ERROR ANALYSIS OF PRESSURE-CORRECTION SCHEMES FOR THE NAVIER-STOKES EQUATIONS WITH OPEN BOUNDARY CONDITIONS

J.L. GUERMOND[1], P. MINEV[2], AND J. SHEN[3]

ABSTRACT. The incompressible Navier-Stokes equations with prescribed normal stress (open) boundary conditions on part of the boundary are considered. It is shown that the standard pressure-correction method is not suitable for approximating the Navier-Stokes equations with open boundary conditions, whereas the rotational pressure-correction method yields reasonably good error estimates. These results appear to be the first for splitting schemes with open boundary conditions. Numerical results in agreement with the error estimates are presented.

1. INTRODUCTION

In this paper we consider the time-dependent Navier-Stokes equations with normal stress boundary conditions prescribed on parts of the boundary. These conditions are usually imposed to model outflow boundaries or free surfaces. For Newtonian flows, the boundary conditions in question take the form

$$\left[\mathsf{p}n^T - \frac{\nu}{2}n^T(\nabla\mathsf{u} + (\nabla\mathsf{u})^T)\right]|_\Gamma = b,$$

where u is the velocity vector field, p is the pressure, Γ is the boundary of the domain Ω, n is the unit outward normal, and b is the prescribed data.

There are numerous ways to discretize the time-dependent incompressible Navier-Stokes equations in time. Undoubtedly, the most popular one consists of using projection methods. Most of these techniques are based on the original ideas of Chorin [1] and Temam [20]. They are usually fractional step methods composed of two substeps such that either the Laplacian of the velocity or the pressure gradient is made explicit in one substep and (implicitly) corrected in the other substep. In both cases, one substep always consists of the projection of some vector field onto

Date: Draft version: March 4, 2003, submitted to *SIAM J.Numer. Anal.*

1991 *Mathematics Subject Classification.* 65N35, 65N22, 65F05, 35J05.

Key words and phrases. Navier-Stokes equations, Incompressibility, Pressure-correction methods, Open boundary conditions, Finite elements, Spectral approximations.

[1]LIMSI (CNRS-UPR 3152), BP 133, 91403, Orsay, France (guermond@limsi.fr). The work of this author is supported by CNRS and Texas Institute for Computational and Applied Mathematics, Austin, Texas, under a TICAM Visiting Faculty Fellowship. Part of this work was completed while this author was visiting University of Alberta and Purdue University.

[2]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada (minev@ualberta.ca). The work of this author is supported by an NSERC research grant.

[3]Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA. (shen@math.purdue.edu). The work of this author is partially supported by NFS grant DMS-0074283.

a divergence-free space. Following the terminology introduced in [9], a scheme is classified as a pressure-correction (resp. velocity-correction) method if the pressure gradient (resp. Laplacian of the velocity) is treated explicitly in one substep and (implicitly) corrected in the other substep. In the present paper we restrict ourselves to pressure-correction methods. Each of the above two classes of methods has a standard form and a rotational form (see [8, 7]), and each of them can be implemented either in algebraic form (cf. [3, 13]) or in differential form. However, to the best of our knowledge, no rigorous error analysis of any of these schemes with open boundary conditions is available in the literature. Moreover, there is some confusion in the literature on the performance of these methods with this type of boundary conditions. The aim of this paper is to discuss some of these issues and to derive error estimates.

We show that the standard pressure-correction schemes, implemented either in algebraic form or in differential form (in fact, they can be shown to be equivalent), are not suitable for approximating the Navier-Stokes equations supplemented with open boundary conditions. However, we show that the rotational pressure-correction schemes yield reasonable error estimates. More precisely, assuming full regularity of the Stokes problem, the second-order rotational pressure-correction method yields $\mathcal{O}(\Delta t^{3/2})$ convergence rate for the velocity in the L^2-norm and $\mathcal{O}(\Delta t)$ convergence rate for both the velocity in the H^1-norm and the pressure in the L^2-norm. These estimates deteriorate if the Stokes problem does not possesses full regularity, as it is probably the case in three-dimensions.

The paper is organized as follows. In the next section, we introduce notations and some mathematical tools which will be needed in the sequel. In section 3, we study the standard pressure-correction schemes. After establishing an error estimate for a semi-discretized scheme, we prove error estimates for fully discretized versions of the algorithm implemented either in algebraic form or in differential form. In section 4, we present the rotational pressure-correction schemes. First, we study a singularly perturbed system of PDE's that shares the essential features of its discrete counterparts. Then, we prove the major result of this paper, *i.e.* Theorem 4.1. Several illustrative numerical experiments are reported in section 5.

2. Preliminaries

We shall consider the time-dependent Navier-Stokes equations on a finite time interval $[0, T]$ and in an open, connected, bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$, or 3) with a boundary Γ sufficiently smooth. We assume that the following non-trivial partition holds: $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\text{meas}(\Gamma_1) \neq \emptyset$, $\text{meas}(\Gamma_2) \neq \emptyset$.

2.1. **Notations.** We denote by $H^m(\Omega)$ and $\|\cdot\|_m$ ($m = 0, \pm 1, \cdots$) the standard Sobolev spaces and norms. In particular, the norm and inner product of $L^2(\Omega) = H^0(\Omega)$ are denoted by $\|\cdot\|_0$ and $(\cdot, \cdot)$ respectively. We shall also make use of fractional Sobolev spaces $H^s(\Omega)$ which are defined by interpolation. To account for homogeneous Dirichlet boundary conditions on Γ_1, we define

$$X = \{v \in H^1(\Omega) : v|_{\Gamma_1} = 0\}. \tag{2.1}$$

Owing to the Poincaré inequality, $\|\nabla v\|_0$ is a norm equivalent to $\|v\|_1$ for all $v \in X$. Henceforth, we redefine the norm $\|\cdot\|_1$ in X such that $\|v\|_1 := \|\nabla v\|_0$.

We introduce two spaces of incompressible vector fields

$$H = \{v \in L^2(\Omega)^d;\ \nabla\cdot v = 0;\ v\cdot n|_{\Gamma_1} = 0\}, \tag{2.2}$$

$$V = \{v \in H^1(\Omega)^d;\ \nabla\cdot v = 0;\ v|_{\Gamma_1} = 0\}, \tag{2.3}$$

and we define P_H to be the L^2-orthogonal projection onto H, *i.e.*

$$(u - P_H u, v) = 0,\ \forall u \in L^2(\Omega)^d,\ \forall v \in H. \tag{2.4}$$

We also denote

$$N = \{q \in H^1(\Omega); q|_{\Gamma_2} = 0\}. \tag{2.5}$$

The following well-known lemma plays a key role in the analysis of projection methods.

Lemma 2.1. *The following orthogonal decomposition of $L^2(\Omega)^d$ holds:*

$$L^2(\Omega)^d = H \oplus \nabla N. \tag{2.6}$$

Since the nonlinear term in the Navier-Stokes equations has a marginal influence on the splitting error, we shall hereafter consider only the time-dependent Stokes equations written in terms of velocity, u, and pressure p:

$$\begin{cases} \partial_t \mathsf{u} + A\mathsf{u} + \nabla \mathsf{p} = f \quad \text{in } \Omega\times[0,T], \\ \nabla\cdot \mathsf{u} = 0 \quad \text{in } \Omega\times[0,T], \\ \mathsf{u}|_{\Gamma_1} = 0, \quad \text{and} \quad (\mathsf{p}n^T - n^T D\mathsf{u})|_{\Gamma_2} = 0 \quad \text{in [0,T]}, \\ \mathsf{u}|_{t=0} = u_0 \quad \text{in } \Omega, \end{cases} \tag{2.7}$$

Henceforth, the operators A and D may assume one of the two following forms:

$$Av = -\nu\nabla\cdot Dv, \tag{2.8}$$

$$Dv = \begin{cases} \nabla v & \text{case 1}, \\ \frac{1}{2}(\nabla v + (\nabla v)^T) & \text{case 2}. \end{cases} \tag{2.9}$$

We recall that the symmetric positive definite bilinear form

$$a(u,v) = \langle Au, v\rangle_{X',X} = (Du, Dv) \tag{2.10}$$

induces a norm on X that is equivalent to the H^1-norm. We denote by α the coercivity constant of a:

$$a(v,v) \geq \alpha\|\nabla v\|_0^2, \quad \forall v \in X. \tag{2.11}$$

To simplify our presentation, we assume that the unique solution (u,p) to the above system is as smooth as needed.

To perform the temporal discretization of the problem, we define $\Delta t > 0$ to be a time step and we set $t^k = k\Delta t$ for $0 \leq k \leq K = [T/\Delta t]$. Let $\phi^0, \phi^1, \dots \phi^K$ be a sequence of functions in some Hilbert space E. We denote by $\phi_{\Delta t}$ this sequence, and we use the following discrete norms:

$$\|\phi_{\Delta t}\|_{\ell^2(E)} := \left(\Delta t \sum_{k=0}^{K} \|\phi^k\|_E^2\right)^{1/2}, \qquad \|\phi_{\Delta t}\|_{\ell^\infty(E)} := \max_{0\leq k\leq K}\left(\|\phi^k\|_E\right). \tag{2.12}$$

We denote by c a generic constant that is independent of small parameters like ϵ, Δt, and h but possibly depends on the data and the solution. We shall use the expression $A \lesssim B$ to say that there exists a generic constant c such that $A \leq cB$.

Let μ be a positive real number. We shall repeatedly make use of the following interpolation result whose proof is given in Appendix A.

Lemma 2.2. *For all $0 \le s \le 1$, there exists an operator $\mathcal{I}_{\mu,s} : H^s(\Omega) \longrightarrow H^1_0(\Omega)$ such that for all r in $H^s(\Omega)$ we have*

$$\|r - \mathcal{I}_{\mu,s} r\|_0 \lesssim \mu^{\frac{s}{2}} \|r\|_{H^s(\Omega)}, \tag{2.13}$$

$$\|\mathcal{I}_{\mu,s} r\|_1 \lesssim \mu^{-1+\frac{s}{2}} \|r\|_{H^s(\Omega)}. \tag{2.14}$$

2.1.1. *The inverse of the Stokes operator and its regularity index.* In this section we recall properties of the inverse of the Stokes operator. Let X' be the dual space of X. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X' and X. The inverse of the Stokes operator, which we shall denote by $S : X' \longrightarrow X$, is defined as follows. For all v in X', $S(v) \in X$ is the solution to the following dual problem

$$\begin{cases} a(w, S(v)) - (r, \nabla \cdot w) = \langle v, w \rangle, & \forall w \in X, \\ (q, \nabla \cdot S(v)) = 0, & \forall q \in L^2(\Omega). \end{cases} \tag{2.15}$$

Obviously, we have

$$\forall v \in X', \qquad \|S(v)\|_1 + \|r\|_0 \le c \|v\|_{X'}. \tag{2.16}$$

It is well-known that when Dirichlet boundary conditions on the velocity are enforced on the entire boundary and Ω is smooth or convex, we have $\|r\|_1 \lesssim \|v\|_0$ (see, for instance, [21]). In the present case, where boundary conditions are mixed, it is a nontrivial task to determine the regularity of r. It is generally expected that the H^1-regularity does not hold in the three-dimensional case. However, it is possible that regularity in some fractional Sobolev space holds. To account for this, we make the following definition:

Definition 2.1 (regularity index of the Stokes operator)**.** *The regularity index of the Stokes operator is the largest number, s, such that for all $v \in L^2(\Omega)^d$, the solution $r \in L^2(\Omega)$ to the dual Stokes Problem (2.15) satisfies*

$$\|r\|_{H^s(\Omega)} \lesssim \|v\|_0.$$

We observe from (2.16) that $s \ge 0$, and it is clear that $s \le 1$. Hence, the case $s = 0$ is referred to as no regularity while the case $s = 1$ is referred to as full regularity. We refer to [12] for techniques to evaluate this index in two dimensions.

The operator S has interesting properties as listed below.

Lemma 2.3. *For all v in X, all $0 < \gamma < 1$, and all $0 < \mu < 1$ we have*

$$a(v, S(v)) \ge (1-\gamma)\|v\|_0^2 - c(\gamma) \left(\mu^{2\alpha_1} \|\nabla \cdot v\|_0^2 + \mu^{-2\alpha_2} \|v - P_H v\|_0^2 \right),$$

with $\alpha_1 = \frac{s}{2}$ and $\alpha_2 = 1 - \frac{s}{2}$ and s being the regularity index of the Stokes operator. In particular,

$$\forall v \in V, \qquad (\nabla S(v), \nabla v) = \|v\|_0^2.$$

Proof. Owing to the definition of $S(v)$ and owing to the fact $\mathcal{I}_{\varepsilon,s}r$ is zero on Γ_2, we have

$$\begin{aligned} a(v,S(v)) &= \|v\|_0^2 + (r,\nabla\cdot v) \\ &= \|v\|_0^2 + (r-\mathcal{I}_{\mu,s}r,\nabla\cdot v) + (\nabla\mathcal{I}_{\mu,s}r, v) \\ &= \|v\|_0^2 + (r-\mathcal{I}_{\mu,s}r,\nabla\cdot v) + (\nabla\mathcal{I}_{\mu,s}r, v-P_H v) \\ &\geq \|v\|_0^2 - \left(\mu^{\alpha_1}\|\nabla\cdot v\|_0 + \mu^{-\alpha_2}\|v-P_Hv\|_0\right)\|r\|_{H^s(\Omega)}. \end{aligned}$$

Then using the fact that s is the regularity index of the Stokes operator, and the definition 2.1 we derive the desired bound. □

Lemma 2.4. *The bilinear form*

$$X\times X' \ni (v,w) \longmapsto \langle S(v),w\rangle := a(S(v),S(w)) \in \mathbb{R}$$

induces a semi-norm on X' that we denote by $|\cdot|_\star$, and

$$\forall v\in X',\qquad |v|_\star = a(S(v),S(v))^{1/2} \lesssim \|v\|_{X'}.$$

Proof. It is clear that the bilinear form is symmetric, $\langle S(v),w\rangle = a(S(v),S(w)) = \langle S(w),v\rangle$, and positive, $\langle S(v),v\rangle = a(S(v),S(v))$; hence, $\langle S(v),w\rangle$ induces a semi-norm on X'. Furthermore,

$$|v|_\star^2 = \langle S(v),v\rangle = a(S(v),S(v)) \lesssim \|v\|_{X'}^2.$$

The proof is complete. □

3. Standard pressure-correction methods

In this section we consider the standard form of the pressure-correction scheme.

