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# GALERKIN FINITE ELEMENT APPROXIMATIONS OF STOCHASTIC ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We describe and analyze two numerical methods for a linear elliptic problem with stochastic coefficients and homogeneous Dirichlet boundary conditions. Here the aim of the computations is to approximate statistical moments of the solution, and in particular we illustrate on the case of the computation of the expected value of the solution. Since the approximation of the stochastic coefficients from the elliptic problem is in general not exact, we derive related a priori error estimates. The first method generates iid approximations of the solution by sampling the coefficients of the equation and using a standard Galerkin finite elements variational formulation. The Monte Carlo method then uses these approximations to compute corresponding sample averages. The second method is based on a finite dimensional approximation of the stochastic coefficients, turning the original stochastic problem into a deterministic parametric elliptic problem. A Galerkin finite element method, of either h or p version, then approximates the corresponding deterministic solution yielding approximations of the desired statistics. We include a comparison of the computational work required by each method to achieve a given accuracy. This comparison suggests intuitive conditions for an optimal selection of these methods.

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#### 1. INTRODUCTION

The use of computational models has increasingly affected the historical development of engineering products, based on first testing prototypes and then building a final version. Computational models aim to reduce the cost of the experimental tests as well as to discard faulty designs before they are built.

Due to the great development in computational resources and scientific computing techniques, more mathematical models can be solved efficiently. Ideally, this artillery could be used to solve many classical partial differential equations, the mathematical models we shall focus on here, to high accuracy. However, in many cases, the information available to solve a given problem is far from complete and is in general very limited. This is the case when solving a partial differential equation whose coefficients depend on material properties that are known to some accuracy. The same may occur with its boundary conditions, and even with the geometry of its domain, see for example the works [6, 7]. Naturally, since the current engineering trends are toward more reliance on computational predictions, the need for assessing the level of accuracy in the results grows accordingly. More than ever, the goal then becomes to represent and propagate the uncertainties from the available data to the desired result through our partial differential equation. By uncertainty we mean either intrinsic variability of physical quantities or simply lack of knowledge about some physical behavior, cf. [47]. If variability is interpreted as randomness then naturally we can apply probability theory. To be fruitful, probability theory requires considerable empirical information about the random quantities in question, generally in the form of probability distributions or their statistical moments. On the other hand, if the only available information comes in the form of some bounds for the uncertain variables, the description and analysis of uncertainty may be based on other methods, like e.g. convexity methods cf. [10, 20]. This approach is closely related to the so called "worst case scenario". Uncertainties may arise at different levels. They could appear in the mathematical model, e.g. if we are not sure about the linear behavior of some material, or in the variables that describe the model, e.g. if the linear coefficient that describes the material is not completely known -observe that this knowledge may depend on a representative scale of the model-.

Here we shall discuss the second alternative, and use a probabilistic description for the coefficient variability, leading us to the study of stochastic differential equations. Although the results presented on this paper can be generalized to linear elliptic stochastic partial differential equations we now focus our study on the standard model problem, a second order linear elliptic equation with homogeneous Dirichlet boundary conditions.

Let D be a convex bounded polygonal domain in  $\mathbb{R}^d$  and  $(\Omega, \mathcal{F}, P)$  be a complete probability space. Here  $\Omega$  is the set of outcomes,  $\mathcal{F} \subset 2^{\Omega}$  is the  $\sigma$ -algebra of events and  $P : \mathcal{F} \to [0, 1]$ is a probability measure. Consider the stochastic linear elliptic boundary value problem: find a stochastic function,  $u : \Omega \times \overline{D} \to \mathbb{R}$ , such that P-almost everywhere in  $\Omega$ , or in other words almost surely (a.s.), we have

(1.1) 
$$-\nabla \cdot (a(\omega, \cdot)\nabla u(\omega, \cdot)) = f(\omega, \cdot) \text{ on } D,$$
$$u(\omega, \cdot) = 0, \quad \text{on } \partial D.$$

Here  $a, f: \Omega \times D \to \mathbb{R}$  are stochastic functions. If we denote by B(D) the Borel  $\sigma$ -algebra generated by the open subsets of D, then a, f are assumed measurable with the  $\sigma$ -algebras ( $\mathcal{F} \otimes B(D)$ ) and  $B(\mathbb{R})$ . Here we work with the natural  $\sigma$ -algebra  $\mathcal{F} = \sigma(a, f)$ , which is the smallest one that makes both a and f measurable functions. On what follows we shall assume that a is bounded and uniformly coercive, i.e.

(1.2) 
$$\exists a_{\min}, a_{\max} \in (0, +\infty): P(\omega \in \Omega: a(\omega, x) \in [a_{\min}, a_{\max}], \forall x \in D) = 1$$

To ensure regularity of the solution u we assume also that a has a uniformly bounded and continuous first derivative, i.e. there exists a real deterministic constant C such that

(1.3) 
$$P(\omega \in \Omega: a(\omega, \cdot) \in C^{1}(\overline{D}) \text{ and } \max_{\overline{D}} |\nabla_{x}a(\omega, \cdot)| < C) = 1.$$

In addition, the right hand side in (1.1) satisfies

(1.4) 
$$\int_{\Omega} \int_{D} f^{2}(\omega, x) dx \, dP(\omega) < +\infty \text{ which implies } \int_{D} f^{2}(\omega, x) dx < +\infty \text{ a.s.}$$

Stochastic differential equations driven by the Wiener process have been widely applied in Mathematical Finance, see [33, 35, 43]. On the other hand, stochastic partial differential equations have been more popular to model physical phenomena, e.g. random vibrations, seismic activity, oil reservoir management, composite materials, etc, see [4, 19, 21, 25, 31, 32, 34, 48, 52] and the references therein. Solving a stochastic partial differential equation entails finding the joint probability distribution of the solution, which is a hard problem. In practice we shall usually be satisfied with much less, namely the computation of some moments, e.g. the expected value of the solution, or some probability related to a given event, e.g. the probability of some eventual failure, cf. [30, 39].

Whenever we approximate a given stochastic partial differential equation, a corresponding discretization error exists. The data uncertainties, the discretization error and the error from solving the discretized equations inaccurately add to the uncertainty in the result. Thus, it seems of little use to make one of them small when the others are relatively large, e.g. to solve the equations with high accuracy if the data is largely inaccurate. Moreover, quantifying the uncertainty in computer based simulations is also important for the eventual validation of the results. Without such quantification, validation becomes arbitrary, cf. [2, 42, 45, 47].

Depending on the structure of the noise that drives an elliptic partial stochastic differential equation, there are different numerical approximations. For example, when the size of the noise is relatively small, a Neumann expansion around the mean value of the equation's operator is a popular alternative. It requires only the solution of standard deterministic partial differential equations, the number of them being equal to the number of terms in the expansion. Equivalently, a Taylor expansion of the solution around its mean value with respect to the noise yields the same result. Similarly, the work [34] uses formal Taylor expansions up to second order of the solution but does not study their convergence properties. Recently, the work [5] proposed a perturbation method with successive approximations. It also proves that uniform coercivity of the diffusion is sufficient for the convergence of the perturbation method.

When only the load is stochastic, it is also possible to derive deterministic equations for the statistical moments of the solution. This case was analyzed in [3, 37] and more recently in the work [49], where a new method to solve these equations with optimal complexity is presented.

On the other hand, the work by Babuška et al. [16, 17] and by Ghanem and Spanos [25] address the general case where all the coefficients are stochastic. Both approaches transform the original stochastic problem into a deterministic one with higher dimensions, and they differ in the choice of the approximating functional spaces. The work [16] uses finite elements to approximate the noise dependence of the solution, while [23, 25] uses a formal expansion in terms of Hermite

polynomials. The approximation error from these numerical methods can be then bounded in terms of deterministic quantitites.

Monte Carlo methods are both general and simple to code and they are naturally suited for parallelization. They generate a set of independent identically distributed (iid) approximations of the solution by sampling the coefficients of the equation, using a spatial discretization of the partial differential equation, e.g. by a Galerkin Finite Elements formulation. Then, using these approximations we can compute corresponding sample averages of the desired statistics. Monte Carlo methods have a rate of convergence that may be considered slow, but its computational work grows only like a polynomial with respect to the number of random variables present in the problem. It is worth mentioning that in particular cases their convergence can be accelerated by variance reduction techniques [33]. The convergence rate of the Monte Carlo method is interpreted in probability sense and a practical estimate of its error needs an a posteriori estimate of the variance of the sampled random variable, which in turns requires an a priori bound on higher statistical moments, cf. the Berry Esseen Theorem in [18]. Besides this, if the probability density of a random variable is smooth, the convergence rate of the Monte Carlo method for the approximation of its expected value can be improved, cf. [41, 55]. Quasi Monte Carlo methods, see [14, 50, 51], offer a way to get a better convergence rate than the one of the Monte Carlo method, although this advantage seems to deteriorate in general when the number of the number of random variables present in the problem becomes large.

Another way to provide a notion of stochastic partial differential equations is based on the Wick product and the Wiener chaos expansion, see [31] and [56]. This approach yields solutions in Kondratiev spaces of stochastic distributions and are based on a different interpretation of (1.1); the solutions proposed in [31] and [56] are not the same as those arising from (2.3). The choice between (2.3) and [31] is a modeling decision, based on the physical situation under study. For example, with the Wick product we have  $E[a \diamond u] = E[a]E[u]$  regardless on the correlation between a and f, whereas this is in general not true with the usual product. A numerical approximation for Wick stochastic linear elliptic partial differential equations is studied in [54], yielding a priori convergence rates.

This work studies the case of stochastic linear elliptic partial differential equations with random diffusion and load coefficients stating and proving conditions for existence and uniqueness of solutions. For example, to obtain a meaningful numerical solution for (1.1) its diffusion coefficient should be uniformly coercive. This work compares a Monte Carlo Galerkin method with the Stochastic Galerkin Finite Element method introduced in [16] and introduces a related *p*-version, providing theoretical a priori convergence analysis in each case. A priori estimates are useful to characterize the convergence, and ultimately they provide information to compare the number of operations required by numerical methods. The conclusion for now is that if the noise is described by a small number of random parameters or if the accuracy requirement is sufficiently strict, then a Stochastic Galerkin method is to be preferred, otherwise a Monte Carlo Galerkin method seems still to be the best choice. It is worth mentioning that the development of numerical methods are expected to appear.

The rest of the paper is organized as follows.

Section 2 introduces the basic notation and proves the existence and uniqueness for solutions, as well as perturbation estimates in energy norm with respect to perturbations in the stochastic coefficients of the equation, i.e. modeling error estimates. The approximation of the coefficients in (1.1) is done by means of the Karhunen-Loève expansion, see [38]. We assume that a finite number of terms of such an expansion have been identified by a statistical procedure, see [40, 57, 58], and base the numerical approximations on such information. Once the coefficient approximation step

is done, the stochastic problem is transformed into a parametric elliptic problem and it is natural to apply finite element techniques to construct related numerical methods.

Section 3 defines some deterministic finite element subspaces and gives their approximation properties. Tensor product of deterministic finite element subspaces are used to construct the Stochastic Galerkin finite element approximation.

Sections 5 and 6 present two versions of Galerkin finite element approximations for the stochastic partial differential equation under study. These versions differ only in the choice of the approximation subspaces, namely in the direction of the parameter related with the noise representation. Section 5 uses polynomial approximation with fixed degree, while Section 6 uses polynomial approximation with increasingly higher degree, proving the standard exponential convergence with respect to degree p, cf. [27]. Beside this, precise implementation details are included in Section 7 to ensure minimal computational cost in both cases.

On the other hand, the Monte Carlo Galerkin finite element approximation is based on first sampling the coefficients in the stochastic partial differential equation and then approximating the corresponding realization of the solution. Section 4 studies convergence of the Monte Carlo Galerkin finite element method for the approximation of the expected value of the solution.

To characterize the convergence of the numerical approximations, a priori error estimates are developed for each of the methods described in this work. Using this information, Section 8 compares the asymptotical computational work required by each of the approximations, giving some intuition on their possible application.

### 2. Theoretical Aspects of the Continuous Problem

2.1. Notation and function spaces. Let  $m \in \mathbb{N}$ , D be an open, connected and convex subset of  $\mathbb{R}^m$  with polygonal boundary  $\partial D$ . Denote the volume of D by  $|D| \equiv \int_D 1 dx$ . For  $s \in \mathbb{N}$  and  $1 \leq p \leq +\infty$ , let  $W^{s,q}(D)$  be the Sobolev space of functions having generalized derivatives up to order s in the space  $L^q(D)$ . Using the standard multi-index notation,  $\alpha = (\alpha_1, \ldots, \alpha_d)$  is a d-tuple of non-negative integers and the length of  $\alpha$  is given by  $|\alpha| = \sum_{i=1}^d \alpha_i$ . The Sobolev norm of  $v \in W^{s,q}(D)$  will be denoted by

$$||v||_{W^{s,q}(D)} = \left\{ \sum_{|\alpha| \le s} \int_{D} |\partial^{\alpha} v|^{q} dx \right\}^{1/q}, \quad 1 \le q < +\infty,$$

and

$$\|v\|_{W^{s,\infty}(D)} = \max_{|\alpha| \le s} \left( \operatorname{ess } \sup_{D} |\partial^{\alpha} v| \right).$$

We shall write  $H^s(D) \equiv W^{s,2}(D)$  and omit the index 2 from the symbol of its norm, i.e.,  $\|\cdot\|_{H^s(D)} = \|\cdot\|_{W^{s,2}(D)}$ . As usual, the function space  $H_0^1(D)$  is the subspace of  $H^1(D)$  consisting of functions which vanish at the boundary of D in the sense of trace, equipped with the norm  $\|v\|_{H_0^1(D)} = \{\int_D |\nabla v|^2 dx\}^{1/2}$ . Whenever s = 0 we shall keep the notation with  $L^q(D)$  instead of  $W^{0,q}(D)$ . For the sake of generality, sometimes we shall let H be a Hilbert space with inner product  $(\cdot, \cdot)_H$ . In that case we shall also denote the dual space of H, H', that contains linear bounded functionals,  $\mathcal{L} : H \to \mathbb{R}$ , and is endowed with the operator norm  $\|\mathcal{L}\|_{H'} = \sup_{0 \neq v \in H} \frac{\mathcal{L}(v)}{\|v\|_H}$ . Since stochastic functions have intrinsically different structure with respect to  $\omega$  and with respect

