The propagation problem in longest-edge refinement

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SUMMARY

In this work we investigate the refinement propagation process in longest-edge based local refinement algorithms for unstructured meshes of triangles. We prove that asymptotically the propagation path extends on average to a few neighbor adjacent triangles. Supporting numerical experiments are provided that demonstrate this asymptotic behavior and explore other characteristics of the mesh sequence generated.

KEY WORDS: mesh refinement; longest edge bisection; propagation path.

1. INTRODUCTION

Triangulating a set of points plays a central role in Computational Geometry, and is a basic tool in many other fields such as Computer Graphics, Geometric Modeling and Finite Element Method [2, 4]. A related problem that is also of considerable interest is refinement of a mesh. The refinement problem can be described as any technique involving the insertion of additional vertices in order to produce meshes with desired features: well shaped triangles, mesh conformity and smoothness. The presence of thin triangles can lead to undesirable behavior affecting numerical stability and accuracy. Mesh conformity refers to the requirement that the
intersection of adjacent triangles is either a common vertex or an entire side. Mesh smoothness implies that the transition between small and large elements should be gradual.

Certain longest-edge refinement algorithms [10, 11, 13] guarantee the construction of non-degenerate and smooth unstructured triangulations. In these schemes the longest edges are progressively bisected and hence all angles in subsequent refined triangulations are greater than or equal to half the smallest angle in the initial triangulation [14]. However, the extent of secondary refinements induced in neighboring† elements by the initiating element edge bisection is not known [7, 13]. One can construct pathological cases where refinement of a single element propagates through the entire mesh (Figure 1 (a)). However experience indicates that this is an exception and that in practice the refinement propagates through only a few neighbors on average. Our goal here is to address this question. We provide both theoretical results and empirical evidence showing that successive application of refinement to an arbitrary unstructured triangular mesh produces meshes in which the average propagation path is reduced in each refinement stage, and asymptotically approaches the constant 5.

Figure 1. Longest edge refinement propagation. The dependencies in the propagation when refining t are indicated by arrows.

†Throughout this work, neighbor triangles are triangles sharing an edge
2. PRELIMINARIES. THE REFINEMENT AND THE PROPAGATION PROBLEM

The refinement of triangular meshes involves two main tasks. The first is the partition of the target triangles and the second is the propagation to successive neighbor triangles to preserve conformity. Several approaches for partitioning triangles have been studied. The simplest is *Bisection* into two subtriangles by connecting the midpoint of one of the edges to the opposite vertex. If the longest edge is chosen for the bisection, then this is called *Longest Edge Bisection*, see Figure 2 (a). The *Four Triangles Longest Edge Partition, (4T-LE)* bisects a triangle into four subtriangles where the original triangle is first subdivided by its longest edge as before and then the two resulting triangles are bisected by joining the new midpoint of the longest edge to the midpoints of the remaining two edges of the original triangle, as in Figure 2 (d), [11].

Bank *et al*. [1] simply connect non-conforming nodes to the opposite vertices of neighbor triangles to enforce conformity. However, a small apposite angle may be bisected in this step. Alternatively, one can connect to the midpoint of the longest edge of the neighbor triangle and continue the longest edge bisection process.

**Definition 1 (Longest Edge Neighbor Triangle)** *The longest edge neighbor of a triangle t is the neighbor triangle t*\(^*\) *which shares with t the longest edge of t.*

In the case of an isosceles or equilateral triangle, we may assume a ‘roundoff level’ perturbation to yield a single longest edge. This can be random and hence uniqueness is not implied. However, in the cases in which one of the longest edges has been already identified for bisection in a neighbor triangle, this edge is chosen as the longest edge to get the refinement as local as possible.
Definition 2 (Longest Edge Propagation Path (LEPP) [11, 12]) The Longest Edge Propagation Path of a triangle $t_0$ is the ordered list of all adjacent triangles $\Lambda = \{t_0, t_1, \ldots, t_n\}$ such that $t_i$ is the longest edge neighbor triangle of $t_{i-1}$.

If the longest edge bisection is used to refine a given triangle $t$, then the LEPP$(t)$ provides the list of triangles to be refined, see Figure 3. Note that if the 4T-LE partition is used to refine a given triangle $t$, then the LEPP’s of the neighbor triangles of $t$ in the mesh $\tau^* = \tau - t$ provide the lists of triangles to be refined (see Figure 4 and Table I). As a consequence, the LEPP’s provide the main adjacency lists used by the algorithms.