3.1. **Semi-discretized case.** For purely Dirichlet boundary conditions, the second-order pressure-correction scheme is known to be one-order more accurate than the original projection scheme of Chorin–Temam (cf. [23, 2, 19, 6]). Using the second-order backward difference formula (BDF2) to discretize the time derivative, the second-order pressure-correction scheme takes the following form:
Set $u^0 = u_0$, $p^0 = \mathsf{p}|_{t=0}$ which can be computed from the data, and compute $(\tilde u^1, u^1, p^1)$ by using the scheme below with BDF2 replaced by the backward Euler formula. Then, for $k\geq 1$, compute $(\tilde u^{k+1}, u^{k+1}, p^{k+1})$ such that

$$\begin{cases} \dfrac{3\tilde u^{k+1} - 4u^k + u^{k-1}}{2\Delta t} + A\tilde u^{k+1} + \nabla p^k = f(t^{k+1}), \\ \tilde u^{k+1}|_{\Gamma_1} = 0, \quad \text{and} \quad (p^k n^T - n^T D\tilde u^{k+1})|_{\Gamma_2} = 0. \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dfrac{3u^{k+1} - 3\tilde u^{k+1}}{2\Delta t} + \nabla(p^{k+1}-p^k) = 0, \\ \nabla\cdot u^{k+1} = 0, \\ u^{k+1}\cdot n|_{\Gamma_1} = 0, \quad \text{and} \quad (p^{k+1}-p^k)|_{\Gamma_2} = 0. \end{cases} \tag{3.2}$$

The first substep accounts for viscous effects, whereas the second one accounts for incompressibility. The second substep is usually referred to as the projection step, for it is a realization of the identity $u^{k+1} = P_H\tilde u^{k+1}$. We emphasize that it is essential, for stability considerations, that $(p^{k+1}-p^k)|_{\Gamma_2} = 0$ is enforced. Otherwise, (3.2) can not be interpreted as a projection step. However, the boundary conditions in (3.2) lead to

$$\begin{aligned}\frac{\partial}{\partial n}p^{k+1}|_{\Gamma_1} = \frac{\partial}{\partial n}p^{k}|_{\Gamma_1} = \ldots = \frac{\partial}{\partial n}p^{1}|_{\Gamma_1},\\ p^{k+1}|_{\Gamma_2} = p^{k}|_{\Gamma_2} = \ldots = p^{1}|_{\Gamma_2},\end{aligned} \tag{3.3}$$

which are certainly inaccurate since they are almost never satisfied by the exact solution. In the purely Dirichlet case, *i.e.* $\Gamma_2 = \emptyset$, it is possible to deduce a reasonably good approximation result for the pressure in the L^2-norm. But when $\Gamma_2 \neq \emptyset$ the pressure approximation is severely degraded.

Not being aware of any published convergence result for the scheme (3.1)–(3.2), we shall prove the following result.

Theorem 3.1. *If* (u,p)*, the solution to (2.7), is smooth enough in space and time, the solution to (3.1)–(3.2) satisfies the following error estimates:*

$$\|\mathsf{p}_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} + \|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^\infty(H^1(\Omega)^d)} \lesssim \Delta t^{\frac{1}{2}},$$
$$\|\mathsf{u}_{\Delta t} - u_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} + \|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} \lesssim \Delta t^{\frac{s+1}{2}},$$

where s *is the regularity index of the Stokes operator.*

Remark 3.1. Note that the error on the pressure in the L^2-norm is $\mathcal{O}(\Delta t^{\frac{1}{2}})$, whereas it is $\mathcal{O}(\Delta t)$ when Dirichlet boundary conditions are enforce on the whole boundary.

Proof of Theorem 3.1. As it will become clear in the course of the proof, using BDF2 instead of the backward Euler formula does not improve the accuracy in the presence of open boundary conditions. So to simplify the presentation, we consider the backward Euler formula for the time derivative.

$$\begin{cases}\dfrac{\tilde{u}^{k+1} - u^k}{\Delta t} + A\tilde{u}^{k+1} + \nabla p^k = f(t^{k+1}),\\ \tilde{u}^{k+1}|_{\Gamma_1} = 0, \quad \text{and} \quad (p^k n^T - n^T D\tilde{u}^{k+1})|_{\Gamma_2} = 0.\end{cases} \tag{3.4}$$

and

$$\begin{cases}\dfrac{u^{k+1} - \tilde{u}^{k+1}}{\Delta t} + \nabla(p^{k+1} - p^k) = 0,\\ \nabla\cdot u^{k+1} = 0,\\ u^{k+1}\cdot n|_{\Gamma_1} = 0, \quad \text{and} \quad (p^{k+1} - p^k)|_{\Gamma_2} = 0.\end{cases} \tag{3.5}$$

Technically, the proof is very similar to those in Shen [19] and Guermond [4], hence we only show the steps where the consistency error is degraded.

Let us introduce the interpolation operator $\mathcal{I}_{\Delta t,1} : H^1(\Omega) \longmapsto H^1_0(\Omega)$ defined in Lemma 2.2. This operator is such that for all r in $H^1(\Omega)$

$$\|\mathcal{I}_{\Delta t,1}r - r\|_0 \lesssim \Delta t^{\frac{1}{2}}\|r\|_1, \tag{3.6}$$

$$\|\nabla\mathcal{I}_{\Delta t,1}r\|_0 \lesssim \Delta t^{-\frac{1}{2}}\|r\|_1. \tag{3.7}$$

Without introducing any essential extra error, we can take $p^0 = \mathcal{I}_{\Delta t,1}\mathsf{p}|_{t=0}$ which would imply $p^k|_{\Gamma_2} = 0$ for all k.

Now we introduce the following notations:

$$\begin{cases} e^k = \mathsf{u}(t^k) - u^k, & \tilde{e}^k = \mathsf{u}(t^k) - \tilde{u}^k, \\ \psi^k = \mathcal{I}_{\Delta t,1}\mathsf{p}(t^{k+1}) - p^k, & q^k = \mathcal{I}_{\Delta t,1}\mathsf{p}(t^k) - p^k. \end{cases}$$

The weak form of the error equation that corresponds to the viscous step (3.4) is given by

$$\begin{aligned} \frac{1}{\Delta t}(\tilde{e}^{k+1} - e^k, v) + a(\tilde{e}^{k+1}, v) - (\psi^k, \nabla\cdot v) = (R(t^{k+1}), v) \\ + (\mathsf{p}(t^{k+1}) - \mathcal{I}_{\Delta t,1}\mathsf{p}(t^{k+1}), \nabla\cdot v), \quad \forall v \in X, \end{aligned}$$

where $R(t^{k+1}) = \frac{1}{\Delta t}(u(t^{k+1}) - u(t^k)) - u_t(t^{k+1}) = \mathcal{O}(\Delta t)$. Taking $v = \tilde{e}^{k+1}$ in the above equation, and using (3.6), we can derive

$$2\Delta t(\mathsf{p}(t^{k+1}) - \mathcal{I}_{\Delta t,1}\mathsf{p}(t^{k+1}), \nabla\cdot\tilde{e}^{k+1}) \lesssim \Delta t^2 + \alpha\Delta t\|\tilde{e}^{k+1}\|_1^2,$$

$$\|\tilde{e}^{k+1}\|_0^2 + \|\tilde{e}^{k+1} - e^k\|_0^2 + \alpha\Delta t\|\tilde{e}^{k+1}\|_1^2 - 2\Delta t(\psi^k, \nabla\cdot\tilde{e}^{k+1}) \le \|e^k\|_0^2 + c\Delta t^2. \tag{3.8}$$

Note that the consistency error is degraded at this step, more precisely, a Δt factor is already missing in the above estimate.

The error equation corresponding to the projection step (3.5) can be written as

$$\begin{cases} \dfrac{1}{\Delta t}e^{k+1} + \nabla q^{k+1} = \dfrac{1}{\Delta t}\tilde{e}^{k+1} + \nabla\psi^{k+1}, \\ \nabla\cdot e^{k+1} = 0, \\ e^{k+1}\cdot n|_{\Gamma_1} = 0, \quad \text{and } q^{k+1}|_{\Gamma_2} = 0. \end{cases}$$

Taking the square of the first relation above and multiplying the result by Δt^2, we infer

$$\|e^{k+1}\|_0^2 + \Delta t^2\|\nabla q^{k+1}\|_0^2 = \|\tilde{e}^{k+1}\|_0^2 + \Delta t^2\|\nabla\psi^k\|_0^2 - 2\Delta t(\psi^{k+1}, \nabla\cdot\tilde{e}^{k+1}). \tag{3.9}$$

Note that integration by parts can be performed on both sides thanks to the fact that $q^{k+1}|_{\Gamma_2} = 0 = \psi^k|_{\Gamma_2}$. Now we have

$$\begin{aligned} \Delta t^2\|\nabla\psi^k\|_0^2 &= \Delta t^2\|\nabla q^k + \nabla(\mathcal{I}_{\Delta t}(p(t^{k+1}) - p(t^k)))\|_0^2, \\ &\le \Delta t^2(\|\nabla q^k\|_0^2 + c\Delta t^{1-\frac{1}{2}}\|\nabla q^k\|_0 + c'\Delta t^{2(1-\frac{1}{2})}) \\ &\le \Delta t^2(1+\Delta t)\|\nabla q^k\|_0^2 + c\Delta t^2, \end{aligned}$$

where the consistency error is also degraded by a factor of $\mathcal{O}(\Delta t)$. Combining this result and the previous one, we have

$$\begin{aligned} \|e^{k+1}\|_0^2 + \Delta t^2\|\nabla q^{k+1}\|_0^2 \le \|\tilde{e}^{k+1}\|_0^2 + \Delta t^2(1+\Delta t)\|\nabla q^k\|_0^2 \\ - 2\Delta t(\psi^{k+1}, \nabla\cdot\tilde{e}^{k+1}) + c\Delta t^2. \end{aligned} \tag{3.10}$$

Thus, we can obtain the first error estimate of the theorem by combining (3.8) and (3.10), using the Discrete Gronwall lemma, and repeating the whole argument for time increments. The second estimate can be derived by a duality argument similar to that used in the proof of Lemma 4.4. □

Remark 3.2. Note that the need to integrate by parts in the term $2\Delta t(\nabla\psi^{k+1}, \tilde{e}^{k+1})$ in (3.9) is critical, and it is made possible by enforcing the homogeneous Dirichlet boundary condition on the pressure at Γ_2 in the projection step (3.2).

3.2. **Fully discretized cases.** It is clear that the artificial Dirichlet boundary condition (3.3) is responsible for the poor convergence property of the algorithm (3.1)–(3.2). Some authors have claimed that this boundary condition can be avoided by considering full discretizations and using an algebraic argument, namely, inexact factorization of the corresponding linear system. It is believed by some that algebraic manipulations may overcome difficulties encountered in functional analysis. We show below that this argument is misleading.