Since stochastic functions have intrinsically different structure with respect to  $\omega$  and with respect to x, the analysis of numerical approximations requires tensor spaces. Let  $H_1, H_2$  be Hilbert spaces. The tensor space  $H_1 \otimes H_2$  is the completion of formal sums  $u(y, x) = \sum_{i=1,...,n} v_i(y)w_i(x)$ ,  $\{v_i\} \subset H_1, \{w_i\} \subset H_2$ , with respect to the inner product  $(u, \hat{u})_{H_1 \otimes H_2} = \sum_{i,j} (v_i, \hat{v}_j)_{H_1} (w_i, \hat{w}_j)_{H_2}$ . For example, let us consider two domains,  $y \in \Gamma, x \in D$  and the tensor space  $L^2(\Gamma) \otimes H^1(D)$ , with

tensor inner product

$$(u,\hat{u})_{L^{2}(\Gamma)\otimes H^{1}(D)} = \int_{\Gamma} \left( \int_{D} u(y,x)\hat{u}(y,x)dx \right) dy + \int_{\Gamma} \left( \int_{D} \nabla_{x} u(y,x) \cdot \nabla_{x} \hat{u}(y,x)dx \right) dy$$

Thus, if  $u \in L^2(\Gamma) \otimes H^k(D)$  then  $u(y, \cdot) \in H^k(D)$  a.e. on  $\Gamma$  and  $u(\cdot, x) \in L^2(\Gamma)$  a.e. on D. Moreover, we have the isomorphism  $L^2(\Gamma) \otimes H^k(D) \simeq L^2(\Gamma; H^k(D)) \simeq H^k(D; L^2(\Gamma))$  with the definitions

$$L^{2}(\Gamma; H^{k}(D)) = \left\{ v: \Gamma \times D \to \mathbb{R} \mid v \text{ is strongly measurable and } \int_{\Gamma} \|v(y, \cdot)\|_{H^{k}(D)}^{2} < +\infty \right\},$$

$$\begin{aligned} H^k(D;L^2(\Gamma)) &= \Big\{ v: \Gamma \times D \to \mathbb{R} \mid v \text{ is strongly measurable, } \forall |\alpha| \le k \exists \ \partial_\alpha v \in L^2(\Gamma) \otimes L^2(D) \text{ and} \\ &\int_{\Gamma} \int_{D} \partial_\alpha v(y,x) \varphi(y,x) dx dy = (-1)^{|\alpha|} \int_{\Gamma} \int_{D} v(y,x) \partial_\alpha \varphi(y,x) dx dy, \ \forall \varphi \in C_0^{\infty}(\Gamma \times D) \Big\}. \end{aligned}$$

Similar constructions can be done for tensor product of Banach spaces, although the norm for the tensor space used to obtain the completion of the formal sums has to be defined explicitly on each case. Here the Banach space  $C(\Gamma; H)$  comprises all continuous functions  $u: \Gamma \to H$  with the norm  $\|u\|_{C(\Gamma;H)} \equiv \sup_{y \in \Gamma} \|u(y)\|_{H}$ . Similar definitions apply to the spaces  $C^{k}(\Gamma; H), k = 1, ...,$ cf. [22] p. 285.

Let Y be an  $\mathbb{R}^{N}$ -valued random variable in  $(\Omega, \mathcal{F}, P)$ . If  $Y \in L^{1}_{P}(\Omega)$  we denote its expected value by

$$E[Y] = \int_{\Omega} Y(\omega) dP(\omega) = \int_{\mathbb{R}^N} y \ d\mu_Y(y).$$

where  $\mu_Y$  is the distribution measure for Y, defined for the Borel sets  $\tilde{b} \in B(\mathbb{R}^N)$ , by  $\mu_Y(\tilde{b}) \equiv P(Y^{-1}(\tilde{b}))$ . If  $\mu_Y$  is absolutely continuous with respect to the Lebesgue measure then there exists a density function  $\rho_Y : \mathbb{R} \to [0, +\infty)$ , such that

$$E[Y] = \int_{\mathbb{R}^N} y \ \rho_Y(y) dy.$$

Analogously, whenever  $Y_i \in L^2_P(\Omega)$  for  $i = 1, \ldots, d$  the covariance matrix of Y,  $Cov[Y] \in \mathbb{R}^{d \times d}$ , is defined by  $Cov[Y](i,j) = Cov(Y_i, Y_j) = E[(Y_i - E[Y_i])(Y_j - E[Y_j])]$ ,  $i, j = 1, \ldots, d$ . Besides this, whenever  $u(\omega, x)$  is a stochastic process the positive semi definite function  $Cov[u](x_1, x_2) =$  $Cov[u(x_1), u(x_2)] = Cov[u(x_2), u(x_1)]$  is the covariance function of the stochastic process u.

To introduce the notion of stochastic Sobolev spaces we first recall the definition of stochastic weak derivatives. Let  $v \in L^2_P(\Omega) \otimes L^2(D)$ , then the  $\alpha$  stochastic weak derivative of  $v, w = \partial_{\alpha} v \in L^2_P(\Omega) \otimes L^2(D)$ , satisfies

$$\int_D v(\omega, x) \partial^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_D w(\omega, x) \phi(x) dx, \quad \forall \phi \in C_0^\infty(D), \ a.s$$

We shall work with stochastic Sobolev spaces  $\widetilde{W}^{s,q}(D) = L_P^q(\Omega, W^{s,q}(D))$  containing stochastic functions,  $v : \Omega \times D \to \mathbb{R}$ , that are measurable with respect to the product  $\sigma$ -algebra  $\mathcal{F} \otimes B(D)$  and equipped with the averaged norms

$$\|v\|_{\widetilde{W}^{s,q}(D)} = E[\|v\|_{W^{s,q}(D)}^q]^{1/q} = E[\sum_{|\alpha| \le s} \int_D |\partial^{\alpha} v|^q dx]^{1/q}, \quad 1 \le q < +\infty$$

and

$$\|v\|_{\widetilde{W}^{s,\infty}(D)} = \max_{|\alpha| \le s} \left( \operatorname{ess } \sup_{\Omega \times D} |\partial^{\alpha} v| \right).$$

Observe that if  $v \in \widetilde{W}^{s,q}(D)$  then  $v(\omega, \cdot) \in W^{s,q}(D)$  a.s. and  $\partial^{\alpha} v(\cdot, x) \in L^q_P(\Omega)$  a.e. on D for  $|\alpha| \leq s$ . Whenever q = 2, the above space is a Hilbert space, i.e.  $\widetilde{W}^{s,2}(D) = \widetilde{H}^s(D) \simeq L^2_P(\Omega) \otimes H^s(D)$ .

2.2. Existence and uniqueness for the solution of a linear stochastic elliptic problem. In this section, we discuss some issues related to the formulation of the elliptic boundary value problem (1.1) and its solution. We also develop perturbation estimates corresponding to a change in the coefficients of (1.1). These perturbation estimates are both useful to analyze modeling errors and to develop discretization error estimates. Let us consider the tensor product Hilbert space  $H = \tilde{H}_0^1(D) \simeq L_P^2(\Omega; H_0^1(D))$  endowed with the inner product  $(v, u)_H \equiv E[\int_D \nabla v \cdot \nabla u dx]$ . Define the bilinear form,  $\mathcal{B} : H \times H \to \mathbb{R}$ , by  $\mathcal{B}(v, w) \equiv E[\int_D a \nabla v \cdot \nabla w dx], \quad \forall v, w \in H$ . The standard assumption (1.2) yields both the continuity and the coercivity of  $\mathcal{B}$ , i.e.

(2.1) 
$$|\mathcal{B}(v,w)| \leq a_{max} ||v||_{H} ||w||_{H}, \forall v, w \in H,$$

and

(2.2) 
$$a_{min} \|v\|_{H}^{2} \leq \mathcal{B}(v, v), \quad \forall v \in H.$$

A direct application of the Lax-Milgram Lemma, cf. [13], implies the existence and uniqueness for the solution to the variational formulation: find  $u \in H$  such that

(2.3) 
$$\mathcal{B}(u,v) = \mathcal{L}(v), \quad \forall v \in H.$$

Here  $\mathcal{L}(v) \equiv E[\int_D f v dx], \forall v \in H$  defines a bounded linear functional since the random field f satisfies (1.4). Since the domain D is convex and bounded and assumptions (1.2), (1.3) on the diffusion a hold, the theory of elliptic regularity, cf. [1, 22], implies that the solution of (1.1) satisfies  $u(\omega, \cdot) \in H^2(D) \cap H_0^1(D)$  a.s. Moreover, standard arguments from measure theory show that the solution to (2.3) also solves (1.1). The formulation (2.3) together with assumption (2.10) on finite dimensional noise give the basis for the Stochastic Galerkin Finite element (SGFEM) method introduced in Sections 5 and 6, while formulation (1.1) is the basis for the Monte Carlo Galerkin Finite element (MCGFEM) method, discussed in Section 4.

2.3. Continuity with respect to the coefficients a and f. Usually in practical problems the information about the stochastic processes a and f is only limited. For example, we may only have approximations for their expectations and covariance functions to use in the implementation of a numerical method for (1.1). Moreover, due to some efficiency considerations, sometimes it may be even useful to use an approximation of a and f. Therefore, an additional approximation error, of modeling type, appears in the computations together with the usual discretization error. For implementation details, see [57, 58, 24, 32, 8]. In the next proposition we consider a weak formulation with perturbed bilinear form and functional, and estimate the size of the corresponding perturbation in the solution with the energy norm.

**Proposition 2.1.** Let  $(H, (\cdot, \cdot)_H)$  be a Hilbert space. Consider two symmetric bilinear forms  $\mathcal{B}, \widehat{\mathcal{B}}$ :  $H \times H \to \mathbb{R}$  that are H-coercive and bounded, i.e. there exist real constants  $0 < a_{min} \leq a_{max}$  such that

(2.4) 
$$a_{min} \|v\|_{H}^{2} \leq \min\{\mathcal{B}(v,v), \widehat{\mathcal{B}}(v,v)\}, \quad \forall v \in H,$$

and

(2.5) 
$$\max\{|\mathcal{B}(v,w)|, |\mathcal{B}(v,w)|\} \le a_{max} \|v\|_{H} \|w\|_{H}, \quad \forall v, w \in H$$

We assume that the above bilinear forms are comparable, i.e. there exist a constant  $\gamma$  such that

(2.6) 
$$|(\mathcal{B} - \widehat{\mathcal{B}})(v, w)| \le \gamma ||v||_H ||w||_H, \quad \forall v, w \in H.$$

Consider two bounded linear functionals,  $\mathcal{L}, \hat{\mathcal{L}} \in H'$  and let  $u, \hat{u} \in H$  be the solutions of the problems

$$\begin{aligned} \mathcal{B}(u,v) &= \mathcal{L}(v), \quad \forall v \in H, \\ \hat{\mathcal{B}}(\hat{u},v) &= \hat{\mathcal{L}}(v), \quad \forall v \in H. \end{aligned}$$

Then there holds

(2.7) 
$$\|u - \hat{u}\|_{H} \leq \frac{1}{a_{min}} (\|\mathcal{L} - \hat{\mathcal{L}}\|_{H'} + \frac{\gamma}{a_{min}} \|\hat{\mathcal{L}}\|_{H'})$$

*Proof.* Since by Lax-Milgram's lemma u and  $\hat{u}$  are well defined we can consider their difference  $e \equiv u - \hat{u}$ . Then  $R(a, \hat{v}) = R(a, \hat{v}) - R(a, \hat{v})$ 

$$\begin{aligned} \mathcal{B}(e,e) &= \mathcal{B}(e,u) - \mathcal{B}(e,\hat{u}) \\ &= \mathcal{L}(e) - \mathcal{B}(e,\hat{u}) \\ &= \mathcal{L}(e) + (\hat{\mathcal{B}} - \mathcal{B})(e,\hat{u}) - \hat{\mathcal{B}}(e,\hat{u}) \\ &= (\mathcal{L} - \hat{\mathcal{L}})(e) + (\hat{\mathcal{B}} - \mathcal{B})(e,\hat{u}) \\ &\leq (\|\mathcal{L} - \hat{\mathcal{L}}\|_{H'} + \gamma \|\hat{u}\|_{H}) \|e\|_{H}. \end{aligned}$$

Since we have  $\|\hat{u}\|_{H} \leq \frac{1}{a_{min}} \|\hat{\mathcal{L}}\|_{H'}$  and  $\|e\|_{H}^{2} \leq \frac{1}{a_{min}} \mathcal{B}(e, e)$  (2.7) follows. **Remark 2.1.** The proof of the previous lemma gives

$$\|u - \hat{u}\|_{H}^{2} \leq \frac{1}{a_{min}} (\|\mathcal{L} - \hat{\mathcal{L}}\|_{H'} \|u - \hat{u}\|_{H} + |(\hat{\mathcal{B}} - \mathcal{B})(u - \hat{u}, \hat{u})|).$$

If in addition we know that there exist Banach spaces,  $V_1, V_2$ , and positive constants,  $C, \gamma'$  such that

$$\|\cdot\|_{V_1} \le C \|\cdot\|_H$$
 and

(2.8) 
$$|(\hat{\mathcal{B}} - \mathcal{B})(u, v)| \le \gamma' ||u||_{V_1} ||v||_{V_2}, \quad \forall u \in V_1, \ v \in V_2$$

then

(2.9) 
$$\|u - \hat{u}\|_{H} \leq \frac{1}{a_{min}} (\|\mathcal{L} - \hat{\mathcal{L}}\|_{H'} + C\gamma' \|u\|_{V_2}).$$

We shall see that there are cases where it is not possible to apply (2.7) but (2.9) is still valid. Next we consider a sequence of problems like (2.3) with approximate coefficients. A direct application of Proposition 2.1 yields the following convergence result.