\begin{figure}
\hspace{1cm} (a) \hspace{4.5cm} (b)
\begin{center}
\includegraphics[width=0.4\textwidth]{figure3.png}
\end{center}
\caption{(a) LEPP$(t)$ = $\{t, t_a, t_b, t_c\}$ (b) LE bisection of $t$ and refinement propagation}
\end{figure}

\begin{figure}
\hspace{1cm} (a) \hspace{4.5cm} (b) \hspace{4.5cm} (c)
\begin{center}
\includegraphics[width=0.8\textwidth]{figure4.png}
\end{center}
\caption{(a) Edge bisection for refining triangle $t$ (b) 4T-LE refinement of $t$ and propagation refinement area, (c) refinement of triangles}
\end{figure}

Definition 3 (Boundary and Interior triangle) $^{\dagger}$ Let $\tau$ be a two dimensional triangulation for a bounded domain $\Omega$. A triangle $t \in \tau$ is said to be a boundary triangle if $t$ has an edge

$^{\dagger}$Throughout this work $\tau$ denotes a 2D triangulation
Table I. Triangles and associated LEPP’s of mesh in Figure 4 (a)

<table>
<thead>
<tr>
<th>Triangle</th>
<th>LEPP</th>
<th>LEPP on mesh $\tau^* = \tau - t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>${t, t_c, t_d}$</td>
<td>-</td>
</tr>
<tr>
<td>$t_a$</td>
<td>${t_a, t, t_c, t_d}$</td>
<td>${t_a}$</td>
</tr>
<tr>
<td>$t_b$</td>
<td>${t_b, t_c}$</td>
<td>${t_b, t_c}$</td>
</tr>
<tr>
<td>$t_c$</td>
<td>${t_c, t_d}$</td>
<td>${t_c, t_d}$</td>
</tr>
</tbody>
</table>

coincident with the boundary $\partial \Omega$ of $\Omega$. Otherwise, $t$ is an interior triangle of $\tau$.

**Definition 4 (Pair of Terminal triangles)** Two neighbor triangles $(t, t^*)$ will be called a ‘pair’ of terminal triangles if they share a common longest edge. If a triangle $t$ does not belong to a pair of terminal triangles, $t$ is said to be a ‘single’ triangle.

For any triangle $t_0$, if $LEPP(t_0) = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ then for triangle $t_n$ either: (i) $t_n$ has its longest edge coincident with the boundary or (ii) $t_{n-1}$ and $t_n$ are a pair of terminal triangles that share a common longest edge, [12].

If all the triangles in a mesh are pairs of terminal triangles, then all the $LEPP$ lists are comprised only of two triangles.

**Definition 5 (LEPP-balanced mesh)** Triangulation $\tau$ is said to be LEPP-balanced if it is comprised of pairs of terminal triangles. Otherwise, it is said to be a non LEPP-balanced mesh.

**Remark:** In [5] the terminology ‘balanced’ is applied to angles and areas. This is relevant to triangle shape but not directly related to our $LEPP$ study here.

**Definition 6 (LEPP-balancing degree)** Let $\tau$ contain $N$ triangles of which $T$ triangles are in pairs of terminal triangles. Then, the LEPP-balancing degree of $\tau$, noted as $B(\tau)$, is defined as follows:

$$B(\tau) = \frac{T}{N}$$  \hspace{1cm} (1)

Note that $0 \leq B(\tau) \leq 1$ and in the case $B(\tau) = 1$, the mesh is LEPP-balanced.
**Remark:** If \( \tau \) is such that the LEPP-balancing degree is 0, then the conformity process when refining any triangle \( t_0 \in \tau \) extends to the boundary of \( \tau \).

![Figure 5](image.png)

Figure 5. (a) LEPP-Balanced mesh, (b) Non balanced LEPP mesh (white triangles are pairs of terminal triangles)

Figure 5 shows a LEPP-balanced mesh in (a) and a non balanced mesh in (b). Here and in subsequent figures we represent the longest-edges with a dashed line.

A simple example of a LEPP-balanced mesh is any mesh comprised entirely of pairs of right triangles sharing the longest-edges. In such a mesh, if one applies uniform 4T-LE refinement, then all triangles are pairs of terminal triangles and they are similar to the original right triangles.

### 3. PROPAGATION PROPERTIES OF RECURSIVE 4T-LE REFINEMENT

We are particulary interested in the average and maximum lengths of the propagation paths generated by longest-edge refinement since they are important in assessing algorithm efficiency.

First we introduce the *Conformity Neighborhood* associated with the application of 4T-LE local refinement. This concept will be useful in the study of the propagation properties.

