MATHEMATICAL PERSPECTIVES ON LARGE EDDY SIMULATION MODELS FOR TURBULENT FLOWS

J.-L. GUERMOND[1,2], J.T. ODEN[2], AND S. PRUDHOMME[2]

Abstract. The main objective of this paper is to review and report on key mathematical issues related to the theory of Large Eddy Simulation of turbulent flows. The principal achievement is to show that most LES models have mathematical counterparts that have been introduced to resolve questions such as uniqueness, existence of a maximum principle, convergence to entropy solutions, and convergence in graph norms. As a conclusion to this review, we suggest mathematical criteria for selecting LES models.

Contents

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1. Introduction

1.1. **Introductory comments.** A common experience in everyday life is to observe the swirling motion of fluids flowing past objects or through conduits as so-called eddies are created and spun off to produce complicated flow patterns. Such phenomena are recognized as examples of turbulent flow, and an understanding of turbulence, its quantification, prediction, simulation, and control have become one of the most elusive and important goals in science and engineering.

It is now generally accepted that the Navier–Stokes equations, modeling the behavior of incompressible viscous fluids, describe accurately what is observed as turbulence. Hence, considering that enormous computing power is available, one may be tempted to think that simulating numerically the Navier–Stokes equations should resolve the turbulence question. Unfortunately, despite steady advances in computing power, attempts at the Direct Numerical Simulation (DNS) of the Navier–Stokes equations have been limited to rather low values of the Reynolds number, Re. At the present times, simulating time-dependent flows at Reynolds numbers greater than a few thousands is a daunting task. The reasons for this very limited success of DNS is rooted in the heuristic Kolmogorov estimate $\mathcal{O}(R_e^{9/4})$ for the total number of degrees of freedom required to simulate flows at a given value of R_e. Considering the current pace of progress in computing power, this estimate undercuts the prospect of DNS of large-Reynolds number flows to some date possibly far in the future. Since the times of Reynolds and Boussinesq and others, approximations of the Navier–Stokes equations through the use of so-called turbulence models based on time-averaged or space-averaged quantities (Reynolds Averaged Navier–Stokes models, k-ϵ models, *etc.*) have been used in engineering applications as a means of overcoming, though often crudely, the formidable and, to date, virtually impossible task of DNS. The situation is further complicated by the absence of a complete mathematical theory of turbulence as described by either the Navier–Stokes equations directly or any of the various turbulence models, with the result

that contemporary methods for studying such phenomena are often based on heuristics, empiricism, and unjustifiable assumptions.

In mathematical terms, the turbulence question is an elusive one. Since the bold definition of turbulence by Leray in the 1930's [39], calling turbulent solution any weak solution of the Navier–Stokes equations, progress has been frustratingly slow. The major obstacle in analyzing the Navier–Stokes equations has to do with the question of uniqueness of solutions, a question not yet solved owing to the possibility that the occurrence of so-called vorticity bursts reaching scales smaller than the Kolmogorov scale cannot be excluded.

In recent years, significant progress toward the development of useful turbulent models has occurred based on the observation that the whole range of flow scales may not be important in many significant engineering applications. In such applications, global information on "large-scale" features of the flow is sought for such physical quantities as momentum or heat transfer. Hence, the notion that global behavior could be well approximated by a turbulence model without having to approximate accurately fine scales is viewed by many as a possible breakthrough in turbulence modeling. This has led modelers to devise artifacts for representing the interaction between the unreachable small scales and the large ones. These models are commonly known as Large Eddy Simulation (LES) models. Many LES models have been proposed, but no satisfactory mathematical theory for LES has yet been found.

Our goal in the present paper is to report important mathematical results on LES modeling and to identify and elucidate several mathematical issues that stand in the way of developing a complete theory. A significant aspect of LES theory, and one we shall address in this exposition, is the close relationship between mathematical properties of LES models and the numerical methods used to implement them in specific applications. This relationship was pointed by Ferziger [13]: "In general, there is a close connection between the numerical methods and the modeling approach used in simulation; this connection has not been sufficiently appreciated by many authors." In response to Ferziger's observation, we attempt to show that most of the LES models are in fact regularization techniques which, either at the continuous or at the discrete level, are mathematical (numerical) tools for solving problems such as uniqueness of solutions, existence of a maximum principle, convergence to entropy solutions, or convergence in graph norms.

The paper is organized as follows. In Section 2, we recall the energy cascade phenomenon and the Kolmogorov scaling theory. We also quote some of the latest outstanding mathematical results confirming rigorously this phenomenon. Then, in Section 3, we review the filtering issue which for many authors seems to be a paradigm for LES. We show that this question yields a paradox that does not yet seem to have been recognized in the literature. We demonstrate also that, if done correctly, filtering falls in the class of regularization techniques that solve the uniqueness question. In Section 4, we

study the Smagorinsky model. We show that this model belongs to the class of the p-Laplacian regularizations and solves also the uniqueness question. In Section 5, we analyze models based on spectral viscosity. We show that, when properly designed, these models are numerical techniques that guarantee convergence to the entropy solution when applied to scalar nonlinear conservation laws. In Section 6 we review models based on scale-similarity and two-level subgrid viscosity. We show that these models are numerical methods that have stabilizing properties guaranteeing optimal convergence in some relevant graph norm when approximating non-coercive equations.

1.2. **Notations and preliminaries.** In this section we introduce notations that will be used throughout the paper. We also recall the definitions of standard functional spaces for the reader's convenience.

Unless explicitly stated, the fluid domain Ω is assumed to be an open bounded subset in $\mathbb{R}^3$ with a Lipschitz regular boundary Γ.

Definition 1.1 (3D-torus)**.** *When $\Omega = (0, 2\pi)^3$ and periodic boundary conditions are enforced in the three space directions, the domain Ω is referred to as the 3D-torus.*

Real and complex-valued vectors/tensors are denoted in bold face. For any real and complex-valued vectors/tensors we denote by $|\cdot|$ the Hermitian norm, *i.e.* the ℓ^2-norm, and for any multi-index $\mathbf{k} \in \mathbb{Z}^d$ we set $|\mathbf{k}|_\infty = \max_{1\le i\le d} |k_i|$, *i.e.* the ℓ^∞-norm.

For $1 \le p \le +\infty$, we denote by $L^p(\Omega)$ the complex vector space of Lebesgue measurable functions such that

1) $\int_\Omega |f(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} < +\infty$, if $1 \le p < \infty$
2) $|f(\mathbf{x})| < +\infty$ for a.e. $\mathbf{x} \in \Omega$, if $p = \infty$.

Partial derivatives of a function v with respect to variable ξ are denoted by $\partial_\xi v$; in the case in which v depends only on ξ, we will write $d_\xi v$. As usual, $W^{m,p}(\Omega)$ is the Sobolev space composed of functions that are in $L^p(\Omega)$ and whose partial derivatives up to order m are in $L^p(\Omega)$. We shall denote by $\|\cdot\|_{m,p}$ the norm of $W^{m,p}(\Omega)$, making no distinction between the norms of scalar-valued and vector-valued function. When no confusion is possible we also denote by $\|\cdot\|_0$ the norm of $L^2(\Omega)$.

1.3. **Navier–Stokes equations.** Throughout this paper, we consider the Navier–Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u}|_\Gamma = 0 \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{1.1}$$

where $\mathbf{u}_0$ is the initial data and $\mathbf{f}$ a source term. Note that the density ρ is chosen equal to unity. The choice of no-slip or periodic boundary conditions simplifies somewhat the mathematical analysis without affecting the

important features inherent to the Navier–Stokes equations. We introduce two spaces of solenoidal vector fields:

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega),\ \nabla\cdot \mathbf{v} = 0,\ \mathbf{v}_{|\Gamma} = 0,\ \text{or } \mathbf{v} \text{ is periodic}\},$$
$$\mathbf{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega),\ \nabla\cdot \mathbf{v} = 0,\ \mathbf{v}\cdot\mathbf{n}_{|\Gamma} = 0,\ \text{or } \mathbf{v} \text{ is periodic}\}.$$

The initial velocity $\mathbf{u}_0$ is assumed to be in $\mathbf{H}$. We denote by $\mathbf{P_H}$ the $\mathbf{L}^2$-orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$.

2. Energy cascade and Kolmogorov's scaling theory

In this section, we review Kolmogorov's scaling theory since it is very often referred to in LES, and to provide a coherent introduction to this theory we also recall the vortex stretching mechanism. In addition, we quote recently proven mathematical results which corroborate the energy cascade phenomenon in turbulent flows and discuss issues related to the existence of attractors for the Navier–Stokes dynamical system. For additional details on the material presented here we refer to Doering, Gibbon [11] and Foias, Manley, Rosa, Temam [18].

This section is meant to convince non-mathematician readers that though the heuristic theory may give the feeling that everything is well understood, the mathematical theory is incomplete due to the uniqueness question still being open. Though this question is often overlooked (even ignored) in practice, we want to show that it has important practical consequences.

2.1. **The vortex stretching mechanism.**

We first rewrite the momentum equation in terms of the vorticity, $\boldsymbol{\omega}$, defined as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

By taking the curl of the momentum equation, we obtain

$$\partial_t \boldsymbol{\omega} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} = \nu\nabla^2\boldsymbol{\omega} + \mathbf{D}\cdot\boldsymbol{\omega} + \nabla\times\mathbf{f}, \tag{2.1}$$

where $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ is the deformation tensor. The evolution equation for $\boldsymbol{\omega}$ resembles in many aspects the momentum equation governing the velocity field $\mathbf{u}$, except for one term, the so-called *vortex stretching term* $\mathbf{D}\cdot\boldsymbol{\omega}$. This term explains some fundamental differences between flows in two or three dimensions.

In two space dimensions, for instance in the (x, y) plane, the velocity $\mathbf{u}$ has components $(u, v, 0)$ which implies that the vorticity has a nonzero component only in the z-direction, *i.e.*, $\boldsymbol{\omega} = (0, 0, \omega)$. It immediately follows that the vortex stretching term $\mathbf{D}\cdot\boldsymbol{\omega}$ vanishes identically and does not contribute to the evolution of the vorticity field.

In three space dimensions, the term $\mathbf{D}\cdot\boldsymbol{\omega}$ does not necessarily vanish and may give rise to a potentially strong local phenomenon referred to as the *vortex stretching mechanism* (a mechanism held responsible for intensifying the circulation in tornadoes). Recall that $\mathbf{D}$ is symmetric, hence it is diagonalizable; furthermore, the trace of $\mathbf{D}$ being zero ($\mathrm{tr}(\mathbf{D}) = \nabla\cdot\mathbf{u} = 0$), $\mathbf{D}$ has

at least one non-negative eigenvalue. If it happens that $\boldsymbol{\omega}$ is aligned with one eigenvector associated with a positive eigenvalue of $\mathbf{D}$, then the vorticity magnitude and the angular velocity increase as long as the diffusive term $\nu\nabla^2\boldsymbol{\omega}$ and the source term $\nabla\times\mathbf{f}$ are not strong enough to counterbalance this mechanism. From a physical point of view, this implies that a fluid element would first contract in the direction perpendicular to the vorticity vector, and then stretch along this direction in order for the angular momentum to be conserved (if dissipation is neglected). The vortex stretching mechanism is held responsible for local amplification of the vorticity magnitude and thus for the production of smaller and smaller scale structures in the flow field. This phenomenon thus implies a transfer of energy from large length scales to smaller ones, usually known as the energy cascade.

2.2. **Fourier analysis and energy cascade.** The study of homogeneous isotropic turbulence is usually performed on the periodic three-dimensional domain $\Omega=(0,L)^3$ using spectral analysis. Instead of considering the 3D torus $(0,2\pi)^3$, we prefer to keep the dimension L as we will subsequently use dimensional analysis. Since $\mathbf{u}(\mathbf{x},t)$ is square integrable in space, the velocity field can be expanded in terms of the Fourier series

$$\mathbf{u}(\mathbf{x},t)=\sum_{\boldsymbol{\ell}}\hat{\mathbf{u}}(\boldsymbol{\ell},t)\,e^{i\boldsymbol{\ell}\cdot\mathbf{x}},$$

where the wave numbers $\boldsymbol{\ell}$ are given by $\boldsymbol{\ell}=2\pi\boldsymbol{n}/L$, $\boldsymbol{n}\in\mathbb{Z}^3$, and where the Fourier coefficients, or modes, satisfy:

$$\hat{\mathbf{u}}(\boldsymbol{\ell},t)=\frac{1}{L^3}\int_\Omega\mathbf{u}(\mathbf{x},t)\,e^{-i\boldsymbol{\ell}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}.$$

The main advantage of such a decomposition is that it permits to distinguish the various length scales in the velocity field, the length scale associated with the wavenumber $\boldsymbol{\ell}$ being defined, for example, as $2\pi/|\boldsymbol{\ell}|_\infty$.

Taking the Fourier Transform of the Navier–Stokes equations, we can derive the time evolution of each mode $\hat{\mathbf{u}}(\boldsymbol{\ell},t)$:

$$\left\{\begin{array}{l} d_t\hat{\mathbf{u}}(\boldsymbol{\ell},t)=\hat{\mathbf{f}}(\boldsymbol{\ell})-\nu|\boldsymbol{\ell}|^2\hat{\mathbf{u}}(\boldsymbol{\ell},t)\\ \qquad\qquad\qquad\qquad+\dfrac{i}{L^3}(\mathbf{I}-\boldsymbol{\ell}\boldsymbol{\ell}^T/|\boldsymbol{\ell}|^2)\cdot\displaystyle\sum_{\boldsymbol{\ell}_1+\boldsymbol{\ell}_2=\boldsymbol{\ell}}[\hat{\mathbf{u}}(\boldsymbol{\ell}_1,t)\cdot\boldsymbol{\ell}_2\hat{\mathbf{u}}(\boldsymbol{\ell}_2,t)]\\ \boldsymbol{\ell}\cdot\hat{\mathbf{u}}(\boldsymbol{\ell},t)=0\end{array}\right. \tag{2.2}$$

where $\mathbf{I}$ is the unit tensor, $\mathbf{I}-\boldsymbol{\ell}\boldsymbol{\ell}^T/|\boldsymbol{\ell}|^2$ the projector onto divergence free vector fields in wavenumber space, and $\hat{\mathbf{f}}(\boldsymbol{\ell})$ the Fourier modes of the body force, assuming $\mathbf{f}$ to be independent of time. The body force term $\hat{\mathbf{f}}(\boldsymbol{\ell})$ supplies the energy to the system at wavenumber $\boldsymbol{\ell}$ in the support of $\hat{\mathbf{f}}$. We suppose here that it is only significant for long wavelengths and provides for sufficient power to maintain the system within a permanent (steady) turbulent regime. This term does not directly excite the short scales present in the flow. The second term in (2.2) is the so-called *viscous dissipation term*

$\nu|\boldsymbol{\ell}|^2\hat{\mathbf{u}}(\boldsymbol{\ell},t)$. Because of the factor $|\boldsymbol{\ell}|^2$, viscous dissipation is more effective at short rather than at large length scales. The last term is the result of the Fourier transform of the nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$. This term actually allows for mode coupling in the wavenumber space, a mechanism which provides for the activation of shorter and shorter length scales in the flow. Equation (2.2) clearly shows that any triad $(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\boldsymbol{\ell}_3)$ are coupled if and only if one of these wavenumbers is the sum of the other two. In other words, the nonlinear term allows for the transfer of energy from the large scales, excited by the body force, to the smallest scales, for which viscous dissipation becomes predominant. This mechanism is usually referred to as the energy cascade.