**Corollary 2.1** (Convergence result). Consider the Hilbert space  $H = \widetilde{H}_0^1(D)$ , two convergent sequences of stochastic processes,  $\{a_n\}, \{f_n\}$ , satisfying

$$0 < a_{\min} \le a_n \le a_{\max} < \infty, \ (P \otimes dx) \ a.e. \ on \ D \times \Omega, \ \|a_n - a\|_{\widetilde{L}^{\infty}(D)} \to 0$$

and

$$\|f_n - f\|_{\widetilde{L}^2(D)} \to 0.$$

Then the stochastic processes u and  $u_n$ , defined by

$$E[\int_{D} a_{n} \nabla u_{n} \cdot \nabla v dx] = E[\int_{D} f_{n} v dx], \ \forall v \in H$$
$$E[\int_{D} a \nabla u \cdot \nabla v dx] = E[\int_{D} f v dx], \ \forall v \in H,$$

satisfy

$$||u - u_n||_{\tilde{H}^1_0(D)} \le \frac{C_D}{a_{min}} (||f_n - f||_{\tilde{L}^2(D)} + \frac{1}{a_{min}} ||a - a_n||_{\tilde{L}^\infty(D)}) ||f||_{\tilde{L}^2(D)}) \to 0.$$

Following a similar line as in the previous Corollary, we now apply (2.9) to produce an alternative convergence estimate. This result requires a weaker approximation of the diffusion coefficient, namely in  $\tilde{L}^{2p}(D)$ ,  $1 \leq p < +\infty$ , but needs more regularity from the solution u.

**Corollary 2.2** (Second convergence result). Let 1 with <math>1/p + 1/q = 1. Assume all the hypothesis from the previous corollary except that related with the convergence of the sequence  $\{a_n\}$ , which satisfies instead

$$||a_n - a||_{\tilde{L}^{2p}(D)} \to 0.$$

Besides this, assume that the solution u belongs to the stochastic Sobolev space  $\widetilde{W}^{1,2q}(D)$ . Then

$$\begin{aligned} \|u - u_n\|_{\widetilde{H}^1_0(D)} &\leq \frac{1}{a_{min}} (C_D \|f_n - f\|_{\widetilde{L}^2(D)} \\ &+ \|a - a_n\|_{\widetilde{L}^{2p}(D)} \|u\|_{\widetilde{W}^{1,2q}(D)}) \to 0. \end{aligned}$$

*Proof.* Let  $V_1 = H$ . Then in order to apply (2.9) it is enough to bound the difference of bilinear forms

$$\begin{split} |\int_{D} E[(a-\hat{a})\nabla u \cdot \nabla v] dx| &\leq \left(\int_{D} E[(a-\hat{a})^{2}|\nabla u|^{2}] dx\right)^{1/2} \left(\int_{D} E[|\nabla v|^{2}] dx\right)^{1/2} \\ &\leq \left(\int_{D} E[(a-\hat{a})^{2p}] dx\right)^{1/2p} \left(\int_{D} E[|\nabla u|^{2q}] dx\right)^{1/2q} \left(\int_{D} E[|\nabla v|^{2}] dx\right)^{1/2}. \end{split}$$

**Remark 2.2** (Sufficient conditions for the hypothesis of the previous corollaries). Here we recall the Karhunen-Loève expansion, a suitable tool for the approximation of stochastic processes. Consider a stochastic process a with continuous covariance function,  $Cov[a]: \overline{D} \times \overline{D} \to \mathbb{R}$ . Besides this, let  $\{(\lambda_i, b_i)\}_{i=1}^{\infty}$  denote the sequence of eigenpairs associated with the compact self adjoint operator that maps

$$f \in L^2(D) \mapsto \int_D Cov[a](x, \cdot)f(x)dx \in L^2(D).$$

Its non-negative eigenvalues,  $\sqrt{\int_{D\times D} (Cov[a](x_1, x_2))^2 dx_1 dx_2} \ge \lambda_1 \ge \lambda_2 \ge \ldots \ge 0$  satisfy  $\sum_{i=1}^{+\infty} \lambda_i = \int_D Var[a](x)dx$ . The corresponding eigenfunctions are orthonormal, i.e.  $\int_D b_i(x)b_j(x)dx = \delta_{ij}$ . The truncated Karhunen-Loève expansion of the stochastic process a, cf. [38, 57, 58], is

$$a_{\hat{N}}(\omega, x) = E[a](x) + \sum_{i=1}^{\hat{N}} \sqrt{\lambda_i} b_i(x) Y_i(\omega)$$

where the real random variables,  $\{Y_i\}_{i=1}^{\infty}$ , are mutually uncorrelated, have mean zero and unit variance. These random variables are uniquely determined by

$$Y_i(\omega) = \frac{1}{\sqrt{\lambda_i}} \int_D (a(\omega, x) - E[a](x))b_i(x)dx.$$

for  $\lambda_i > 0$ . Then, by Mercer's theorem cf. [46], p. 245, we have

$$\sup_{x \in D} E[(a - a_{\hat{N}})^2](x) = \sup_{x \in D} \sum_{i=\hat{N}+1}^{+\infty} \lambda_i b_i^2(x) \to 0, \quad as \quad \hat{N} \to \infty.$$

If in addition,

- the images  $Y_i(\Omega)$ , i = 1, ..., are uniformly bounded in  $\mathbb{R}$
- the eigenfunctions b<sub>i</sub> are smooth, which is the case when the covariance function is smooth, and uniformly bounded functions, and
- the eigenvalues have the decay  $\lambda_i = \mathcal{O}(\frac{1}{1+i^s})$  for some s > 1,

then  $||a - a_{\hat{N}}||_{\tilde{L}^{\infty}(D)} \to 0$ . Notice that for larger values of s we can also obtain the convergence of higher spatial derivatives of  $a_{\hat{N}}$  in  $\tilde{L}^{\infty}(D)$ . The last two conditions can be readily verified once the covariance function of a is known. However, observe that it is also necessary to verify the uniform coercivity of  $a_{\hat{N}}$ , which depends on the probability distributions of  $Y_i$ , i = 1, ...

2.4. Finite dimensional noise case. In many problems the source of the randomness can be approximated using just a small number of mutually uncorrelated, sometimes mutually independent, random variables. Take for example the case of a truncated Karhunen-Loève expansion described previously. Whenever the coefficients a and f are independent their corresponding truncated Karhunen-Loève expansions can be computed as in Remark 2.2. Otherwise a joint Karhunen-Loève expansion of a and f has to be computed, taking into account their joint covariance structure.

Assumption 2.1 (Finite dimensional and independent noise). Whenever we apply some numerical method to solve (1.1) we assume that the coefficients used in the computations satisfy

(2.10) 
$$a(\omega, x) = a(Y_1(\omega), \dots, Y_N(\omega), x) \text{ and } f(\omega, x) = f(Y_1(\omega), \dots, Y_N(\omega), x)$$

where  $\{Y_j\}_{j=1}^N$  are real random variables with mean value zero, unit variance, are mutually independent, and their images,  $\Gamma_{i,N} \equiv Y_i(\Omega)$  are bounded intervals in  $\mathbb{R}$  for  $i = 1, \ldots, N$ . Moreover, we assume that each  $Y_i$  has a density function  $\rho_i : \Gamma_{i,N} \to \mathbb{R}^+$  for  $i = 1, \ldots, N$ .

In the sequel we use the notations  $\rho(y) = \prod_{i=1}^{N} \rho_i(y_i) \ \forall y \in \Gamma$  for the joint probability density of  $(Y_1, \ldots, Y_N)$  and  $\Gamma \equiv \prod_{i=1}^{N} \Gamma_{i,N} \subset \mathbb{R}^N$  for the support of such probability density.

After making assumption (2.10), we have by Doob-Dynkin's lemma, cf. [43], that u, the solution corresponding to the stochastic partial differential equation (1.1) can be described by just a finite number of random variables, i.e.  $u(\omega, x) = u(Y_1(\omega), \ldots, Y_N(\omega), x)$ . The number N has to be large enough so that the approximation error is sufficiently small. Now the goal is to approximate the function u(y, x). In addition, the stochastic variational formulation (2.3) has a deterministic equivalent in the following: find  $u \in L^2_{\rho}(\Gamma) \otimes H^1_0(D)$  such that

$$(2.11) \int_{\Gamma} \rho(y) \int_{D} a(y,x) \nabla u(y,x) \cdot \nabla v(y,x) dx dy = \int_{\Gamma} \rho(y) \int_{D} f(y,x) v(y,x) dx dy, \ \forall \ v \in L^{2}_{\rho}(\Gamma) \otimes H^{1}_{0}(D).$$

In this work the gradient notation,  $\nabla$ , always means differentiation with respect to  $x \in D$  only unless otherwise stated. The corresponding strong formulation for the problem now becomes an elliptic partial differential equation with an N-dimensional parameter, i.e.

(2.12) 
$$\begin{aligned} -\nabla \cdot (a(y,x) \, \nabla u(y,x)) &= f(y,x) \quad \forall (y,x) \in \Gamma \times D, \\ u(y,x) &= 0 \qquad \forall (y,x) \in \Gamma \times \partial D. \end{aligned}$$

Making Assumption (2.10) is a crucial step, turning the original stochastic elliptic equation (1.1) into a deterministic parametric elliptic one and allowing the use of finite element and finite difference techniques to approximate the solution of the resulting deterministic problem.

### Truncation of the outcomes set, $\Gamma$

For computational reasons, it may be useful to compute the solution of (2.12) in a subdomain with strictly positive probability,  $\Gamma_0 \subset \Gamma$ , e.g. to save computational work. Beside this, for the sake of the definition and analysis of the numerical methods, we assume the probability density of Y to be strictly positive in  $\Gamma_0$ . In that case, we shall approximate the function

$$E[u(Y, \cdot) \ \mathbf{1}_{\{Y \in \Gamma_0\}}] = E[u(Y, \cdot)|Y \in \Gamma_0] \ P(Y \in \Gamma_0)$$

instead of the original E[u]. If  $\bar{u}$  is an approximation of u in  $\Gamma_0$  then we have the splitting

(2.13) 
$$\begin{split} \|E[u(Y,\cdot)] - E[\bar{u}(Y,\cdot) \ 1_{\{Y \in \Gamma_0\}}]\| \\ \leq \|E[u(Y,\cdot)] - E[u(Y,\cdot) \ 1_{\{Y \in \Gamma_0\}}]\| + \|E[u(Y,\cdot) - \bar{u}(Y,\cdot)|Y \in \Gamma_0]\| P(Y \in \Gamma_0) \\ \\ 10 \end{split}$$

Property 2.2 below gives an estimate for the first error contribution, related to the truncation of  $\Gamma$ . The second error contribution in (2.13) is the discretization error and it will be analyzed for each numerical approximation, see Sections 4, 5 and 6. In those Sections we shall simplify the notation by writing  $\Gamma = \Gamma_0$  and work with the corresponding conditional probability space.

**Property 2.2.** Let u be the solution of the problem (2.12), then there exists a constant C such that

(2.14) 
$$\|E[u(Y,\cdot)] - E[u(Y,\cdot) \ 1_{\{Y \in \Gamma\}}]\|_{H^1_0(D)} \le C\sqrt{P(Y \notin \Gamma_0)} \|f\|_{L^2_\rho(\Gamma \setminus \Gamma_0) \otimes L^2(D)}$$

*Proof.* To derive (2.14), it is enough to estimate

$$\begin{split} \int_{D} \left| \int_{\Gamma \setminus \Gamma_{0}} \rho(y) \nabla u(y, x) dy \right|^{2} dx &\leq P(Y \in \Gamma \setminus \Gamma_{0}) \int_{D} \int_{\Gamma \setminus \Gamma_{0}} \rho(y) |\nabla u(y, x)|^{2} dy dx \\ &\leq P(Y \in \Gamma \setminus \Gamma_{0}) \frac{C_{D}}{a_{min}} \int_{\Gamma \setminus \Gamma_{0}} \rho(y) \int_{D} |f(y, x)|^{2} dx \, dy \end{split}$$

**Remark 2.3.** Let us assume that the components of the vector Y,  $\{Y_j\}_{j=1}^N$ , are random variables with bounded  $m_0$ -th moment. Also, we assume that  $\Gamma_0 = \prod_{j=1}^N [-\bar{y}_j, \bar{y}_j]$  where  $\{\bar{y}_j\}_{j=1}^N$  are positive real numbers. Using Markov's inequality (cf. [11]) we have

$$P(Y \notin \Gamma_0) = 1 - \prod_{j=1}^{N} P(|Y_j| \le \bar{y}_j) \le 1 - \prod_{j=1}^{N} \left( 1 - \frac{1}{(\bar{y}_j)^{m_0}} E[|Y_j|^{m_0}] \right).$$

**Remark 2.4.** If each of the independent random variables  $Y_i$ , i = 1, ..., N has a continuous probability density,  $\rho_i$ , let  $F_i(x) = \int_{-\infty}^x \rho_i(y) dy$  and define the mutually independent and uniformly distributed random variables  $Z_i(\omega) \equiv F_i(Y_i(\omega)) \sim U(0,1)$ , i = 1, ..., N. Consider for i = 1, ..., N,  $F_i^{-1}$ :  $[0,1] \rightarrow \Gamma_i$  and let  $y(z) \equiv (F_1^{-1}(z_1), ..., F_N^{-1}(z_N))$ ,  $\tilde{a}(z,x) \equiv a(y(z),x)$  and  $\tilde{f}(z,x) \equiv f(y(z),x)$ . Now we can equivalently reformulate (2.11) to: find  $\tilde{u} \in L^2([0,1]^N) \otimes H_0^1(D)$  such that

$$\int_{[0,1]^N} \int_D \tilde{a}(z,x) \nabla \tilde{u}(z,x) \cdot \nabla v(z,x) dx dz = \int_{[0,1]^N} \int_D \tilde{f}(z,x) v(z,x) dx dz, \ \forall \ v \in L^2([0,1]^N) \otimes H^1_0(D).$$

Thus, depending on the features of a particular problem, one may discretize (2.11) or (2.15), and apply the numerical methods and the analysis included in this work.

## 3. The finite element spaces

In this section, first we consider standard finite element spaces on the spatial set  $D \subset \mathbb{R}^d$  and the outcomes set  $\Gamma \subset \mathbb{R}^N$ , separately. Then we define tensor product finite element spaces on the set  $\Gamma \times D$  which we will use to construct approximations of the solution of the parametric boundary value problem (2.12). The section concludes with some useful approximation properties of the tensor finite element spaces.

3.1. Finite element spaces on the spatial set  $D \subset \mathbb{R}^d$ : h-version. Consider a family of piecewise linear (continuous) finite element approximation spaces,  $X_h^d \subset H_0^1(D)$ , based on conforming triangulations (of simplices),  $\mathcal{T}_h^d$ , of the convex polyhedral domain,  $D \subset \mathbb{R}^d$ , with a maximum mesh spacing parameter h > 0. We shall always assume that the triangulations are nondegenerate

(sometimes also called regular), cf. [13] p.106. Then, (cf. [13], [15], [44]) the finite element spaces  $X_h^d$  satisfy the standard approximation estimate, namely that for all  $v \in H^2(D) \cap H_0^1(D)$ 

(3.1) 
$$\min_{\chi \in X_h^d} \|v - \chi\|_{H^1_0(D)} \le C h \|v\|_{H^2(D)},$$

where C > 0 is a constant independent from v and h.