**Definition 7** (Conformity Neighborhood \( V_c \)) *When refining a triangle \( t \in \tau \), the Conformity Neighborhood \( V_c(t) \) of \( t \), is the set of triangles in \( \tau^* = \tau - t \) that need to be refined due to the conformity process for \( t \).*
The propagation problem in longest-edge refinement

Definition 8 (M1) When refining a triangle \( t \in \tau \), \( M_1(t) \) is said to be the size of \( V_c(t) \): 
\[ M_1(t) = |V_c(t)|. \]

Proposition 1. For each \( t \in \tau \), \( M_1(t) \) is the sum of the lengths of the LEPP’s of the neighbors of \( t \) in the mesh \( \tau^* = \tau - t \). \hfill \Box

Figure 1 shows that it always is possible to construct meshes in which \( M_1(t) \) is \( \Theta(N) \), where \( N \) is the number of elements. Here, the average of \( M_1 \) is 
\[ \mu(M_1) = \frac{\sum M_1(t)}{N} = \frac{\sum_{k=1}^{N-1} k}{N} = \frac{N^2 - N}{2}. \] On the other hand, if \( B(\tau) = 1 \) as in Figure 5 (a), then \( M_1(t) \leq 5 \forall t \in \tau \).

Definition 9 (M2) For each \( t \in \tau \), \( M_2(t) \) is the maximum length of the LEPP’s of the neighbor triangles of \( t \) in the mesh \( \tau^* = \tau - t \): 
\[ M_2 = \max\{|V_c(t) \cap \text{LEPP}(t_a)|, t_a \text{ neighbor to } t\}. \]

Since the conformity process extends at most by the three edges of \( t \) the propagation defines at most three lists of ordered triangles. \( M_2(t) \) is the maximum number of triangles of the three resulting lists. For example, in Figure 4, \( M_2(t) = 2 \) because the maximum number of triangles among \( \{t_b, t_c\}, \{t_c, t_d\}, \{t_a\} \) is 2, see Table I.

Proposition 2. Let \( \tau \) be LEPP-balanced. Then, for each interior triangle \( t \in \tau \), \( M_1(t) = 5 \) and \( M_2(t) = 2 \). Moreover, \( M_1(t) = 5 \iff M_2(t) = 2 \).

Proof: Let \( t \) be an interior triangle of \( \tau \). Since \( \tau \) is a LEPP-balanced mesh, \( t \) is adjacent to another triangle \( t_1 \) by their common longest edge. Let \( t_2 \) and \( t_3 \) be the two other adjacent triangles to \( t \). Again, since \( \tau \) is a LEPP-balanced mesh, \( t_2 \) and \( t_3 \) are adjacent to other triangles \( t'_2 \) and \( t'_3 \) by their respective common longest edges, and \( t'_2 \neq t \neq t'_3 \). Considering the mesh \( \tau^* = \tau - t \) we have that \( \text{LEPP}(t_1) = \{t_1\}, \text{LEPP}(t_2) = \{t_2, t'_2\} \) and \( \text{LEPP}(t_3) = \{t_3, t'_3\} \). Hence \( V_c(t) = \{t_1, t_2, t_3, t'_1, t'_3\} \) so \( M_1(t) = 5 \) and \( M_2(t) = 2 \). \hfill \Box

Our next goal is to prove that the uniform application of the 4T-LE partition will produce a sequence of meshes with increasing LEPP-balancing degree approaching 1. As a consequence,
the mean of $M_1$ and the mean of $M_2$ tend to 5 and 2 respectively, when the number of refinements applied tends to infinity.

**Proposition 3.** \[10\] (a) The first application of the 4T-LE partition to a given triangle $t_0$ introduces two new triangles that are similar to the original triangle $t_0$. Moreover, these two triangles have their longest edges coincident with the longest edge of the original triangle. And two new triangles $t_1$, that are similar to each other but not necessarily to the original triangle $t_0$. Triangles $t_1$ may be a terminal pair or not.

(b) The iterative application of the 4T-LE partition to a given triangle $t_0$ introduces at most one new distinct (up to similarity) triangle in each iteration.

Proof: Follows immediately from angle properties, see Figure 6 (a).

**Proposition 4.** If the 4T-LE partition to an initial triangle $t_0$ introduces a pair of terminal triangles $t_1$, then the iterative application of the 4T-LE partition introduces pairs of terminal triangles excepting the triangles located at the longest edge of $t_0$. Moreover, in this case only two classes of similar triangles are generated, corresponding to $t_0$ and $t_1$ respectively (see Figure 6).

Proof: The situation in the above hypothesis is depicted in Figure 6 (b). The result follows trivially from the angle properties of parallel lines in the nested triangles.