The energy cascade was perhaps intuitively imagined by Leonardo da Vinci as early as the sixteen century when he wrote: "... the small eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by both small eddies and large." An alternative description of the energy cascade is given by Lesieur [40] in the following terms: the flow reaches an equilibrium state where the vortex stretching mechanism produces an "infinite hierarchy of eddies; each of them sucking the energy of the bigger ones on which they ride, while they are being sucked by the smaller eddies riding on them" the feast stopping at the viscous dissipation scale.

Note that the nonlinear term does not participate to the global kinetic energy balance since

$$\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{u}\cdot\nabla(\tfrac{1}{2}\mathbf{u}^2)\,\mathrm{d}\mathbf{x} = \int_\Omega \nabla\cdot(\tfrac{1}{2}\mathbf{u}^2\mathbf{u})\,\mathrm{d}\mathbf{x} = 0. \tag{2.3}$$

Once again, it is clear that the role of the nonlinear term is to redistribute the energy from the large scales to the small ones.

We will refer to this energy cascade mechanism again in the next section to present Kolmogorov's scaling theory. Meanwhile, we demonstrate how to decompose the kinetic energy of the flow field into contributions from each wavenumber or length scale. Indeed, the instantaneous mean value of the kinetic energy, after making use of Parseval's equality, is given by

$$\frac{1}{L^3}\int_\Omega \frac{1}{2}\mathbf{u}^2(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \sum_{\boldsymbol{\ell}} \frac{1}{2}\,|\hat{\mathbf{u}}(\boldsymbol{\ell},t)|^2 = \sum_k \left\{ \sum_{|\boldsymbol{\ell}|_\infty = k} \frac{1}{2}\,|\hat{\mathbf{u}}(\boldsymbol{\ell},t)|^2 \right\}$$

where $k = 2\pi n/L$, $n\in\mathbb{N}$. A possible decomposition of the kinetic energy is therefore suggested as

$$\frac{1}{2L^3}\|\mathbf{u}(\cdot,t)\|_0^2 = \frac{2\pi}{L}\sum_k E(k,t)$$

where the quantity

$$E(k,t) := \frac{L}{2\pi}\sum_{|\boldsymbol{\ell}|_\infty = k} \frac{1}{2}\,|\hat{\mathbf{u}}(\boldsymbol{\ell},t)|^2 \tag{2.4}$$

defines the kinetic energy associated with wavenumbers $\boldsymbol{\ell}$, $|\boldsymbol{\ell}|_\infty = k$, or equivalently, associated with the length scale $2\pi/k$.

2.3. **Kolmogorov's scaling theory.** When such a flow reaches a permanent steady-state regime, the energy cascade mechanism suggests the following hypothesis:

(HK) *The rate of energy cascading from the largest scales to the finest ones is assumed to be constant and independent of time.*

This hypothesis seems reasonable because the energy is supplied to the system by the body force at large length scales, and is consumed by viscous dissipation at the short scales. However, due (2.3), we observe that the average energy input rate is the same as the average energy dissipation rate, which we denote by ϵ:

$$\epsilon = \frac{\nu}{L^3}\langle \|\nabla \mathbf{u}\|_0^2 \rangle, \tag{2.5}$$

where $\langle\cdot\rangle$ denotes the time average, and it is assumed that $\langle \|\nabla \mathbf{u}\|_0^2 \rangle$ exists! In the intermediate range, the energy is transferred by nonlinear effects. Supposing that viscous effects are negligible for these scales, it follows that the energy $E(k)$ should depend only of k and ϵ and not directly on the viscosity; supposing that it assumes the form

$$E(k) \sim \epsilon^a k^b$$

and invoking the dimensional relations (L for length and T for time)

$$[k] = [L]^{-1}, \quad [\epsilon] = [L]^2[T]^{-3}, \quad [E(k)] = [L]^3[T]^{-2},$$

we obtain $a = \frac{2}{3}$ and $b = -\frac{5}{3}$, giving the well-known formula:

$$E(k) = C_K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \tag{2.6}$$

where C_K is believed to be a "universal" dimensionless constant.

We now derive the Kolmogorov length scale λ_K, defined as the scale at which inertial effects actually balance viscous dissipation. Since the Reynolds number measures the energetic ratio between inertial terms and viscous terms, the Kolmogorov length scale can be determined by:

$$\frac{U_{k_K}\lambda_K}{\nu} \sim 1 \tag{2.7}$$

where U_{k_K} is the velocity scale associated with $k_K = 2\pi/\lambda_K$. The velocity U_{k_K} is obtained from the kinetic energy $E(k_K)$; that is, using a scaling argument and relation (2.6), we get:

$$U_{k_K} = (k_K E(k_K))^{1/2} \sim (\epsilon k_K^{-1})^{1/3} \sim (\epsilon \lambda_K)^{1/3}$$

Hence we deduce

$$\lambda_K = c_K \nu^{3/4} \epsilon^{-1/4}, \tag{2.8}$$

where c_K is a dimensionless constant which can be related to C_K (see e.g. [11, p. 55]). The Kolmogorov length scale λ_K is the smallest active scale in the flow; that is, the smallest energetically relevant scale.

Finally, we relate the Kolmogorov length scale to a global Reynolds number. As for U_{k_K}, we first define a macroscopic velocity scale U as

$$\frac{1}{2}U^2 := \frac{2\pi}{L}\sum_{k=2\pi/L}^{2\pi/\lambda_K} E(k).$$

Using (2.6), we observe that this sum is approximately equal to $\epsilon^{2/3}L^{2/3}$, which yields:

$$\epsilon \sim LU^3. \tag{2.9}$$

Now, from the definition of the Reynolds number $R_e = UL/\nu$, we have, using (2.9) and (2.8):

$$R_e \sim \frac{\epsilon^{1/3}L^{4/3}}{\nu} \sim \frac{\epsilon^{1/3}L^{4/3}}{\epsilon^{1/3}\lambda_K^{4/3}},$$

or simply

$$\frac{\lambda_K}{L} \sim R_e^{-3/4}. \tag{2.10}$$

This relation is widely used in computational fluid mechanics to estimate the number of cells needed to fully resolve turbulent flows. For example, if one is interested in simulating a turbulent flow at $R_e = 1000$ in a unit cube domain ($L = 1$) using finite difference or finite element methods, one would need to consider grid sizes of order $\lambda_K/L \sim R_e^{-3/4} \sim 5.6 \times 10^{-3}$ in each direction, that is, approximately $(L/\lambda_K)^3 \sim R_e^{9/4} \sim 5.6$ million cells in total.

Kolmogorov's scaling argument is both a simple and a surprising result. Astonishingly, the argument does not take into account the complex structure of the nonlinear term of the Navier–Stokes equations. It is merely based on an intuitive interpretation of the vortex stretching and energy cascade mechanisms. However, it is noteworthy that the scaling law (2.6) has been actually observed in numerous laboratory experiments, especially in the small scale range (see *e.g.* Lesieur [40, p. 87] or the review of Sreenivasan [58]). It is even more remarkable that a mathematical result, proposed by Foias *et al.* [17] and presented below, corroborates the correctness of Kolmogorov's scaling theory.

2.4. **Mathematical justification of Kolmogorov's scaling theory.** A quite simple mathematical justification of the energy cascade mechanism and hence of the Kolmogorov's scaling theory has been recently proposed by Foias *et al.* in [17]. We summarize here the principal ideas.

The authors consider either periodic boundary conditions in the 3D-torus or no-slip boundary conditions, assuming in the second case that Γ is $\mathcal{C}^2$.

They introduce the Stokes operator $A = -\mathbf{P_H}\nabla^2 : \mathbf{V} \cap \mathbf{H}^2(\Omega) \to \mathbf{H}$. Being positive and self-adjoint, and its inverse compact, the Stokes operator possesses a basis of eigenvectors $(\mathbf{w}_j)_{j>0}$, which is orthonormal and complete in $\mathbf{H}$, and the corresponding eigenvalues are such that $0 < \lambda_0 \leq \lambda_1 \leq \ldots \lambda_j \to +\infty$, as $j \to \infty$. In this basis, any weak solution of the Navier–Stokes equations can be expanded as

$$\mathbf{u} = \sum_{j=0}^{+\infty} \hat{u}_j \mathbf{w}_j.$$

At this stage of the exposition, it is convenient to introduce the following notations:

$$k_j := \lambda_j^{1/2} \qquad \mathbf{u}_k := \sum_{\lambda_j = k^2} \hat{u}_j \mathbf{w}_j \qquad \mathbf{u}_{k',k''} := \sum_{k' \leq k < k''} \mathbf{u}_k.$$

In other words, k_j is the wavenumber associated with the eigenvalue λ_j; $\mathbf{u}_k$ is the sum of all components of $\mathbf{u}$ that are indexed by the same wavenumber k; $\mathbf{u}_{k',k''}$ is the partial expansion of $\mathbf{u}$ between wavenumbers k' and k''.

The Navier–Stokes equations (1.1) then reduce to

$$\left\{ \begin{array}{l} \partial_t \mathbf{u}_k + \nu A \mathbf{u}_k + (\mathbf{P_H}(\mathbf{u} \cdot \nabla \mathbf{u}))_k = \mathbf{f}_k, \quad \forall\ k \geq k_0, \\ \mathbf{u}_{k|t=0} = \mathbf{u}_{0,k}, \end{array} \right. \tag{2.11}$$

where the source term $\mathbf{f}$ is assumed to be time independent, to belong to $\mathbf{H}$, and to be localized in the spectral space: *i.e.*, it is assumed that there exist $\underline{k}_f$ and $\overline{k}_f$ such that $0 < \underline{k}_f \leq \overline{k}_f < +\infty$ and $\mathbf{f} = \mathbf{f}_{\underline{k}_f, \overline{k}_f}$. We also introduce the medium wavenumber k_f as

$$k_f = \frac{\|A^{1/2}\mathbf{f}\|_0}{\|A^{-1/2}\mathbf{f}\|_0}$$

which naturally satisfies $\underline{k}_f \leq k_f \leq \overline{k}_f$.

One of the key ingredients now consists in defining the time average of quantities of interest (kinetic energy, enstrophy, *etc.*). Being given a function of time $g(t)$, our goal here is to give a sense to the limit:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g(\tau)\, \mathrm{d}\tau$$

for it may happen that this limit does not exist (think of $g(t) = \sin(\log(1+t)) + \cos(\log(1+t))$). Let G be the set of bounded real-valued functions on $[0, +\infty)$ and denote by G_0 the subset of G whose functions admit a limit at $+\infty$. Define $p : G \longrightarrow \mathbb{R}$ such that $p(g) = \limsup_{t \to +\infty} g(t)$ and $\Lambda_0 : G_0 \longrightarrow \mathbb{R}$ such that $\Lambda_0(g) = \lim_{t \to +\infty} g(t)$. It is clear that for all g and f in G, $p(g+f) \leq p(g) + p(f)$ and $p(\gamma g) = \gamma p(g)$ for all $\gamma > 0$. Furthermore, for all g in G we have $\Lambda_0(g) \leq p(g)$; as a result, owing to the Hahn-Banach theorem, there exists $\Lambda : G \longrightarrow \mathbb{R}$, such that $\Lambda(g) \leq \Lambda_0(g)$ for all g in G_0 and for all g in G $\Lambda(g) \leq p(g)$; *i.e.* Λ is an extension of Λ_0 (see [18, p. 225]

and [16] for more details). The operator Λ is the generalized limit and we hereafter denote $\Lambda(g) = \text{LIM}_{t\to+\infty}\, g(t)$. Equipped with this new tool, it is now possible to define time-averages for any function g in G by

$$\langle g\rangle = \underset{t\to+\infty}{\text{LIM}}\, \frac{1}{t}\int_0^t g(\tau)\,\mathrm{d}\tau. \tag{2.12}$$

Actually, this definition can be broadened even more as shown in [18, p. 191].

Now we can define the average kinetic energy $e = k_0^3\langle \mathbf{u}^2\rangle/2$ and the average dissipation rate per unit time and mass $\epsilon = \nu k_0^3\langle \|\nabla\times\mathbf{u}\|_0^2\rangle$. We introduce $k_K = (\epsilon/\nu^3)^{1/4}$ and $k_T = (\epsilon/2\nu e)^{1/2}$, the Kolmogorov and Taylor wavenumbers, respectively. Introducing the characteristic length $L_0 = k_0^{-1}$ and the characteristic velocity U such that $e = U^2/2$, we can define the Reynolds number $R_e = UL_0/\nu$. Using standard energy estimates, it is easy to derive the following bound:

$$R_e = \frac{k_0^{1/2}\langle \|\mathbf{u}\|_0^2\rangle}{\nu} \le \frac{\langle \|A^{1/2}\mathbf{u}\|_0^2\rangle^{1/2}}{\nu k_0^{1/2}} \le \frac{\|A^{-1/2}\mathbf{f}\|_0}{\nu^2 k_0^{1/2}}. \tag{2.13}$$

Then, the following theorem is proved in [17].

Theorem 2.1. *Provided R_e is sufficiently large so that $R_e \ge (k_0/k_f)^{1/2}$, there exists a constant c such that*

$$\epsilon \le ck_0U^3, \quad k_K \le ck_0R_e^{3/4}, \quad k_T \le ck_0R_e^{1/2}.$$

This theorem is a rigorous justification, though still incomplete, of the estimates on ϵ and k_K established in (2.9) and (2.10) and of another standard estimate on k_T that can be obtained by Kolmogorov's scaling theory.