3.2. Tensor product finite element spaces on the outcomes set  $\Gamma \subset \mathbb{R}^{N}$ : k-version. Let  $\Gamma = \prod_{i=1}^{N} \Gamma_i$  be as in Subsection 2.4. Consider a partition of  $\Gamma$  consisting of a finite number of disjoint  $\mathbb{R}^N$ -boxes,  $\gamma = \prod_{j=1}^N (a_j^{\gamma}, b_j^{\gamma})$ , with  $(a_j^{\gamma}, b_j^{\gamma}) \subseteq \Gamma_j$  for  $j = 1, \dots, N$ . The mesh spacing parameters,  $k_j > 0$ , are defined by  $k_j \equiv \max_{\gamma} |b_j^{\gamma} - a_j^{\gamma}|$ , for  $1 \le j \le N$ . For every non-negative integer random  $q \in \mathbb{N}$  consider the finite element approximation space of (discontinuous) piecewise polynomials with degree at most q on each direction,  $Y_k^{N,q} \subset L^2(\Gamma)$ . Thus, if  $\varphi \in Y_k^{N,q}$  its restriction to each of the partition boxes satisfies  $\varphi|_{\gamma} \in \operatorname{span}\left(\prod_{j=1}^N y_j^{\alpha_j} : \alpha_j \in \mathbb{N} \text{ and } \alpha_j \leq q, j = 1, \dots, N\right)$ .

The finite element spaces  $Y_k^{N,q}$  have (cf. Section 4.6 in [13]) the following approximation property: for all  $v \in H^{q+1}(\Gamma)$ 

(3.2) 
$$\min_{\varphi \in Y_k^{N,q}} \|v - \varphi\|_{L^2(\Gamma)} \le C \sum_{j=1}^N (k_j)^{q+1} \|\partial_{y_j}^{q+1}v\|_{L^2(\Gamma)}$$

where C > 0 is a constant independent of v and  $k_i > 0$ .

3.3. Tensor product finite element spaces on  $\Gamma \times D$ :  $k \times h$ -version. Here, we shall discuss some approximation properties of the following tensor product finite element spaces

$$(3.3) Y_k^{N,q} \otimes X_h^d \equiv \left\{ \psi = \psi(y,x) \in L^2(\Gamma \times D) : \psi \in \operatorname{span}\left(\varphi(y)\,\chi(x) : \varphi \in Y_k^{N,q}, \,\chi \in X_h^d\right) \right\}.$$

with  $X_h^d$  and  $Y_k^{N,q}$  as in Subsections 3.1 and 3.2. For later use we recall the standard  $L^2$ - projection operators  $\Pi_k^{N,q}: L^2(\Gamma) \to Y_k^{N,q}$  by

 $(\Pi_k^{N,q}w - w, \varphi)_{L^2(\Gamma)} = 0, \quad \forall \varphi \in Y_k^{N,q}, \quad \forall w \in L^2(\Gamma),$ (3.4)

and the  $H_0^1$  projection operator  $\mathcal{R}_h^d: H_0^1(D) \to X_h^d$  by

(3.5) 
$$(\nabla \mathcal{R}_h^d v - v, \nabla \chi)_{L^2(D)} = 0, \quad \forall \chi \in X_h^d, \quad \forall v \in H_0^1(D).$$

Estimates (3.1) and (3.2) imply

(3.6)  
$$\|v - \mathcal{R}_{h}^{d}v\|_{H_{0}^{1}(D)} \leq C h \|v\|_{H^{2}(D)},$$
$$\|w - \Pi_{k}^{N,q}w\|_{L^{2}(\Gamma)} \leq C \sum_{j=1}^{N} (k_{j})^{q+1} \|\partial_{y_{j}}^{q+1}w\|_{L^{2}(\Gamma)}.$$

for all  $v \in H^2(D) \cap H^1_0(D)$  and  $w \in H^{q+1}(\Gamma)$ .

We now present an approximation property for the tensor product finite element spaces defined in (3.3) which is a direct implication of the approximation properties of the spaces  $Y_k^{N,q}$  and  $X_h^d$ . **Proposition 3.1.** Let q be a nonnegative integer. Then, there exists a constant C > 0 depending only on d, N and q, such that

$$(3.7) \quad \inf_{\psi \in Y_k^{N,q} \otimes X_h^d} \|v - \psi\|_{L^2(\Gamma; H_0^1(D))} \le C \left\{ h \|v\|_{L^2(\Gamma; H^2(D))} + \sum_{j=1}^N (k_j)^{q+1} \|\partial_{y_j}^{q+1}v\|_{L^2(\Gamma; H_0^1(D))} \right\}$$

for all  $v \in C^{q+1}(\Gamma; H^2(D) \cap H^1_0(D))$ .

*Proof.* Since  $\Pi_k^{N,q}(\mathcal{R}_h v) \in Y_k^q \otimes X_h^d$ , using (3.6) we obtain

(3.8)  
$$\begin{aligned}
&\inf_{\psi \in Y_k^q \otimes X_h^d} \|v - \psi\|_{L^2(\Gamma; H_0^1(D))} \leq \|v - \Pi_k^{N, q}(\mathcal{R}_h v)\|_{L^2(\Gamma; H_0^1(D))} \\
&\leq \|v - \mathcal{R}_h v\|_{L^2(\Gamma; H_0^1(D))} + \|\mathcal{R}_h v - \Pi_k^{N, q}(\mathcal{R}_h v)\|_{L^2(\Gamma; H_0^1(D))} \\
&\leq C h \|v\|_{L^2(\Gamma; H^2(D))} + \|\mathcal{R}_h v - \Pi_k^{N, q}(\mathcal{R}_h v)\|_{L^2(\Gamma; H_0^1(D))}.
\end{aligned}$$

Applying the estimate (3.2) and using the boundedness of  $\mathcal{R}_h$  in  $H^1_0(D)$  yields

$$\|\mathcal{R}_{h}v - \Pi_{k}^{N,q}(\mathcal{R}_{h}v)\|_{L^{2}(\Gamma;H_{0}^{1}(D))} \leq \|v - \Pi_{k}^{N,q}v\|_{L^{2}(\Gamma;H_{0}^{1}(D))} \leq \sum_{j=1}^{N} (k_{j})^{q+1} \|\partial_{y_{j}}^{q+1}v\|_{L^{2}(\Gamma)\otimes H_{0}^{1}(D)}.$$

The estimate (3.7) follows combining (3.8) with the last estimate.  $\Box$ 

3.4. Tensor product finite element spaces on  $\Gamma \times D$ :  $p \times h$ -version. Consider again the set  $\Gamma = \Gamma_1 \times \ldots \times \Gamma_N \subset \mathbb{R}^N$ , a multi-index  $p = (p_1, \ldots, p_N)$  and the tensor finite element space

$$Z^p = \bigotimes_{i=1}^N Z_i^{p_i}$$

where the one dimensional global polynomial subspaces,  $Z_i^{p_i}$ , are defined by

$$Z_i^{p_i} = \left\{ v : \Gamma_i \to \mathbb{R} : v \in \operatorname{span}(y^s, s = 0, \dots, p_i) \right\}, \quad i = 1, \dots, N.$$

Here, the basic tool for approximation are the following tensor product finite element spaces

$$(3.9) Z^p \otimes X^d_h \equiv \left\{ \psi = \psi(y, x) \in L^2(\Gamma \times D) : \psi \in \operatorname{span}\left(\varphi(y)\,\chi(x) : \varphi \in Z^p, \, \chi \in X^d_h\right) \right\}.$$

Similarly to Subsection 3.3, the  $p \times h$ -version has the following spatial approximation property **Proposition 3.2.** Let  $v \in L^2(\Gamma; H^2(D) \cap H^1_0(D))$  then there exists a constant C > 0 independent of v and h such that

$$\inf_{\psi \in L^2(\Gamma) \otimes X_h^d} \|v - \psi\|_{L^2(\Gamma; H^1_0(D))} \le \, C \, h \, \|v\|_{L^2(\Gamma; H^2(D))}$$

The approximation properties in the y-direction will be studied in Section 6.

### 4. The Monte Carlo Galerkin Finite Element Method

In this section we describe the use of the standard Monte Carlo Galerkin Finite Element Method (MCGFEM) to construct approximations to some solution statistics. These approximations are based on sample averages of independent, identically distributed realizations corresponding to sample coefficient functions. For each realization of the coefficients a realization of the approximate solution is computed with standard finite element spaces. Our discussion will be based on the approximation of the expected value function  $E[u(Y, \cdot)]$  by a sample average of M realizations of an approximate solution  $u_h(Y, \cdot)$ . The stochastic coefficients depend on the noise as described in Subsection 2.4 and we simply use the notation  $a(\omega, \cdot)$ , meaning  $a(Y(\omega), \cdot)$ .

# Formulation of the Monte Carlo Galerkin Finite Element Method (MCGFEM):

- Give a number of realizations, M, a piecewise linear finite element space on D,  $X_h^d$ , as defined in subsection 3.1.

- For each j = 1, ..., M sample iid realizations of the diffusion  $a(\omega_j, \cdot)$ , the load  $f(\omega_j, \cdot)$  and find a corresponding approximation  $u_h(\omega_j; \cdot) \in X_h^d$  such that

(4.1) 
$$B_{\omega_j}(u_h(\omega_j, \cdot), \chi) \equiv \int_D a(\omega_j, \cdot) \nabla u_h(\omega_j, \cdot) \cdot \nabla \chi dx = (f(\omega_j, \cdot), \chi), \quad \forall \chi \in X_h^d.$$

- Finally use the sample average  $\frac{1}{M} \sum_{j=1}^{M} u_h(\omega_j; \cdot)$  to approximate E[u].  $\Box$ 

Here we only consider the case where  $X_h^d$  is the same for all realizations, i.e. the spatial triangulation is deterministic. The computational error naturally separates into the two parts

(4.2) 
$$E[u] - \frac{1}{M} \sum_{j=1}^{M} u_h(\omega_j, \cdot) = \left( E[u] - E[u_h] \right) + \left( E[u_h] - \frac{1}{M} \sum_{j=1}^{M} u_h(\omega_j, \cdot) \right) \equiv \mathcal{E}_h + \mathcal{E}_S.$$

The size of the spatial triangulation controls the space discretization error  $\mathcal{E}_h$ , cf. Proposition 4.2, while the number of realizations, M of  $u_h$ , controls the statistical error  $\mathcal{E}_S$ .

To study the behavior of the statistical error, let us first consider the random variable  $\|\mathcal{E}_S\|_{L^2(D)}$ which, due to the independence of the realizations  $\omega_j$ ,  $j = 1, \ldots, M$ , satisfies the estimate

(4.3) 
$$M E \left[ \| \mathcal{E}_S \|_{H_0^1(D)}^2 \right] \le \| u_h \|_{\tilde{H}_0^1(D)}^2 \le \left( \frac{C_D}{a_{\min}} \right)^2 \| f \|_{\tilde{L}^2(D)}^2$$

and a similar result also holds in  $L^2(D)$ . Then, thanks to (4.3) we have that, for either  $H = L^2(D)$ or  $H = H_0^1(D)$ , the statistical error  $\mathcal{E}_S$  tends a.s. to zero as we increase the number of realizations, i.e.

**Proposition 4.1.** Suppose that there exist a constant C > 0 independent from M and h such that the statistical error in H norm satisfies

(4.4) 
$$M E[||\mathcal{E}_S||_H^2] \le C, \quad \forall M, h.$$

Then, taking the number of realizations increasingly from the set  $\{2^k : k \in \mathbb{N}\}$ , we have for any  $\alpha \in (0, 1/2)$  and any choice of mesh size h

$$\lim_{M \to \infty} M^{\alpha} \| \mathcal{E}_S \|_H = 0 \ a.s$$

*Proof.* Let  $\epsilon > 0$ . Then (4.4) and Markov's inequality give

$$P(M^{\alpha} \| \mathcal{E}_S \|_H > \epsilon) \le \frac{E[M^{2\alpha} \| \mathcal{E}_S \|_H^2]}{\epsilon^2} \le \frac{C}{\epsilon^2 M^{1-2\alpha}}.$$

Thus

$$\sum_{k=1}^{\infty} P\left(M_k^{\alpha} \| \mathcal{E}_S \|_H > \epsilon\right) \le \frac{C}{\epsilon^2} \sum_{k=1}^{\infty} \frac{1}{M_k^{1-2\alpha}} \le \frac{C}{\epsilon^2} \sum_{k=1}^{\infty} \frac{1}{(2^{1-2\alpha})^k} < \infty$$

and Borel-Cantelli's lemma implies that for any given  $\epsilon > 0$  and  $\alpha \in (0, 1/2)$ 

 $P\left(M^{\alpha} \| \mathcal{E}_S \|_H > \epsilon \text{ infinitely often}\right) = 0$ 

which finishes the proof.  $\Box$ 

Under the same assumptions as in Proposition 4.1 we have that for any given  $\epsilon > 0$  there exist a constant C > 0 independent from  $\epsilon$ , M and h such that

(4.5) 
$$P\left(\|\mathcal{E}_S\|_H \ge \frac{\epsilon}{\sqrt{M}}\right) \le \frac{C}{\epsilon^2}.$$

Thus, within a given confidence level we have the usual convergence rate for the Monte Carlo method, which is independent from the mesh size h. Next we present some error estimates for the space discretization error,  $||E[u-u_h]||_{H_0^1(D)}$  and  $||E[u-u_h]||_{L^2(D)}$ . The first two lemmata estimate strong errors, which yield in Remark 4.2 a bound for the spatial discretization error in E[u]. The proofs are done assuming piecewise linear finite element spaces on regular meshes and follow the standard arguments for the deterministic case, cf. [13].