To demonstrate that recursive uniform 4T-LE refinement introduces meshes with relatively more pairs of terminal triangles for any arbitrary triangular mesh we consider right, acute and obtuse triangles respectively. We begin in the next Proposition with the right and acute triangle cases:

**Proposition 5.** (Right and acute triangle cases) The application of the 4T-LE partition to an initial right or acute triangle $t_0$ produces two new single triangles similar to the original one (located at the longest edge of $t_0$) and a pair of terminal triangles $t_1$. These triangles $t_1$ are also similar to the original one $t_0$ in the case of right triangle $t_0$, and they are similar to
each other but non-similar to the initial one in the case of acute triangle $t_0$. (See Figure 6).

\[ \square \]

![Figure 6. 4T-LE partition. Acute triangle](image)

The obtuse triangle case offers a different situation:

**Proposition 6.** (Obtuse triangle case) The application of the 4T-LE partition to an initial obtuse triangle $t_0$, produces two new single subtriangles similar to the original one (located at the longest edge of $t_0$) and a pair of subtriangles $t_1$. These subtriangles $t_1$ either

1. are a pair of similar terminal triangles, as in Figure 6 (b) ($t_0$ is said to be a Type 1 obtuse triangle), or

2. a pair of similar single triangles, as in Figure 7 (b) ($t_0$ is said to be a Type 2 obtuse triangle).

![Figure 7. (a) Type 2 obtuse triangle $t_0$ (b) 4T-LE partition of $t_0$](image)

**Proof:**

Let $\alpha_0 \leq \beta_0 \leq \gamma_0$ be the angles of the initial obtuse triangle $t_0$ and let $a, b, c$ be the sides of $t_0$ respectively opposite to $\alpha_0, \beta_0$ and $\gamma_0$. For the new non-similar subtriangles generated, we
denote by $\epsilon$ the opposite angle to $MN$ and $\sigma$ the opposite angle to $CN$. Since $MN \leq CN$ and $\epsilon \leq \sigma$ the longest edge of $t_1$ is either the new edge $CM$ or $CN$. In the first situation (point 1 of the Proposition), triangles $t_1$ are a pair of terminal triangles sharing edge $CM$ as the longest edge.

In the second case, the largest angle of $t_1$ is $\sigma$ (see triangles $t_1$ in Figure 7). The new triangles $t_1$ are not a pair of terminal triangles (point 2 of the Proposition).

It should be noted that the $4T$-LE partition always produces two new single triangles similar to the original one (located at the longest edge of $t_0$) and excepting for Type 2 obtuse triangles, a pair of terminal triangles (similar or non similar to the original one). Moreover, in this scenario, the single triangles generated by the iterative $4T$-LE partition are those located at the longest edge of the initial triangle, Proposition 4 (see Figure 6).

The following Proposition states the recursive improvement property of the $4T$-LE partition for obtuse triangles [10]:

**Proposition 7.** If the $4T$-LE partition of an obtuse triangle $t_0$ introduces a pair of similar single triangles $t_1$, (Type 2 obtuse triangles), then

1. $\gamma_1 = \sigma$
2. $\gamma_1 = \gamma_0 - \epsilon \leq \gamma_1 - \alpha_0$

hold for the new angles of the new triangle $t_1$, see Figure 7.

It is worth noting that in repeated $4T$-LE refinement, Type 2 obtuse triangles are less obtuse than in the preceding mesh, and after a finite number $k$ of $4T$-LE partitions the new generated triangles will be no longer obtuse. Proposition 5, for right or acute triangle cases then applies.

In view of the previous properties, we have:

**Proposition 8.** Let $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$ be a sequence of nested meshes obtained by repeated application of $4T$-LE partition to the previous mesh. Then, the LEPP-balancing degree of the meshes tends to 1 as $n \to \infty$. 
Proof:
It suffices to prove the result for the case in which the initial mesh $\tau_0$ only contains a single triangle $t_0$. Then, the number of generated triangles associated with the $4T$-$LE$ partition at stage $n$ of refinement is:

$$N_n = 4^n$$  \hspace{1cm} (2)

First, we prove the proposition for initial right, acute, and the Type 1 obtuse triangles. In this situation, the number of triangles in pairs of terminal triangles $T_n$ generated at stage $n$ of uniform $4T$-$LE$ partition satisfies (see Proposition 4 and Figure 6):

$$T_n = 4T_{n-1} + 2(N_{n-1} - T_{n-1}) = 2(T_{n-1} + N_{n-1})$$  \hspace{1cm} (3)

with $N_0 = 1$ and $T_0 = 0$.
We simplify the recurrence Equations 2 and 3 to get:

$$T_n = 2^nN_0 + (4^n - 2^n)N_0$$  \hspace{1cm} (4)

Therefore,

$$\lim_{n \to \infty} B(\tau_n) = \lim_{n \to \infty} \frac{T_n}{N_n} = 1$$

To complete the proof, we now consider the case of an initial Type 2 obtuse triangle $t_0$. Table II presents the number of distinct types of triangles generated by the $4T$-$LE$ iterative refinement of $t_0$. We denote by $t^n_j$ the number of triangles of similarity class $t_j, j = 0, 1, 2, \cdots, k$ at stage $n$ of refinement. For example, after the second refinement 4 triangles are similar to $t_0$, 8 triangles similar to $t_1$ and 4 new triangles similar to $t_2$.