3.2.1. *The algebraic viewpoint.* Let us introduce a discrete setting to approximate (2.7) in space. Let $X_h \subset X$ and $M_h \subset L^2_0(\Omega)$ be two finite-dimensional spaces with suitable interpolation properties and satisfying the inf-sup condition. Let $N_u = \dim(X_h)$, $N_p = \dim(M_h)$ and $\{v_i\}_{i=1,\dots,N_u}$, $\{q_k\}_{k=1,\dots,N_p}$ be basis functions for X_h and M_h respectively. We define the following matrices:

$$\mathcal{M} = \left[\int_\Omega v_i \cdot v_j\right], \quad \mathcal{K} = \left[\int_\Omega Dv_i : Dv_j\right], \quad \mathcal{D} = \left[-\int_\Omega q_k \nabla\cdot v_j\right]. \tag{3.11}$$

Denoting by U and P the coefficient vectors of $u_h \in X_h$ and $p_h \in M_h$ in the considered bases, we consider the following *coupled* BDF2 scheme in matrix form:

$$\begin{bmatrix} \frac{3}{2\Delta t}\mathcal{M} + \mathcal{K} & \mathcal{D}^T \\ \mathcal{D} & 0 \end{bmatrix} \begin{bmatrix} U^{k+1} \\ P^{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\Delta t}\mathcal{M}(4U^k - U^{k-1}) + F^{k+1} \\ 0 \end{bmatrix} \tag{3.12}$$

where we have set $F^{k+1} = \left[\int_\Omega v_i \cdot f(t^{k+1})\right]$. The main idea underlying the so-called inexact factorization method is to perform an incomplete block LU factorization for the matrix in the left-hand side of (3.12). One of the simplest incomplete factorization is

$$\begin{bmatrix} \frac{3}{2\Delta t}\mathcal{M} + \mathcal{K} & \mathcal{D}^T \\ \mathcal{D} & 0 \end{bmatrix} \approx \begin{bmatrix} \frac{3}{2\Delta t}\mathcal{M} + \mathcal{K} & 0 \\ \mathcal{D} & -\frac{2\Delta t}{3}\mathcal{D}\mathcal{M}^{-1}\mathcal{D}^T \end{bmatrix} \begin{bmatrix} \mathcal{I} & \frac{2\Delta t}{3}\mathcal{M}^{-1}\mathcal{D}^T \\ 0 & \mathcal{I} \end{bmatrix} \tag{3.13}$$

Then, one can solve (3.12) approximately as follows:

$$\begin{cases} \text{Step 1:} \quad (\dfrac{3}{2\Delta t}\mathcal{M} + \mathcal{K})\tilde{U}^{k+1} = \dfrac{1}{2\Delta t}\mathcal{M}(4U^k - U^{k-1}) + F^{k+1} \\ \text{Step 2:} \quad \mathcal{D}\mathcal{M}^{-1}\mathcal{D}^T\Phi^{k+1} = \dfrac{3}{2\Delta t}\mathcal{D}\tilde{U}^{k+1} \\ \text{Step 3:} \quad U^{k+1} = \tilde{U}^{k+1} - \dfrac{2\Delta t}{3}\mathcal{M}^{-1}D^T\Phi^{k+1} \\ \text{Step 4:} \quad P^{k+1} = \Phi^{k+1}. \end{cases} \tag{3.14}$$

Note that we can also write (3.12) in the pressure-correction form

$$\begin{bmatrix} \frac{3}{2\Delta t}\mathcal{M} + K & \mathcal{D}^T \\ \mathcal{D} & 0 \end{bmatrix} \begin{bmatrix} U^{k+1} \\ P^{k+1} - P^k \end{bmatrix} = \begin{bmatrix} \frac{1}{2\Delta t}\mathcal{M}(4U^k - U^{k-1}) + F^{k+1} - \mathcal{D}^T P^k \\ 0 \end{bmatrix},$$

and using again the incomplete block factorization (3.13), we obtain the pressure-correction form of the algorithm (3.14):

$$(3.15)\quad \begin{cases} \text{Step 1:} & (\frac{3}{2\Delta t}\mathcal{M}+\mathcal{K})\tilde{U}^{k+1} = \frac{1}{2\Delta t}\mathcal{M}(4U^k - U^{k-1}) + F^{k+1} - \mathcal{D}^T P^k \\ \text{Step 2:} & \mathcal{D}\mathcal{M}^{-1}\mathcal{D}^T\Phi^{k+1} = \frac{3}{2\Delta t}\mathcal{D}\tilde{U}^{k+1} \\ \text{Step 3:} & U^{k+1} = \tilde{U}^{k+1} - \frac{2\Delta t}{3}\mathcal{M}^{-1}D^T\Phi^{k+1} \\ \text{Step 4:} & P^{k+1} = \Phi^{k+1} + P^k. \end{cases}$$

These schemes and more elaborate ones have been introduced in [3, 13]. This idea is also the basis for very similar works presented in [14, 15, 10].

At this point, one may observe that the artificial Dirichlet condition in (3.3) is not directly enforced in (3.14) and (3.15), and one may be led to believe that this algebraic manipulation has solved an essential difficulty, and that this type of method should behave better than the algorithm based on (3.1)–(3.2). To show that this argument is misleading, let us come back to the functional analysis.

3.2.2. *Functional analysis viewpoint.* Let $A_h : X_h \longrightarrow X_h'$ be the operator such that $(A_h u_h, v_h) = a(u_h, v_h)$ for every couple (u_h, v_h) in $X_h \times X_h$. A_h is the discrete counterpart of A.

Let us introduce now the discrete divergence operator $B_h : X_h \longrightarrow M_h$ and its adjoint $B_h^{\mathrm{T}} : M_h \longrightarrow X_h'$ such that for every couple (v_h, q_h) in $X_h \times M_h$ we have $(B_h v_h, q_h) = -(\nabla\cdot v_h, q_h) = (v_h, B_h^{\mathrm{T}} q_h)$.

It is clear that the algorithm (3.15) is strictly equivalent to the following one (*i.e.* both schemes yield exactly the same solution):

$$(3.16)\quad \begin{cases} \text{Step 1:} & (\frac{3}{2\Delta t} + A_h)\tilde{u}_h^{k+1} = \frac{1}{2\Delta t}(4u_h^k - u_h^{k-1}) + f_h^{k+1} - B_h^{\mathrm{T}} p_h^k \\ \text{Step 2:} & B_h B_h^{\mathrm{T}}\phi_h^{k+1} = \frac{3}{2\Delta t}B_h\tilde{u}_h^{k+1} \\ \text{Step 3:} & u_h^{k+1} = \tilde{u}_h^{k+1} - \frac{2\Delta t}{3}B_h^{\mathrm{T}}\phi_h^{k+1} \\ \text{Step 4:} & p_h^{k+1} = \phi_h^{k+1} + p_h^k, \end{cases}$$

where f_h^{k+1} is the L^2-projection of $f(t^{k+1})$ onto X_h. It is also clear that the abstract error analysis developed in [4, 6] applies here. By carrying out an analysis similar to that in [4, 6], one realizes that the critical step, overlooked in the algebraic viewpoint, is the continuity of the operator B_h^T in step 2 of (3.16) (*i.e.* the projection step). To make this point clear while keeping the analysis simple, let us assume that $M_h \subset H^1(\Omega)$. This hypothesis can be appropriately weakened upon additional non-essential technical details. The continuity of B_h^{T} is clarified by the following lemma.

Lemma 3.1. *For all q_h in M_h we have*

$$\|B_h^{\mathrm{T}} q_h\|_0 \lesssim (1 + s(h))\|q_h\|_1$$

where $s(h)$ is the constant such that

$$s(h) = \max_{v_h\in X_h} \frac{\|v_h\|_{L^2(\Gamma_2)}}{\|v_h\|_0}.$$

Proof. From the definition of B_h we deduce

$$\begin{aligned}(B_h^T q_h, B_h^T q_h) &= -(\nabla\cdot B_h^T q_h, q_h)\\ &= (B_h^T q_h, \nabla q_h) - \int_{\Gamma_2} q_h\,(B_h^T q_h)\cdot n\\ &\le \|B_h^T q_h\|_0\,\|\nabla q_h\|_0 + \|q_h\|_{L^2(\Gamma_2)}\|B_h^T q_h\|_{L^2(\Gamma_2)}\\ &\le \|B_h^T q_h\|_0\,\|q_h\|_1 + s(h)\|q_h\|_{H^{1/2}(\Gamma)}\|B_h^T q_h\|_0\\ &\lesssim (1+s(h))\,\|q_h\|_1\,\|B_h^T q_h\|_0.\end{aligned}$$

The conclusion follows easily. □

Remark 3.3. Note that in general $s(h)\to+\infty$ when $h\to 0$. In particular for finite elements we have $s(h)\eqsim h^{-1/2}$.

Now, to state the final convergence result, let us make the following usual assumptions.

Hypothesis 3.1. *There exist two spaces $W\subset X$, $Z\subset L^2(\Omega)$ and two continuous functions $\epsilon_1(h)$, $\epsilon_2(h)$ vanishing at 0 such that for all $v\in W$ and $q\in Z$, the solution to the following Stokes problem*

$$\begin{cases} a(v_h,w_h)-(q_h,\nabla\cdot w_h) = a(v,aw_h)-(q,\nabla\cdot w_h), & \forall w_h\in X_h\\ (r_h,\nabla\cdot v_h) = (r_h,\nabla\cdot v), \quad \forall r_h\in M_h, \end{cases} \tag{3.17}$$

satisfies the following error estimates

$$\begin{aligned} \|v-v_h\|_0 &\le \epsilon_1(h)\|v\|_W,\\ \|v-v_h\|_1+\|q-q_h\|_0 &\le \epsilon_2(h)(\|v\|_W+\|q\|_Z). \end{aligned} \tag{3.18}$$

Henceforth we write $v_h=\Pi_{1,h}v$ and $q_h=\Pi_{2,h}q$

Hypothesis 3.2. *There exists a positive constant β independent of h, such that*

$$\inf_{q_h\in M_h}\sup_{v_h\in X_h}\frac{\int_\Omega q_h\nabla\cdot v_h}{\|q\|_0\|v_h\|_1}\ge\beta. \tag{3.19}$$

We assume that the scheme is initialized such that the following estimates holds.

Hypothesis 3.3. *We assume that $\tilde u_h^0$, $\tilde u_h^1$, and p_h^1 are computed such that the following estimates hold:*

$$\begin{cases} \|\Pi_{1,h}\mathsf{u}(\Delta t)-u_h^1\|_0 \lesssim \min(\epsilon_1(h),\Delta t\epsilon_2(h)),\\ \|\Pi_{1,h}\mathsf{u}(\Delta t)-u_h^1\|_1 \lesssim \Delta t^{1/2}\epsilon_2(h),\\ \|\Pi_{2,h}\mathsf{p}(\Delta t)-p_h^1\|_1 \lesssim \epsilon_2(h). \end{cases}$$

Then, using arguments similar to those in [5, 6], we can prove the following result:

Theorem 3.2. *Under the Hypotheses 3.1, 3.2, 3.3, and provided the solution to (2.7) is regular enough in time and space, the solution of (3.16), or equivalently of (3.15), satisfies the following error estimates*

$$\begin{aligned} \|\mathsf{u}_{\Delta t}-u_{h,\Delta t}\|_{\ell^2(L^2(\Omega)^d)}+\|\mathsf{u}_{\Delta t}-\tilde u_{h,\Delta t}\|_{\ell^2(L^2(\Omega)^d)} &\lesssim s(h)^2\Delta t^2+\epsilon_1(h)\\ \|\mathsf{u}_{\Delta t}-\tilde u_{h,\Delta t}\|_{\ell^\infty(H^1(\Omega)^d)}+\|\mathsf{p}_{\Delta t}-p_{h,\Delta t}\|_{\ell^\infty(L^2(\Omega))} &\lesssim s(h)\Delta t+\epsilon_2(h) \end{aligned} \tag{3.20}$$

Remark 3.4. Note that the error estimates are not uniform in space. For finite elements we have $s(h) \approx h^{-1/2}$; consequently, the error estimate on the velocity in the L^2-norm is $\mathcal{O}(\Delta t^2/h)$, while the error estimate is $\mathcal{O}(\Delta t/h^{1/2})$ for the velocity in H^1-norm and for the pressure in L^2-norm.

Let us finish this section by stating a convergence result for the fully discretized case where the Dirichlet condition in (3.3) is enforced essentially, i.e., we assume now that $M_h \subset N$, where N is defined in (2.5). Accordingly, we modify the interpolation Hypothesis 3.1 as follows.

Hypothesis 3.4. *The interpolation operators* $\Pi_{1,h} : Z \longmapsto X_h$, $\Pi_{2,h} : Z \longmapsto M_h$ *are such that for all* (v, q) *in* $W \times Z$

$$\|v - \Pi_{1,h}v\|_1 + \|q - \Pi_{2,h}q\|_0 \leq \epsilon_2'(h)(\|v\|_W + \|q\|_Z). \tag{3.21}$$

Remark 3.5. For finite elements we have $\epsilon_2'(h) \approx h^{1/2}$ (see Lemma 2.2).

In this case, the operator B_h^T is clearly continuous, but the interpolation properties are no longer optimal and we have the following result:

Theorem 3.3. *Under the Hypotheses 3.4, 3.2, 3.3, and provided the solution to (2.7) is regular enough in time and space, the solution of (3.16) in the case of* $M_h \subset N$, *satisfies the following error estimates*

$$\|\mathsf{u}_{\Delta t} - \tilde{u}_{h,\Delta t}\|_{\ell^\infty(H^1(\Omega)^d)} + \|\mathsf{p}_{\Delta t} - p_{h,\Delta t}\|_{\ell^\infty(L^2(\Omega))} \lesssim \Delta t^{1/d} + \epsilon_2'(h) \tag{3.22}$$

Proof. This is an easy extension of Theorem 3.1. □

Remark 3.6. Note that the error estimate (3.22) is now uniform with respect to time and space, but the convergence rate is too poor to be recommendable in practice. This estimate is confirmed by numerical results reported in § 5.1.

In conclusion, whether the boundary condition (3.3) is enforced weakly as done in the so-called inexact factorization approach (the user being possibly unaware of it), or essentially, there is a price to pay for this "variational crime." In the first case, the price is non-uniformity of the error estimates, and in the second case it is non-optimality of the interpolation properties. Restated in other words, the inexact factorization involves an unbounded operator but optimal pressure interpolation, and the differential formulation involves bounded operators but suboptimal pressure interpolation.

3.3. **Alternative boundary conditions.** We finish this section by recalling that to simulate outflow boundary conditions, an alternative set of conditions is $p|_{\Gamma_2} = 0$, $u \times n|_{\Gamma_2} = 0$. This set of conditions is not equivalent to the zero normal stress conditions studied above. Nevertheless, an interesting property of these boundary conditions is that they are compatible with the pressure-correction algorithm (3.1)-(3.2). *i.e.* they yield near optimal convergence rates. We refer to Guermond-Quartapelle [5] for other technical details on this matter.

4. Rotational pressure-correction methods

In this section, we show that the rotational pressure-correction scheme introduced in [22] improves, by a factor of $\Delta t^{1/2}$, the error estimates of the standard pressure-correction scheme. It is proved in [9, 7] that, when Dirichlet boundary conditions are enforced on the entire boundary, the same improvement holds. The main result is stated in Theorem 4.1.