Lemma 4.1. There holds

(4.6) 
$$\|\sqrt{a}\nabla(u-u_h)\|_{L^2(D)} \le C h \|u\|_{H^2(D)}, \quad a.s.$$

(4.7) 
$$\|u - u_h\|_{L^2(D)} \le C h^2 \|u\|_{H^2(D)}, \quad a.s.$$

*Proof.* Let  $\omega \in \Omega$ . Then

(4.8)  
$$B_{\omega}(e,e) = B_{\omega}(e,u - \mathcal{R}_{h}^{d}u) \\ \leq a_{\max} \|e\|_{H_{0}^{1}(D)} \|\nabla(u - \mathcal{R}_{h}^{d}u)\|_{L^{2}(D)} \\ \leq C h \|e\|_{H_{0}^{1}(D)} \|u\|_{H^{2}(D)}$$

and the use of the uniform coercivity assumption (1.2) yields (4.6). To prove (4.7) we follow a standard duality argument. Let  $\omega \in \Omega$ , then consider the continuous dual problem

$$B_{\omega}(\varphi, v) = (e, v), \ \forall \ v \in H_0^1(D)$$

and then we use elliptic regularity,  $\|\varphi\|_{H^2(D)} \leq C \|e\|_{L^2(D)}$ . Notice that the constant of elliptic regularity depends in general on  $\omega$  and we have used assumption (1.3) to control it uniformly.

Therefore,

$$\begin{split} \|e\|_{L^{2}(D)}^{2} = B_{\omega}(\varphi, e) \\ = B_{\omega}(\varphi - \mathcal{R}_{h}^{d}\varphi, e) \\ \leq a_{max} \|e\|_{H_{0}^{1}(D)} \|\varphi - \mathcal{R}_{h}^{d}\varphi\|_{H_{0}^{1}(D)} \\ \leq \|e\|_{H_{0}^{1}(D)} Ch\|\varphi\|_{H^{2}(D)} \\ \leq Ch^{2} \|f\|_{L^{2}(D)} \|e\|_{L^{2}(D)}. \end{split}$$

A direct application of Lemma 4.1 gives

**Proposition 4.2** (Spatial discretization error estimates). There holds

$$||E[u] - E[u_h]||_{H^1_0(D)} \le E[||u - u_h||^2_{H^1_0(D)}]^{1/2} \le Ch||f||_{\tilde{L}^2(D)}$$

and similarly

$$||E[u] - E[u_h]||_{L^2(D)} \le E[||u - u_h||^2_{L^2(D)}]^{1/2} \le Ch^2 ||f||_{\tilde{L}^2(D)}$$

The results from Proposition 4.2 and estimate (4.5) will be used in Section 8 to compare the MCGFEM with other discretizations for (1.1).

#### 5. The Stochastic Galerkin Finite Element Method: $k \times h$ -version

This section defines and analyzes the  $k \times h$ -version of the stochastic Galerkin finite element method  $(k \times h$ -SGFEM) which, via a Galerkin variational formulation, yields approximations,  $u_{kh} \in Y_k^{\Gamma,q} \otimes X_h^d$ , of the solution u of the parametric elliptic boundary value problem (2.12). Then, the section ends by showing how to use  $u_{kh}$  to construct approximations of the expected value of u, analyzing the corresponding approximation error.

Formulation of the  $k \times h$ -SGFEM: Let  $q \in \mathbb{N}$  and  $\Gamma$  be a bounded box in  $\mathbb{R}^{N}$ . The  $k \times h$ -SGFEM approximation is the tensor product,  $u_{kh} \in Y_{k}^{\Gamma,q} \otimes X_{h}^{d}$ , such that

(5.1) 
$$\int_{\Gamma} \rho \left( a \nabla_x u_{kh}, \nabla_x \psi \right)_{L^2(D)} dy = \int_{\Gamma} \rho \left( f, \psi \right)_{L^2(D)} dy, \quad \forall \ \psi \in Y_k^{\Gamma, q} \otimes X_h^d.$$

Recall that  $N \in \mathbb{N}$ ,  $\rho : \Gamma \to (0, +\infty)$  is the density function of the vector-valued random variable  $Y : \Omega \to \Gamma \subset \mathbb{R}^N$  which has mutually independent components. Hence, the assumption (1.2) on the random function  $a(\omega, x) \equiv a(Y(\omega), x)$  reads

(5.2) 
$$a(y,x) \in [a_{\min}, a_{\max}], \quad \forall (y,x) \in \overline{\Gamma \times D}$$

For the case where a is a truncated Karhunen Loève expansion Section 7 discusses how to compute efficiently  $u_{kh}$ , the solution of (5.1), by a double-orthogonal polynomials technique. To carry out the analysis of the  $k \times h$ -version, assume that

(5.3) 
$$a \in C^{q+1,1}(\overline{\Gamma \times D}) \text{ and } f \in C^{q+1}(\Gamma; L^2(D)),$$

which is trivially satisfied whenever a and f have the form of a truncated Karhunen Loève expansion (cf. Section 6). Hence, by Lemma 4.1 in [36], the solution u of (2.12) satisfies

$$u \in C^{q+1}(\Gamma; H^2(D) \cap H^1_0(D))$$

Use (5.2) and (2.12) to obtain

(5.4) 
$$\|u(y,\cdot)\|_{H^1_0(D)} \leq \frac{C_D}{a_{\min}} \|f(y,\cdot)\|_{L^2(D)}, \quad \forall y \in \Gamma,$$

where  $C_D$  is the constant of the Poincaré–Friedrichs inequality on D. Also, elliptic regularity yields

(5.5) 
$$\|u(y,\cdot)\|_{H^2(D)} \le C_{0,B} \|f(y,\cdot)\|_{L^2(D)}, \quad \forall y \in \Gamma,$$

where  $C_{0,B}$  is a constant which depends on D and  $||a||_{L^{\infty}(\Gamma;W^{1,\infty}(D))}$ . Finally, take derivatives with respect to  $y_j$  in (2.12), proceed as in the derivation of (5.4), and follow an induction argument arriving at

(5.6) 
$$\|\partial_{y_j}^{q+1} u(y, \cdot)\|_{H^1_0(D)}^2 \le C_{q+1,B} \sum_{0 \le \beta \le q+1} \|\partial_{y_j}^\beta f(y, \cdot)\|_{L^2(D)}^2, \quad \forall y \in \Gamma, \text{ and } 1 \le j \le N$$

where  $C_{q+1,B}$  is a constant which depends on q,  $C_D$ ,  $a_{\min}$  and  $||a||_{L^{\infty}(D;W^{q+1,\infty}(\Gamma))}$ .

We now derive an a priori error estimate for the  $k \times h$ -SGFEM in the energy norm, that will be later used to derive an error estimate for  $E[u - u_{kh}]$ .

**Proposition 5.1.** Let u be the solution of the problem (2.12) and  $u_{kh} \in Y_k^{\Gamma,q} \otimes X_h^d$  be the  $k \times h$ -SGFEM approximations of u defined in (5.1). If (5.3) holds and  $\rho \in L^{\infty}(\Gamma)$ , then

(5.7) 
$$\left( \int_{\Gamma} \rho \| \sqrt{a} \nabla_x (u - u_{kh}) \|_{L^2(D)}^2 dy \right)^{\frac{1}{2}} \leq C \sqrt{\|a\rho\|_{L^{\infty}(\Gamma \times D)}} \left( h \|f\|_{L^2(\Gamma; L^2(D))} + \sum_{j=1}^N (k_j)^{q+1} \sqrt{\sum_{0 \le \beta \le q+1} \|\partial_{y_j}^{\beta} f\|_{L^2(\Gamma; L^2(D))}^2} \right)$$

where the constant C depends on q, D,  $\Gamma$  and a, and is independent of k, h and u.

*Proof.* Let  $V_{kh} = Y_k^{\Gamma,q} \otimes X_h^d$  and  $e = u - u_{kh}$ . Combining (5.1) with (2.12) gives the standard Galerkin orthogonality

(5.8) 
$$\int_{\Gamma} \rho \ (a \ \nabla_x e, \nabla_x \psi)_{L^2(D)} \ dy = 0, \quad \forall \ \psi \in V_{kh}$$

and thus

(5.9) 
$$\left( \int_{\Gamma} \rho \|\sqrt{a} \nabla_{x} e\|_{L^{2}(D)}^{2} dy \right)^{\frac{1}{2}} \leq \inf_{\psi \in V_{kh}} \left( \int_{\Gamma} \rho \|\sqrt{a} \nabla_{x} (u - \psi)\|_{L^{2}(D)}^{2} dy \right)^{\frac{1}{2}} \\ \leq \|a\rho\|_{L^{\infty}(\Gamma \times D)}^{\frac{1}{2}} \inf_{\psi \in V_{kh}} \|u - \psi\|_{L^{2}(\Gamma; H^{1}(D))}.$$

Using (5.9) and the estimate (3.7) implies

$$\left(\int_{\Gamma} \rho \|\sqrt{a} \nabla_{x} e\|_{L^{2}(D)}^{2} dy\right)^{\frac{1}{2}} \leq C \|a\rho\|_{L^{\infty}(\Gamma \times D)}^{\frac{1}{2}} \left\{h\|u\|_{L^{2}(\Gamma; H^{2}(D))} + \sum_{j=1}^{N} (k_{j})^{q+1} \|\partial_{y_{j}}^{q+1}u\|_{L^{2}(\Gamma; H^{1}_{0}(D))}\right\}.$$

The last estimate combined with (5.5) and (5.6) yields (5.7).

As a direct result from (5.7) we obtain

Corollary 5.1 (Convergence result). Under the assumptions of Proposition 5.1 we have

$$||E[u(Y,\cdot)] - E[u_{kh}(Y,\cdot)]||_{H_0^1(D)} \le C\left(h + \sum_{j=1}^N (k_j)^{q+1}\right)$$

with C > 0 independent from u, h and  $k_j$ .

The next step is to use Proposition 5.1 to estimate the  $L^2(D)$  error in the approximation of the expected value of  $u(Y, \cdot)$ .

**Theorem 5.2.** Let u be the solution of the problem (2.12) and  $u_{kh} \in Y_k^{\Gamma,q} \otimes X_h^d$  be the  $k \times h$ -SGFEM approximations of u defined in (5.1). If (5.3) holds and  $\rho \in L^{\infty}(\Gamma)$ , then

(5.10) 
$$\left\| E[u(Y,\cdot)] - E[u_{kh}(Y,\cdot)] \right\|_{L^2(D)} \le C \left( h^2 + \sum_{j=1}^N (k_j)^{2(q+1)} \right) \|a\rho\|_{L^\infty(\Gamma \times D)} \|f\|_{L^2(D;H^{q+1}(\Gamma))}.$$

The constant C depends on q, D,  $\Gamma$  and a, and it is independent from k, h and u.

*Proof.* Let  $V_{kh} = Y_k^{\Gamma,q} \otimes X_h^d$ ,  $e \equiv u - u_{kh}$ , and  $g \equiv E[e(Y, \cdot)] \in H_0^1(D)$ . Then, we consider  $\hat{u} \in C^{q+1}(\Gamma; H^2(D) \cap H_0^1(D))$  defined, for  $y \in \Gamma$ , as the solution of the following elliptic boundary value problem

(5.11) 
$$-\nabla_x \cdot (a(y,\cdot) \nabla_x \hat{u}(y,\cdot)) = g(\cdot) \text{ in } D,$$
$$\hat{u}(y,\cdot) = 0 \text{ on } \partial D.$$

Then (5.5) reads

(5.12) 
$$\|\hat{u}(y,\cdot)\|_{H^2(D)} \le C_{0,B} \|g\|_{L^2(D)}, \quad \forall y \in \Gamma,$$

and since g is independent from y, the estimate (5.6) for the problem (5.11) reads

(5.13) 
$$\sum_{|\beta| \le q+1} \|\partial_y^\beta \hat{u}(y, \cdot)\|_{H^1(D)}^2 \le C_{q+1,B} \|g\|_{L^2(D)}^2, \quad \forall y \in \Gamma.$$

Use (5.11) and the orthogonality property (5.8) to obtain

$$\int_{\Gamma} \rho (g, e)_{L^2(D)} dy = \int_{\Gamma} \rho \left( a \nabla_x e, \nabla_x (\hat{u} - \psi) \right)_{L^2(D)} dy, \quad \forall \psi \in V_{kh},$$

which yields, by Cauchy Schwartz inequality,

$$(5.14) ||g||_{L^2(D)}^2 \le \widetilde{B}_1 \ \widetilde{B}_2$$

with

$$\widetilde{B}_1 \equiv \left(\int_{\Gamma} \rho \|\sqrt{a} \nabla_x e\|_{L^2(D)}^2 dy\right)^{\frac{1}{2}}$$

and

$$\widetilde{B}_2 \equiv \inf_{\psi \in V_{kh}} \left( \int_{\Gamma} \rho \| \sqrt{a} \nabla_x (\hat{u} - \psi) \|_{H^1(D)}^2 \, dy \right)^{\frac{1}{2}}.$$

Next, observe that the energy estimate (5.7) gives

(5.15) 
$$\widetilde{B}_{1} \leq C \left( h + \sum_{j=1}^{N} (k_{j})^{q+1} \right) \|a\rho\|_{L^{\infty}(\Gamma \times D)}^{\frac{1}{2}} \|f\|_{L^{2}(D; H^{q+1}(\Gamma))}$$

Finally, use (3.7), (5.12) and (5.13), to bound  $B_2$  as follows

(5.16)  

$$\widetilde{B}_{2} \leq C \|a\,\rho\|_{L^{\infty}(\Gamma\times D)}^{\frac{1}{2}} \left\{h\left(\int_{\Gamma} \|\hat{u}\|_{H^{2}(D)}^{2}\,dy\right)^{1/2} + \sum_{j=1}^{N} (k_{j})^{q+1} \left(\int_{\Gamma} \|\partial_{y_{j}}^{q+1}\hat{u}\|_{H^{1}_{0}(D)}^{2}\,dy\right)^{\frac{1}{2}}\right\}$$

$$\leq C \|a\,\rho\|_{L^{\infty}(\Gamma;L^{\infty}(D))}^{\frac{1}{2}} \left(h + \sum_{j=1}^{N} (k_{j})^{q+1}\right) \|g\|_{L^{2}(D)}.$$

Combining (5.14), (5.15) and (5.16), the estimate (5.10) follows.  $\Box$ 

**Remark 5.1.** Let  $k \equiv \max_{1 \le j \le N} k_j$  and assume that  $u \in C^{q+1}(\Gamma; H^{s+1}(D) \cap H_0^1(D))$  for some integer  $s \ge 1$ , and that  $X_h^d$  are standard finite element spaces consisting of continuous piecewise polynomial functions of degree less or equal to s (cf., e.g., [13], [15], [44]). Then, proceeding in a similar way we get the estimate  $\|E[(u(Y, \cdot) - u_{kh}(Y, \cdot))]\|_{L^2(D)} \le C(h^{s+1} + k^{\frac{(q+1)(s+1)}{s}})$  which is a superconvergence result with respect to k that generalizes the estimate (5.10).