The most striking result from [17] deals with energy fluxes. Taking the inner product of (2.11) with $\mathbf{u}_{k'k''}$ in $\mathbf{H}$, taking the sum on k from some k' to some $k'' > k'$, yields

$$\tfrac{1}{2}d_t\|\mathbf{u}_{k',k''}\|_0^2 + \nu\|\nabla\mathbf{u}_{k',k''}\|_0^2 = (\mathbf{f}_{k',k''}, \mathbf{u}_{k',k''}) + e_{k'}(\mathbf{u}) - e_{k''}(\mathbf{u}), \tag{2.14}$$

where we have set

$$e_k(\mathbf{u}) = (\mathbf{u}_{k,\infty}\cdot\nabla\mathbf{u}_{k,\infty}, \mathbf{u}_{k_0,k}) - (\mathbf{u}_{k_0,k}\cdot\nabla\mathbf{u}_{k_0,k}, \mathbf{u}_{k,\infty}).$$

The quantity $e_k(\mathbf{u})$ can be interpreted physically as being the flux per unit time of kinetic energy into the higher modes $\mathbf{u}_{k,\infty}$ induced by vortex stretching. Taking the time average of (2.14) and assuming $k'' > k' > \overline{k}_f$, we deduce

$$\langle e_{k'}(\mathbf{u})\rangle - \langle e_{k''}(\mathbf{u})\rangle = \nu\|\nabla\mathbf{u}_{k',k''}\|_0^2. \tag{2.15}$$

Modulo some technical details pertaining to the possible loss of regularity as $k'' \to +\infty$ (the technical issue that is responsible for the uniqueness question), it is possible to pass to the limit $k'' \to +\infty$, so that equality (2.15) is replaced by the inequality

$$\langle e_{k'}(\mathbf{u})\rangle \ge \nu\|\nabla\mathbf{u}_{k',\infty}\|_0^2. \tag{2.16}$$

As a result, we conclude from (2.15)–(2.16) that for $k > \overline{k}_f$, the energy flux $\langle e_k(\mathbf{u})\rangle$ is nonnegative and monotone decreasing. Hence, energy flows down from the scales enforced by the forcing term to the small scales, as suggested by the arguments on vortex stretching. From (2.15) it is now easy to derive the following bounds:

Theorem 2.2 (Foias *et al.* [17])**.** *Provided $k'' > k' > \overline{k}_f$, we have*

$$0 \le 1 - \frac{\langle e_{k''}(\mathbf{u})\rangle}{\langle e_{k'}(\mathbf{u})\rangle} \le \left(\frac{k''}{k_T}\right)^2 \left(1 - \left(\frac{k'}{k_T}\right)^2\right)^{-1}$$

and, provided $k > \overline{k}_f$, then

$$1 - \left(\frac{k}{k_T}\right)^2 \le \frac{k_0^3 \nu \langle \|\nabla \mathbf{u}_{k,\infty}\|_0^2\rangle}{\epsilon} \le 1.$$

The two bounds established in this theorem have the following physical interpretation. The first one implies that

$$\text{if } k_T \gg k'' > k' > \overline{k}_f, \text{ then } \langle e_{k''}(\mathbf{u})\rangle \approx \langle e_{k'}(\mathbf{u})\rangle;$$

that is, the energy flux through wavenumbers k within the range $k_T \gg k > \overline{k}_f$ is nearly constant, confirming the energy cascade scenario as imagined by Kolmogorov. The second bound yields

$$\text{if } k_T \gg k > \overline{k}_f, \text{ then } k_0^3 \nu \langle \|\nabla \mathbf{u}_{k,\infty}\|_0^2\rangle \approx \epsilon;$$

that is, for scales such that the corresponding wavenumbers k are within the range $k_T \gg k > \overline{k}_f$, the energy lost by viscous dissipation below these scales is nearly independent of k and approximately equal to the energy dissipation rate ϵ, confirming the well-known hypothesis by Kolmogorov stated in (HK).

2.5. **Attractors of the Navier–Stokes equations.** The attractors of autonomous dissipative dynamical systems characterize the long time behavior of flows and are closely related to their stability properties. The attractor is a compact subset of the phase space (the normed space $\boldsymbol{L}^2(\Omega)$ here) toward which flow solutions converge after long time periods. Although there exist various definitions of the dimension of a set, an upper bound on the dimension of the attractor is obtained by taking the smallest N for which all N-dimensional subsets of $\boldsymbol{L}^2(\Omega)$ (sets of initial states at $t = 0$ for example) contract to zero volume as $t \to 0$ (see [11] or [18, p. 117–118] for more details). For example, the dimension of a steady-state flow (fixed point) is zero while the dimension of a periodic flow (periodic orbit) is one. For more complex flows, the attractor dimension is not necessarily a positive integer and the estimate (2.10) from Kolmogorov's scaling theory yields the widely accepted conjecture that the Navier–Stokes equations should have a global attractor of dimension $d \le \mathcal{O}(R_e^{9/4})$.

However, the question of whether a global attractor of the Navier–Stokes equations in 3D exists is an issue that is not yet settled. This question has important practical implications, for knowing that an attractor exists

and that this attractor is finite dimensional would guarantee that long-time behavior of Navier–Stokes solutions can reasonably be numerically approximated (*i.e.* using a finite number of degrees of freedom).

The question of the existence of an attractor is intimately linked with the problem of knowing whether time-dependent Navier–Stokes solutions are unique, or, equivalently, of ascertaining that the time evolution governing the solutions is deterministic. As surprising as it may be, this simple question is still open. The main obstacle in the way is that it cannot be yet proven that the solutions to the 3D Navier–Stokes equations are smooth for arbitrarily long times. Up to now, no *a priori* estimate has been found that guarantees that the enstrophy, usually defined as $\|\nabla\times\mathbf{u}\|_0^2$, remains finite for all times; that is, no *a priori* estimate guarantees that the vorticity does not blow up somewhere in the domain in finite time. Note, however, that for a given forcing term $\mathbf{f}$ and a given initial data $\mathbf{u}_0$, the quantity $\frac{1}{t}\int_0^t \|\nabla\times\mathbf{u}\|_0^2\,\mathrm{d}\tau$ is bounded; hence, the enstrophy is bounded in the mean. Nevertheless, this bound does not preclude the enstrophy to blow-up intermittently like $(t-t_0)^\alpha$ with $-1<\alpha<0$. Hence, we cannot exclude *a priori* the possible occurrence of rare (intermittent) vorticity bursts driving the energy deep down to scales much shorter than the standard Kolmogorov scale. At the present state of the art, one cannot prove or disprove mathematically that the energy cascade scenario does indeed stop at the Kolmogorov scale everywhere in the domain. In other words, in the present level of understanding, we cannot disprove the possibility that the nonlinear term (the vortex stretching mechanism) may be so strong that we are not guaranteed the linear viscosity is strong enough to stop the cascade everywhere in the fluid domain.

If such a blow-up were to occur, the time-evolution of the solution would not be unique; this would be unacceptable as deterministic Newtonian time evolution would be lost. Moreover, finite-time singularities would mean that the flow would develop arbitrarily small-scale structures, violating the axiom of continuum mechanics assuming a scale separation between individual atomic evolution and collective hydrodynamics motions. In conclusion, though the uniqueness question may seem to be an irrelevant issue, it is "actually intimately tied up with the efficiency of the Navier–Stokes equations as a model for fluid turbulence" [11, p. xii].

As a result, information on the attractor, or equivalently on the smallest active scale of the flow, can be obtained only by assuming *a priori* some regularity on the enstrophy. For instance, in the 3D-torus we have the following result proved in Constantin, Foias, and Temam [10].

Theorem 2.3. *Assuming that the quantity*

$$\epsilon_1 = \limsup_{t\to+\infty}\ \sup_{\mathbf{u}_0\in\mathbf{H}}\ \frac{1}{t}\int_0^t \nu\|\nabla\times\mathbf{u}(\tau,\cdot)\|_{0,\infty}^2\,\mathrm{d}\tau,$$

is finite and defining the corresponding Kolmogorov scale $\lambda_1 = \nu^{3/4}\epsilon_1^{-1/4}$, *then the Lyapunov dimension,* d_L, *of the global attractor of the 3D Navier–Stokes equations is bounded from above as follows:*

$$d_L \leq c\left(\frac{L}{\lambda_1}\right)^3. \tag{2.17}$$

Note that the fractal dimension d_F (box counting) and the Hausdorff dimension d_H are such that $d_H \leq d_F \leq d_L$, so the estimate (2.17) is actually a bound on these three dimensions. This bound shows that the Kolmogorov scale, λ_1, is the smallest active scale of the flow if ϵ_1 is finite.

Any attempt trying to weaken the smoothness hypotheses on the vorticity has so far yielded suboptimal results. For instance, we have the following result proved by Gibbon and Titi [21].

Theorem 2.4. *Assuming that* $\epsilon_2 = \nu L^{-3}\sup_t \|\nabla\times\mathbf{u}\|_{0,2}^2$ *is finite and defining the corresponding Kolmogorov scale* $\lambda_2 = \nu^{3/4}\epsilon_2^{-1/4}$, *we have the following bound on* d_L*:*

$$d_L \leq c\left(\frac{L}{\lambda_2}\right)^{4.8}. \tag{2.18}$$

Hence, assuming that the enstrophy is finite for all times, the best estimate available up to now for the smallest length scale ℓ of the flow is

$$\frac{\ell}{L} \geq c\left(\frac{\lambda_2}{L}\right)^{1.6}.$$

In other words, weakening the assumption on the regularity of the vorticity translates immediately into the possible existence of active scales, ℓ, smaller than the Kolmogorov scale λ_2.

We conclude this section by recalling a consequence of the ladder theorem of Doering *et al.* (see [2] and [11, p. 143]): if for some $\delta > 0$, $\sup_t \|u\|_{0,3+\delta}$ is finite then no singularity can occur. The reader will note that the gap between what is known to be bounded, *i.e.* $\sup_t \|u\|_{0,2}$, and what should be bounded for the whole Navier–Stokes building to fall neatly in place, $\sup_t \|u\|_{0,3+\delta}$, is frustratingly small.

3. The filtering concept in LES

Large Eddy Simulation, or simply LES, an acronym coined in the ground breaking paper of Leonard [38], has as its primary goal to modify the Navier–Stokes equations in order to obtain a new system of equations which is easier to approximate while retaining all the most energetic features of the unperturbed problem. The classical idea is to use a filter which allows for the separation of large and small length scales in the flow-field. Applying the filtering operator to the Navier–Stokes equations provides a new equation governing the large scales, except for one term still given in terms of the

small velocity scales. Modeling this term in an appropriate manner, a procedure commonly referred to as the *closure problem*, one can arrive at a set of equations with only the large velocity (and pressure) scales as the unknown. In this section, we first show that the closure problem associated with the filtering procedure actually gives way to a paradox. In addition, we show another interesting result establishing that adequate filtering is actually equivalent to a regularization of the Navier–Stokes equations. In terms of mathematical analysis, regularization of the equations allows for solving the uniqueness issue.

3.1. **Filtering operator and closure problem.** Let the filtering operator be denoted by $\overline{(\cdot)} : w \longmapsto \overline{w}$. Note that filtering can be performed either in space or in time, or both in space and time. For examples of filters, we refer the reader to [53]. Although many types of filters could be selected, we will simply assume here that the filtering operator is linear and commutes with differential operators (properties shared by most filters of interest).

Applying the filtering operator to the Navier–Stokes equations yields the new system of equations:

$$\left\{\begin{array}{l} \partial_t \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} + \nabla \overline{p} - \nu \nabla^2 \overline{\mathbf{u}} = \overline{\mathbf{f}} - \nabla \cdot \mathbb{T}, \\ \nabla \cdot \overline{\mathbf{u}} = 0 \\ \overline{\mathbf{u}}|_\Gamma = 0, \quad \text{or } \overline{\mathbf{u}} \text{ is periodic}, \\ \overline{\mathbf{u}}|_{t=0} = \overline{\mathbf{u}}_0, \end{array}\right. \tag{3.1}$$

where

$$\mathbb{T} = \overline{\mathbf{u} \otimes \mathbf{u}} - \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} \tag{3.2}$$

is usually referred to as the subgrid-scale tensor. In order to be able to solve (3.1) for $\overline{\mathbf{u}}$ without having to resort to $\mathbf{u}$, *i.e.* to close the problem, the tensor $\mathbb{T}$ needs to be expressed in term of $\overline{\mathbf{u}}$ only. The closure problem which consists of finding an accurate model $\mathbb{T}(\overline{\mathbf{u}})$ for the subgrid-scale tensor $\mathbb{T}$ certainly represents the main difficulty of LES.

3.2. **The closure paradox.** It is legitimate to believe that if Problem (3.1) can be closed exactly, *i.e.* without invoking *ad hoc* hypotheses, the Holy Grail for turbulence would then be uncovered. We show in the following that exact closure is actually possible.