4.1. **Rotational form.** When applied to problems with open boundary conditions on Γ_2, the rotational pressure-correction scheme takes the following form:
Set $u^0 = u_0$, $p^0 = \mathsf{p}|_{t=0}$ which can be computed from the data, and compute $(\tilde{u}^1, u^1, p^1)$ by using the scheme shown below with BDF2 replaced by the backward Euler formula. Then, for $k \geq 1$, compute $(\tilde{u}^{k+1}, u^{k+1}, p^{k+1})$ such that

$$\begin{cases} \dfrac{3\tilde{u}^{k+1} - 4u^k + u^{k-1}}{2\Delta t} + A\tilde{u}^{k+1} + \nabla p^k = f(t^{k+1}), \\ \tilde{u}^{k+1}|_{\Gamma_1} = 0, \quad (p^k n^T - n^T D\tilde{u}^{k+1})|_{\Gamma_2} = 0, \end{cases} \tag{4.1}$$

$$\begin{cases} \dfrac{3u^{k+1} - 3\tilde{u}^{k+1}}{2\Delta t} + \nabla \phi^{k+1} = 0, \\ \nabla \cdot u^{k+1} = 0, \\ u^{k+1} \cdot n|_{\Gamma_1} = 0, \quad \phi^{k+1}|_{\Gamma_2} = 0. \end{cases} \tag{4.2}$$

$$\phi^{k+1} = p^{k+1} - p^k + \chi \nabla \cdot \tilde{u}^{k+1}, \tag{4.3}$$

where χ is a tunable positive coefficient.

Remark 4.1. As originally introduced in [22], the coefficient χ was taken to be equal to α, defined in (2.11), which is simply ν in the Newtonian case. The analysis performed in [9, 7] shows that this choice is sufficient to guarantee stability and convergence when Dirichlet boundary conditions are enforced. However, when natural boundary conditions are enforced on parts of the boundary, the analysis (see below) shows that χ should be chosen such that

$$0 < \chi < 2\alpha \inf_{v \in X} \frac{\|\nabla v\|^2}{\|\nabla \cdot v\|^2}. \tag{4.4}$$

Owing to the inequality $\|\nabla \cdot v\|^2 \leq d\|\nabla v\|^2$, where d is the space dimension, it is sufficient to choose

$$0 < \chi < \frac{2}{d}\alpha. \tag{4.5}$$

4.2. **A corresponding singularly perturbed system.** To better understand the behaviour of the scheme (4.1)-(4.2)-(4.3), we examine first a singularly perturbed system corresponding to the limiting case as $\Delta t \to 0$ (with $\varepsilon \sim \Delta t$). This system of PDEs is obtained by eliminating u^k from (4.1)-(4.2) and dropping some higher-order terms in ε:

$$\begin{cases} \partial_t u^\varepsilon + Au^\varepsilon + \nabla p^\varepsilon = f, \quad u^\varepsilon|_{\Gamma_1} = 0, \quad (p^\varepsilon n^T - n^T D u^\varepsilon)|_{\Gamma_2} = 0, \\ \nabla \cdot u^\varepsilon - \varepsilon \nabla^2 \phi^\varepsilon = 0, \quad \dfrac{\partial \phi^\varepsilon}{\partial n}|_{\Gamma_1} = 0, \quad \phi^\varepsilon|_{\Gamma_2} = 0 \\ \varepsilon \partial_t p^\varepsilon = \phi^\varepsilon - \chi \nabla \cdot u^\varepsilon, \end{cases} \tag{4.6}$$

with $u^\varepsilon|_{t=0} = u(0)$ and $p^\varepsilon(0) = p(0)$.

4.2.1. *An estimate on* $\nabla \cdot u^\varepsilon$. The following lemma is the key to obtain improved error estimates.

Lemma 4.1. *Provided* u *and* p *are smooth enough in time and space, we have*

$$\|\nabla \cdot u^\varepsilon\|_{L^\infty(L^2(\Omega)^d)} + \sqrt{\varepsilon}\|\nabla \phi^\varepsilon\|_{L^\infty(L^2(\Omega))} \lesssim \varepsilon^{\frac{5}{4}}.$$

Proof. We set $e = u^\varepsilon - \mathsf{u}$ and $q = p^\varepsilon - \mathsf{p}$. Subtracting (4.6) from (2.7), we find

$$e_t + Ae + \nabla q = 0; \quad e|_{\Gamma_1} = 0, \quad (qn^T - n^T De)|_{\Gamma_2} = 0, \tag{4.7}$$

$$\nabla\cdot e - \varepsilon\nabla^2\phi^\varepsilon = 0, \quad \frac{\partial\phi^\varepsilon}{\partial n}|_{\Gamma_1} = 0, \quad \phi^\varepsilon|_{\Gamma_2} = 0, \tag{4.8}$$

$$\varepsilon q_t = \phi^\varepsilon - \chi\nabla\cdot e - \varepsilon\mathsf{p}_t. \tag{4.9}$$

with $e(0) = 0$ and $q(0) = 0$.

Taking the inner product of the time derivative of (4.7) with e_t, we find

$$\frac{1}{2}\partial_t\|e_t\|_0^2 + \alpha\|\nabla e_t\|_0^2 - (q_t, \nabla\cdot e_t) \le 0. \tag{4.10}$$

The inner product of (4.9) with $\nabla\cdot e_t$ yields

$$(q_t, \nabla\cdot e_t) = \frac{1}{\varepsilon}(\phi^\varepsilon, \nabla\cdot e_t) - (\mathsf{p}_t, \nabla\cdot e_t) - \frac{\chi}{2\varepsilon}\partial_t\|\nabla\cdot e\|^2, \tag{4.11}$$

and the inner product of the time derivative of (4.8) with ϕ^ε yields

$$\frac{1}{\varepsilon}(\phi^\varepsilon, \nabla\cdot e_t) = -(\nabla\phi_t^\varepsilon, \nabla\phi^\varepsilon). \tag{4.12}$$

The above two relations lead to

$$(q_t, \nabla\cdot e_t) = -\frac{1}{2}\partial_t\|\nabla\phi^\varepsilon\|_0^2 - (p_t, \nabla\cdot e_t) - \frac{\chi}{2\varepsilon}\partial_t\|\nabla\cdot e\|^2. \tag{4.13}$$

Substituting this expression into (4.10) we obtain

$$\frac{1}{2}\partial_t\|e_t\|_0^2 + \alpha\|\nabla e_t\|_0^2 + \frac{1}{2}\partial_t\|\nabla\phi^\varepsilon\|_0^2 + \frac{\chi}{2\varepsilon}\partial_t\|\nabla\cdot e\|_0^2 = -(\mathsf{p}_t, \nabla\cdot e_t). \tag{4.14}$$

At this point, one would like to replace $\nabla\cdot e_t$ by $\varepsilon\nabla^2\phi_t^\varepsilon$ in $(\mathsf{p}_t, \nabla\cdot e_t)$ and integrate by parts. The integration by parts is not possible since neither p_t nor $\partial_n\phi_t^\varepsilon$ is zero at the boundary Γ_2. To account for this fact, we introduce the interpolation operator $\mathcal{J}_\varepsilon : H^1(\Omega) \longmapsto H_0^1(\Omega) \subset N$ such that $\mathcal{J}_\varepsilon = \mathcal{I}_{\sqrt{\varepsilon},1}$ where $\mathcal{I}_{\mu,s}$ has been defined in Lemma 2.2. Recall that for all r in $H^1(\Omega)$ we have from Lemma 2.2 that

$$\|\mathcal{J}_\varepsilon r - r\|_0 \lesssim \varepsilon^{\frac{1}{4}}\|r\|_1, \qquad \|\nabla\mathcal{J}_\varepsilon r\|_0 \lesssim \varepsilon^{-\frac{1}{4}}\|r\|_1. \tag{4.15}$$

We rewrite (4.14) as

$$\frac{1}{2}\partial_t(\|e_t\|_0^2 + \|\nabla\phi^\varepsilon\|^2 + \frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2) + \alpha\|\nabla e_t\|_0^2 = -(\mathsf{p}_t - \mathcal{J}_\varepsilon\mathsf{p}_t, \nabla\cdot e_t) + \varepsilon(\nabla\mathcal{J}_\varepsilon\mathsf{p}_t, \nabla\phi_t^\varepsilon).$$

Note that we used the fact that $\mathcal{J}_\varepsilon\mathsf{p}_t$ is zero at Γ_2 to integrate by parts. This is the key argument in this proof. Now we integrate in time between 0 and t. Since

$e(0) = 0$ and $q(0) = 0$, we infer $e_t(0) = 0$. Note also that $\phi^\varepsilon(0) = 0$, we obtain

$$\begin{aligned}\frac{1}{2}(\|e_t\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2 + \frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2) + \alpha\int_0^t \|\nabla e_t\|_0^2 d\tau \\ = -(\mathsf{p}_t - \mathcal{J}_\varepsilon \mathsf{p}_t, \nabla\cdot e) + \int_0^t (\mathsf{p}_{\tau\tau} - \mathcal{J}_\varepsilon \mathsf{p}_{\tau\tau}, \nabla\cdot e) d\tau \\ + \varepsilon(\nabla\mathcal{J}_\varepsilon \mathsf{p}_t, \nabla\phi^\varepsilon) - \int_0^t \varepsilon(\nabla\mathcal{J}_\varepsilon \mathsf{p}_{\tau\tau}, \nabla\phi^\varepsilon) d\tau \\ \leq \frac{1}{4}(\frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2) + \int_0^t (\frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2) d\tau \\ + c\,\varepsilon\|\mathsf{p}_t - \mathcal{J}_\varepsilon \mathsf{p}_t\|^2_{L^\infty(0,t;L^2(\Omega))} + c'\,\varepsilon^2\|\mathcal{J}_\varepsilon \mathsf{p}_t\|^2_{L^\infty(0,t;H^1(\Omega))} \\ + c\,\varepsilon\|\mathsf{p}_{tt} - \mathcal{J}_\varepsilon \mathsf{p}_{tt}\|^2_{L^2(0,t;L^2(\Omega))} + c'\,\varepsilon^2\|\mathcal{J}_\varepsilon \mathsf{p}_{tt}\|^2_{L^2(0,t;H^1(\Omega))}\end{aligned}$$

Using the estimates (4.15), we infer

$$\frac{1}{4}(\|e_t\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2 + \frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2) + \alpha\int_0^t \|\nabla e_t\|_0^2 d\tau \leq \int_0^t (\frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2) d\tau + c\,\varepsilon^{\frac{3}{2}}.$$

An application of the Gronwall lemma leads to

$$\|e_t\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2 + \frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2 + \int_0^t \|\nabla e_\tau\|_0^2 d\tau \lesssim \varepsilon^{\frac{3}{2}}. \tag{4.16}$$

The proof is complete. □

4.2.2. *L^2-estimate on the velocity.*

Lemma 4.2. *Provided* u *and* p *are smooth enough in time and space, we have*

$$\|\mathsf{u} - u^\varepsilon\|_{L^2(L^2(\Omega)^d)} \lesssim \varepsilon^{\frac{5+s}{4}}, \tag{4.17}$$

where s is the regularity index of the Stokes operator.

Proof. We multiply (4.7) by $S(e)$. Owing to Lemma 2.4 we infer

$$\frac{1}{2}\partial_t|e|_\star^2 + a(e, S(e)) = 0.$$

Using Lemma 2.3 with $\mu = \sqrt{\varepsilon}$, we obtain

$$\frac{1}{2}\partial_t|e|_\star^2 + \frac{1}{2}\|e\|_0^2 \lesssim \varepsilon^{\alpha_1}\|\nabla\cdot e\|_0^2 + \varepsilon^{-\alpha_2}\|e - P_H e\|_0^2.$$

From the definition of ϕ^ε, it is clear that $\varepsilon\nabla\phi^\varepsilon = e - P_H e$, we then derive from the estimates in Lemma 4.1 that

$$\begin{aligned}\frac{1}{2}\partial_t|e|_\star^2 + \frac{1}{2}\|e\|_0^2 &\lesssim \varepsilon^{\alpha_1}\|\nabla\cdot e\|_0^2 + \varepsilon^{1-\alpha_2}\varepsilon\|\nabla\phi^\varepsilon\|_0^2 \\ &\lesssim \varepsilon^{\frac{5}{2}}(\varepsilon^{\alpha_1} + \varepsilon^{1-\alpha_2}).\end{aligned}$$

Since $\alpha_1 = 1 - \alpha_2$, we find

$$\frac{1}{2}\partial_t|e|_\star^2 + \frac{1}{2}\|e\|_0^2 \lesssim \varepsilon^{\frac{5}{2}+\alpha_1} = \varepsilon^{\frac{5+s}{2}}.$$

One completes the proof by integrating in time. □

4.3. **Error estimates for the time discrete case.** The main result in this paper is the following:

Theorem 4.1. *Let $0 < \chi < \frac{2\alpha}{d}$. Assuming that the solution to (2.7) is smooth enough in time and space, the solution $(u^k, \tilde{u}^k, p^k)$ to (4.1)-(4.2)-(4.3) satisfies the estimates*

$$\|\mathsf{u}_{\Delta t} - u_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} + \|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} \lesssim \Delta t^{\frac{5+s}{4}},$$
$$\|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)} + \|\mathsf{p}_{\Delta t} - p_{\Delta t}\|_{\ell^2(L^2(\Omega))} \lesssim \Delta t^{\frac{3+s}{4}},$$

where s is the regularity index of the Stokes operator.