# 6. The Stochastic Galerkin Finite Element Method: $p \times h$ -version

The goal of this section is to introduce and analyze the  $p \times h$ -version of the SGFEM method, that uses global polynomials in the y direction instead of piecewise discontinuous ones. This method yields, cf. Theorem 6.2, an exponential rate of convergence with respect to p, the degree of the polynomials used for approximation. The work by Gui and Babuška [27] on the h - p versions of the Finite Element in one dimension inspired the current analysis of the p-version tensor product approximation in the y direction. The application of the p-version in the y direction is motivated by the fact that u is infinitely differentiable with respect to  $y \in \Gamma$ , cf. Lemma 6.1, and the comparison of computational work in Section 8. As the  $k \times h$ -version studied in Section 5, the p-version also gives a super-convergent approximation of the expected value of the solution, cf. Theorem 6.3.

The basic assumptions for this section are summarized as follows:

Assumption 6.1. Assume that the density function,  $\rho: \Gamma \to \mathbb{R}$  of the N dimensional random vector Y, is bounded and that the components of Y are mutually independent. Beside this, the functions  $a, f: \Gamma \times D \to \mathbb{R}$  are finite Karhunen-Loève expansions, i.e.,  $a(y,x) = E[a](x) + \sum_{i=1}^{N} b_i(x)y_i$  and  $f(y,x) = E[f](x) + \sum_{i=1}^{N} \hat{b}_i(x)y_i$ . Moreover, the uniform coercivity assumption (1.2) and the right hand side assumption (1.4) yield  $b_i \in C^1(\overline{D})$ ,  $\hat{b} \in L^2(D)$  for  $i = 1, \ldots, N$ . Use the notation

$$\hat{\Gamma}_i \equiv \prod_{1 \le j \le N, j \ne i} \Gamma_j$$

and let  $\hat{y}_i$  be an arbitrary element of  $\hat{\Gamma}_i$ . Then, for each  $\hat{y}_i \in \hat{\Gamma}_i$  let

$$\tilde{a}_i(\hat{y}_i) \equiv \min_{x \in D} \left\{ E[a](x) + \sum_{\substack{1 \le j \le N, j \ne i}} b_j(x)y_j \right\}$$
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and assume a slightly stronger uniform coercivity requirement, i.e. there exists a constant  $\tilde{\nu} > 0$  such that

$$\nu_i(\hat{y}_i) \equiv \tilde{a}_i(\hat{y}_i) - \|b_i\|_{L^{\infty}(D)} \max_{y \in \Gamma_i} |y| \ge \tilde{\nu} > 0, \quad \forall \hat{y}_i \in \hat{\Gamma}_i, \quad i = 1, \dots, N.$$

Observe that with the above construction we have  $0 < \tilde{\nu} \leq a_{min}.\Box$ 

 $p \times h$ -version of the SGFEM method: Let and  $\Gamma$  be a bounded box in  $\mathbb{R}^{N}$ . Then for  $h \in (0, h_0)$  the  $p \times h$ -version SGFEM approximation is the tensor product  $u_h^p \in Z^p \otimes X_h^d$ , recall (3.9), that satisfies

(6.1) 
$$\int_{\Gamma} \rho \left( a \nabla_x u_h^p, \nabla_x \chi \right)_D dy = \int_{\Gamma} \rho \left( f, \chi \right)_D dy \quad \forall \ \chi \in Z^p \otimes X_h^d.$$

6.1. Error estimates. To illustrate the convergence of the p-version here we study for the particular case of the energy error, i.e. consider

$$\begin{split} \|u - u_h^p\|_H = & \sqrt{\int_{\Gamma} \rho(y) \int_{D} a(y, x) |\nabla(u - u_h^p)(y, x)|^2 dx dy} \\ \leq & \sqrt{\|\rho a\|_{L^{\infty}(\Gamma \times D)}} \min_{v \in Z^p \otimes X_h^d} \|u - v\|_{L^2(\Gamma) \otimes H_0^1(D)} \\ \leq & \sqrt{\|\rho a\|_{L^{\infty}(\Gamma \times D)}} \left\{ \min_{v \in Z^p \otimes H_0^1(D)} \|u - v\|_{L^2(\Gamma) \otimes H_0^1(D)} + \min_{v \in L^2(\Gamma) \otimes X_h^d} \|u - v\|_{L^2(\Gamma) \otimes H_0^1(D)} \right\}. \end{split}$$

This bound splits the error into an  $L^2(\Gamma)$  approximation error and a standard  $H_0^1(D)$  FEM approximation error. The rest of this section studies the first one, since for the second we can apply the results from Proposition 3.2. The minimizer

$$\|u - u^p\|_{L^2(\Gamma) \otimes H^1_0(D)} = \min_{v \in Z^p \otimes H^1_0(D)} \|u - v\|_{L^2(\Gamma) \otimes H^1_0(D)}$$

is the projection

$$u^p = (\Pi_1 \dots \Pi_N)u$$

with  $\Pi_i : L^2(\Gamma) \otimes H^1_0(D) \to L^2(\Gamma) \otimes H^1_0(D)$  being the natural extension of the  $L^2$  projection  $\overline{\Pi}_i : L^2(\Gamma_i) \to Z_i^{p_i}$ , so the difference  $u - u^p$  splits into

$$u - u^p = (1 - \Pi_1)u + \ldots + (\Pi_1 \ldots \Pi_{N-1})(1 - \Pi_N)u.$$

In addition, the boundedness of the projections  $\Pi_i$  yields

(6.2) 
$$\|u - u^p\|_{L^2(\Gamma) \otimes H^1_0(D)} \le \sum_{i=1}^N \|(1 - \Pi_i)u\|_{L^2(\Gamma) \otimes H^1_0(D)}.$$

Without loose of generality we now consider the first term on the right hand side of (6.2), since the other terms have a completely similar behavior. Moreover, since

$$\|(1-\Pi_1)u\|_{L^2(\Gamma)\otimes H^1_0(D)}^2 = \int_{\Gamma_2 \times \ldots \times \Gamma_N} \left( \int_{\Gamma_1} \|(1-\Pi_1)u(y_1, y_2, \ldots, y_N, \cdot)\|_{H^1_0(D)}^2 dy_1 \right) dy_2 \ldots dy_N$$

it is enough to estimate

(6.3) 
$$(E_1)^2(y_2, \dots, y_N) \equiv \int_{\Gamma_1} \|(1 - \Pi_1)u(y_1, y_2, \dots, y_N, \cdot)\|_{H^1_0(D)}^2 dy_1$$

and thus our analysis requires only one dimensional arguments in the y-direction. Let  $\Gamma_1 = (y_{min}, y_{max})$  and consider the map  $\Psi : (-1, 1) \to H_0^1(D)$  defined by

$$\Psi(t) = u(y_1(t), y_2, \dots, y_N, \cdot) \in H_0^1(D)$$
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with the affine transformation,  $y_1 : [-1,1] \to \Gamma_1$ ,  $y_1(t) \equiv \left(\frac{y_{max}+y_{min}}{2}\right) + \left(\frac{y_{max}-y_{min}}{2}\right) t$ . In the upcoming estimate of the quantities  $||d_n||_{H_0^1(D)}$ , to be proved in Lemma 6.2, we need to consider a continuation of  $\Psi$  to the complex plane, namely that

**Lemma 6.1** (Complex continuation). The function  $\Psi : [-1,1] \to H_0^1(D)$  can be analytically continuated to the complex domain.

*Proof.* Let  $t_0 \in (-1, 1)$ . We shall prove that the real function  $\Psi$  can be represented as a power series for  $|t - t_0| < r_{t_0}$ , for some  $r_{t_0} > 0$ . Since  $\Psi$  depends linearly on f, let us assume that f(y, x) = f(x) only, without loose of generality. Let  $y(t) = (y_1(t), y_2, \ldots, y_N)$  and consider the formal series

$$u_F(t) \equiv \sum_{j=0}^{+\infty} \left(\frac{|\Gamma_1|(t-t_0)}{2}\right)^j u_j$$

with  $u_j \in H_0^1(D)$  satisfying

(6.4) 
$$\begin{aligned} -\nabla \cdot \left(a(y(t_0), x) \ \nabla u_0(x)\right) &= f(x), \quad \forall x \in D, \\ u_0(x) &= 0 \qquad \quad \forall x \in \partial D, \end{aligned}$$

and for  $j \ge 0$ 

(6.5) 
$$\begin{aligned} -\nabla \cdot \left( a(y(t_0), x) \ \nabla u_{j+1}(x) \right) &= \nabla \cdot \left( b_1(x) \ \nabla u_j(x) \right) \quad \forall x \in D, \\ u_{j+1}(x) &= 0 \qquad \quad \forall x \in \partial D. \end{aligned}$$

This construction implies

$$\|u_{j+1}\|_{H_0^1(D)} \le \frac{C_D \|b_1\|_{L^{\infty}(D)}}{a_{min}} \|u_j\|_{H_0^1(D)}, \quad j \ge 1$$

and then

$$||u_F||_{H_0^1(D)} \le \frac{C_D ||f||_{L^2(D)}}{a_{min}} \frac{1}{1-q} < \infty$$

for  $q \equiv \frac{|t-t_0||\Gamma_1|C_D||b_1||_{L^{\infty}(D)}}{2a_{min}} < 1$ . Thus for any  $t_0 \in (-1, 1)$  and  $|t-t_0| < r_{t_0} \equiv \frac{2a_{min}}{|\Gamma_1|C_D||b_1||_{L^{\infty}(D)}}$ , the function  $u_F$  can be represented as a power series in  $t-t_0$ . At the same time, we have the equality  $u_F(t) = \Psi(t)$  for  $t \in (-1, 1)$  since both functions solve the linear elliptic equation

$$\nabla \cdot (a(y(t), x) \nabla u(y(t), x)) = f(x), \quad \forall x \in D,$$
  
$$u(y(t), x) = 0 \qquad \qquad \forall x \in \partial D.$$

which has a unique solution. Then  $u_F$  is the analytic continuation of  $\Psi$  and the proof is complete.

**Remark 6.1.** Consider the natural extension of the variable t to the complex  $\eta$ . Observe that  $\Psi(\eta)$  from Lemma 6.1 solves

(6.6) 
$$\begin{aligned} -\nabla \cdot \left(a(y(\eta), x) \ \nabla \Psi(\eta, x)\right) &= f(x), \quad \forall x \in D, \\ \Psi(\eta, x) &= 0 \qquad \quad \forall x \in \partial D. \end{aligned}$$

Following [27], we use the Legendre polynomials to prove approximation estimates for the  $p \times h$ -version of the SGFEM. Since the Legendre polynomials,

$$p_n(t) \equiv \frac{1}{2^n n!} \frac{d^n}{dt^n} ((t^2 - 1)^n), \ n = 0, 1, \dots$$

are orthogonal with respect to the  $L^{2}(-1,1)$  inner product we have the error representation

(6.7) 
$$(E_1)^2(y_2, \dots, y_N) = \frac{|\Gamma_1|}{2} \sum_{n=p_1+1}^{+\infty} \frac{2}{2n+1} ||d_n||^2_{H^1_0(D)}$$

with the corresponding Fourier coefficients

$$d_n \equiv \frac{2n+1}{2} \int_{-1}^{1} \Psi(t) p_n(t) dt \in H_0^1(D).$$

Therefore, to obtain an estimate for  $E_1$  we shall study the convergence of the tail series in (6.7). Notation 6.1. Consider the natural extension of the variable t to the complex  $\eta$  and introduce the real function,  $\tilde{a} : \mathbb{C} \to \mathbb{R}$ ,

$$\tilde{a}(\eta) \equiv \min_{x \in D} \operatorname{Re} \left\{ a(y_1(\eta), y_2, \dots, y_N, x) \right\}$$
$$= \min_{x \in D} a(\operatorname{Re} \left\{ y_1(\eta) \right\}, y_2, \dots, y_N, x)$$
$$= \min_{x \in D} \left[ a(0, y_2, \dots, y_N, x) + \operatorname{Re} \left\{ y_1(\eta) \right\} b_1(x) \right]$$

with  $\operatorname{Re}\{c\}$  being the real part of  $c \in \mathbb{C}$ . Whenever  $\tilde{a}(\eta) \neq 0$  the continuated function  $\Psi$ , the solution of (6.6), satisfies the bound

(6.8)  
$$\|\Psi(\eta)\|_{H^{1}_{0}(D)} = \|u(y_{1}(\eta), y_{2}, \dots, y_{N}, \cdot)\|_{H^{1}_{0}(D)}$$
$$\leq C_{D} \frac{\|f(y_{1}(\eta), y_{2}, \dots, y_{N}, \cdot)\|_{L^{2}(D)}}{\tilde{a}(\eta)}$$

with  $C_D$  being the Poincaré constant for the domain D. Beside this, define

$$\tilde{a}_1(y_2,\ldots,y_N) \equiv \min_{x \in D} a(0,y_2,\ldots,y_N,x)$$
$$= \min_{x \in D} \left[ E[a](x) + \sum_{i=2}^N b_i(x)y_i \right] \ge a_{min}, \quad \forall (y_2,\ldots,y_N) \in \prod_{2 \le j \le N} \Gamma_j$$

and observe that

(6.9) 
$$\tilde{a}(\eta) \ge \tilde{a}_1 - |y_1(\eta)| \quad ||b_1||_{L^{\infty}(D)}.$$

We are now ready to estimate the Fourier coefficients in (6.7).