From Proposition 6 (2) and Figure 7 we derive Table II, in which the following relation holds:

$$t^n_j = 2(t^n_{j-1} + t^n_{j-1})$$, \hspace{1cm} j = 1, 2, 3, \cdots, k$$  \hspace{1cm} (5)

The solution to Equation (5) with initial condition $t^n_0 = 1$ can be easily expressed in terms of binomial coefficients as follows:

$$t^n_j = 2^n \binom{n}{k}$$  \hspace{1cm} (6)
On the other hand, from Proposition 7, the iterative 4T-LE partition of any obtuse triangle \( t_0 \) produces a finite sequence of distinct (up to similarity) triangles, \( t_j \), \( 0 < j \leq k \). After \( k \) refinement stages there will no longer be distinct new generated triangles different from those already generated, (see proof of Proposition 6). Therefore, the number of triangles in pairs of terminal triangles \( T_n \) after the \( k \) refinement stage with \( n > k \) satisfy:

\[
T_n \geq 2^n \sum_{m=k}^{n} \binom{n}{m}
\]

### Table II. Triangles evolution in the 4T-LE partition

<table>
<thead>
<tr>
<th>Ref.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>\cdots</th>
<th>k</th>
<th>\cdots</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>\cdots</td>
<td>( t^k_0 )</td>
<td>\cdots</td>
<td>( t^n_0 )</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>2</td>
<td>8</td>
<td>24</td>
<td>64</td>
<td>\cdots</td>
<td>( t^k_1 )</td>
<td>\cdots</td>
<td>( t^n_1 )</td>
<td></td>
</tr>
<tr>
<td>( t_2 )</td>
<td>4</td>
<td>24</td>
<td>96</td>
<td>\cdots</td>
<td>( t^k_2 )</td>
<td>\cdots</td>
<td>( t^n_2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t_3 )</td>
<td>8</td>
<td>64</td>
<td>\cdots</td>
<td>( t^k_3 )</td>
<td>\cdots</td>
<td>( t^n_3 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t_4 )</td>
<td>\cdots</td>
<td>16</td>
<td>\cdots</td>
<td>( t^k_4 )</td>
<td>\cdots</td>
<td>( t^n_4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \cdots )</td>
<td>\cdots</td>
<td>\cdots</td>
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<td>\cdots</td>
<td>\cdots</td>
<td>\cdots</td>
<td>\cdots</td>
<td></td>
</tr>
<tr>
<td>( t_k )</td>
<td>\cdots</td>
<td>( t^k_k )</td>
<td>\cdots</td>
<td>( t^n_k )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It follows that:

\[
1 \geq B(\tau_n) \geq \frac{2^n \sum_{m=k}^{n} \binom{n}{m}}{2^n \sum_{m=0}^{n} \binom{n}{m}} = \frac{\sum_{m=k}^{n} \binom{n}{m}}{2^n}
\]

Taking limits:

\[
1 \geq \lim_{n \to \infty} B(\tau_n) \geq \lim_{n \to \infty} \frac{2^n \sum_{m=k}^{n} \binom{n}{m}}{2^n \sum_{m=0}^{n} \binom{n}{m}} = \lim_{n \to \infty} \frac{\sum_{m=k}^{n} \binom{n}{m}}{2^n}
\]

Since

\[
\sum_{m=k}^{n} \binom{n}{m} = 2^n - \sum_{m=0}^{k-1} \binom{n}{m} \geq 2^n - \binom{n}{k-1}(k)
\]

we have

\[
1 \geq \lim_{n \to \infty} B(\tau_n) \geq \lim_{n \to \infty} \frac{2^n - \binom{n}{k-1}(k)}{2^n} = 1
\]

So, \( \lim_{n \to \infty} B(\tau_n) = 1 \). \( \Box \)
Corollary 1. For iterative application of 4T-LE uniform refinement to an initial triangular mesh $\tau_0$, the means of $M_1$ and of $M_2$ tend to 5 and 2 respectively, as the number of refinements tend to infinity.

Proof: This result follows directly from Propositions 2 and 8. □

The iterative application of uniform 4T-LE refinement to an arbitrary triangular mesh reveals a geometric structure of the edges. Often, a fractal set is convenient to describe the result of such an iterative process, see [3, 4, 8]. We use a fractal set to define the geometric structure as follows:

Definition 10. Let $S(\tau)$ be the set of single triangles (not pairs of terminal triangles) in a mesh $\tau$. A polyline $P(\tau)$ is the set of connected longest-edges of triangles in $S(\tau)$. The set of polylines of $\tau$ is a fractal set and represented by $C(\tau)$.