Remark 4.2. With full Stokes regularity, *i.e.* $s = 1$, the L^2-norm of the error on the velocity is $\mathcal{O}(\Delta t^{\frac{3}{2}})$, and the H^1-norm of the error on the velocity and the L^2-norm of the error on the pressure are $\mathcal{O}(\Delta t)$. In view of Lemma 4.1 and of the first estimate in Lemma 4.3, we believe that the H^1 estimates can be improved up to $\mathcal{O}(\Delta t^{\frac{5}{4}})$ by a sophisticated argument using weighted seminorms in time as in [16, 18]. However, the details of this proof are beyond the scope of this paper. Numerical results reported in Section 5 seem to confirm this conjecture, at least in 2D.

The proof of Theorem 4.1 is carried out in a way similar to that of Theorem 4.1 in [7], but since there are several important differences in the proofs of the underlying lemmas we give all the details. In particular the error analysis reveals why a homogeneous Dirichlet boundary condition must be enforced on ϕ^{k+1} on Γ_2; it explains also the origin of the factor χ in (4.3).

Let us first introduce some notations. For any sequence $\varphi^0, \varphi^1, \ldots$, we set

$$\delta_t\varphi^k = \varphi^k - \varphi^{k-1}, \quad \delta_{tt}\varphi^k = \delta_t(\delta_t\varphi^k), \quad \delta_{ttt}\varphi^k = \delta_t(\delta_{tt}\varphi^k),$$

and

$$\begin{cases} e^k = \mathsf{u}(t^k) - u^k, & \tilde{e}^k = \mathsf{u}(t^k) - \tilde{u}^k, \\ \psi^k = \mathsf{p}(t^{k+1}) - p^k, & q^k = \mathsf{p}(t^k) - p^k. \end{cases} \tag{4.18}$$

It is straightforward to show that $(\tilde{u}^1, u^1, p^1)$ obtained by using the scheme (4.1)–(4.2)–(4.3), with BDF2 replaced by backward Euler, satisfies the following estimates:

$$\begin{aligned} \|e^1\|_0 + \|\tilde{e}^1\|_0 + \Delta t^{\frac{1}{2}}(\|\nabla e^1\|_0 + \|\nabla \tilde{e}^1\|_0) &\lesssim \Delta t^2, \\ \|q^1\|_0 &\lesssim \Delta t. \end{aligned} \tag{4.19}$$

Note that for any bilinear form $(\cdot,\cdot)$ and any sequences $a^0, a^1, \ldots$, and $b^0, b^1, \ldots$, we have

$$\delta_t(a^{k+1}, b^{k+1}) = (\delta_t a^{k+1}, b^{k+1}) + (a^k, \delta_t b^{k+1}). \tag{4.20}$$

The error estimates of Theorem 4.1 are proved through a succession of lemmas. The following result is the discrete counterpart of Lemma 4.1.

Lemma 4.3. *Under the hypotheses of theorem 4.1, we have*

$$\begin{aligned} \|\nabla\cdot \tilde{u}_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} + \sqrt{\Delta t}\|\nabla \phi_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} &\lesssim \Delta t^{\frac{5}{4}}, \\ \|\delta_t \tilde{e}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)} &\lesssim \Delta t^{\frac{7}{4}}, \\ \|\delta_t \tilde{e}_{\Delta t} - \delta_t e_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} &\lesssim \Delta t^{\frac{9}{4}}. \end{aligned}$$

Proof. If we define

$$R^k = \partial_t \mathsf{u}(t^k) - \frac{3\mathsf{u}(t^k) - 4\mathsf{u}(t^{k-1}) + \mathsf{u}(t^{k-2})}{2\Delta t}, \tag{4.21}$$

then, for $k \geq 2$, the equations that control the time increments of the errors are given by

$$\begin{cases} \dfrac{3\delta_t\tilde{e}^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}}{2\Delta t} + A\delta_t\tilde{e}^{k+1} + \nabla\delta_t\psi^k = \delta_t R^{k+1}, \\ \delta_t\tilde{e}^{k+1}|_{\Gamma_1} = 0, \quad (\delta_t\psi^k n^T - n^T D\delta_t\tilde{e}^{k+1})|_{\Gamma_2} = 0, \end{cases} \tag{4.22}$$

and

$$\begin{cases} \dfrac{3}{2\Delta t}\delta_t e^{k+1} - \nabla\phi^{k+1} = \dfrac{3}{2\Delta t}\delta_t\tilde{e}^{k+1} - \nabla\phi^k, \\ \nabla\cdot\delta_t e^{k+1} = 0, \\ \delta_t e^{k+1}\cdot n|_{\Gamma_1} = 0, \quad \phi^{k+1}|_{\Gamma_2} = \phi^k|_{\Gamma_2} = 0. \end{cases} \tag{4.23}$$

We take the inner product of (4.22) with $4\Delta t\,\delta_t\tilde{e}^{k+1}$ to get

$$\begin{aligned} 2(\delta_t\tilde{e}^{k+1}, 3\delta_t\tilde{e}^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}) + 4\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 \\ - 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t\psi^k) = 4\Delta t(\delta_t\tilde{e}^{k+1}, \delta_t R^{k+1}) \\ \leq \gamma\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 + c\Delta t^7, \end{aligned} \tag{4.24}$$

where γ will be chosen later, and we have used the coercivity of the bilinear form a together with the fact that $\|\delta_t R^{k+1}\|_0 \lesssim \Delta t^3$. Note also that we have used the inequality $2ab \leq \gamma a^2 + b^2/\gamma$ which holds for all $\gamma > 0$. We shall repeatedly use this standard trick hereafter without mentioning it any more.

Let us denote $I = 2(\delta_t\tilde{e}^{k+1}, 3\delta_t\tilde{e}^{k+1} - 4\delta_t e^k + \delta_t e^{k-1})$, then we have

$$\begin{aligned} I =& 6(\delta_t\tilde{e}^{k+1}, \delta_t\tilde{e}^{k+1} - \delta_t e^{k+1}) + 2(\delta_t\tilde{e}^{k+1} - \delta_t e^{k+1}, 3\delta_t e^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}) \\ &+ 2(\delta_t e^{k+1}, 3\delta_t e^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}). \end{aligned}$$

Let I_1, I_2, and I_3 be the three terms in the right-hand side. Using the following algebraic identities

$$2(a^{k+1}, a^{k+1} - a^k) = |a^{k+1}|^2 + |a^{k+1} - a^k|^2 - |a^k|^2, \tag{4.25}$$

$$\begin{aligned} 2(a^{k+1}, 3a^{k+1} - 4a^k + a^{k-1}) = & |a^{k+1}|^2 + |2a^{k+1} - a^k|^2 + |\delta_{tt}a_{k+1}|^2 \\ & -|a^k|^2 - |2a^k - a^{k-1}|^2, \end{aligned} \tag{4.26}$$

we derive

$$\begin{aligned} I_1 &= 3\|\delta_t\tilde{e}^{k+1}\|_0^2 + 3\|\delta_t e^{k+1} - \delta_t\tilde{e}^{k+1}\|_0^2 - 3\|\delta_t e^{k+1}\|_0^2, \\ I_3 &= \|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1} - \delta_t e^k\|_0^2 + \|\delta_{ttt}e^{k+1}\|_0^2 - \|\delta_t e^k\|_0^2 - \|2\delta_t e^k - \delta_t e^{k-1}\|_0^2. \end{aligned}$$

Owing to (4.23) and using the fact that $e^k \in H$, we derive the following equality:

$$\frac{3}{2\Delta t}I_2 = -2(\nabla\delta_t\phi^{k+1}, 3\delta_t e^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}) = 0.$$

Collecting all the above results, we obtain

$$
\begin{aligned}
3\|\delta_t\tilde{e}^{k+1}\|_0^2 - 3\|\delta_t e^{k+1}\|_0^2 + \|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1} - \delta_t e^k\|_0^2 \\
+ 3\|\delta_t e^{k+1} - \delta_t\tilde{e}^{k+1}\|_0^2 + \|\delta_{ttt} e^{k+1}\|_0^2 \\
+ (4-\gamma)\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 - 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t\psi^k) \\
\leq c\,\Delta t^7 + \|\delta_t e^k\|_0^2 + \|2\delta_t e^k - \delta_t e^{k-1}\|_0^2.
\end{aligned}
\tag{4.27}
$$

Taking the square of (4.23) and integrating over the domain we obtain

$$
\begin{aligned}
3\|\delta_t e^{k+1}\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^{k+1}\|_0^2 = 3\|\delta_t\tilde{e}^{k+1}\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^k\|_0^2 \\
+ 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \phi^k).
\end{aligned}
\tag{4.28}
$$

Note that integration by parts on $(\delta_t e^{k+1}, \nabla\phi^{k+1})$ and $(\delta_t\tilde{e}^{k+1}, \nabla\phi^k)$ is legitimate because both $\phi^{k+1}|_{\Gamma_2}$ and $\phi^k|_{\Gamma_2}$ are zero. Since $\phi^k = p^k - p^{k-1} - \chi\nabla\cdot\tilde{e}^k$, we can bound the inner product in the right-hand side of (4.28) as follows

$$
\begin{aligned}
4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \phi^k) &= 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, p^k - p^{k-1} - \chi\nabla\cdot\tilde{e}^k) \\
&= 2\chi\Delta t(-\|\nabla\cdot\tilde{e}^{k+1}\|_0^2 + \|\nabla\cdot\tilde{e}^k\|_0^2 + \|\nabla\cdot\delta_t\tilde{e}^{k+1}\|_0^2) \\
&\quad - 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t\psi^k) + 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t p(t^{k+1}))
\end{aligned}
\tag{4.29}
$$

To control the troublesome term $\Delta t\|\nabla\cdot\delta_t\tilde{e}^{k+1}\|_0^2$ we use

$$
\chi\|\nabla\cdot v\|_0^2 \leq 2\gamma'\alpha\|\nabla v\|_0^2, \quad \forall v \in X.
\tag{4.30}
$$

Due to the condition χ, (4.4), we know that the constant γ' is such that $0 < \gamma' < 1$. Summing (4.27), (4.28) and (4.29), and using the inequality (4.30), we finally obtain

$$
\begin{aligned}
&\|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1} - \delta_t e^k\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^{k+1}\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^{k+1}\|_0^2 \\
&\quad + (4 - 4\gamma' - \gamma)\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 + 3\|\delta_t(e^{k+1} - \tilde{e}^{k+1})\|_0^2 + \|\delta_{ttt}e^{k+1}\|_0^2 \\
&\leq \|\delta_t e^k\|_0^2 + \|2\delta_t e^k - \delta_t e^{k-1}\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^k\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^k\|_0^2 \\
&\quad + 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t p(t^{k+1})) + c\Delta t^7.
\end{aligned}
\tag{4.31}
$$

At this point, we are formally at the same stage as (4.14). To integrate by parts in time the term $(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t p(t^{k+1}))$ we use (4.20) as follows

$$
(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t p(t^{k+1})) = \delta_t(\nabla\cdot\tilde{e}^{k+1}, \delta_t p(t^{k+1})) - (\nabla\cdot\tilde{e}^k, \delta_{tt}p(t^{k+1})).
$$

Next, we use the interpolation operator defined in (4.15). Let us denote $\mathcal{R}^{k+1} = p(t^{k+1}) - \mathcal{J}_{\Delta t}(p(t^{k+1}))$ (where $\mathcal{J}_{\Delta t} = \mathcal{I}_{\sqrt{\Delta t},1}$). Then, we have

$$
\frac{1}{\Delta t}\|\delta_{tt}\mathcal{R}^{k+1}\|_0^2 + \|\nabla\delta_{tt}\mathcal{J}_{\Delta t}(p(t^{k+1}))\|_0^2 \lesssim \Delta t^{\frac{7}{2}}.
$$

Since $\mathcal{J}_{\Delta t}(p(t^{k+1}))$ is zero on Γ_2, we have

$$\begin{aligned}(\nabla\cdot\delta_t\tilde{e}^{k+1},\delta_t p(t^{k+1})) &= \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) + \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{J}_{\Delta t}(p(t^{k+1})))\\ &\quad - (\nabla\cdot\tilde{e}^{k},\delta_{tt}\mathcal{R}^{k+1}) - (\nabla\cdot\tilde{e}^{k},\delta_{tt}\mathcal{J}_{\Delta t}(p(t^{k+1})))\\ &= \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) + \frac{2\Delta t}{3}\delta_t(\nabla\phi^{k+1},\nabla\delta_t\mathcal{J}_{\Delta t}(p(t^{k+1})))\\ &\quad - (\nabla\cdot\tilde{e}^{k},\delta_{tt}\mathcal{R}^{k+1}) - \frac{2\Delta t}{3}(\nabla\phi^{k},\nabla\delta_{tt}\mathcal{J}_{\Delta t}(p(t^{k+1})))\\ &\le \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) + \frac{2\Delta t}{3}\delta_t(\nabla\phi^{k+1},\nabla\delta_t\mathcal{J}_{\Delta t}(p(t^{k+1})))\\ &\quad \frac{\chi\Delta t}{2}\|\nabla\cdot\tilde{e}^{k}\|_0^2 + \frac{\Delta t^2}{3}\|\nabla\phi^{k}\|_0^2 + c\Delta t^{\frac{7}{2}}.\end{aligned}$$

By inserting this bound into (4.31), we obtain

$$\begin{aligned}&\|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1} - \delta_t e^{k}\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^{k+1}\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^{k+1}\|_0^2\\ &\qquad + (4-4\gamma'-\gamma)\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 + 3\|\delta_t(e^{k+1}-\tilde{e}^{k+1})\|_0^2 + \|\delta_{ttt}e^{k+1}\|_0^2\\ &\quad\le \|\delta_t e^{k}\|_0^2 + \|2\delta_t e^{k} - \delta_t e^{k-1}\|_0^2\\ &\qquad + \frac{4}{3}\Delta t^2(1+\Delta t)\|\nabla\phi^{k}\|_0^2 + 2\chi\Delta t(1+\Delta t)\|\nabla\cdot\tilde{e}^{k}\|_0^2\\ &\qquad 4\Delta t\delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) + \frac{8\Delta t^2}{3}\delta_t(\nabla\phi^{k+1},\nabla\delta_t\mathcal{J}_{\Delta t}(p(t^{k+1})))\\ &\qquad + c\,\Delta t^{\frac{9}{2}}.\end{aligned}$$