**Lemma 6.2.** Let  $\tau \in (0,1)$ . Then there exists positive constants C > 0 and  $\theta_f(\hat{y}_1, \tau) > 0$  such that

$$\|d_n\|_{H^1_0(D)} \le \frac{C_D \quad \theta_f(2n+1)}{\tau \tilde{a}_1 2^n} \int_{-1}^1 \left(\frac{1-t^2}{t+1+C(1-\tau)}\right)^n dt$$

Proof. Consider

$$d_n = \frac{2n+1}{2} \int_{-1}^{1} \Psi(t) p_n(t) dt = \frac{(2n+1)(-1)^n}{n! \ 2^{n+1}} \int_{-1}^{1} \frac{d^n}{dt^n} \Psi(t) (1-t^2)^n dt.$$

Use the analytic continuation of the real function  $\Psi$  to the complex domain as in Lemma 6.1. The application of Cauchy's formula gives

$$\frac{d^n}{dt^n}\Psi(t) = \frac{n!(-1)^n}{2\pi i} \int_{\gamma_t} \frac{\Psi(\eta)}{(\eta-t)^{n+1}} d\eta,$$

where  $\gamma_t$  is a positively oriented closed circumference with center at the real point  $t \in (-1, 1)$ , radius R(t), and such that all singularities from  $\Psi$  are exterior to  $\gamma_t$ . Denote the complex closed ball  $B(t, R(t)) = \{z \in \mathbb{C} : |z - t| \le R(t)\}$ . Estimate (6.8) implies

$$\|\frac{d^{n}}{dt^{n}}\Psi(t)\|_{H^{1}_{0}(D)} \leq \frac{C_{D} \ n!}{2\pi} \int_{\gamma_{t}} \frac{\|f(y_{1}(\eta), \hat{y}_{1}, \cdot)\|_{L^{2}(D)}}{\tilde{a}(\eta)|\eta - t|^{n+1}} |d\eta|$$

$$\leq \frac{C_{D} \ n!}{2\pi} \left( \sup_{\eta \in \gamma_{t}} \|f(y_{1}(\eta), \hat{y}_{1}, \cdot)\|_{L^{2}(D)} \right) \int_{\gamma_{t}} \frac{|d\eta|}{\tilde{a}(\eta)|\eta - t|^{n+1}}$$

$$\leq \frac{C_{D} \ n!}{(R(t))^{n}} \left( \sup_{\eta \in \gamma_{t}} \|f(y_{1}(\eta), \hat{y}_{1}, \cdot)\|_{L^{2}(D)} \right) \sup_{\eta \in \gamma_{t}} \frac{1}{\tilde{a}(\eta)}.$$

Let

$$\theta_f \equiv \sup_{t \in [-1,1]} \sup_{\eta \in \gamma_t} \|f(y_1(\eta), \hat{y}_1, \cdot)\|_{L^2(D)}$$

then estimate (6.10) implies

(6.11) 
$$\|d_n\|_{H^1_0(D)} \le \frac{(2n+1)C_D\theta_f}{2^{n+1}} \int_{-1}^1 \left(\frac{1}{\inf_{\eta \in \gamma_t} \tilde{a}(\eta)}\right) \left(\frac{1-t^2}{R(t)}\right)^n dt$$

Now choose R(t) to control the above integral as follows. Let  $\tau \in (0, 1)$  and consider the interval  $I = \{y \in \mathbb{R} : |y| \le (1 - \tau) \frac{\nu}{\|b_1\|_{L^{\infty}(D)}}\}$ . By construction, we have the strict inclusion  $\Gamma_1 \subset I$  with  $dist(\Gamma_1, I) > \frac{(1 - \tau)\tilde{\nu}}{C_1} > 0$ .

Then, the corresponding preimages satisfy the inclusion  $(y_1)^{-1}(\Gamma_1) = [-1,1] \subset (y_1)^{-1}(I)$  and  $(y_1)^{-1}(I) = (-1 - \delta_1, 1 + \delta_2)$  with  $\delta_1, \delta_2 > 0$ . Choose the radius

(6.12) 
$$R(t) \equiv \min\{t+1+\delta_1, -t+1+\delta_2\} > 0$$

which implies, by virtue of (6.9),

(6.13) 
$$\inf_{\eta \in \gamma_t} \tilde{a}(\eta) \ge \tau \tilde{a}_1.$$

Let

(6.14) 
$$\delta(\tau, y_2, \dots, y_N) \equiv \min\{\delta_1, \delta_2\} \ge \frac{(1-\tau)\tilde{\nu}}{C_1} \frac{2}{y_{min} - y_{max}}$$

and observe that (6.11-6.14) imply

$$\|d_n\|_{H^1_0(D)} \le \frac{(2n+1)C_D\theta_f}{\tau \tilde{a}_1 2^n} \int_{-1}^1 \left(\frac{1-t^2}{t+1+\delta}\right)^n dt$$

which is what we wanted to prove.

Now we use a result from [27], namely that we have **Lemma 6.3** (Integral estimate). Let  $\xi < -1$  and define

$$r \equiv \frac{1}{|\xi| + \sqrt{\xi^2 - 1}}, \ 0 < r < 1$$

Then there holds

$$(-1)^n \int_{-1}^1 \left(\frac{t^2 - 1}{t + \xi}\right)^n dt = (2r)^n 2^{n+1} \frac{n!}{(2n+1)!!} \Phi_{n,0}(r^2)$$

where  $\Phi_{n,0}(r^2)$  is the Gauss hypergeometric function. Moreover, we have

$$\Phi_{n,0}(r^2) = \sqrt{1 - r^2} + \mathcal{O}\left(\frac{1}{n^{1/3}}\right)$$
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uniformly with respect to 0 < r < 1.

**Lemma 6.4.** Let  $\tau \in (0,1)$ . Under Assumption 6.1 there exist positive constants  $C, \theta_f > 0$  such that

$$\|d_n\|_{H^1_0(D)} \le \frac{2C_D\theta_f}{\tau \tilde{a}_1} \sqrt{\frac{\pi n}{2}} \ \left(\sqrt{1-r^2} + \mathcal{O}\left(\frac{1}{n^{1/3}}\right)\right) r^n$$

with

$$r \equiv \frac{1}{|\xi| + \sqrt{\xi^2 - 1}}, \ 0 < r < 1$$

and  $\xi < -1 - C(1 - \tau) < -1$ .

Proof. The result follows from the last two lemmata and the asymptotic equivalence

$$\frac{(2n)!!}{(2n-1)!!} \sim \sqrt{\frac{\pi n}{2}}, \quad n \to \infty.$$

 $\Box$  Finally, we can state the estimate for the size of the series in (6.7).

**Lemma 6.5.** Let  $\tau \in (0,1)$ . Under assumption 6.1 there exist positive constants  $C, \theta_f > 0$  such that

$$(E_1)(y_2,\ldots,y_N) \le \frac{C_D \theta_f \sqrt{|\Gamma_1|}}{\tau \tilde{a}_1} (\sqrt{1-r^2} + \mathcal{O}\left(\frac{1}{(p_1)^{1/3}}\right)) \sqrt{\pi} \quad \frac{r^{p_1+1}}{\sqrt{1-r^2}}$$

*Proof.* Use Lemma 6.4 to estimate the tail of the series

$$(E_1)^2(y_2,\ldots,y_N) = \frac{|\Gamma_1|}{2} \sum_{n=p_1+1}^{+\infty} \frac{2}{2n+1} ||d_n||^2_{H^1_0(D)}.$$

The main result of this section, namely the exponential convergence with respect to the multiindex p as in [27], follows from the above lemmata, i.e.

**Theorem 6.2.** Let  $\tau \in (0,1)$  and u be the solution of (2.11),  $u \in L^2(\Gamma) \otimes H^1_0(D)$ , which is analytic with respect to y, onto the subspace  $Z^p \otimes H^1_0(D)$ . Then there exist positive constants,  $0 < C, C_f$ , such that

(6.15)  

$$E_{p} \equiv \min_{v \in Z^{p} \otimes H_{0}^{1}(D)} \|u - v\|_{L^{2}(\Gamma) \otimes H_{0}^{1}(D)}$$

$$\leq \frac{C_{D}C_{f}}{\tau} \sqrt{\pi} \sqrt{\sum_{i=1}^{N} |\Gamma_{i}| \int_{\hat{\Gamma}_{i}} \left( \left(1 + \frac{1}{\sqrt{1 - r_{i}^{2}}} \mathcal{O}\left(\frac{1}{(p_{i})^{1/3}}\right)\right) \frac{r_{i}^{p_{i}+1}}{\tilde{a}_{i}} \right)^{2} d\hat{y}_{i}}$$
with

with

$$r_i(\hat{y}_i) \equiv \frac{1}{|\xi_i| + \sqrt{\xi_i^2 - 1}}, \ 0 < r_i < 1, i = 1, \dots, N$$

and  $\xi_i(\hat{y}_i) < -1 - C(1 - \tau) < -1$ .

Similarly as the  $k \times h$ -version, cf. (5.10), the *p*-version has a superconvergence result for the approximation of expected value of the solution.

**Theorem 6.3** (Superconvergence of the *p*-version with piecewise linear FEM in space.). There holds/ 3.7

$$\|E[u-u_h^p]\|_{L^2(D)} \le C\left(h^2 + \frac{1}{\tau}\sum_{i=1}^N (\|r_i\|_{L^{\infty}(\hat{\Gamma}_i)})^{2p_i+2}\right)$$

with  $0 < r_i(\hat{y}_i) < 1$  as in Theorem 6.2 and C > 0 is independent from  $u, h, p_i$  and  $r_i$ .

The proof of the previous Theorem uses Theorem 6.2 and and is completely similar to the proof of Theorem 5.2.

# 7. Double orthogonal polynomials

Here we explain the use of double orthogonal polynomials to compute efficiently the solution of the  $k \times h$ -version and the  $p \times h$ -version studied in Sections 5 and 6 respectively. The idea is to use a special basis to decouple the system in the *y*-direction, yielding just a number of undecoupled systems, each one with the size and structure of one Monte Carlo realization of (4.1).

Without any loss of generality we focus on the *p*-version, i.e. find  $u_h^p \in Z^p \otimes X_h^d$  such that

(7.1) 
$$\int_{\Gamma} \rho(y) \int_{D} a(y,x) \nabla u_{h}^{p}(y,x) \cdot \nabla v(y,x) dx dy = \int_{\Gamma} \rho(y) \int_{D} f(y,x) v(y,x) dx dy \quad \forall v \in Z^{p} \otimes X_{h}^{d}.$$

Let  $\{\psi_j(y)\}\$  be a basis of the subspace  $Z^p \subset L^2(\Gamma)$  and  $\{\varphi_i(x)\}\$  be a basis of the subspace  $X^d_h \subset H^1_0(D)$ . Write the approximate solution as

(7.2) 
$$u_h^p(y,x) = \sum_{j,i} u_{ij} \psi_j(y) \varphi_i(x)$$

and use test functions  $v(y, x) = \psi_k(y)\varphi_\ell(x)$  to find the coefficients  $u_{ij}$ . Then (7.1) gives

$$\sum_{j,i} \left( \int_{\Gamma} \rho(y) \psi_k(y) \psi_j(y) \int_D a(y,x) \nabla \varphi_i(x) \cdot \nabla \varphi_\ell(x) dx dy \right) u_{ij} = \int_{\Gamma} \rho(y) \psi_k(y) \int_D f(y,x) \varphi_\ell(x) dx dy, \forall k, \ell \in \mathbb{C}$$

which can be rewritten as  $\sum_{j,i} \left( \int_{\Gamma} \rho(y) \psi_k(y) \psi_j(y) K_{i,\ell}(y) dy \right) u_{ij} = \int_{\Gamma} \rho(y) \psi_k(y) f_\ell(y) dy \quad \forall k, \ell \text{ with } K_{i,\ell}(y) \equiv \int_D a(y,x) \nabla \varphi_i(x) \cdot \nabla \varphi_\ell(x) dx \text{ and } f_\ell(y) \equiv \int_D f(y,x) \varphi_\ell(x) dx.$  Now, if we have that the diffusion coefficient, a, is a truncated Karhunen-Loève expansion,  $a(y,x) = E[a](x) + \sum_{n=1}^N b_n(x) y_n$ , and by the independence of the Y components its joint probability density decouples into the product  $\rho(y) = \prod_{m=1}^N \rho_m(y_m)$  then we have a corresponding "Karhunen-Loève" like expression for the stiffness matrix

$$K_{i,\ell}(y) \equiv \int_D (E[a](x) + \sum_{n=1}^N b_n(x)y_n) \nabla \varphi_i(x) \cdot \nabla \varphi_\ell(x) dx = K_{i,\ell}^0 + \sum_{n=1}^N y_n K_{i,\ell}^n$$

with deterministic coefficients

$$K_{i,\ell}^0 \equiv \int_D E[a](x) \nabla \varphi_i(x) \cdot \nabla \varphi_\ell(x) dx$$

and

$$K_{i,\ell}^n \equiv \int_D b_n(x) \nabla \varphi_i(x) \cdot \nabla \varphi_\ell(x) dx.$$

By the same token we have,

$$\int_{\Gamma} \rho(y)\psi_k(y)\psi_j(y)K_{i,\ell}(y)dy = K_{i,\ell}^0 \int_{\Gamma} \rho(y)\psi_k(y)\psi_j(y)dy + \sum_{n=1}^N K_{i,\ell}^n \int_{\Gamma} y_n \rho(y)\psi_k(y)\psi_j(y)dy$$

Since  $\psi_k \in Z^p$ , with multiindex  $p = (p_1, \ldots, p_N)$ , it is enough to take it as the product

$$\psi_k(y) = \prod_{\substack{r=1\\24}}^N \psi_{kr}(y_r)$$

where  $\psi_{kr}: \Gamma_r \to \mathbb{R}$  is a basis function of the subspace

$$Z^{p_r} = \text{span}[1, y, \dots, y^{p_r}] = \text{span}[\psi_{hr} : h = 1, \dots, p_r + 1]$$

Keeping this choice of  $\psi_k$  in mind,

$$\begin{split} \int_{\Gamma} \rho(y)\psi_{k}(y)\psi_{j}(y)K_{i,\ell}(y)dy = & K_{i,\ell}^{0} \int_{\Gamma} \prod_{m=1}^{N} \rho_{m}(y_{m})\psi_{km}(y_{m})\psi_{jm}(y_{m})dy \\ &+ \sum_{n=1}^{N} K_{i,\ell}^{n} \int_{\Gamma} y_{n} \prod_{m=1}^{N} \rho_{m}(y_{m})\psi_{km}(y_{m})\psi_{jm}(y_{m})dy \end{split}$$

Now, for every set  $\Gamma_n$ , n = 1, ..., N choose the polynomials,  $\psi_j(y) = \prod_{n=1}^N \psi_{jn}(y_n)$ , to be biorthogonal, i.e. for n = 1, ..., N they must satisfy