The interest in $C(\tau)$ comes from the fact that the location of polylines in $\tau$ represents the geometric place where the set of single triangles are located. Therefore, $C(\tau)$ may be useful in post-processing generated meshes so that single triangles could be modified to produce pairs of terminal triangles, or it could be useful for applying local refinement by similar partition, simple bisection etc.

The following proposition summarizes some important properties of $C(\tau)$.

Proposition 9. Let $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_n\}$ be a sequence of nested meshes obtained by uniform 4T-LE refinement. Let $\tau_k$ the first mesh of $\Gamma$ in which there are no new classes of triangles (up to similarity). Then:

1. The number of edges in $C(\tau_i)$ is exactly double the number of edges in $C(\tau_{i-1})$, $k+1 \leq i \leq n$.

2. Excepting the case of Type 2 obtuse triangles, $C(\tau_i)$, $k \leq i \leq n$, are shape and length invariant over the respective meshes. We say that $C(\tau)$ is stable.

3. Excepting the case of Type 2 obtuse triangles, the sets of edges in $C(\tau_i)$ $k \leq i \leq n$ are
located at the longest edges of single triangles of mesh $\tau_k$.

4. Excepting the case of Type 2 obtuse triangles, the number of edges in $C(\tau_i)$ is exactly the number of single triangles in $\tau_i$.

Proof:
Point 1 follows from the fact that each edge is bisected in uniform refinement. To prove point 2 it suffices to see that a polyline in $C(\tau_{i-1})$ is exactly the same polyline as in $C(\tau_i)$ excepting that the latter has double number of edges; then the shape and length of any polyline is invariant over the respective meshes, $k \leq i \leq n$. From Definition 10, a polyline in $C(\tau_i)$ is the set of connected longest-edges of single triangles in $S(\tau_i)$, and as $C(\tau)$ is stable then the longest-edges are referred to single triangles in mesh $\tau_k$ (point 3 of the proposition). Point 4 of the proposition easily follows from the fact that any edge in a polyline comes from a single triangle in $S(\tau_i)$. □

4. NUMERICAL EXPERIMENTS

In this section we present numerical results showing that the practical behavior of the 4T-LE partition is in concordance with the reported theory in this work, mainly Propositions 8, 9 and Corollary 1.

4.1. 4T-LE REFINEMENT. TRIANGLE CASES STUDY

We treat two different cases: (1) right, acute and Type 1 obtuse triangles and (2) Type 2 obtuse triangles. To these initial meshes we apply seven stages of uniform 4T-LE refinement. The goal in this first experiment is to calculate the number of triangles in terminal pairs and compare to the amount of single triangles in each stage of the refinement (Table III). By stage 7 the number of triangles in terminal pairs is clearly larger than the number of the single triangles and this is in agreement with Proposition 8. In Figures 8 and 10 the initial triangle meshes and four stages of uniform refinement for acute and Type 2 obtuse triangles are presented. The shaded triangles are single triangles and the other are terminal pairs. The
subtriangles in Figure 10 (b) are single triangles. For the successive refinements, Figures 10 (c)-(e), an increasing number of pairs of terminal triangles are generated and these triangles tend to cover the area of the initial triangle. Figures 9 and 11 depict the fractal sets $C(\tau)$ for several refined meshes. The set of polylines of $\tau$ corresponds to the longest edges of the same refined triangles (point 3 of Proposition 9) and indicated by straight line segments.

Table III. Triangles in terminal pairs (T) and single triangles (S) evolutions.

<table>
<thead>
<tr>
<th>Triangle case</th>
<th>Ref. Stage:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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</thead>
<tbody>
<tr>
<td>Right, Acute, Type 1 obtuse</td>
<td>T</td>
<td>0</td>
<td>2</td>
<td>56</td>
<td>240</td>
<td>992</td>
<td>4032</td>
<td>16256</td>
<td></td>
</tr>
<tr>
<td></td>
<td>S</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
</tr>
<tr>
<td>Type 2 obtuse</td>
<td>T</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>26</td>
<td>162</td>
<td>802</td>
<td>3586</td>
<td>15234</td>
</tr>
<tr>
<td></td>
<td>S</td>
<td>1</td>
<td>4</td>
<td>14</td>
<td>38</td>
<td>94</td>
<td>222</td>
<td>510</td>
<td>1150</td>
</tr>
</tbody>
</table>

Figure 8. 4T-LE refinement. Acute triangle case

Figure 9. Fractal set $C(\tau)$. Acute triangle case
4.2. 4T-LE REFINEMENT. ARBITRARY MESHES STUDY

We next consider a Delaunay mesh in a rectangle (Figure 12 (a)) and an irregular mesh in a pentagon (Figure 14 (a)) with five stages of uniform refinement. It should be noted that the triangles in the Delaunay mesh are almost regular in terms of the angles, moreover, the mean of the minimum angles and of the maximum angles are 48.91 and 72.91 degrees respectively. The initial value of $B(\tau_0)$ is 0.4833. On the other hand, for the irregular mesh in the pentagon the mean of the minimum angles and of the maximum angles are 9.18 and 120.41 degrees respectively, and $B(\tau_0) = 0$.