Summing up the relation above for $l = 2, ..., k$ and taking into account (4.19), we obtain

$$\begin{aligned}&\|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1} - \delta_t e^{k}\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^{k+1}\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^{k+1}\|_0^2\\ &\qquad + (4-4\gamma'-\gamma)\alpha\Delta t\sum_{l=2}^{k}\|\nabla\delta_t\tilde{e}^{l+1}\|_0^2 + 3\sum_{l=2}^{k}\|\delta_t e^{l+1} - \delta_t\tilde{e}^{l+1}\|_0^2\\ &\quad\le c\left(\|\delta_t e^2\|_0^2 + \|2\delta_t e^2 - \delta_t e^1\|_0^2 + \Delta t^2\|\nabla\phi^2\|_0^2 + \Delta t\|\nabla\cdot\tilde{e}^2\|_0^2 + \Delta t^{\frac{7}{2}}\right)\\ &\qquad + \Delta t\sum_{l=2}^{k}\left(\frac{4}{3}\Delta t^2\|\nabla\phi^{l}\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^{l}\|_0^2\right)\\ &\qquad - 4\Delta t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) - \frac{8\Delta t^2}{3}(\nabla\phi^{k+1},\nabla\delta_t\mathcal{J}_{\Delta t}(p(t^{k+1})))\\ &\qquad + 4\Delta t(\nabla\cdot\tilde{e}^2,\delta_t\mathcal{R}^2) + \frac{8\Delta t^2}{3}(\nabla\phi^2,\nabla\delta_t\mathcal{J}_{\Delta t}(p(t^2)))\\ &\quad\le c\Delta t^{\frac{7}{2}} + \frac{2}{3}\Delta t^2\|\nabla\phi^{k+1}\|_0^2 + \chi\Delta t\|\nabla\cdot\tilde{e}^{k+1}\|_0^2\\ &\qquad + \Delta t\sum_{l=2}^{k}\left(\frac{4}{3}\Delta t^2\|\nabla\phi^{l}\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^{l}\|_0^2\right).\end{aligned}$$

Since $0 < \gamma' < 1$, we can choose γ such that $4 - 4\gamma' - \gamma \ge 0$. Then, an application of the discrete Gronwall lemma yields the desired result. □

Remark 4.3. Note that to balance the term $-(\nabla\cdot\delta_t\tilde{e}^{k+1},\psi^k)$ in (4.27) it is necessary to integrate by parts the term $(\delta_t\tilde{e}^{k+1},\nabla\phi^k)$ in (4.28). This is possible only because the Dirichlet boundary condition $\phi^k|_{\Gamma_2}=0$ is enforced. This fact is the main reason why we enforce an homogeneous Dirichlet boundary condition on ϕ^{k+1} in (4.2). This argument shows the importance of the error analysis (or stability analysis) performed in the proof of Lemma 4.3. The necessity of the Dirichlet boundary condition also becomes clear when one understands that (4.2) is a realization of $u^{k+1}=P_H\tilde{u}^{k+1}$, since the orthogonal complement of H is ∇N according to Lemma 2.1.

Remark 4.4. The introduction of the parameter χ together with the bound (4.4) is justified by step (4.30). Whether the bound (4.4) is sharp is not yet clear.

Lemma 4.4. *Under the hypotheses of Theorem 4.1, we have*

$$\|\mathsf{u}_{\Delta t}-\tilde{u}_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)}+\|\mathsf{u}_{\Delta t}-u_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)}\lesssim \Delta t^{\frac{5+s}{4}}.$$

Proof. By using the relation $e^l=\tilde{e}^l+\frac{2\Delta t}{3}\nabla\phi^l$, for all $l\geq 2$, one obtains

$$\begin{cases}\dfrac{3\tilde{e}^{k+1}-4\tilde{e}^k+\tilde{e}^{k-1}}{2\Delta t}+A\tilde{e}^{k+1}+\nabla\gamma^k=R^{k+1},\\ \tilde{e}^{k+1}|_{\Gamma_1}=0,\quad (\gamma^k n-n^TD\tilde{e}^{k+1})|_{\Gamma_2}=0,\end{cases}\tag{4.32}$$

where $\nabla\gamma^k$ stands for the collection of all the gradient terms.

As in the time continuous case, we make use of the inverse Stokes operator. By taking the inner product of (4.32) with $4\Delta t S(\tilde{e}^{k+1})$ and using the identity (4.26), we obtain

$$\begin{aligned}|\tilde{e}^{k+1}|_\star^2+|2\tilde{e}^{k+1}-\tilde{e}^k|_\star^2+|\delta_{tt}\tilde{e}^{k+1}|_\star^2+4\Delta t\,a(\tilde{e}^{k+1},S(\tilde{e}^{k+1}))\\ =4\Delta t\,(R^{k+1},S(\tilde{e}^{k+1}))+|\tilde{e}^k|_\star^2+|2\tilde{e}^k-\tilde{e}^{k-1}|_\star^2.\end{aligned}$$

Using Lemma 2.3 with $\mu=\sqrt{\Delta t}$ and Lemma 4.3, we infer

$$\begin{aligned}4a(\tilde{e}^{k+1},S(\tilde{e}^{k+1}))&\geq 2\|\tilde{e}^{k+1}\|_0^2-c(\Delta t^{\alpha_1}\,\|\nabla\cdot\tilde{e}^{k+1}\|_0^2+\Delta t^{-\alpha_2}\,\|\tilde{e}^{k+1}-e^{k+1}\|^2)\\ &\geq 2\|\tilde{e}^{k+1}\|_0^2-c(\Delta t^{\alpha_1}\,\|\nabla\cdot\tilde{e}^{k+1}\|_0^2+\Delta t^{1-\alpha_2}\,\Delta t\|\nabla\phi^{k+1}\|^2\\ &\geq 2\|\tilde{e}^{k+1}\|_0^2-c\Delta t^{\alpha_1+\frac{5}{2}}\\ &\geq 2\|\tilde{e}^{k+1}\|_0^2-c\Delta t^{\frac{5+s}{2}}\end{aligned}$$

We also have

$$4\Delta t(R^{k+1},S(\tilde{e}^{k+1}))\leq c\Delta t\|R^{k+1}\|_{-1}^2+\Delta t\|\tilde{e}^{k+1}\|_0^2\leq c'\Delta t^5+\Delta t\|\tilde{e}^{k+1}\|_0^2$$

Combining these two estimates, we obtain

$$|\tilde{e}^{k+1}|_\star^2+|2\tilde{e}^{k+1}-\tilde{e}^k|_\star^2+\Delta t\|\tilde{e}^{k+1}\|_0^2\leq|\tilde{e}^k|_\star^2+|2\tilde{e}^k-\tilde{e}^{k-1}|_\star^2+c\Delta t^{1+\frac{5+s}{2}}.$$

The desired result is now an easy consequence of the discrete Gronwall Lemma. The estimate on $\|\mathsf{u}_{\Delta t}-u_{\Delta t}\|_0$ is obtained by using the triangular inequality $\|\mathsf{u}_{\Delta t}-u_{\Delta t}\|_0\leq\|\mathsf{u}_{\Delta t}-\tilde{u}_{\Delta t}\|_0+\frac{2\Delta t}{3}\|\nabla\phi_{\Delta t}\|_0$ (derived from (4.2)) and Lemma 4.3 . ☐

The key for obtaining improved estimates on $\|\tilde{e}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)}$ and $\|q_{\Delta t}\|_{\ell^2(L^2(\Omega))}$ is to derive an improved estimate on $\frac{1}{2\Delta t}(3\delta_t\tilde{e}^{k+1}-4\delta_t\tilde{e}^k+\delta_t\tilde{e}^{k-1})$. To this end, for any sequence of functions $\phi^0,\phi^1,\ldots$, we define

$$D_t\phi^{k+1}:=\frac{1}{2}(3\phi^{k+1}-4\phi^k+\phi^{k-1}).$$

Lemma 4.5. *Under the hypotheses of theorem 4.1, we have*

$$\Delta t^{-1}\|(D_t\tilde{e})_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} \lesssim \Delta t^{\frac{3+s}{4}}.$$

Proof. We use the same argument as in the proof of the L^2-estimate but we use it on the time increment $\delta_t\tilde{e}^{k+1}$. For $k \geq 2$ we have

$$\frac{3\delta_t\tilde{e}^{k+1} - 4\delta_t\tilde{e}^k + \delta_t\tilde{e}^{k-1}}{2\Delta t} + A\delta_t\tilde{e}^{k+1} + \nabla\delta_t\gamma^{k+1} = \delta_t R^{k+1}.$$

By taking the inner product of the above relation with $4\Delta t S(\delta_t\tilde{e}^{k+1})$ and repeating the same arguments as in the previous lemma, we obtain

$$\begin{aligned} |\delta_t\tilde{e}^{k+1}|_\star^2 + &|2\delta_t\tilde{e}^{k+1} - \delta_t\tilde{e}^k|_\star^2 + |\delta_{ttt}\tilde{e}^{k+1}|_\star^2 + \Delta t\|\delta_t\tilde{e}^{k+1}\|_0^2 \\ &\leq c\Delta t\|\delta_t R^{k+1}\|_0^2 + c\Delta t(\Delta t^{\alpha_1}\|\nabla\cdot\delta_t\tilde{e}^{k+1}\|_0^2 + \Delta t^{-\alpha_2}\|\delta_t\tilde{e}^{k+1} - \delta_t e^{k+1}\|_0^2) \\ &\quad + |\delta_t\tilde{e}^k|_\star^2 + |2\delta_t\tilde{e}^k - \delta_t\tilde{e}^{k-1}|_\star^2. \end{aligned}$$

Applying the discrete Gronwall lemma, and using the initial estimates and Lemma 4.3, we obtain

$$\|\delta_t\tilde{e}_{\Delta t}\|_{l^2(L^2(\Omega)^d)}^2 \lesssim \Delta t^{\frac{7+s}{2}}.$$

We conclude by using the fact that $2D_t\tilde{e}^{k+1} = 3\delta_t\tilde{e}^{k+1} - \delta_t\tilde{e}^k$. □

We are now in position to prove the remaining claims in Theorem 4.1.

Lemma 4.6. *Under the hypotheses of Theorem 4.1, we have*

$$\|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)} + \|\mathsf{p}_{\Delta t} - p_{\Delta t}\|_{\ell^2(L^2(\Omega))} \lesssim \Delta t^{\frac{3+s}{4}}.$$

Proof. By adding the viscous step and the projection step, it is clear that we have

$$\begin{cases} A\tilde{e}^{k+1} + \nabla(q^{k+1} + \chi\nabla\cdot\tilde{e}^{k+1}) = h^{k+1}, \\ \nabla\cdot\tilde{e}^{k+1} = g^{k+1}, \quad \tilde{e}^{k+1}|_{\Gamma_1} = 0, \quad ((q^{k+1} + \chi\nabla\cdot\tilde{e}^{k+1})n - n^T\tilde{e}^{k+1})|_{\Gamma_2} = 0, \end{cases} \tag{4.33}$$

where

$$\begin{aligned} h^{k+1} &= R^{k+1} - \frac{D_t e^{k+1}}{\Delta t} \\ g^{k+1} &= -\frac{2\Delta t}{3}\nabla^2\phi^{k+1}. \end{aligned} \tag{4.34}$$

Thanks to Lemma 4.3, we have

$$\|g^{k+1}\|_0 = \|\nabla\cdot\tilde{e}^{k+1}\|_0 \lesssim \Delta t^{\frac{5}{4}}, \ \forall k. \tag{4.35}$$

Since $e^k = P_H\tilde{e}^k$, owing to Lemma 4.5, we infer

$$\Delta t^{-1}\|\delta_t\tilde{e}_{\Delta t}\|_{l^2(L^2(\Omega)^d)} \leq \Delta t^{-1}\|\delta_t e_{\Delta t}\|_{l^2(L^2(\Omega)^d)} \lesssim \Delta t^{\frac{3+s}{4}}.$$

Hence, we have

$$\begin{aligned} \|h_{\Delta t}\|_{\ell^2(H^{-1}(\Omega)^d)} &\lesssim \|R_{\Delta t}\|_{\ell^2(H^{-1}(\Omega)^d)} + \Delta t^{-1}\|D_t\tilde{e}_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)}, \\ &\lesssim \Delta t^{\frac{3+s}{4}}. \end{aligned} \tag{4.36}$$

Now, we apply the following standard stability result for non-homogeneous Stokes systems to (4.33) (*cf.* [21]),

$$\|\tilde{e}^{k+1}\|_1 + \|(q^{k+1} + \chi\nabla\cdot\tilde{e}^{k+1})\|_0 \lesssim \|h^{k+1}\|_{-1} + \|g^{k+1}\|_0. \tag{4.37}$$

Thanks to (4.35) and (4.36), we derive

$$\|\tilde{e}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)} + \|(q+\chi\nabla\cdot\tilde{e})\|_{\ell^2(L^2(\Omega))} \lesssim \Delta t^{\frac{3+s}{4}}.$$

Then, from

$$\|q^{k+1}\|_0 \le \|q^{k+1} + \chi\nabla\cdot\tilde{e}^{k+1}\|_0 + \chi\|\nabla\cdot\tilde{e}^{k+1}\|_0,$$

we derive

$$\|q_{\Delta t}\|_{l^2(L^2(\Omega))} \lesssim \Delta t^{\frac{3+s}{4}}.$$

□

Thus, all the results in Theorem 4.1 have been proved.