Proposition 3.1. *Assuming Ω to be the 3D-torus, exact closure of (3.1) is then achievable.*

Proof. This result was pointed out by Germano in [19, 20]. Let $\varepsilon > 0$ be a cutoff scale. We consider the following filter (later referred to as the Helmholtz filter): for any given function $\mathbf{v}$, the filtered function $\overline{\mathbf{v}}$ is defined as the solution to the elliptic PDE:

$$\overline{\mathbf{v}} - \varepsilon^2 \nabla^2 \overline{\mathbf{v}} = \mathbf{v}, \tag{3.3}$$

that is, $\overline{\mathbf{v}} := (I-\varepsilon^2\nabla^2)^{-1}\mathbf{v}$. From a theorem by Agmon–Douglis–Nirenberg, it is shown that this filtering operator is continuous from $\mathbf{L}^q(\Omega)$, $1 < q < +\infty$, onto $\mathbf{W}^{2,q}(\Omega)$ (see *e.g.* [3, 56]). Furthermore, it can be shown that $\overline{(\cdot)}$ commutes with space and time derivatives. Hence, $\overline{(\cdot)}$ is intuitively an acceptable filter. Then, using the fact that

$$\mathbf{u}\otimes\mathbf{u} = (\overline{\mathbf{u}} - \varepsilon^2\nabla^2\overline{\mathbf{u}})\otimes(\overline{\mathbf{u}} - \varepsilon^2\nabla^2\overline{\mathbf{u}}),$$

and since

$$\overline{\mathbf{u}}\otimes\overline{\mathbf{u}} = \overline{\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}} - \varepsilon^2\nabla^2\overline{\overline{\mathbf{u}}\otimes\overline{\mathbf{u}}},$$

it follows that

$$\begin{aligned}
\mathbb{T}_{ij} &= \overline{(\overline{u}_i - \varepsilon^2\nabla^2\overline{u}_i)(\overline{u}_j - \varepsilon^2\nabla^2\overline{u}_j)} - \overline{u}_i\overline{u}_j,\\
&= \overline{\overline{u}_i\overline{u}_j} - \varepsilon^2\overline{\overline{u}_j\nabla^2\overline{u}_i + \overline{u}_i\nabla^2\overline{u}_j} + \varepsilon^4\overline{\nabla^2\overline{u}_i\nabla^2\overline{u}_j} - \overline{u}_i\overline{u}_j,\\
&= \varepsilon^2\overline{\nabla^2(\overline{u}_i\overline{u}_j) - \overline{u}_j\nabla^2\overline{u}_i - \overline{u}_i\nabla^2\overline{u}_j} + \varepsilon^4\overline{\nabla^2\overline{u}_i\nabla^2\overline{u}_j}\\
&= 2\varepsilon^2\overline{\nabla\overline{u}_i\cdot\nabla\overline{u}_j} + \varepsilon^4\overline{\nabla^2\overline{u}_i\nabla^2\overline{u}_j},
\end{aligned}$$

which is actually a closed-form of the subgrid-scale tensor. □

Remark 3.1. Another way to derive an exact closure involves defining the filter by means of a mollifier. Let ϕ be a bounded positive function in $\mathbb{R}^3$, fast decreasing at infinity, such that $\int_{\mathbb{R}^3}\phi\,\mathrm{d}\mathbf{x} = 1$, and such that its Fourier transform does not vanish. For instance, the Gaussian kernel $\phi(\mathbf{x}) = \pi^{-3/2}\exp(-|\mathbf{x}|^2)$ satisfies these hypotheses. Then for $\varepsilon > 0$, define $\phi_\varepsilon(\mathbf{x}) = \varepsilon^{-3}\phi(\mathbf{x}/\varepsilon)$ and set

$$\overline{\mathbf{v}} := \phi_\varepsilon * \mathbf{v}. \tag{3.4}$$

Denoting by $\mathcal{F}$ the Fourier transform, we infer that

$$\forall\ \mathbf{u}\in\mathbf{L}^1(\Omega),\quad \mathbf{u} = \mathcal{F}^{-1}(\mathcal{F}(\overline{\mathbf{u}})/\mathcal{F}(\phi)).$$

Hence, in this case also, the subgrid-scale tensor can be expressed in terms of the only filtered field.

The preceding results can actually be generalized by observing that exact closure is achieved whenever the filtering operator induces an isomorphism. Indeed, filters (3.3) and (3.4) are isomorphisms. This is intuitively verified by the fact that they do not remove information from the field they are applied to; they simply deform the spectrum of the field. For instance, given some $t > 0$, filter (3.3) induces isomorphisms between $L^\infty(0,t;\mathbf{H})$ and $L^\infty(0,t;\mathbf{H}\cap\mathbf{H}^2(\Omega))$ and between $L^2(0,t;\mathbf{V})$ and $L^2(0,t;\mathbf{V}\cap\mathbf{H}^3(\Omega))$. Hence, this filter induces an isomorphism between the set of weak solutions of (1.1) and that of (3.1). As a result, filtering and achieving exact closure is unlikely to improve the situation since, roughly speaking, given that the two sets of weak solutions are isomorphic, one should expect to use the same number of degrees of freedom for approximating the Navier–Stokes equations as for approximating the filtered equations.

Formally speaking, we are faced with the following paradox:

Filtering and achieving exact closure may not reduce the number of degrees of freedom.

In conclusion, we conjecture that filtering the Navier–Stokes equations is an efficient approach only if *inexact* closure is performed.

3.3. **Leray regularization.** The first outstanding result which involves filtering is the proof of existence of weak solutions to the Navier–Stokes equation by Leray [39]. We assume here that Ω is the 3D-torus $(0, 2\pi)^3$. Considering a sequence of mollifying functions $(\phi_\varepsilon)_{\varepsilon>0}$ satisfying:

$$\phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^3), \quad \mathrm{supp}(\phi_\varepsilon) \in B(0,\varepsilon), \quad \int_{\mathbb{R}^3} \phi_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1, \tag{3.5}$$

and the convolution product $\phi_\varepsilon * \mathbf{v}$ such that:

$$\phi_\varepsilon * \mathbf{v}(\mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y})\phi_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y},$$

Leray suggested to regularize the Navier–Stokes equations as follows:

$$\begin{cases} \partial_t \mathbf{u} + (\phi_\varepsilon * \mathbf{u}) \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \phi_\varepsilon * \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \phi_\varepsilon * \mathbf{u}_0, \end{cases} \tag{3.6}$$

where, although the same variable names are used, it is understood that the solution of (3.6) is different from the solution of the Navier–Stokes equation (1.1). Leray proved the following theorem:

Theorem 3.1 (Leray [39])**.** *For all $\mathbf{u}_0 \in \mathbf{H}$, $\mathbf{f} \in \mathbf{H}$, and $\varepsilon > 0$, (3.6) has a* unique *$\mathcal{C}^\infty$ solution. This solution is also bounded in $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ and one subsequence converges weakly in $L^2(0,T;\mathbf{V})$ to a weak Navier–Stokes solution as $\varepsilon \to 0$.*

The striking result here is that the solution of (3.6) is unique. Hence, moderate filtering of the advection velocity (and if necessary, of the data $\mathbf{u}_0$ and $\mathbf{f}$) is sufficient to guarantee uniqueness of $\mathcal{C}^\infty$ solutions; that is, it takes only a small amount of smoothing to ascertain that the energy cascade stops everywhere in the domain and for all times.

Rewriting the momentum equation of (3.6) as:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \phi_\varepsilon * \mathbf{f} - (\phi_\varepsilon * \mathbf{u}) \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \tag{3.7}$$

and introducing the tensor $\mathbb{T}_L$ such that:

$$\nabla \cdot \mathbb{T}_L = (\phi_\varepsilon * \mathbf{u}) \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot (\mathbf{u} \otimes (\phi_\varepsilon * \mathbf{u}) - \mathbf{u} \otimes \mathbf{u}), \tag{3.8}$$

it is reasonable to think of the Leray regularization as a LES model. Equation (3.7) is indeed the same as the momentum equation of (3.1) except for the fact that the subgrid-scale tensor $\mathbb{T}$ is now approximated by $\mathbb{T}_L$. However, this interpretation is debatable, for the model is not frame invariant. We elaborate on this issue in the following section.

3.4. **The Navier–Stokes–alpha model.** Owing to the identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla(\mathbf{u}^2/2),$$

the Navier–Stokes problem can be rewritten in the form

$$\left\{\begin{array}{l} \partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\pi - \nu\nabla^2\mathbf{u} = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}|_\Gamma = 0, \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{array}\right. \tag{3.9}$$

where $\pi = p + \frac{1}{2}\mathbf{u}^2$ is the total pressure. This form of the equations obviously raises the same uniqueness problem as the original form. This issue can nevertheless be resolved by resorting to the regularization technique proposed by Leray. Following the same strategy as in (3.6) and introducing the notation $\overline{\mathbf{u}} = \phi_\varepsilon * \mathbf{u}$, we are led to consider the following regularized problem

$$\left\{\begin{array}{l} \partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\overline{\mathbf{u}} + \nabla\pi - \nu\nabla^2\mathbf{u} = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{array}\right. \tag{3.10}$$

Using the identities,

$$(\nabla\times\mathbf{u})\times\overline{\mathbf{u}} = \overline{\mathbf{u}}\cdot\nabla\mathbf{u} - (\nabla\mathbf{u})^T\overline{\mathbf{u}},$$
$$\nabla(\mathbf{u}\cdot\overline{\mathbf{u}}) = (\nabla\mathbf{u})^T\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T\mathbf{u},$$

we can recast system (3.10) into the following equivalent form,

$$\left\{\begin{array}{l} \partial_t\mathbf{u} + \overline{\mathbf{u}}\cdot\nabla\mathbf{u} + (\nabla\overline{\mathbf{u}})^T\cdot\mathbf{u} - \nu\nabla^2\mathbf{u} + \nabla\pi' = \mathbf{f}, \\ \nabla\cdot\overline{\mathbf{u}} = 0, \\ \mathbf{u},\ \overline{\mathbf{u}} \text{ are periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{array}\right. \tag{3.11}$$

where $\pi' = \pi - \mathbf{u}\cdot\overline{\mathbf{u}}$.

Upon choosing the Helmholtz filter, introduced in the proof of Proposition (3.1) and defined as

$$\overline{\mathbf{v}} := (I - \varepsilon^2\nabla^2)^{-1}\mathbf{v}, \tag{3.12}$$

we recognize the model thoroughly analyzed in Chen *et al.* [8], Foias, Holm and Titi [15, 14]. Regarding the notational choice $\alpha = \varepsilon$, the authors refer to this particular regularization as the "Navier–Stokes–alpha model". Once again, regularization yields uniqueness and the expected regularity properties as stated in the following theorem:

Theorem 3.2. *Problem (3.11) with the Helmholtz filter (3.12) has a* unique $\mathcal{C}^\infty$ *solution* $\mathbf{u}$. *The solution* $\mathbf{u}$ *is bounded in* $L^\infty(0,T;\mathbf{H})\cap L^2(0,T;\mathbf{V})$ *and one subsequence converges weakly in* $L^2(0,T;\mathbf{V})$ *to a weak Navier–Stokes solution as* $\varepsilon\to 0$.

Numerical simulations reported in [15] show that the energy spectrum of the solution of (3.11) follows the $k^{-5/3}$ law for $k \lesssim 1/\varepsilon$ and rolls off to k^{-3} for $k \gtrsim 1/\varepsilon$. Hence, below the scale ε the regularization has replaced the $k^{-5/3}$ tail of the spectrum, which is difficult to approximate numerically, by a more gentle k^{-3} tail.

At this point of the analysis, it seems natural to ask which of $\mathbf{u}$ or $\overline{\mathbf{u}}$ is the most "physically relevant" quantity. Here, by "physically relevant", we mean that the quantity is frame-invariant. By a change of frame of reference in (3.11) with respect to $\mathbf{u}$, it can be shown that $\mathbf{u}$ is not physically relevant in this sense (see [25]). In order to analyze the frame-invariance with respect to the filtered velocity, we first rewrite (3.11) in terms of $\overline{\mathbf{u}}$. Introducing the strain rate $\mathbf{D} = \frac{1}{2}(\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T)$, the vorticity tensor $\mathbf{\Omega} = \frac{1}{2}(\nabla\overline{\mathbf{u}} - (\nabla\overline{\mathbf{u}})^T)$, and the Jaumann derivative

$$\mathring{\mathbf{D}} = \partial_t\mathbf{D} + \overline{\mathbf{u}}\cdot\nabla\mathbf{D} + \mathbf{D}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{D}$$

we show that (3.11) is equivalent to the following system:

$$\left\{\begin{array}{l} \partial_t\overline{\mathbf{u}} + \overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}} = \nabla\cdot\mathbf{T}, \\ \qquad\qquad \text{with } \mathbf{T} = -p\mathbf{I} + 2\nu(1-\varepsilon^2\nabla^2)\mathbf{D} + 2\varepsilon^2\mathring{\mathbf{D}}, \\ \nabla\cdot\overline{\mathbf{u}} = 0, \\ \overline{\mathbf{u}} \text{ is periodic}, \\ \overline{\mathbf{u}}|_{t=0} = \overline{\mathbf{u}}_0 \end{array}\right. \tag{3.13}$$

We recognize here the constitutive law of a rate-dependent incompressible homogeneous fluid of second grade in which the dissipation is slightly modified by composition with the Helmholtz operator (see [15] for more details). This set of equations is frame-invariant; hence, according to our definition, $\overline{\mathbf{u}}$ is the quantity that is "physically relevant."

In conclusion, the regularization procedure using the Helmholtz filter (3.12), as performed in (3.10), is strictly equivalent to LES where the subgrid tensor is now modeled as follows

$$\mathbb{T} = 2\nu\varepsilon^2\nabla^2\mathbf{D} - 2\varepsilon^2\mathring{\mathbf{D}}. \tag{3.14}$$

The first term amounts to adding some hyper-viscosity (*i.e.* a bilaplacian) whereas the second one introduces dispersion effects.

Finally, we note that the Leray-regularized equations (3.6) can be rewritten in terms of only the filtered velocity $\overline{\mathbf{u}}$, where the regularization is performed this time using the Helmholtz filter. In this case, we obtain a set of equations that are frame-invariant with respect to $\overline{\mathbf{u}}$ except for the term $\varepsilon^2(\nabla\overline{\mathbf{u}})^T\nabla^2\overline{\mathbf{u}}$. If this term of order $\mathcal{O}(\varepsilon^2)$ is simply neglected from the momentum equation, we then surprisingly recover the Navier–Stokes–alpha model (3.11) (see [25]). Hence, the Navier–Stokes–alpha model can be viewed as a frame-invariant version of the Leray regularization. The progress made from the ground-breaking work of Leray in 1934 is deceivingly small.

3.5. **The local energy equilibrium.** Another intriguing problem related to the uniqueness question is that of local energy balance. Though it is quite simple to show that weak solutions of the Navier–Stokes equations satisfy a global energy balance (2.3), it has not yet been possible to prove that a local energy balance holds. For instance, the following result is proved in Duchon and Robert [12]

Proposition 3.2. *Let* $\mathbf{u} \in L^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$ *be a weak solution of the Navier–Stokes equation in the 3D-torus with no source term. Let* $\phi_\varepsilon(\mathbf{x}) = (1/\varepsilon^3)\phi(\mathbf{x}/\varepsilon)$ *be a mollifying sequence (i.e.,* ϕ *is even,* $\mathcal{C}^\infty$ *with compact support in* $\mathbb{R}^3$*, non-negative and normalized). Set* $\mathcal{D}_\varepsilon(\mathbf{u})(\mathbf{x}) = \frac{1}{4}\int \nabla\phi_\varepsilon(\mathbf{y})\cdot\delta\mathbf{u}(\delta\mathbf{u})^2\,\mathrm{d}y$*, where* $\delta\mathbf{u} = \mathbf{u}(\mathbf{x}+\mathbf{y})-\mathbf{u}(\mathbf{x})$*. Then* $\mathcal{D}_\varepsilon(\mathbf{u})$ *converges in* $\mathcal{D}'(]0,T[\times\Omega)$ *to a distribution* $\mathcal{D}(\mathbf{u})$ *that does not depend on* ϕ *and such that the following energy balance holds:*

$$\partial_t(\tfrac{1}{2}\mathbf{u}^2) + \nabla\cdot(\mathbf{u}(\tfrac{1}{2}\mathbf{u}^2+p)) - \nu\nabla^2\tfrac{1}{2}\mathbf{u}^2 + \nu(\nabla\mathbf{u})^2 + \mathcal{D}(\mathbf{u}) = 0.$$

For each solution $\mathbf{u}$, "$\mathcal{D}(\mathbf{u})$ measures a possible dissipation or production of energy caused by lack of smoothness in the velocity field." This quantity is zero only if $\mathbf{u}$ is smooth enough. For instance, it is possible to prove that $\mathcal{D}(\mathbf{u}) = 0$ if $\int|\mathbf{u}(t,\mathbf{x}+\mathbf{y})-\mathbf{u}(t,\mathbf{x})|^3\,\mathrm{d}y \le c(t)|\mathbf{y}|\sigma(|\mathbf{y}|)$ where $\int_0^T c(t)\,\mathrm{d}t < +\infty$ and σ is continuous in 0 with $\sigma(0) = 0$ (see [12]).