The refined meshes for the initial Delaunay mesh are presented in Figures 12 and 13. The shading in Figure 13 (a)-(b) also illustrate the structure of the fractal set $C(\tau)$. In Table IV can be noted that the number of triangles in terminal pairs increase as the refinement stage grows, and as result, as does the LEPP-balancing degree. Table V reports the means and standard deviations of $M_1$ and $M_2$ and respective histograms are graphed in Figure 16. It is observed that both means tend to 5 and 2 respectively, as the refinement continues. The asymptotic behavior is graphed in Figure 15.

Similarly, the refined meshes for the ‘pentagonal’ domain are shown in Figure 14 and the asymptotic behavior for the means $\mu(M1)$, $\mu(M2)$ graphed in Figure 17. The evolution of the LEPP-balancing degree is summarized in Table VI and a comparison graphed in

![Figure 10. 4T-LE refinement. Type 2 obtuse triangle case](image-url)
Figure 11. Fractal set $C(\tau)$. Type 2 obtuse triangle case.

Figure 19. Note that in both meshes the LEPP-balancing degree tends to 1 when the number of refinements increases, even in the Pentagonal mesh, which exhibits an initial LEPP-balancing degree $B(\tau_0) = 0$ (see Figure 19). Table VII reports the means and standard deviations of $M_1$ and $M_2$ and respective histograms are graphed in Figure 18.

These results are also applicable to local refinement. In order to demonstrate this we consider application of $4T$-$LE$ local refinement on a domain corresponding to the Gran Canaria Island (Figure 20). The initial mesh is a Delaunay mesh and local refinement is applied on disjoint subregions $S_1$, $S_2$ and $S_3$ with $S = S_1 \cup S_2 \cup S_3$, for innermost region $S_3$, intermediate region $S_2$ and outermost region $S_1$. Table VIII and Figure 21 confirm similar behavior to that observed for uniform refinement with $\mu(M_1)$ and $\mu(M_2)$ approaching 5 and 2 respectively and the LEPP-balancing degree approaching 1. Figure 22 graphs $M_1$ and $M_2$ histograms for the initial mesh and refinement steps 3 and 6.
Table IV. Delaunay mesh. LEPP-balancing degree

<table>
<thead>
<tr>
<th>Ref. Step</th>
<th>Triangles in terminal pairs</th>
<th>Triangles</th>
<th>LEPP-balancing degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (Initial mesh)</td>
<td>58</td>
<td>120</td>
<td>0.48333</td>
</tr>
<tr>
<td>1</td>
<td>356</td>
<td>480</td>
<td>0.74166</td>
</tr>
<tr>
<td>2</td>
<td>1672</td>
<td>1920</td>
<td>0.87083</td>
</tr>
<tr>
<td>3</td>
<td>7184</td>
<td>7680</td>
<td>0.93541</td>
</tr>
<tr>
<td>4</td>
<td>29728</td>
<td>30720</td>
<td>0.96770</td>
</tr>
<tr>
<td>5</td>
<td>121447</td>
<td>122880</td>
<td>0.98821</td>
</tr>
</tbody>
</table>

Table V. Delaunay mesh. M1 and M2 statistics

<table>
<thead>
<tr>
<th>Ref. Step</th>
<th>Triangles</th>
<th>(\mu(M1))</th>
<th>(\mu(M2))</th>
<th>(\sigma(M1))</th>
<th>(\sigma(M2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (Initial mesh)</td>
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<td>0.8886</td>
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<td>1</td>
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<td>0.5821</td>
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<tr>
<td>2</td>
<td>1920</td>
<td>5.0573</td>
<td>2.1953</td>
<td>0.7727</td>
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<tr>
<td>3</td>
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<td>5.0289</td>
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<td>0.3031</td>
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<td>4</td>
<td>30720</td>
<td>5.0145</td>
<td>2.0481</td>
<td>0.3747</td>
<td>0.2165</td>
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<td>5</td>
<td>122880</td>
<td>5.0076</td>
<td>2.0286</td>
<td>0.2754</td>
<td>0.0407</td>
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</table>