5. Numerical results and discussions

5.1. **Standard pressure-correction scheme.** First, we test the convergence properties of the standard form of the pressure-correction scheme.

Let us consider a square domain $\Omega = (0,1)^2$. We take the exact solution $(\mathsf{u}_1, \mathsf{u}_2, \mathsf{p})$ of the linearized Navier-Stokes equations to be

$$\mathsf{u}_1(x,y,t) = \sin x \sin(y+t),\ \mathsf{u}_2(x,y,t) = \cos x\cos(y+t),\ \mathsf{p}(x,y,t) = \cos x \sin(y+t).$$

Setting $\mathsf{u} = (\mathsf{u}_1, \mathsf{u}_2)^t$, the source term f is given by $f = \mathsf{u}_t - \nabla^2\mathsf{u} + \nabla\mathsf{p}$. We set $\Gamma_2 = \{(x,y) \in \Gamma, x = 1\}$ and $\Gamma_1 = \Gamma\backslash\Gamma_2$. This set of exact solutions satisfies the following open boundary conditions:

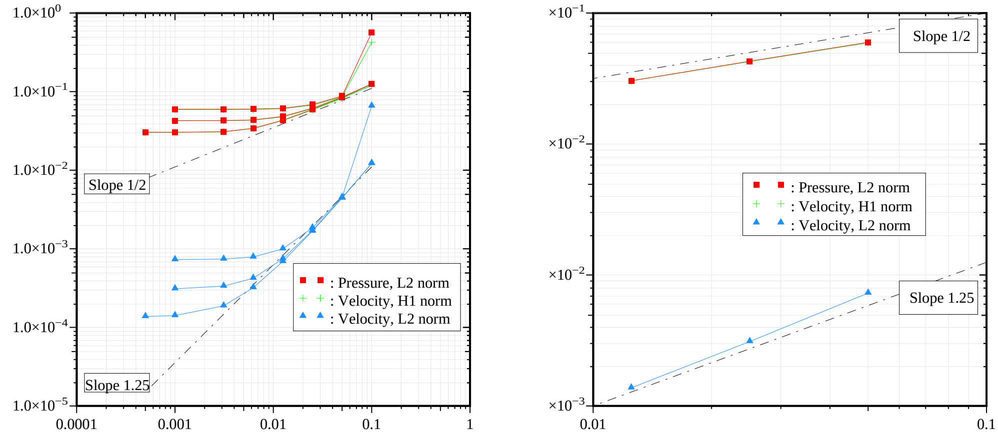
$$-\partial_x \mathsf{u}_2|_{\Gamma_2} = 0, \qquad \mathsf{p} - \partial_x\mathsf{u}_1|_{\Gamma_2} = 0.$$

To confirm the results in Theorem 3.3 we first carry out convergence tests in space and time using $\mathbb{P}_2/\mathbb{P}_1$ finite elements. We use the standard BDF2 pressure-correction scheme which enforces a homogeneous Dirichlet boundary condition on the pressure increment at the open boundary in the projection step. We make tests on three meshes with $h = 1/20$, $1/40$, $1/80$ and $5.10^{-4} \le \Delta t \le 10^{-1}$. The results are reported in Figure 1. On the left panel of Figure 1, we show the error on the velocity in the L^2-norm and the H^1-norm and the error on the pressure in the L^2-norm. The errors are computed at time $t = 1$ and are represented as functions of the time step. On the right panel of Figure 1, we show the errors as functions of h. The errors are computed at $t = 1$ using $\Delta t = 5.10^{-4}$.

Note that on the left panel, the errors at small Δt are dominated by the spacial discretization error, so the reference slope represents the asymptotic convergence rate as $h \to 0$.

One observes from Figure 1 that the error on the velocity in the L^2-norm is $\mathcal{O}(\Delta t + h)$ whilst the other errors are $\mathcal{O}(\Delta t^{1/2} + h^{1/2})$. The $\mathcal{O}(\Delta t^{1/2} + h^{1/2})$ global rate is in full agreement with Theorem 3.3.

We have also implemented the standard BDF2 pressure-correction scheme with a Legendre-Galerkin approximation [17]. The approximation spaces for the velocity and the pressure are $\mathbb{P}_N \times \mathbb{P}_N$ and $\mathbb{P}_{N-2}$ respectively, where $\mathbb{P}_N$ denotes the space of polynomials of degree less than or equal to N. The results are represented in Figure 2. We show the time discretization errors on the left panel of Figure 2 using $N = 40$. For the range of time steps explored, the spatial discretization error is negligible compared to the time discretization error. These tests clearly indicates that the L^2-error of the velocity (resp. the pressure) is of order Δt (resp. $\Delta t^{\frac{1}{2}}$) which are consistent with Theorem 3.3. In the right panel of Figure 2, we show the spatial error using $\Delta t = 10^{-4}$; in this case the time discretization error is negligible

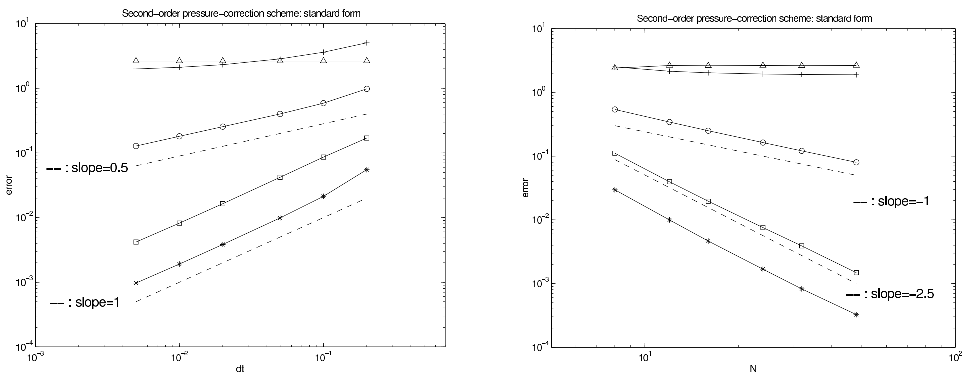


Errors vs. Δt for three meshes: $h = 1/20$, $1/40$, and $1/80$.

Errors vs. h with $\Delta t = 5.10^{-4}$.

FIGURE 1. Standard pressure-correction scheme: finite elements; errors at $t = 1$. Velocity: (▲) L^2-norm; (+) H^1-norm. Pressure: (■) L^2-norm. Note that the curves corresponding to the error on the velocity in H^1-norm and the pressure in L^2-norm almost coincide.

compared to the spatial discretization error. These tests indicate that the L^2-error on the pressure is of order $\mathcal{O}(N^{-1})$ and the L^2-error on the velocity is of order $\mathcal{O}(N^{-5/2})$. Note that the convergence rate w.r.t. N^{-1} in the spectral method is double that of the finite elements w.r.t. h. This is due to the fact that the grid spacing near the open boundary is N^{-2} (resp. h) in the spectral method (resp. finite elements).



Errors vs. Δt (with $N = 40$ fixed).

Errors vs. N (with $\Delta t = 10^{-4}$ fixed).

FIGURE 2. Standard pressure-correction scheme: spectral method. Error on velocity: (*) L^2-norm; (□) H^1-norm. Error on pressure: (○) L^2-norm; (+) H^1-norm; (△) L^∞-norm.

5.2. **Rotational pressure-correction scheme.** We now use the same example to test the time accuracy of the rotational pressure-correction scheme described in (4.1)–(4.2)-(4.3).

Let us first report the results with $\mathbb{P}_2/\mathbb{P}_1$ finite elements. We use $h = 1/80$ to guarantee that the error in space is significantly smaller than the splitting error. The results are reported in Figure 3. The convergence rate of the error on the velocity in the L^2-norm is close to $\mathcal{O}(\Delta t^{3/2})$ and that in the H^1-norm behaves like $\mathcal{O}(\Delta t^{5/4})$, which is higher than the $\mathcal{O}(\Delta t)$ rate predicted by Theorem 4.1 (see Remark 4.2 and Lemma 4.3). The convergence rate of the error on the pressure in the L^∞-norm is $\mathcal{O}(\Delta t)$ and that in the L^2-norm is between $\mathcal{O}(\Delta t)$ and $\mathcal{O}(\Delta t^{\frac{3}{2}})$. These rates are mostly consistent with the error estimates in Theorem 4.1. The accuracy saturation observed for small time steps comes from the spatial discretization error.

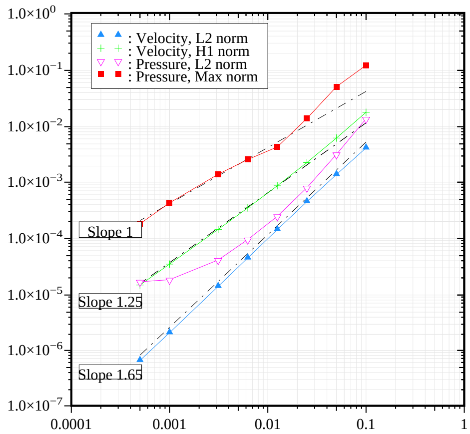


FIGURE 3. Rotational pressure-correction scheme: finite elements; errors at $t = 1$ vs. Δt (using $h = 1/80$). Velocity: ($\blacktriangle$) L^2-norm; ($+$) H^1-norm. Pressure: ($\triangledown$) L^2-norm; ($\blacksquare$) L^∞-norm.

The results using the Legendre-Galerkin method are reported in Figure 4. We note that the convergence rate for the error on the velocity in the L^2-norm is of order $O(\Delta t^{\frac{3}{2}})$ as predicted by Theorem 4.1. The convergence rates on all the other quantities are also close to $O(\Delta t^{\frac{3}{2}})$ which are higher than what Theorem 4.1 predicts (see Remark 4.2).

To complete this series of tests, we have performed convergence tests in 3D using $\mathbb{P}_2/\mathbb{P}_1$ finite elements. The boundary conditions and the source term in the Stokes equations are fixed so that the solution is given by

$$\begin{aligned}
\mathsf{u}_1(x,y,z,t) &= \sin x \sin(y+z+t),\\
\mathsf{u}_2(x,y,z,t) &= \cos x \cos(y+z+t),\\
\mathsf{u}_3(x,y,z,t) &= \cos(x)\sin(y+t),\\
\mathsf{p}(x,y,t) &= \cos x \sin(y+z+t).
\end{aligned}$$

Both the standard and the rotational form of the BDF2 pressure-correction scheme were tested. We show in Figure 5 the maximum in time of the L^2-norm of the errors on the velocity and the pressure for both schemes. On the left panel we

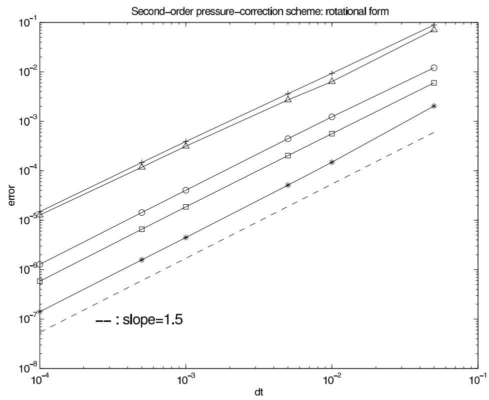
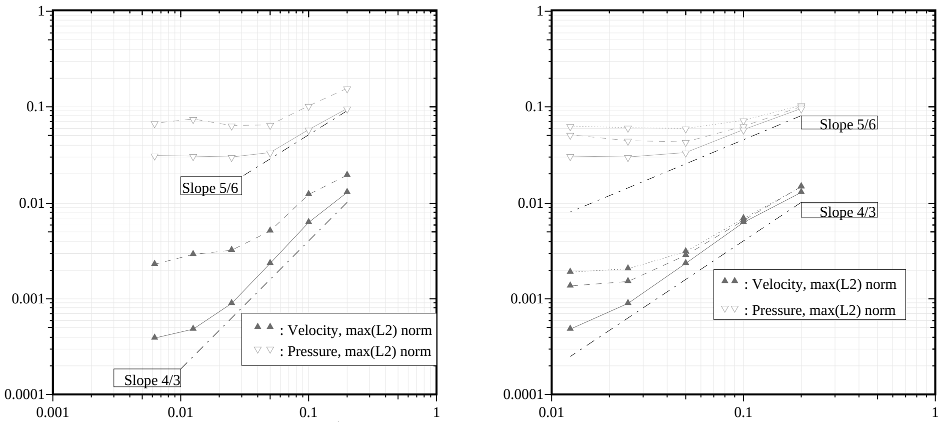


FIGURE 4. Rotational pressure-correction scheme: spectral method. error vs. dt with $N = 40$ fixed. Error on velocity: (*) L^2-norm; (□) H^1-norm. Error on pressure: (○) L^2-norm; (+) H^1-norm; (△) L^∞-norm.



Errors vs. Δt for $h = 1/40$. Standard form (dashes) vs. rotational form (solid line)

Errors vs. Δt for $h = 1/10$, $1/20$, $1/40$. Rotational form, $\chi = 2/3$.

