsf{P}_{\mathsf{g}}^{n}} \boldsymbol{e}_i \cdot \boldsymbol{h}_g^h \, d\mathsf{P} + \int_{\mathsf{Q}^n} \boldsymbol{e}_i \cdot \boldsymbol{f} \, d\mathsf{Q} \quad (173)$$

which proves that for the time-dependent case conservation of momentum is attained on space-time slabs.

4 Conclusions

The conservation properties of Galerkin and some common stabilised Galerkin schemes for advection-diffusion and Navier-Stokes equations have been examined. It was shown why advection-diffusion and Navier-Stokes problems based on advective weak forms, computed from Galerkin and stabilised Galerkin finite element methods, are not generally conservative. It was however shown that, through multiscale considerations, this deficiency can be easily overcome for stabilised methods by the addition of a residual term to the weak equation which ensures global conservation. Surprisingly, conservation can be restored more easily to stabilised Galerkin methods than traditional Galerkin methods. The reason for this is that conservation is in conflict with the inf-sup condition associated with incompressibility in the Galerkin formulation, whereas in stabilised formulations the inf-sup condition is circumvented.

Stabilised, conservative formulations have been described and presented for: steady advection-diffusion equation (equation (64)); unsteady advection-diffusion equation (equation (92)); steady incompressible Navier-Stokes equations (equation (149)); and unsteady incompressible Navier-Stokes equations (equation (165)). The techniques described herein have been extensively tested, and verified in several academic and commercial flow codes, including Spectrum [13] and Acusolve [14].

Similar issues are being investigated in the discontinuous Galerkin method literature. The interested reader is referred to Cockburn et al. [15, 16].

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