Table VI. Pentagonal mesh. LEPP-balancing degree

<table>
<thead>
<tr>
<th>Ref. Step</th>
<th>Triangles in terminal pairs</th>
<th>Triangles</th>
<th>LEPP-balancing degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (Initial mesh)</td>
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<td>125</td>
<td>0</td>
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<tr>
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<td>246</td>
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<td>2000</td>
<td>0.54400</td>
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<tr>
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<td>8000</td>
<td>0.59725</td>
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<td>21240</td>
<td>32000</td>
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Table VII. Pentagonal mesh. M1 and M2 statistics

<table>
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<tr>
<th>Ref. Step</th>
<th>Triangles</th>
<th>(\mu(M1))</th>
<th>(\mu(M2))</th>
<th>(\sigma(M1))</th>
<th>(\sigma(M2))</th>
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</thead>
<tbody>
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<td>0.8833</td>
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<tr>
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<td>2.4123</td>
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<tr>
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<td>128000</td>
<td>5.3706</td>
<td>2.2041</td>
<td>0.9475</td>
<td>0.7576</td>
</tr>
</tbody>
</table>
Figure 12. (a) Delaunay mesh. (b) Uniform 4T-LE refinement
(a) Refinement step 2, 1920 triangles, 1672 pairs of terminal triangles (white triangles)

(b) Refinement step 3, 7680 triangles, 7184 pairs of terminal triangles (white triangles)

Figure 13. Uniform 4T-LE refinement. Delaunay mesh
Figure 14. Uniform 4T-LE refinement. Pentagonal mesh.
Figure 15. Delaunay mesh. Evolution of $\mu(M_1)$ and $\mu(M_2)$.

Figure 16. Delaunay mesh. M1 and M2 histograms.
Figure 17. Pentagonal mesh. Evolution of $\mu(M1)$ and $\mu(M2)$

Figure 18. Pentagonal mesh. M1 and M2 histograms.
Figure 19. LEPP-balancing degree evolution. Delaunay and Pentagonal mesh

Table VIII. Gran Canaria mesh. M1 and M2 statistics

<table>
<thead>
<tr>
<th>Ref. Step</th>
<th>Triangles</th>
<th>$\mu(M1)$</th>
<th>$\mu(M2)$</th>
<th>$\sigma(M1)$</th>
<th>$\sigma(M2)$</th>
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</thead>
<tbody>
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Table IX. Gran Canaria mesh. LEPP-balancing degree

<table>
<thead>
<tr>
<th>Ref. Step</th>
<th>Triangles in terminal pairs</th>
<th>Triangles</th>
<th>LEPP-balancing degree</th>
</tr>
</thead>
<tbody>
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<td>0.41891</td>
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<td>736</td>
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</table>
(a) Refinement step 1, 326 terminal triangles (blue color), 736 total triangles. (b) Refinement step 3, 1588 terminal triangles (blue color), 2624 total triangles. Interior zoom. (c) Refinement step 4, 7020 terminal triangles (blue color), 9258 total triangles.

Figure 20. Local 4T-LE refinement. Gran Canaria mesh.
Figure 21. Gran Canaria mesh. Statistics of M1 and M2 evolution.

Figure 22. Gran Canaria mesh. M1 and M2 histograms.
5. CONCLUSIONS

In this work we have studied the propagation problem associated with the longest edge based refinement algorithms in 2D. We have theoretically proved in the paper that the propagation path asymptotically extends on average to a few neighbor adjacent triangles. This result has also been numerically demonstrated for repeated local refinement. The extent of refinement for triangle \( t \) defines a Conformity Neighborhood characterized by two parameters \( (M_1(t) \) and \( M_2(t)) \). When repeated uniform refinement is applied to an initial arbitrary triangular mesh, the average of the parameters \( M_1(t) \) tends to 5 and the average of \( M_2(t) \) tends to 2. This implies for local refinement of practical applications that on average the propagation of secondary refinements induced by specified refinements will be limited to a proportionally small number of elements with a confined limit. We also have introduced the concept of LEPP-balancing degree (ratio between triangles in terminal pairs and total triangles in a mesh) for longest edge refinement of meshes and have proved that the LEPP-balancing degree asymptotically tends to 1. These results are also a global measure of the improvement of the generated meshes obtained by \( 4T-LE \) iterative refinement. The counterpart 3D propagation problem is a more complex study because the number of connectivity patterns are considerably higher than in 2D, more than fifty of partial divisions in the respective \( 8T-LE \) partition, [9].

Finally, the polyline \( P(\tau) \) and the fractal set \( C(\tau) \) of a triangulation has been introduced and some of their properties have been established. This latter concept may be of value in mesh smoothing, since the fractal set delineates the region comprised of single triangles.

ACKNOWLEDGEMENTS

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REFERENCES