(7.3) 
$$\int_{\Gamma_n} \rho_n(z)\psi_{kn}(z)\psi_{jn}(z)dz = \delta_{kj}$$
$$\int_{\Gamma_n} z\rho_n(z)\psi_{kn}(z)\psi_{jn}(z)dz = c_{kn}\delta_{kj}.$$

To find the polynomials  $\psi$  we have to solve N eigenproblems, each of them with size  $(1 + p_n)$ . The computational work required by these eigenproblems is negligible with respect to the one required to solve for  $u_{ij}$ , cf. [26], section 8.7.2. The orthogonality properties (7.3) for  $\psi$  imply the decoupling  $\int_{\Gamma} \rho \psi_k \psi_j dy = \delta_{kj}$ ,  $\int_{\Gamma} y_n \rho \psi_k \psi_j dy = c_{kn} \delta_{kj}$ . By means of this decoupling we now conclude that  $\sum_{j,i} \left( \int_{\Gamma} \rho(y) \psi_k(y) \psi_j(y) K_{i,\ell}(y) dy \right) = K_{i,\ell}^0 \int_{\Gamma} \rho(y) \psi_k(y) \psi_j(y) dy + \sum_{n=1}^N K_{i,\ell}^n \int_{\Gamma} y_n \rho(y) \psi_k(y) \psi_j(y) dy = \left( K_{i,\ell}^0 + \sum_{n=1}^N c_{kn} K_{i,\ell}^n \right) \delta_{kj}.$ 

The structure of the linear system that determines  $u_{ij}$  now becomes block diagonal, which each block being coercive and with the sparsity structure identical to one deterministic FEM stiffness matrix, i.e.

$$\begin{bmatrix} \left(K^{0} + \sum_{n=1}^{N} c_{1n} \ K^{n}\right) & 0 & \dots & 0 \\ 0 & \left(K^{0} + \sum_{n=1}^{N} c_{2n} \ K^{n}\right) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & 0 & \left(K^{0} + \sum_{n=1}^{N} c_{Nn} \ K^{n}\right) \end{bmatrix}$$

The conclusion is that the work to find the coefficients  $u_{ij}$  in (7.2) is the same as the one needed to compute  $\prod_{i=1}^{N} (1+p_i)$  Monte Carlo realizations of  $u_h$  defined in (4.1). Beside this, observe that as a consequence of the uniform coercivity assumption, each of the diagonal blocks in the system above is symmetric and strictly positive definite.

#### 8. Asymptotical efficiency comparisons

In this section we compare the asymptotical numerical complexity for the Monte Carlo Galerkin finite element method, cf. Section 4, with the Stochastic Galerkin finite element method introduced in Sections 5 and 6. The quantity of interest, i.e. the goal of the computation, is the expected value of the solution, E[u], and its approximation is studied in both  $L^2(D)$  and  $H_0^1(D)$  sense. The Monte Carlo Galerkin finite element method controls the computational error in probability sense, yielding a computational work that grows only like a polynomial with respect to the number of random variables decribing the problem. On the other hand, the Stochastic Galerkin finite element method yields deterministic bounds for the error. Thanks to the analytic dependence of E[u] with respect to the random variables  $Y_i$ , cf. Section 6, the p-version of the Stochastic Galerkin finite element method has a computational complexity that can compete with the Monte Carlo approach. A posteriori error analysis and adaptivity issues will not be treated here but will be the subject of forthcoming papers. In all the cases, the spatial discretization is done by piecewise linear finite elements on globally quasiuniform meshes. For the  $k \times h$ -SGFEM the  $\Gamma$  partitions are also assumed to be globally quasiuniform. Beside this, the diffusion function a is assumed to be a truncated Karhunen-Loève expansion.

8.1. MCGFEM versus  $k \times h$ -SGFEM. Here we consider the computational work to achieve a given accuracy bounded by a positive constant TOL, for both the MCGFEM and the  $k \times h$ -SGFEM methods. This optimal computational work indicates under which circumstances one method may be best suited. When using the MCGFEM method to approximate the solution of (1.1) in energy norm the error becomes, applying Proposition 4.2 together with (4.5) that given a confidence level,  $0 < c_0 < 1$ , there exists a constant C > 0 depending only on  $c_0$  such that

(8.1) 
$$P\left(\|E[u] - \frac{1}{M}\sum_{j=1}^{M} u_h(\cdot;\omega_j)\|_{H^1_0(D)} \le C(h + \frac{1}{\sqrt{M}})\right) \ge c_0$$

Then, in the sense of (8.1) we write  $E_{MCGFEM} = O(h) + O(1/\sqrt{M})$ . The corresponding computational work for the MCGFEM method is  $Work_{MCGFEM} = O((1/h^d)^r + 1/h^d)M$ , where the parameter  $1 \le r \le 3$  relates to the computational effort devoted to solve one linear system with with *n* unknowns,  $O(n^r)$ . From now on we continue the discussion with the optimal r = 1 that can be achieved by means of the Multigrid method, cf. [12, 13, 29]. Thus, choosing *h* and *M* to minimize the computational work for a given desired level of accuracy TOL > 0 yields the optimal work

(8.2) 
$$Work_{MCGFEM}^* = O\left(\mathrm{TOL}^{-(2+d)}\right).$$

On the other hand, if we apply a  $k \times h$ -SGFEM with piecewise polynomials of order q in the y-direction the computational error in  $H_0^1(D)$  norm is, cf. Corollary 5.1,

$$E_{SGFEM} = O(h) + NO(k^{q+1})$$

and the corresponding computational work for the  $k \times h$ -version is

$$Work_{SGFEM} = O(h^{-d}(1+q)^N k^{-N}).$$

Here N is the number of terms in the truncated Karhunen-Loève expansion of the coefficients a and f and k is the discretization parameter in the y direction. Similarly as before, we can compute the optimal work for the  $k \times h$ -SGFEM method, yielding

$$Work_{SGFEM}^* = O((1+q)^N \left(\frac{\text{TOL}}{N}\right)^{-\frac{N}{q+1}} \text{TOL}^{-d}).$$
<sup>26</sup>

Therefore, a  $k \times h$ -version SGFEM is likely to be preferred whenever TOL is sufficiently small and N/2 < 1 + q, i.e. if the number of terms in the Karhunen-Loève expansion of a is large then the degree of approximation in the y direction, q, has to become correspondingly large. We summarize the comparison results in Table 1, where we also include corresponding results from the  $p \times h$ -version, to be derived in Subsection 8.2.

	MCGFEM	$k \times h$ -version SGFEM	$p \times h$ -version SGFEM
Work	$M/h^d$	$\frac{(1+q)^N}{h^d k^N}$	$\frac{(1+p)^N}{h^d}$
$H_0^1(D)$ Error	$h + \frac{1}{\sqrt{M}}$	$h + k^{q+1}$	$h + r^{(p+1)}$
$H_0^1(D) Work^*$	$\mathrm{TOL}^{-(2+d)}$	$\mathrm{TOL}^{-\frac{N}{q+1}}\mathrm{TOL}^{-d}$	$(\log_r(\mathrm{TOL}))^N \mathrm{TOL}^{-d}$

TABLE 1. Approximation of the function E[u] in  $H_0^1(D)$ . Asymptotical numerical complexity for the MCGFEM and SGFEM methods.

Similarly, if we are interested in controlling the difference  $||E[u] - E[u_h]||_{L^2(D)}$  the application of (4.2) and Properties 4.1 and 4.2 for the MCGFEM method and Theorem 5.2 on the superconvergence of the  $k \times h$ -SGFEM method imply the results shown in Table 2. In this case  $k \times h$ -SGFEM is likely to be preferred whenever N/4 < (q + 1) and TOL is sufficiently small. In addition, the

	MCGFEM	$k \times h$ -version SGFEM	$p \times h$ -version SGFEM
Work	$M/h^d$	$\frac{(1+q)^N}{h^d k^N}$	$\frac{(1+p)^N}{h^d}$
$L^2(D)$ Error	$h^2 + \frac{1}{\sqrt{M}}$	$h^2 + k^{2(q+1)}$	$h^2 + r^{2(p+1)}$
$L^2(D) Work^*$	$\mathrm{TOL}^{-(2+d/2)}$	$(\text{TOL})^{-\frac{N}{2(q+1)}} \text{TOL}^{-d/2}$	$(\log_r(\mathrm{TOL}))^N \mathrm{TOL}^{-d/2}$

TABLE 2. Approximation of the function E[u] in  $L^2(D)$ . Asymptotical numerical complexity for the MCGFEM and SGFEM methods.

comparison tells us that to be able to be competitive with the Monte Carlo method when the number of relevant terms in the Karhunen-Loève expansion is not so small, an optimal method should have a high order of approximation and should avoid as much as possible the coupling between the different components of the numerical solution to preserve computational efficiency. The approach proposed by Ghanem and Spanos [25] based on orthogonal polynomials has, whenever the approximate diffusion satisfies (1.2), a high order of approximation but introduces coupling between the different components of the numerical solution. The uncoupling can be achieved for linear equations using double orthogonal polynomials, see the description in Section 7. With this motivation, Section 6 studies the convergence of the  $p \times h$ -SGFEM.

8.2. MCGFEM versus  $p \times h$ -SGFEM. Here we consider the computational work to achieve a given accuracy, for both the  $p \times h$ - version of SGFEM defined in (6.1) and the MCGFEM method for the approximation of E[u] defined in Section 4, i.e. we are interested in controlling the difference  $||E[u] - E[u_h^p]||_{L^2(D)}$  or  $||E[u] - \frac{1}{M} \sum_{j=1}^M u_h(\cdot; \omega_j)||_{L^2(D)}$ , respectively. This computational work indicates under which circumstances one method may be better suited than the other. Besides this, let us assume that we use in our computations a piecewise linear FEM space in D. When using the MCGFEM method to approximate the expected value of the solution of (1.1), we have the optimal work required to achieve a given desired level of accuracy TOL > 0, cf. (8.2),

$$Work^*_{MCGFEM} = O(1/\text{TOL}^{2+d/2}).$$
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On the other hand, if we apply a  $p \times h$ -version of the SGFEM, with  $p_i = p, i = 1, ..., N$ , the computational error is, cf. Theorem 6.3,

$$E_{SGFEM} = O(h^2) + O(r^{2(p+1)}), \ 0 < r < 1,$$

and the corresponding computational work is, cf. Section 7,

$$Work_{SGFEM} = O\left(\frac{(1+p)^N}{h^d}\right).$$

Recall that N is the number of terms in the truncated Karhunen-Loève expansion of the coefficients a and f and k is the discretization parameter in the y direction. As before, we can compute the optimal work for the SGFEM method, yielding

$$Work_{SGFEM}^* \le O\left( (\log_r(\text{TOL}))^N \text{TOL}^{-d/2} \right)$$

and the asymptotical comparison

$$\lim_{\text{TOL}\to 0} \frac{Work^*_{SGFEM}}{Work^*_{MCGFEM}} = \lim_{\text{TOL}\to 0} (\log_r(\text{TOL}))^N \text{TOL}^2 = 0.$$

Therefore, for sufficiently strict accuracy requirements, i.e. sufficiently small TOL, in the computation of E[u], SGFEM requires less computational effort than MCGFEM. The work of Bahvalov and its subsequent extensions, cf. [9, 28, 55, 41], generalizes the standard Monte Carlo method, taking advantage of the available integrand's smoothness and yielding a faster order of convergence. The optimal work of such a method is for our case, i.e. the approximation of E[u] in  $L^2(D)$ ,  $\mathcal{O}(C(N)\mathrm{TOL}^{-\frac{1}{1/2+q/N}}\mathrm{TOL}^{-d/2})$ , where it is assumed that the integrand u has bounded derivatives up to order q with respect to y and the integral is done in the N-dimensional unit cube.

The result on the computational work of the  $p \times h$ -version of the SGFEM presented in this work is then related to the case  $q = \infty$ , since u is analytic with respect to y. This analyticity allows the exponential convergence with respect to p, cf. Theorem 6.3.

Notice that we only discussed the optimal asymptotical computational work required by both methods, but in practice the constants involved in the asymptotic approximations makes these comparisons just indicative, and not conclusive. In addition, we have only studied the case where the integrals  $\int_{\Gamma_i} \rho_i y^k dy$  can be computed exactly for  $k = 0, 1, \ldots$ , and not considered the more general case where quadrature rules are needed to approximate such integrals.

**Remark 8.1** (Use of higher order FEM approximations on *D*). Based on Remark 5.1 and following the approach from Sections 8.1 and 8.2 we can discuss the possible use of higher order FEM approximations on *D*. This use seems a priori always useful for the  $p \times h$ -version of the SGFEM while for the  $k \times h$ -version higher order FEM on *D* are attractive provided d >> N/(q+1). On the other hand a MCGFEM piecewise linear approximation on *D* with error TOL requires the same work as the optimal higher order method with error slightly larger than TOL<sup>1+d/4</sup>. See [53], p. 1884, for a similar discussion on the weak approximation of ordinary SDEs.

**Remark 8.2** (Combination of MCGFEM with SGFEM). It is possible to combine in a natural way the Monte Carlo method with an SGFEM version, partitioning  $\Gamma$  in order to take advantage of their different convergence rates. Split the domain  $\Gamma$  into  $\Gamma = \Gamma_{MC} \cup \Gamma_G$ ,  $\Gamma_{MC} \cap \Gamma_G = \phi$  and approximate

(8.3) 
$$E[u(Y,\cdot)] = E[u(Y,\cdot)|Y \in \Gamma_{MC}]P(Y \in \Gamma_{MC}) + E[u(Y,\cdot)|Y \in \Gamma_{G}]P(Y \in \Gamma_{G}).$$

using SGFEM in  $\Gamma_G$  and MCGFEM in  $\Gamma_{MC}$ .

**Remark 8.3** (Successive approximation method). The work [5] proposes a successive approximation for the solution of (1.1). As any Neuman expansion with K terms, whenever it converges it has an error of the form  $Error_{BC} \leq C(h + \frac{\xi^{K+1}}{\xi^{K+1}})$  for some  $0 < \xi < 1$ . The computational work to achieve this error is  $Work_{BC} \leq C \frac{N^{K+1}}{(N-1)h^d}$ . Then, a comparison between this method and the MCGFEM yields that if  $\xi < 1/\sqrt{N}$  then the method from [5] is likely to be preferred. On the other hand, for sufficiently small tolerances the  $p \times h$ -version of the SGFEM is more efficient than the successive approximation.

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