As pointed out by Duchon and Robert [12], it is remarkable that every solution of the Navier–Stokes equations obtained as a limit of (a subsequence) of solutions $\mathbf{u}_\varepsilon$ of the regularized equations introduced by Leray (3.6) is "dissipative" in the sense that $\mathcal{D}(\mathbf{u}) \ge 0$; that is, these solutions of the Navier–Stokes equations satisfy

$$\partial_t(\tfrac{1}{2}\mathbf{u}^2) + \nabla\cdot(\mathbf{u}(\tfrac{1}{2}\mathbf{u}^2+p)) - \nu\nabla^2\tfrac{1}{2}\mathbf{u}^2 + \nu(\nabla\mathbf{u})^2 \le 0. \tag{3.15}$$

When $\mathbf{u}$ is obtained as a limit of a finite dimensional Galerkin approximation, the sign of $\mathcal{D}(\mathbf{u})$ is unknown; that is, the lack of smoothness of Galerkin solutions might lead to local energy creation. This counter-intuitive result sheds doubt on the physical relevance of Galerkin solutions, for one would expect lack of smoothness to always dissipate energy. In other words, assuming we compute an approximate solution of the Navier–Stokes equations using the Galerkin technique on a finite element mesh and we make the meshsize go to zero, then we are not guaranteed that the limit solution satisfy (3.15), whereas if we regularize the advection term *à la* Leray, make the meshsize go to zero, and make the regularization parameter go to zero afterward, then the limit solution necessarily satisfies (3.15).

It is also remarkable that the notion of "dissipative solution" introduced by Duchon and Robert coincides with the notion of "suitable weak solutions" introduced by Cafarelli, Khon, and Nirenberg [5] (based on the work of Scheffer [55]) for which the best partial regularity theorem to date has been proved [5].

The argument above can be summarized as follows.

(i) Limit solutions of the Navier–Stokes equations computed by means of LES models like the Leray regularization or the NS-α model are "dissipative" (or "suitable weak solutions") whereas limits of Galerkin approximations may not be so.
(ii) Limits of regularized solutions are possibly more regular than limits of Galerkin approximations.

Hence, limits of LES solutions may be physically more relevant than limits of Galerkin approximations, thus justifying LES strategies *à la* Leray.

4. The p-Laplacian models

In this section, we review LES models based on nonlinear viscosity. We study the Smagorinsky model, the p-Laplacian regularization, and models introduced by Ladyženskaja. We show that the main characteristics of this class of models is that they are all regularizations of the Navier–Stokes equations in the sense that they solve the uniqueness question. We finish this section by showing that the p-Laplacian regularization has also interesting numerical properties that permit the establishment of L^∞-error estimates.

4.1. **The Smagorinsky model.** Possibly one of the most popular models for Large Eddy Simulations is that proposed by Smagorinsky [57]. The model consists in adding to the stress tensor an additional nonlinear viscous term that depends on a small length scale ε to be fixed in some *ad hoc* way. Denoting the deformation tensor by $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, the additional tensor is written in the form

$$c_s\varepsilon^2|\mathbf{D}|\mathbf{D}$$

where c_s is some *ad hoc* constant. Upon introducing the notation $\mathbf{T}(\nabla\mathbf{u}) = c_s|\mathbf{D}|\mathbf{D}$, the perturbed Navier–Stokes equations are

$$\begin{cases} \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nabla\cdot(\nu\nabla\mathbf{u} + \varepsilon^2\mathbf{T}(\nabla\mathbf{u})) = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}|_\Gamma = 0, \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases} \tag{4.1}$$

In the literature, another way of presenting the method consists in stating that (4.1) is the filtered Navier–Stokes equations, although the filter is never specified, where the subgrid tensor is modeled by $\mathbb{T} = \varepsilon^2\mathbf{T}(\nabla\mathbf{u})$. This view point is questionable, for, contrary to the Leray regularization and the NS–alpha model, no filter is specified, and no theoretical motivation to introduce one seems to be necessary.

The constant c_s is usually evaluated so that the model reproduces the $k^{-5/3}$ cascade when simulating isotropic turbulence in the 3D-torus. We refer to [43, 44] for tentative evaluations of this parameter.

When it comes to approximate the solution to (4.1), the free parameter ε is generally chosen to be the mesh size, h, therefore mixing in some *ad*

hoc way the mathematical model with the computational one, generally a serious crime.

While Smagorinsky's turbulence model is acclaimed for its remarkable ability to reproduce the $k^{-5/3}$ energy spectrum provided the constant c_s is tuned adequately [43], this model has serious drawbacks. One concern often raised is that the artificial dissipation does not go to zero as one approaches no-slip boundaries, whereas it is well-known that turbulence vanishes in this case. Note also from the numerical point of view that, when choosing ε to be equal to the meshsize h, the model induces a $\mathcal{O}(h^2)$ consistency error with respect to the Navier–Stokes equations that definitely renders the use of high order approximation methods useless, for accuracy can no longer exceed $\mathcal{O}(h^2)$ even in smooth regions of the flow. In other words, Smagorinsky's model is limited to second order approximation methods in space.

Though Smagorinsky [57] is very often credited for being at the origin of the trend that consists in introducing nonlinear viscosity for regularization purposes, this idea is possibly rooted in the pioneering work of von Neumann and Richtmyer [48] and in the analogy Rivlin[1] made between flows of non-Newtonian fluids and turbulent Newtonian fluids [50].

We now come to a striking result due to O. Ladyženskaja [35, 34] that sheds an original light on Smagorinsky's model.

4.2. **The Ladyženskaja model.** Recalling that the Navier–Stokes equations are based on Newton's linear hypothesis, Ladyženskaja proposed, in a series of papers [35, 34], to modify the incompressible Navier–Stokes equations to take into account possible large velocity gradients. She introduced a nonlinear viscous tensor $\mathbf{T}_{ij}(\nabla\mathbf{u})$, $1 \le i,j \le 3$ satisfying the following conditions:

L1. $\mathbf{T}$ is continuous and there exists some $\mu \ge \frac{1}{4}$ such that

$$\forall\, \boldsymbol{\xi} \in \mathbb{R}^{3\times 3}, \quad |\mathbf{T}(\boldsymbol{\xi})| \le c(1+|\boldsymbol{\xi}|^{2\mu})|\boldsymbol{\xi}|. \tag{4.2}$$

L2. $\mathbf{T}$ has the following coercivity property:

$$\forall\, \boldsymbol{\xi} \in \mathbb{R}^{3\times 3}, \quad \mathbf{T}(\boldsymbol{\xi}) : \boldsymbol{\xi} \ge c|\boldsymbol{\xi}|^2(1+c'|\boldsymbol{\xi}|^{2\mu}). \tag{4.3}$$

L3. $\mathbf{T}$ possesses the following monotonicity property: there exists a constant $c > 0$ such that for all solenoidal fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ in $\mathbf{W}^{1,2+2\mu}(\Omega)$ either coinciding on the boundary Γ or being periodic,

$$\int_\Omega (\mathbf{T}(\nabla\boldsymbol{\xi}) - T(\nabla\boldsymbol{\eta})) : (\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\eta}) \ge c\int_\Omega |\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\eta}|^2. \tag{4.4}$$

[1]According to Rivlin [50], "It has been reported [...] that when a Newtonian fluid flows down a straight pipe of non-circular cross-section, under conditions for which the fluid has become fully turbulent, the [mean] flow is no longer rectilinear, but a secondary flow exists in the cross-sectional planes of a type similar to that [that can be calculated for some non-Newtonian fluids]. This fact suggests that the turbulent Newtonian liquid may, for certain purposes, be regarded as a non-Newtonian fluid."

These conditions are satisfied in the case where

$$\mathbf{T}(\boldsymbol{\xi}) = \beta(|\boldsymbol{\xi}|^2)\boldsymbol{\xi} \tag{4.5}$$

provided the viscosity function $\beta(\tau)$ is a positive monotonically-increasing function of $\tau \geq 0$ and for large values of τ the following inequality holds

$$c\tau^{\mu} \leq \beta(\tau) \leq c'\tau^{\mu},$$

with $\mu \geq \frac{1}{4}$ and c, c' are some strictly positive constants. Smagorinsky's model obviously falls in the admissible category with $\beta(\tau) = \tau^{1/2}$.

Introducing now a (possibly small) positive constant $\varepsilon > 0$, the modified Navier–Stokes equations take the form

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nabla\cdot(\nu\nabla\mathbf{u} + \varepsilon\mathbf{T}(\nabla\mathbf{u})) = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}|_{\Gamma} = 0, \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0. \end{cases} \tag{4.6}$$

The striking result from [35, 34] is the following theorem

Theorem 4.1. *Provided conditions* L1*,* L2*, and* L3 *are satisfied,* $\mathbf{u}_0 \in \mathbf{H}$ *and* $\mathbf{f} \in \mathbf{L}^2(]0,+\infty[;\mathbf{L}^2(\Omega))$*, then (4.6) has a* unique *weak solution in* $L^2(]0,t[;\mathbf{W}^{1,2+2\mu}(\Omega)\cap V)\cap C^0([0,t];\mathbf{H})$ *for all* $t > 0$*.*

Note that uniqueness in ensured for times t possibly arbitrarily large. This result states that a small appropriate amount of nonlinear viscosity is actually sufficient to ascertain that the energy cascade stops, which automatically translates into uniqueness of solutions for arbitrary times.

In conclusion, perturbing the Navier–Stokes equations with a term like Smagorinsky's model solves the uniqueness question.

4.3. **The p-Laplacian.** The p-Laplacian operator is a simple version of both Smagorinsky's and Ladyženskaja's models. This operator is well-known to mathematicians for being a prototype for monotone operators, see *e.g.* Lions [45] or Showalter [56]. We show in this section how this operator can be used for approximating advection dominated advection-diffusion equations, hence giving a second mathematical interpretation of Smagorinsky's model.

Assuming $\Omega \subset \mathbb{R}^d$ and $p \geq 2$, the p-Laplacian is the operator defined as follows:

$$T_p : W_0^{1,p}(\Omega) \ni u \longmapsto T_p(u) = -\nabla\cdot(|\nabla u|^{p-2}\nabla u) \in W_0^{1,p}(\Omega)'.$$

It is clearly bounded in $W_0^{1,p}(\Omega)$ and satisfies the following monotonicity property

$$\exists \alpha > 0, \forall\, u, v \in W_0^{1,p}(\Omega), \qquad \langle T_p(u) - T_p(v), u - v\rangle \geq \alpha\|\nabla(u-v)\|_{1,p}^p,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing (the reader not familiar with this notion can view it as the L^2-scalar product). Note that when $p = 2$, T_p is

nothing else but the Laplacian. Note also that the p-Laplacian counterpart of Smagorinsky's model corresponds to $p = 3$.

This operator has interesting properties owing to the following standard Sobolev embedding: there exists $c > 0$ such that for all $u \in W^{1,p}(\Omega)$

$$\|u\|_{L^q(\Omega)} \leq c\|u\|_{W^{1,p}(\Omega)} \begin{cases} \frac{1}{q} = \frac{1}{p} - \frac{1}{d} & \text{if } 1 \leq p < d \\ p \leq q < \infty & \text{if } p = d \\ q = +\infty & \text{if } p > d \end{cases} \tag{4.7}$$

where d is the space dimension (see *e.g.* [3, 56]). Hence the monotonicity property yields *a priori* estimates in the norm of $W^{1,p}(\Omega)$, which not only controls the standard $H^1(\Omega)$-norm, but also controls the norm of $L^q(\Omega)$. For instance, for $p > 3$ this term allows for a control in the $L^\infty(\Omega)$ norm in three dimensions.

To illustrate these ideas, we consider an advection diffusion equation dominated by advection:

$$\begin{cases} -\varepsilon\nabla^2 u + \boldsymbol{\beta}\cdot\nabla u = f, \\ u|_\Gamma = 0, \end{cases} \tag{4.8}$$

where $\varepsilon > 0$. For the sake of simplicity, we assume that $\boldsymbol{\beta}$ is a smooth solenoidal vector field with zero normal trace on Γ, *i.e.* $\boldsymbol{\beta} \in \mathbf{H}$, and ε is some positive real number. It is well-known that approximating this type of equation is a nontrivial task when the ratio $h\|\boldsymbol{\beta}\|_{0,\infty}/\varepsilon$ is large, h being the typical mesh-size. Standard Galerkin (*i.e.* centered) approximation yields spurious node-to-node oscillations, the heuristic reason for this being that the grid is not fine enough for the viscous effects to dampen gradients. This problem is to be put in parallel with the difficulties in approximating fluid flows at high Reynolds numbers. In the same spirit as that of the Smagorinsky model, one may try to slightly modify the problem by adding some nonlinear viscosity.

Let $X_h \subset H_0^1(\Omega)$ be a finite dimensional space having standard interpolation properties (for instance a continuous $\mathbb{P}_k$ finite element space), *i.e.* there are $c > 0$ and $k > 0$ such that for all $v \in W^{1,p}(\Omega)$

$$\inf_{v_h \in X_h} \left(\|v - v_h\|_{0,p} + h\|v - v_h\|_{1,p}\right) \leq ch^{k+1}\|v\|_{k+1,p} \tag{4.9}$$

Let us set

$$a(u,v) = (\nabla u, \nabla v), \qquad b(u,v) = (\boldsymbol{\beta}\cdot\nabla u, v), \tag{4.10}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$. We consider the following approximate problem

$$\begin{cases} \text{Find } u_h \text{ in } X_h \text{ s.t.} \\ \varepsilon a(u_h, v_h) + b(u_h, v_h) + h^\sigma (T_p(u_h), v_h) = (f, v_h), \quad \forall\, v_h \in X_h, \end{cases} \tag{4.11}$$

where σ and $p \geq 2$ are yet to be determined. It is clear that we perturb the original problem by a term which is $\mathcal{O}(h^\sigma)$. Hence, to preserve optimal convergence estimates on gradients one should choose σ such that $\sigma \geq k$.