FIGURE 5. Pressure-correction scheme with $\mathbb{P}_2/\mathbb{P}_1$ finite elements in 3D. Errors vs. Δt. Velocity: (▲) L^2-norm; Pressure: (▽) L^2-norm.

compare the standard and rotational forms of the scheme using $h = 1/40$. Unfortunately, using a higher uniform resolution in space was not possible due to the high cost of the computations. The grid with a stepsize $h = 1/40$ already contains close to 500000 nodes. On the right panel we show the errors for the rotational form of the scheme using three different meshes: $h = 1/10, 1/20, 1/40$. The convergence rates of the standard version of the scheme are clearly lower than those of the rotational form. The slopes for both, the velocity and the pressure errors obtained with the rotational form of the scheme are slightly lower than the best possible estimate following from the claim of Theorem 4.1. The rates $\mathcal{O}(\Delta t^{\frac{4}{3}})$ and $\mathcal{O}(\Delta t^{\frac{5}{6}})$ seem to correspond to a regularity index $s < 1$.

6. Concluding remarks

In this paper, we have analyzed pressure-correction schemes for approximating the incompressible Navier-Stokes equations with prescribed normal stress boundary conditions enforced on parts of the boundary. Our conclusions are twofold.

First, we have shown that the convergence rates of standard pressure-correction methods are too poor to be recommendable for approximating the Navier-Stokes equations in these circumstances. The main reason for the poor accuracy is that an *artificial* homogeneous Dirichlet boundary condition on the pressure has to be imposed to ensure stability. It is also shown that, contrary to what is often claimed in the literature, the inexact algebraic factorization techniques cannot shortcut the issue on the artificial pressure boundary condition (see Theorem 3.2).

Second, we have shown that the rotational pressure-correction method leads to reasonably good error estimates. More precisely, assuming full regularity of the Stokes problem, we have shown that the second-order rotational pressure-correction method yields $\mathcal{O}(\Delta t^{3/2})$ accuracy for the velocity in the L^2-norm and $\mathcal{O}(\Delta t)$ accuracy for the velocity in the H^1-norm and the pressure in the L^2-norm. To the best of our knowledge, the results presented in this paper are the first convergence estimates for a splitting method to the Navier-Stokes equations with open boundary conditions.

Finally, it is clear that even though the second-order rotational pressure-correction method yields the best error estimates to date, these are still suboptimal and more research is needed to find a splitting scheme with better properties.

Appendix A

Proof of Lemma 2.2. Given $\mu > 0$, we define the function $M_\mu : \Omega \longrightarrow \mathbb{R}$ by $M_\mu(x) = \min(1, \mathrm{dist}(x,\Gamma)/\mu)$. Let us denote by $\Delta\Omega_\mu$ the set of points in Ω whose distance to the boundary is less than μ. It is clear that if Ω is Lipschitz, then we have $\mathrm{meas}(\Delta\Omega_\mu) \lesssim \mu$. Furthermore, we have

$$\|M_\mu\|_{0,\infty} = 1, \qquad \text{and} \qquad \|\nabla M_\mu\|_{0,\infty} = \mu^{-1}.$$

Let u be a smooth function in Ω. We have

$$\begin{aligned} \|u - M_\mu u\|_0^2 &= \int_{\Delta\Omega_\mu} |u(x) - M_\mu(x)u(x)|^2 dx, \\ &\lesssim (1 + \|M_\mu\|_{0,\infty}^2) \int_{\Delta\Omega_\mu} |u|^2 dx. \end{aligned} \tag{A.1}$$

At this point we must find a bound on $\|u\|_{L^2(\Delta\Omega_\mu)}$. Let us first assume that $\Delta\Omega_\mu =]0,\mu[\times]0,1[^{d-1}$. Then, we have

$$u(x_1, x_2, \ldots, x_d) = u(0, x_2, \ldots, x_d) + \int_0^{x_1} \partial_{x_1} u(t, x_2, \ldots, x_d) dt.$$

That is,

$$\begin{aligned}\int_0^\mu u(x_1,x_2,\ldots,x_d)^2 dx_1 &\lesssim \mu u(0,x_2,\ldots,x_d)^2 + \int_0^\mu \big(\int_0^{x_1} \partial_{x_1} u(t,x_2,\ldots,x_d)dt\big)^2 dx_1,\\ &\lesssim \mu u(0,x_2,\ldots,x_d)^2 + \int_0^\mu \big(\int_0^{x_1} dt\big)\big(\int_0^{x_1}(\partial_{x_1}u)^2 dt\big)dx_1,\\ &\lesssim \mu u(0,x_2,\ldots,x_d)^2 + \mu^2\int_0^\mu (\partial_{x_1}u(x_1,x_2,\ldots,x_d))^2 dx_1.\end{aligned}$$

Now, by integrating with respect to the other variables, we obtain

$$\begin{aligned}\|u\|^2_{L^2(\Delta\Omega_\mu)} &\lesssim \mu\|u\|^2_{L^2(\Gamma)} + \mu^2\|\nabla u\|^2_{L^2(\Omega)},\\ &\lesssim \mu\|u\|^2_{H^1(\Omega)} + \mu^2\|\nabla u\|^2_{L^2(\Omega)},\\ &\lesssim \mu\|u\|^2_{H^1(\Omega)}.\end{aligned}$$

That is to say

$$\|u\|_{L^2(\Delta\Omega_\mu)} \lesssim \mu^{\frac12}\|u\|_{H^1(\Omega)}, \tag{A.2}$$

since we may assume $\mu \le 1$. The bound (A.2) is extended to arbitrary domains by means of the usual partition of unity and mapping technique, and it is extended to H^1 functions by a density argument.

Now, combining (A.1) and (A.2), we infer

$$\|u - M_\mu u\|_0 \lesssim \mu^{\frac12}\|u\|_1 \tag{A.3}$$

Let us now introduce $\rho_\mu \in \mathcal{C}^\infty(\mathbb{R}^d)$ a standard non negative mollifier. We extend u outside Ω by $\tilde u$ such that $\|\tilde u\|_1 \lesssim \|u\|_1$. Such an extension always exists provided Ω is smooth enough. Let us also extend M_μ by zero outside Ω and define the operator $\mathcal{I}_\mu u(x) := M_\mu(x)\,\rho_\mu * \tilde u(x)$. Then, by using the estimate (A.3) and standard results on mollifiers, it is clear that

$$\begin{aligned}\|u - \mathcal{I}_\mu u\|_0 &\lesssim \|u - M_\mu u\|_0 + \|M_\mu(\tilde u - \rho_\mu * \tilde u)\|_0,\\ &\lesssim \|u - M_\mu u\|_0 + \|\tilde u - \rho_\mu * \tilde u\|_0,\\ &\lesssim \mu^{\frac12}\|u\|_1 + \mu\|\tilde u\|_1,\\ &\lesssim \mu^{\frac12}\|u\|_1.\end{aligned}$$

It is also clear that $\|u - \mathcal{I}_\mu u\|_0 \lesssim \|u\|_0$. Hence, (2.13) follows from Lions–Petree's interpolation theorem [11].

Now we evaluate $\|\nabla\mathcal{I}_\mu u\|_0$ as follows.

$$\begin{aligned}\|\nabla\mathcal{I}_\mu u\|_0 &\lesssim \|M_\mu\nabla(\rho_\mu * \tilde u)\|_0 + \|\rho_\mu * \tilde u\nabla M_\mu\|_0,\\ &\lesssim \|\nabla(\rho_\mu * \tilde u)\|_0 + \|\rho_\mu * \tilde u\|_{L^2(\Delta\Omega_\mu)}\,\|\nabla M_\mu\|_{0,\infty},\\ &\lesssim \|\nabla(\rho_\mu * \tilde u)\|_0 + \mu^{-1}\|\rho_\mu * \tilde u\|_{L^2(\Delta\Omega_\mu)}\end{aligned}$$

Using the standard inequality $\|f * g\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}\,\|g\|_{L^2(\mathbb{R}^d)}$, it is clear that

$$\|(\nabla\rho_\mu) * \tilde u\|_0 \lesssim \mu^{-1}\|\tilde u\|_0, \qquad \text{and} \qquad \|\rho_\mu * \tilde u\|_{L^2(\Delta\Omega_\mu)} \lesssim \|\tilde u\|_0.$$

That is to say

$$\|\nabla\mathcal{I}_\mu u\|_0 \lesssim \mu^{-1}\|u\|_0. \tag{A.4}$$

On the other hand, owing to (A.2), we also have

$$\|\rho_\mu * \nabla\tilde{u}\|_0 \lesssim \|\tilde{u}\|_1, \qquad \text{and} \qquad \|\rho_\mu * \tilde{u}\|_{L^2(\Delta\Omega_\mu)} \lesssim \mu^{\frac{1}{2}}\|\rho_\mu * \tilde{u}\|_1 \lesssim \mu^{\frac{1}{2}}\|\tilde{u}\|_1.$$

That is to say

$$\|\nabla\mathcal{I}_\mu u\|_0 \lesssim \mu^{-\frac{1}{2}}\|u\|_1. \tag{A.5}$$

In conclusion, using Lions–Petree's interpolation theorem [11] together with estimates (A.4)–(A.5), we infer

$$\|\nabla\mathcal{I}_\mu u\|_0 \lesssim \mu^{-1+\frac{s}{2}}\|u\|_s,\ 0 \le s \le 1.$$

The estimate (2.14) is proved.

References

[1] A. J. Chorin. On the convergence of discrete approximations to the Navier-Stokes equations. *Math. Comp.*, 23:341–353, 1969.

[2] W. E and J.G. Liu. Projection method I: Convergence and numerical boundary layers. *SIAM J. Numer. Anal.*, 32:1017–1057, 1995.

[3] P. M. Gresho and S. T. Chan. On the theory of semi-implicit projection methods for viscous incompressible flow and its implementation via finite element method that also introduces a nearly consistent mass matrix. part I and part II. *Int. J. Numer. Methods Fluids*, 11:587–620, 621–659, 1990.

[4] J.-L. Guermond. Some practical implementations of projection methods for Navier-Stokes equations. *Modél. Math. Anal. Num.*, 30:637–667, 1996. Also in *C. R. Acad. Sci. Paris, Série I*, 319:887–892, 1994.

[5] J.-L. Guermond and L. Quartapelle. On the approximation of the unsteady Navier–Stokes equations by finite element projection methods. *Numer. Math.*, 80(5):207–238, 1998.

[6] Jean-Luc Guermond. Un résultat de convergence d'ordre deux en temps pour l'approximation des équations de Navier-Stokes par une technique de projection incrémentale. *M2AN Math. Model. Numer. Anal.*, 33(1):169–189, 1999. Also in *C. R. Acad. Sci. Paris, Série I*, 325:1329–1332, 1997.

[7] Jean-Luc Guermond and Jie Shen. On the error estimates for the rotational pressure-correction projection methods. *Submitted to Math. Comp.*

[8] Jean-Luc Guermond and Jie Shen. Velocity-correction projection methods for incompressible flows. *To appear in SIAM J. Numer. Anal.*

[9] J.L. Guermond and Jie Shen. Quelques résultats nouveaux sur les méthodes de projection. *C. R. Acad. Sci. Paris, Série I*, 333:1111–1116, 2001.

[10] Moon J. Lee, Byung Do Oh, and Young Bae Kim. Canonical fractional-step methods and consistent boundary conditions for the incompressible Navier–Stokes equations. *J. Comput. Phys.*, 168:73–100, 2001.

[11] J.-L. Lions and J. Peetre. Sur une classe d'espaces d'interpolation. *Inst. Hautes Études Sci. Publ. Math.*, (19):5–68, 1964.

[12] Matthias Orlt and Anna-Margarete Sändig. Regularity of viscous Navier-Stokes flows in non-smooth domains. In S. Nicaise M. Costabel, M. Dauge, editor, *Boundary value problems and integral equations in nonsmooth domains (Luminy, 1993)*, volume 167 of *Lecture Notes in Pure and Appl. Math.*, pages 185–201. Dekker, New York, 1995.

[13] J. Blair Perot. An analysis of the fractional step method. *J. Comput. Phys.*, 108(1):51–58, 1993.

[14] Alfio Quarteroni, Fausto Saleri, and Alessandro Veneziani. Analysis of the Yosida method for the incompressible Navier-Stokes equations. *J. Math. Pures Appl. (9)*, 78(5):473–503, 1999.

[15] Alfio Quarteroni, Fausto Saleri, and Alessandro Veneziani. Factorization methods for the numerical approximation of Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 188(1-3):505–526, 2000.

[16] Rolf Rannacher. On Chorin's projection method for the incompressible Navier-Stokes equations. In *The Navier-Stokes equations II—theory and numerical methods (Oberwolfach, 1991)*, volume 1530 of *Lecture Notes in Math.*, pages 167–183. Springer, Berlin, 1992.

[17] Jie Shen. Efficient spectral-Galerkin method. i. Direct solvers of second- and fourth-order equations using Legendre polynomials. *SIAM J. Sci. Comput.*, 15(6):1489–1505, 1994.

[18] Jie Shen. A new pseudo-compressibility method for the Navier-Stokes equations. *Appl. Numer. Math.*, 21:71–90, 1996.

[19] Jie Shen. On error estimates of projection methods for the Navier-Stokes equations: second-order schemes. *Math. Comp.*, 65(215):1039–1065, July 1996.

[20] R. Temam. Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires ii. *Arch. Rat. Mech. Anal.*, 33:377–385, 1969.

[21] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. North-Holland, Amsterdam, 1984.

[22] L. J. P. Timmermans, P. D. Minev, and F. N. Van De Vosse. An approximate projection scheme for incompressible flow using spectral elements. *Int. J. Numer. Methods Fluids*, 22:673–688, 1996.

[23] J. van Kan. A second-order accurate pressure-correction scheme for viscous incompressible flow. *SIAM J. Sci. Stat. Comput.*, 7(3):870–891, 1986.