Theorem 4.2. *Under above hypotheses and provided u is smooth enough, we have the following error estimate*

$$\|u - u_h\|_{1,p} \leq (h^{\frac{1}{p-1}} + h^{\frac{k+1-\sigma}{p-1}})c(u),$$

where $c(u) = c\max(\|u\|_{1,p}, h^k\|u\|_{k+1,p}, \|u\|_{k+1,p'}^{\frac{1}{p-1}})$.

The interesting aspect of such an estimate is that it is uniform with respect to ε, though ε is implicitly accounted for in $c(u)$. Note also that it is necessary to impose $k+1 > \sigma$ in order to achieve convergence. As a result, consistency and $W^{1,p}$-convergence are guaranteed if

$$k \leq \sigma < k+1.$$

Of course, for the case $\sigma = 1$ and $p = 2$, which corresponds to the crude first order linear viscosity, we obtain $\|u - u_h\|_{1,2} \leq ch$. But a more interesting situation arises if $p > d$, because convergence in the L^∞-norm is then guaranteed due to the Sobolev inequality (4.7). In this case, convergence occurs without unbounded spurious oscillations. Note finally that in three dimensions and for a second order method, *i.e.* $k = 1$, the limit case to obtain L^∞-convergence is $p = 3$ and $\sigma = k + 1 = 2$. That is, the limit case is formally equivalent to the Smagorinsky model.

5. Spectral viscosity methods

In this section we review the so-called spectral viscosity methods. These LES techniques are frequently used in conjunction with spectral approximation methods. We first present the heuristic approach of Kraichnan[33]; then we present the method developed by Tadmor [60] for solving nonlinear scalar conservation equations, and we underline the striking similarities between the two approaches. Contrary to the filtering techniques and the nonlinear viscosity regularizations, these methods are essentially numerical and do not solve the uniqueness question. As far as fluid mechanics is concerned, their main justification is that when applied to scalar conservation laws, they guarantee the convergence of the approximate solution to the unique entropy solution.

5.1. **Kraichnan's vanishing eddy viscosity.** A Fourier approximation of the Navier–Stokes equations in the 3D-torus is built as follows. Given a cut-off wavenumber, $k_c > 1$, we denote by $\mathbf{P}_{k_c}$ the $\mathbf{L}^2$-projection onto $\mathbf{H}$ of vector-valued k_c-trigonometric polynomials:

$$\mathbf{P}_{k_c}(\mathbf{v}) = \hat{\mathbf{v}}_0 + \sum_{\substack{\mathbf{k}\in\mathbb{Z}^3 \\ |\mathbf{k}|_\infty \leq k_c, \mathbf{k}\neq 0}} \left(\mathbf{I} - \frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|}\right)\hat{\mathbf{v}}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Note that the operator $\mathbf{P}_{k_c}$ commutes with derivatives. The approximate solution $\mathbf{u}_{k_c}(\mathbf{x},t) = \sum_{|\mathbf{k}|_\infty \le k_c} \hat{u}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}$, satisfies

$$\begin{cases} \partial_t \mathbf{u}_{k_c} - \nu \mathbf{P}_{k_c} \nabla^2 \mathbf{u}_{k_c} + \mathbf{P}_{k_c}(\mathbf{u}_{k_c} \cdot \nabla \mathbf{u}_{k_c}) = \mathbf{P}_{k_c}\mathbf{f}, \\ \mathbf{u}_{k_c}|_{t=0} = \mathbf{P}_{k_c}\mathbf{u}_0. \end{cases} \tag{5.1}$$

Of course, if k_c is not large enough, *i.e.* if the Kolmogorov scale $\lambda \sim \nu^{3/4}$ is much smaller than the grid size $2\pi k_c^{-1}$, then energy accumulates at the cut-off scale. This translates in practice into spurious node-to-node oscillations in the approximate solution. The main purpose of LES is to avoid this phenomenon by adding extra terms to the Navier–Stokes equations so that the energy dissipates through the Kolmogorov cascade at the right rate. Assuming that the cut-off is large enough so that k_c is in the inertial range of the cascade, then Kraichnan [33] drew upon *ad hoc* statistical hypotheses that an eddy viscosity should be added to (5.1) and he proposed to modify the approximation scheme as follows:

$$\begin{cases} \partial_t \mathbf{u}_{k_c} - \mathbf{P}_{k_c} \nabla^2 ((\nu + \nu_t(k_c) Q_{k_c}) * \mathbf{u}_{k_c}) + \mathbf{P}_{k_c}(\mathbf{u}_{k_c} \cdot \nabla \mathbf{u}_{k_c}) = \mathbf{P}_{k_c}\mathbf{f}, \\ \mathbf{u}_{k_c}|_{t=0} = \mathbf{P}_{k_c}\mathbf{u}_0, \end{cases} \tag{5.2}$$

with the vanishing viscosity $\nu_t(k_c)$ being such that

$$\nu_t(k_c) = E(k_c)^{1/2} k_c^{-1/2}, \tag{5.3}$$

where $E(k_c) = \frac{L}{2\pi}\frac{1}{2}\sum_{|\mathbf{k}|_\infty = k_c} |\hat{\mathbf{u}}_\mathbf{k}|^2$ is the kinetic energy at the cut-off scale. The viscosity kernel assumes the form

$$Q_{k_c}(\mathbf{x}) = \sum_{|\mathbf{k}|_\infty \le k_c} \hat{\nu}_t(|\mathbf{k}|_\infty) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.4}$$

where $\hat{\nu}_t$ is a non-dimensional function which is constant for $|\mathbf{k}|_\infty / k_c \lesssim 0.3$, but increases for higher values of the ratio $|\mathbf{k}|_\infty / k_c$ so that the graph of $\hat{\nu}_t$ makes an upward cusp in the vicinity of k_c (see Figure 1). Note that the convolution is easily evaluated as follows:

$$\nabla^2 (Q_{k_c} * u_{k_c}) = - \sum_{|\mathbf{k}|_\infty \le k_c} \hat{\nu}_t(|\mathbf{k}|_\infty) |\mathbf{k}|_\infty^2 \hat{\mathbf{u}}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.5}$$

Many authors have elaborated on this theory, among which are Chollet and Lesieur [9], who proposed exponential forms of the cusp and power laws like

$$\hat{\nu}_t(|\mathbf{k}|_\infty) = \left[\phi_0 + \phi_1 |\mathbf{k}|_\infty^n k_c^{-n}\right], \tag{5.6}$$

where ϕ_0, ϕ_1 and n are *ad hoc* constants to be determined (see also Lesieur and Métais [41]). The reader is also referred to [13, 53] for reviews of these models.

In order to cast a new light upon this type of model, we now recall the spectral viscosity technique of Tadmor [60] for approximating nonlinear conservation laws. We shall come back to Kraichnan's eddy viscosity model in Section 5.3.

5.2. **Tadmor's spectral viscosity.** The concept of spectral viscosity introduced by Tadmor [60] is a very powerful tool for proving convergence of spectral approximations of nonlinear conservation laws.

Let us consider a scalar conservation law in the d-dimensional torus $\Omega = (0, 2\pi)^d$,

$$\begin{cases} u|_{t=0} = u_0 \in L^\infty(\Omega)^d, \\ \partial_t u + \nabla\cdot(\mathbf{f}(u)) = 0, \quad \text{in the distribution sense}, \\ \partial_t U + \nabla\cdot(\mathbf{F}(u)) \le 0, \quad \forall \text{ convex } U,\ \mathbf{F}(U) = \int_0^u U'(w)\mathbf{f}'(w)dw, \end{cases} \tag{5.7}$$

where $\mathbf{f}$ is a flux a class $\mathcal{C}^s$, s being sufficiently large.

In order to build a spectral approximation of (5.7), we introduce $k_c > 1$ and we denote by P_{k_c} the L^2-projection onto k_c-trigonometric polynomials. The standard Galerkin technique consists in projecting (5.7) onto these k_c-trigonometric polynomials. The resulting scheme is spectrally accurate, but Gibbs oscillations triggered by shocks prevent the approximate solution to converge in $L^1(\Omega)$ to the entropy solution; that is, to the solution that satisfies the entropy conditions in (5.7). To suppress these oscillations while retaining spectral accuracy, Tadmor [60] proposed to augment the Fourier approximation by a spectral viscosity as follows. We look for $u_{k_c}(\mathbf{x}, t) = \sum_{|\mathbf{k}|_\infty \le k_c} \hat{u}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, so that

$$\begin{cases} \partial_t u_{k_c} + \nabla\cdot(P_{k_c}\mathbf{f}(u_{k_c}(\mathbf{x}, t)) = \varepsilon_{k_c}\nabla^2(Q_{k_c} * u_{k_c}(\mathbf{x}, t)), \\ u_{k_c}|_{t=0} = P_{k_c}u_0. \end{cases} \tag{5.8}$$

where the vanishing viscosity ε_{k_c} and the viscosity kernel Q_{k_c} satisfies the following conditions:

T1. There is some $0 < \theta < 1$, such that the coefficient ε_{k_c} satisfies

$$\varepsilon_{k_c} \sim {k_c}^{-\theta}, \quad \text{and} \quad \varepsilon_{k_c}^s \|\partial_x^s P_{k_c} u_0\|_{L^2(\Omega)} \le \text{const}, \quad \forall\ s \ge 0. \tag{5.9}$$

T2. There exists an integer $k_i \gg 1$

$$k_i \sim \frac{{k_c}^{\frac{\theta}{2}}}{(\log k_c)^{\frac{d}{2}}}, \tag{5.10}$$

such that in the spectrum range $0 \le |\mathbf{k}|_\infty \le k_i$ there is no artificial viscosity.

T3. The viscosity kernel is given by its Fourier expansion $Q_{k_c}(\mathbf{x}, t) = \sum_{|\mathbf{k}|_\infty = k_i}^{k_c} \hat{\mathbb{Q}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$, and the coefficients are spherically symmetric, that is, $\hat{\mathbb{Q}}_{\mathbf{k}} = \hat{Q}_p$ for all $|\mathbf{k}|_\infty = p$. Furthermore, The coefficients are monotonically increasing with respect to p and satisfy the estimate

$$|\hat{Q}_p - 1| \le c\frac{k_i^2}{p^2}, \qquad \forall\ p \ge k_i. \tag{5.11}$$

Note that viscosity coefficients like

$$\hat{Q}_{|\mathbf{k}|_\infty} = 1 - \frac{{k_i}^2}{\max(k_i, |\mathbf{k}|_\infty)^2}, \tag{5.12}$$

are acceptable, though it is probably better to have $\mathcal{C}^\infty$ smoothness with respect to k in applications (see [46, p. 336] and §5.3). Note also that in applications one should use $\theta = 1^-$, for this yields the highest accuracy, though theoretically the limit case $\theta = 1$ is excluded. Note also that the artificial viscosity term is easy to implement in the Fourier space since

$$\nabla^2(Q_{k_c} * u_{k_c}) = - \sum_{|\mathbf{k}|_\infty = k_i}^{k_c} \hat{Q}_{|\mathbf{k}|_\infty} |\mathbf{k}|_\infty^2 \hat{u}_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.13}$$

The major result proved in [60] for the 1D case and in [7] for the d-dimensional case is as follows.

Theorem 5.1. *Under the above assumptions, u_{k_c} converges to the unique entropy solution of (5.7) and the following error estimate holds:*

$$\|u - u_{k_c}\|_{L^\infty([0,T];L^1(\Omega))} \leq c\sqrt{\varepsilon_{k_c}}. \tag{5.14}$$

5.3. **Vanishing eddy viscosity versus spectral viscosity.** When looking at (5.2) and (5.8) the similarity of the approaches is striking. In particular the way the artificial viscosity operator is implemented in (5.5) and (5.13) are identical. The major difference lies in the scalings of the viscosities (5.3) and (5.9), and in the definition of the non-dimensional viscosity laws (5.6) and (5.12).

We now compare the graphs of two possible definitions of $\hat{Q}_{|\mathbf{k}|_\infty}$ and $\hat{\nu}_t(|\mathbf{k}|_\infty)$. For Tadmor *et al.*'s spectral viscosity we choose the following definition

$$\begin{gathered} \hat{Q}_{|\mathbf{k}|_\infty} = \tfrac{1}{2}(\tanh(\lambda x) + 1)\left[1 - \frac{1}{f(x)^2}\right], \\ \text{with } \begin{cases} x = \frac{|\mathbf{k}|_\infty - k_i}{k_i} \\ f(x) = \frac{1}{2}(\sqrt{\varepsilon + x^2} + x) + 1 \end{cases} \end{gathered} \tag{5.15}$$

where $\varepsilon = 0.01$, and denoting by ε_m the machine accuracy, we have set $\lambda = \frac{1}{2}\log(\varepsilon/2\varepsilon_m)$. For the eddy viscosity $\hat{\nu}_t(|\mathbf{k}|_\infty)$, we choose one of the expression proposed by Chollet and Lesieur [9] (see also [36, 41, 42]):

$$\hat{\nu}_t(|\mathbf{k}|_\infty) = (2.1)^{-3/2}\big[0.441 + 15.2\exp(-3.03\,k_c/|\mathbf{k}|_\infty)\big]. \tag{5.16}$$

The normalized graph of $\hat{Q}_{|\mathbf{k}|_\infty}$ and $\hat{\nu}_t(|\mathbf{k}|_\infty)$ is shown in Figure 1 for $k_c = 64$ and $k_c = 128$. For $\hat{Q}_{|\mathbf{k}|_\infty}$ we use $\varepsilon_m = 10^{-16}$, $\theta = 1$, and $k_i = 5\sqrt{k_c}$. The spectral viscosity $\hat{Q}_{|\mathbf{k}|_\infty}$ converges exponentially to zero for $|\mathbf{k}|_\infty \leq k_i$. Furthermore, its graph is concave for $|\mathbf{k}|_\infty \gg k_i$ and reaches $\mathcal{O}(k_c^{-1})$ in the vicinity of k_c with a descending slope. This feature corresponds to the well-known fact that to be numerically efficient any artificial viscosity must be

at least of order k_c^{-1} in the high modes. Note that contrary to the spectral viscosity, $\hat{Q}_{|\mathbf{k}|_\infty}$, the graph of the eddy viscosity, $\hat{\nu}_t(|\mathbf{k}|_\infty)$, is convex in the high mode region, and the function does not vanish in the low mode region, destroying *de facto* spectral accuracy.

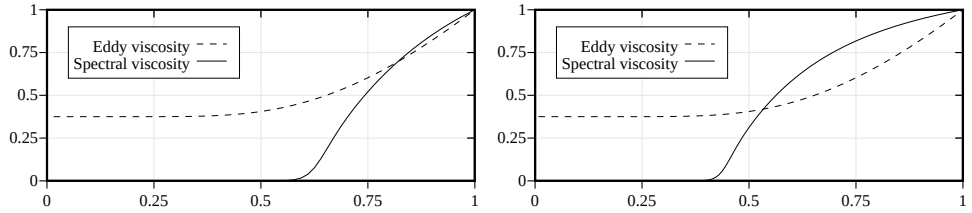

FIGURE 1. Spectral viscosity versus Kraichnan–Chollet–Lesieur's model; Left: $k_c = 64$; Right: $k_c = 128$.

Let us come back to definition (5.3). Assuming that the cut-off is realized in the inertial range of the Kolmogorov cascade, then $E(k_c) \sim k_c^{-5/3}$. If ϕ_0 in (5.6) is not zero, then (5.3) implies that $\nu_t(k_c)\hat{\nu}_t(\mathbf{k})$ can be bounded from below by $c\phi_0 k_c^{-4/3}$. Hence, in first approximation, Kraichnan–Chollet–Lesieur's eddy viscosity model is a standard (almost constant) artificial viscosity scaling like $k_c^{-4/3}$. This result, which does not seem to be stressed in the literature, is odd. Formally, it should limit the convergence of the method to $\mathcal{O}(k_c^{-4/3})$, which is by far not spectral. This type of viscosity seems to be definitely incompatible with spectral accuracy. Furthermore, when looking closely at computation reported in Lesieur and Rogallo [42, Fig. 2], see Figure 2, where the authors deduce numerically an eddy viscosity $\nu_t(k/k_c)/\nu$ from Direct Numerical Simulations, one observes that the graph of the function is slightly concave, a feature of Tadmor *et al.*'s spectral viscosity that cannot be reproduced by a law like (5.6) or (5.16) (see bottom figure).

Numerical DNS simulations reported in McComb and Young [47, Fig. 15], see Figure 3, seem also to contradict (5.6) and (5.16). Figure 3 shows $\nu_t(k/k_c)/\nu$ computed from Direct Numerical Simulations. This Figure seems to indicate that if a law like (5.6) was to hold, then ϕ_0 should be a fast decreasing function of k_c, corroborating the spectral viscosity postulate that the artificial viscosity should be zero in the low mode region.

Note also that the scaling in $k_c^{-4/3}$ asymptotically makes the vanishing viscosity technique inefficient in the high modes region, where it should be of order k_c^{-1} to be numerically efficient. The upward cusp behavior advocated in [9] does not seem to be asymptotically strong enough to achieve this goal. Upon introducing $E_\Sigma = \frac{1}{2}\|\mathbf{u}\|_{0,2}^2$, the total kinetic energy of the flow, a vanishing viscosity defined as follows

$$\nu_t'(k_c) = E(k_c)^{3/10} E_\Sigma^{1/5} k_c^{-1/2},$$

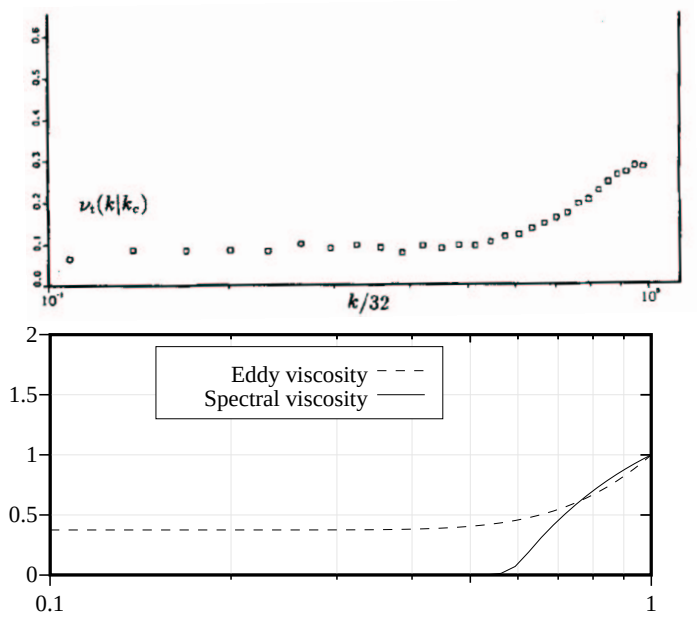

FIGURE 2. Top, eddy viscosity computed from DNS, from Lesieur and Rogallo [42, Fig. 2] (Courtesy of M. Lesieur and *Phys. Fluids*), ν_t vs. k/k_c, $k_c = 32$. Bottom, spectral viscosity and eddy viscosity, $k_c = 32$ and $k_i = 3.5\sqrt{k_c}$.

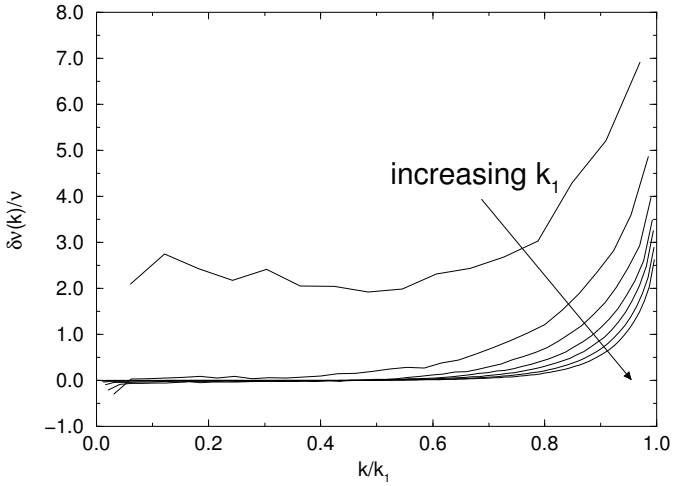

FIGURE 3. Spectral viscosity from McComb and Young [47, Fig. 15], ν_t/ν vs. k/k_c for various cutoffs $k_c = 16$, 32, 48, 64, 80, 96, 112 (Courtesy of D. McComb).

would scale like k_c^{-1} if k_c is in the inertial range. Note that this observation is speculative, since for resolutions commonly used, a $k_c^{1/3}$ factor can easily

be hidden in tunable constants; for instance, for $128 \leq k_c \leq 512$ we have $5 < k_c^{1/3} \leq 8$.

Let us finish this section by recalling an interesting result from [7].

Lemma 5.1. *Under hypotheses* T1, T2, T3*, there exists a constant c that does not depend on k_c s.t.*

$$\forall\ p \geq 1, \qquad \sup_{v_{k_c} \neq 0} \varepsilon_{k_c} \frac{\|\nabla^2 (Q_{k_c} * v_{k_c}) - \nabla^2 v_{k_c}\|_{0,p}}{\|v_{k_c}\|_{0,p}} \leq c$$

This lemma sates that the spectral viscosity is an L^p-bounded perturbation of the standard first order vanishing viscosity (the lemma holding true for $\theta = 1$ also). Intuitively, this means that the spectral viscosity does exactly the same job as the vanishing viscosity on the high wavenumber modes, the only difference being that it does not spoils the consistency error by unnecessarily dampening the low wavenumber modes. This argument proves that, modulo the scaling factor discussed above, the spectral viscosity really works like the vanishing viscosity while retaining the spectral accuracy.

One may wonder whether a spectral viscosity perturbation of the Navier–Stokes equation could solve the uniqueness issue in the same way as the Ladyženskaja's nonlinear viscosity did. To answer this question, let us extend the operator Q_{k_c} to $L^2(\Omega)$ by defining $\tilde{Q}_{k_c} = Q_{k_c} + \sum_{|\mathbf{k}|_\infty > k_c} e^{i\mathbf{k}\cdot\mathbf{x}}$, and let $\mathbf{P}_\infty$ be the $\mathbf{L}^2$-projection onto $\mathbf{H}$. Then, let us modify the Navier–Stokes equations as follows

$$\left\{ \begin{array}{l} \partial_t \mathbf{u}_\varepsilon - \mathbf{P}_\infty \nabla^2 \big((\nu + \varepsilon \tilde{Q}_{k_c}) * \mathbf{u}_\varepsilon(x,t) \big) + \mathbf{P}_\infty (\mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon) = \mathbf{P}_\infty \mathbf{f}, \\ \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0, \end{array} \right. \tag{5.17}$$

where ε is some vanishing viscosity. Then, it is clear that

$$\forall\ \mathbf{v} \in \mathbf{H}_0^1(\Omega), \qquad \nabla^2 (\nu + \varepsilon \tilde{Q}_{k_c}) * (1 - \mathbf{P}_{k_c}) \mathbf{v} = (\nu + \varepsilon) \nabla^2 (1 - \mathbf{P}_{k_c}) \mathbf{v}.$$

That is, except for the finite-dimensional vector space of the k_c-trigonometric solenoidal polynomials, adding a spectral viscosity just amounts to replacing the viscosity ν by $\nu + \varepsilon$, which is clearly not enough for solving the uniqueness question (at the present time). Hence, modifying the Navier–Stokes equations by adding a spectral viscosity does not resolve the uniqueness problem. It is sometimes claimed in the literature that Kraichnan-Chollet-Lesieur's viscosity is "equal to" the Smagorinsky model (see *e.g.* Lesieur [40, p. 237]). In view of the radically different behavior of both techniques concerning the uniqueness issue, this claim is dubious.

The comparison we have made between spectral and eddy viscosity techniques should be tempered somewhat by realizing that, though the Navier–Stokes equations and scalar conservation laws share some important features, these two sets of PDE's are quite different. The difference in behavior of the spectral viscosity regularization (or vanishing viscosity) between scalar conservation laws and the Navier–Stokes equations is rooted in the fact that an *a priori* bound in $L^\infty(\Omega)$-norm is available for the scalar case whereas

such a bound is unknown for the Navier–Stokes equations. Adding an artificial viscosity in the discrete scalar case guarantees $L^\infty(\Omega)$-boundedness, which in turns ensures convergence to the entropy solution (see [7, 46]).

In conclusion, we do not claim that spectral viscosity is adequate for LES (though we stress that it should not be less relevant than the vanishing viscosity technique for it is an L^p-bounded perturbation of it), but we want to draw the attention on the fact that there is an issue here that should probably be addressed. For preliminary steps in this direction we refer to Karamanos, Karniadakis [32] and Pasquetti, Xu [49].

6. Two-scale methods

Hierarchical multilevel settings are very often credited for providing suitable frameworks for LES. We review this point of view in the present section. First we present some popular two-scale LES models; then we show how these models are related to a subgrid stabilization technique that solve non-coercive PDE's, hence giving them some partial mathematical support.

6.1. **The two-scale approximation framework.** Throughout this section, we assume that we have at hand two finite dimensional spaces, $\mathbf{X}_h$ and M_h, for approximating the velocity and the pressure respectively. To avoid irrelevant stability issues on the pressure, we assume that the two spaces satisfy the LBB condition. We assume also that we are given a linear operator $\mathbf{P}_H : \mathbf{X}_h \longrightarrow \mathbf{X}_h$ that we shall call a filter. We call $\mathbf{X}_h$ the fine scale space and $\mathbf{X}_H = \mathbf{P}_H(\mathbf{X}_h)$ the resolved scale space. Here H and h refer to the characteristic meshsize of $\mathbf{X}_H$ and $\mathbf{X}_h$, respectively. It is usual in the literature to take $H \approx 2h$. We assume that $\mathbf{X}_H$ has interpolating properties of $\mathcal{O}(H^{k+1})$ in $\mathbf{L}^2(\Omega)$ and of $\mathcal{O}(H^k)$ in $\mathbf{H}^1(\Omega)$.

6.2. **Two-scale subgrid viscosity methods.** The robustness of artificial viscosity techniques has led authors to adapt this class of methods to the two-scale approximation framework.

6.2.1. *Subgrid viscosity.* Subgrid viscosity models involve modeling the subgrid tensor in the form of a dissipative operator like $\mathbb{T} = \nabla\cdot(\nu_t\nabla\mathbf{u})$ where the so-called turbulent viscosity, ν_t, is assumed to depend only on the fluctuating component of the velocity.

Adopting, formally for the time being, a decomposition of the velocity field in the form $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, one possibility for defining ν_t consists in assuming that the turbulent viscosity depends only on the turbulent kinetic energy $e' = \frac{1}{2}\mathbf{u}'^2$ and a turbulence mixing length scale ε s.t. $\nu_t \sim \varepsilon e'^{1/2}$. Numerically, this idea can be implemented using two-scale approximation techniques by identifying $\mathbf{u}$ with $\mathbf{u}_h$ and $\overline{\mathbf{u}}$ with $\mathbf{P}_H\mathbf{u}_h$; then, the discrete turbulent kinetic energy is $\frac{1}{2}(\mathbf{u}_h - \mathbf{P}_H(\mathbf{u}_h))^2$ and the turbulence length scale at hand is the coarse meshsize H. Hence, the turbulent kinetic energy model, usually referred to as the TKE model (see *e.g.* [61]), consists of setting

$$\nu_t \approx cH|\mathbf{u}_h - \mathbf{P}_H(\mathbf{u}_h)| \tag{6.1}$$

Another possibility for defining ν_t consists in using a mixed approach combining the Smagorinsky model and the TKE eddy viscosity introduced above as in Ta Phuoc Loc and Sagaut [54]:

$$\nu_t(\mathbf{u}_h) = cH^{1+\alpha}|\mathbf{P}_H(\mathbf{u}_h) - \mathbf{u}_h|^{1-\alpha} \begin{cases} |\mathbf{D}(\mathbf{u}_h)|^\alpha \\ \quad \text{or} \qquad\qquad 0 \le \alpha \le 1. \\ |\nabla \times \mathbf{P}_H \mathbf{u}_h|^\alpha \end{cases} \tag{6.2}$$

We refer to Sagaut [53] for a thorough review on this class of models. One major pragmatic interest in this kind of formula is that it combines the stabilizing effects of the Smagorinsky model (for $\alpha > 0$) while allowing for high order consistency. For instance, if the filter is such that for smooth solutions $\|\mathbf{P}_H(\mathbf{u}_h) - \mathbf{u}_h\| = \mathcal{O}(H^{k+1})$, then the subgrid term is $\mathcal{O}(H^{k(1-\alpha)+2})$, which is at most $\mathcal{O}(H^{k+1})$ if $\alpha \le k^{-1}$. In particular, in viscous layers, the solution being smooth in these regions, the subgrid term is of the same order as the consistency error of the underlying numerical scheme.

In weak form, the two subgrid viscosity techniques above amount to adding the following subgrid semilinear form to the momentum equation:

$$a_{sgs}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = (\nu_t(\mathbf{u}_h)\nabla \mathbf{u}_h, \nabla \mathbf{v}_h), \qquad \mathbf{u}_h \in \mathbf{X}_h,\ \mathbf{v}_h \in \mathbf{X}_h. \tag{6.3}$$

6.2.2. *The variational multiscale method.* An alternative strategy proposed by Hughes *et al.* [27, 26, 28], called the LES "variational multiscale" method, is based on the idea that the additional dissipation term should act only on the turbulent part of the velocity field and should leave the resolved part unchanged. This idea amounts to writing the subgrid tensor in the form $\mathbb{T} = (\nabla \cdot (\nu_t \nabla \mathbf{u}'))'$. Within a two-scale approximation framework, the corresponding semilinear form that is added to the momentum equation is as follows:

$$a_{sgs}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) = \big(\nu_t(\mathbf{u}_h)\nabla(\mathbf{u}_h - \mathbf{P}_H\mathbf{u}_h), \nabla(\mathbf{v}_h - \mathbf{P}_H\mathbf{v}_h)\big), \tag{6.4}$$

where the turbulent viscosity may take various forms. In [27], the authors propose the following two possible choices:

$$\nu_t(\mathbf{u}_h) = cH^2 \begin{cases} |\mathbf{D}(\mathbf{u}_h)| \\ \quad \text{or} \\ |\mathbf{D}(\mathbf{u}_h - \mathbf{P}_H\mathbf{u}_h)| \end{cases} \tag{6.5}$$

From numerical tests reported in [26, 28], it is not clear which form of ν_t performs the best, but from the approximation point of view, we note that in the regions where the solution is smooth, the first form of the subgrid term is $\mathcal{O}(H^{k+1})$, which is exactly the order of the consistency error of the approximation method, whereas the other form is unnecessarily much smaller, *i.e.* $\mathcal{O}(H^{2k+1})$.

6.2.3. *The discrete viewpoint.* Denoting by $\mathcal{I}_h$ an $\mathbf{L}^2$-stable interpolation operator in $\mathbf{X}_h$, all the techniques described above, once rewritten in weak form, boil down to solving the following approximate Navier–Stokes equations:

Find $\mathbf{u}_h \in \mathcal{C}^1(0,T;\mathbf{X}_h)$ and $p_h \in \mathcal{C}^0(0,T;M_h)$ such that

$$\left\{\begin{array}{l} (d_t\mathbf{u}_h,\mathbf{v}_h)+(\mathbf{u}_h\cdot\nabla\mathbf{u}_h,\mathbf{v}_h)-(p_h,\nabla\cdot\mathbf{v}_h)+\nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) \\ \qquad\qquad +a_{sgs}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h), \quad \forall\mathbf{v}_h\in\mathbf{X}_h \\ (q_h,\nabla\cdot\mathbf{u}_h)=0, \quad \forall q_h\in M_h \\ \mathbf{u}_h|_{t=0}=\mathcal{I}_h\mathbf{u}_0, \end{array}\right. \tag{6.6}$$

where $a_{sgs}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)$ is the semilinear form accounting for the modeling of the subgrid tensor.

6.3. **Scale similarity models.** Besides the approaches that consist in approximating the subgrid tensor as a dissipative operator, one original class of techniques, introduced by Bardina, Ferziger, and Reynolds [1], consists of modeling the subgrid tensor as a dispersive operator by assuming scale-similarity. More precisely, still decomposing the flow field into an average velocity $\overline{\mathbf{u}}$ and a fluctuation $\mathbf{u}'$, the scale-similarity hypothesis consists in assuming

$$\mathbb{T}=\overline{\mathbf{u}\oplus\mathbf{u}}-\overline{\mathbf{u}}\oplus\overline{\mathbf{u}}\sim\overline{\overline{\mathbf{u}}\oplus\overline{\mathbf{u}}}-\overline{\overline{\mathbf{u}}}\oplus\overline{\overline{\mathbf{u}}}.$$

Denoting by ε the length scale of the filter, this hypothesis can be rephrased as follows: "the eddies smaller than $\mathcal{O}(\varepsilon)$ interact with those $\mathcal{O}(\varepsilon)$ in the mean in the same way those $\mathcal{O}(\varepsilon)$ interact with those $\mathcal{O}(2\varepsilon)$."

Within a two-scale approximation framework, the approximate subgrid tensor assumes the following form:

$$\mathbb{T}_h(\mathbf{u}_h)=\mathbf{P}_H(\mathbf{u}_h\otimes\mathbf{u}_h)-\mathbf{P}_H(\mathbf{u}_h)\otimes\mathbf{P}_H(\mathbf{u}_h),$$

and the corresponding weak form is

$$a_{sgs}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h)=(\nabla\cdot\mathbb{T}_h(\mathbf{u}_h),\mathbf{v}_h), \qquad \mathbf{u}_h\in\mathbf{X}_h,\ \mathbf{v}_h\in\mathbf{X}_h. \tag{6.7}$$

When compared to DNS, this model shows very high correlations with the actual subgrid tensor [1, 51]. However, in practice, according to Ferziger [13], "it is found that this model hardly dissipates any energy and cannot serve as a stand alone SGS model". Actually, put more bluntly, when used alone this model yields unsustainable numerical instabilities. The main difficulty is that the semilinear form a_{sgs} is not positive, hence yielding serious stability problems. The problem has been analyzed by Layton [37], and the author proposed slight modifications yielding coherent energy estimates.

Noting that the scale-similarity hypothesis can be restated as follows:

$$\overline{\mathbf{u}\oplus\mathbf{u}}\sim\overline{\mathbf{u}}\oplus\overline{\mathbf{u}}+\overline{\overline{\mathbf{u}}\oplus\overline{\mathbf{u}}}-\overline{\overline{\mathbf{u}}}\oplus\overline{\overline{\mathbf{u}}}, \tag{6.8}$$

and using the notation $\overline{\mathbf{u}}'=\overline{\overline{\mathbf{u}}}-\overline{\mathbf{u}}$, Layton in [37] proposes to further develop the right-hand side of (6.8) as follows

$$\overline{\mathbf{u}\oplus\mathbf{u}}\sim\overline{\overline{\mathbf{u}}}\oplus\overline{\overline{\mathbf{u}}}+\overline{\overline{\overline{\mathbf{u}}}\oplus\overline{\mathbf{u}}'+\overline{\mathbf{u}}'\oplus\overline{\overline{\mathbf{u}}}}+\mathbf{R}(\overline{\mathbf{u}}),$$

where the residual $\mathbf{R}(\overline{\mathbf{u}})$ is defined by

$$\mathbf{R}(\overline{\mathbf{u}}) = \overline{\mathbf{u}} \oplus \overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}} \oplus \overline{\overline{\mathbf{u}}} + \overline{\overline{\overline{\mathbf{u}}} \oplus \overline{\overline{\mathbf{u}}}} - \overline{\overline{\mathbf{u}}} \oplus \overline{\overline{\mathbf{u}}} + \overline{\overline{\mathbf{u}}' \oplus \overline{\mathbf{u}}'}.$$

The tensor $\mathbf{R}$ is then modeled by using one of the dissipative subgrid viscosity models presented in the previous section 6.2.1, and we shall denote by $a_{sgs}(\mathbf{u}; \mathbf{u}, \mathbf{v})$ the corresponding semilinear weak from.

Now let us turn our attention to the remaining tensor. The original result pointed out in [37] is that this tensor is dispersive; *i.e.*, its contribution to the global kinetic energy balance is zero. Actually, owing to the incompressibility constraint and provided the filter is self-adjoint and commutes with differentiation, it can be proved that the following semilinear form is positive:

$$\int_\Omega \nabla\cdot \left(\overline{\overline{\mathbf{u}}} \oplus \overline{\overline{\mathbf{u}}} + \overline{\overline{\mathbf{u}}} \oplus \overline{\mathbf{u}}' + \overline{\mathbf{u}}' \oplus \overline{\overline{\mathbf{u}}}\right) \cdot \overline{\mathbf{v}}\, \mathrm{d}x.$$

From the approximation point of view, it is not realistic to expect the incompressibility, the commutation, and the self-adjointness hypotheses to be satisfied exactly. Hence, we now propose to further modify the formulation by rewriting the transport term in rotational form using the identity

$$\mathbf{w}\cdot\nabla\mathbf{w} = (\nabla\times\mathbf{w})\times\mathbf{w} + \tfrac{1}{2}\nabla\mathbf{w}^2.$$

Then, changing the definition of the pressure accordingly, we write the transport term as follows:

$$\int_\Omega \left[(\nabla\times\overline{\overline{\mathbf{u}}})\times\overline{\overline{\mathbf{u}}}\right]\cdot\overline{\mathbf{v}} + \left[(\nabla\times\overline{\overline{\mathbf{u}}})\times\overline{\mathbf{u}}' + \overline{\mathbf{u}}'\times(\nabla\times\overline{\overline{\mathbf{u}}})\right]\cdot\overline{\overline{\mathbf{v}}}\,\mathrm{d}x$$

It is clear now that setting $\overline{\mathbf{v}} = \overline{\mathbf{u}}$ in this new semilinear form yields zero without any additional assumption. In this form, the contribution of the combination of the transport term and the subgrid term to the kinetic energy balance is zero pointwise.

At the discrete level we replace $\overline{\mathbf{u}}$ by $\mathbf{u}_h$ and $\overline{\overline{\mathbf{u}}}$ by $\mathbf{u}_H$. Denoting the subgrid component of $\mathbf{u}_h$ by $\mathbf{u}_h^H = \mathbf{u}_h - \mathbf{u}_H$, the discrete two-scale Navier–Stokes problem can now be written as follows:

Find $\mathbf{u}_h \in \mathcal{C}^1(0,T;\mathbf{X}_h)$ and $p_h \in \mathcal{C}^0(0,T;M_h)$ such that

$$\left\{\begin{array}{l} (d_t\mathbf{u}_h,\mathbf{v}_h) + \big((\nabla\times\mathbf{u}_H)\times\mathbf{u}_H,\mathbf{v}_h\big) - (\pi_h,\nabla\cdot\mathbf{v}_h) + \nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) \\ \qquad\qquad + \big((\nabla\times\mathbf{u}_H)\times\mathbf{u}_h^H + (\nabla\times\mathbf{u}_h^H)\times\mathbf{u}_H, \mathbf{P}_H\mathbf{v}_h\big) \\ \qquad\qquad + a_{sgs}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{X}_h \\ (q_h,\nabla\cdot\mathbf{u}_h) = 0, \quad \forall q_h \in M_h \\ \mathbf{u}_h|_{t=0} = \mathcal{I}_h\mathbf{u}_0, \end{array}\right. \tag{6.9}$$

where π_h is an approximation of the total pressure.

6.4. **Subgrid viscosity stabilization.** The goal of this section is to give a partial mathematical justification for the two-scale LES discrete models introduced above. We show that these models are closely related to a class of stabilizing techniques that solves non-coercive PDE's. More precisely,

we show that adding a two-scale subgrid viscosity to the Galerkin formulation yields a method that is optimal for approximating linear contraction semi-groups of class $\mathcal{C}^0$. Throughout this section we shall use the following definition:

Definition 6.1 (Graph of a function and graph norm)**.** *Let A and B two normed spaces and let $f : A \longrightarrow B$. The graph of f is defined as the subset of $A \times B$ such as*

$$\textit{graph } f = \{(x, f(x));\ x \in A\}.$$

The graph norm on A, associated with the function f, is defined as:

$$\|x\|_f = \|x\|_A + \|f(x)\|_B$$

where $\|\cdot\|_A$ and $\|\cdot\|_B$ are norms defined on A and B respectively.

6.4.1. *The continuous setting.* To illustrate our point of view, we consider hereafter the following linear problem.

For $f \in \mathcal{C}^1([0, +\infty[; L)$ and $u_0 \in D(A)$,
find $u \in \mathcal{C}^1([0, +\infty[; L) \cap \mathcal{C}^0([0, +\infty[; D(A))$ such that

$$\begin{cases} u_{|t=0} = u_0, \\ d_t u + Au = f, \end{cases} \tag{6.10}$$

where L is a separable Hilbert space and $A : D(A) \subset L \longrightarrow L$ is a linear operator. We assume that A is monotone:

$$\forall v \in D(A), \qquad (Av, v)_L \geq 0, \tag{6.11}$$

and A is maximal

$$\forall f \in L,\ \exists v \in D(A), \qquad v + Av = f. \tag{6.12}$$

The reader may think of $L = L^2(\mathbb{R})$ and $A = \partial_x$. Now let us set $V = D(A)$ and let us equip V with the graph norm: $\|v\|_V = (\|v\|_L^2 + \|Av\|_L^2)^{1/2}$. It can be shown that, the graph of A being closed, V is a Hilbert space when equipped with the scalar product $(u, v)_L + (Au, Av)_L$. For the sake of the simplicity of the presentation, we assume hereafter that V is a space of vector-valued functions on Ω in $\mathbb{R}^m$, $m \geq 1$.

Owing to the Hille–Yosida theorem (see *e.g.* Brezis [3, p. 110] or Yosida [62, p. 248]), problem (6.10) is well-posed and admits the following stability properties

$$\begin{cases} \|u\|_{\mathcal{C}^0([0,T];L)} \leq c\,(\|u_0\|_L + T\|f\|_{\mathcal{C}^0([0,T];L)}), \\ \|u\|_{\mathcal{C}^1([0,T];L)} + \|u\|_{\mathcal{C}^0([0,T];V)} \leq c\,(\|u_0\|_V + T\|f\|_{\mathcal{C}^1([0,T];L)}). \end{cases} \tag{6.13}$$

The fact that A is maximal is a key ingredient for proving the wellposedness of (6.10). This property can be better understood in the light of the following proposition:

Proposition 6.1. *Let $E \subset F$ be two Hilbert spaces with dense and continuous embedding, and let $T \in \mathcal{L}(E; F)$ be a monotone operator. The following two properties are equivalent.*

(i) T is maximal.
(ii) There exists two constants $c_1 > 0$, $c_2 \geq 0$ such that

$$\forall u \in E, \qquad \sup_{v \in F} \frac{(Tu, v)_F}{\|v\|_F} \geq c_1 \|u\|_E - c_2 \|u\|_F. \tag{6.14}$$

The key to the theory developed herein is to build a discrete framework for which a discrete counterpart to (6.14) holds.

When it comes to approximating the solution to (6.10), it is known that the Galerkin technique is not appropriate if A is not coercive. In general, it is not possible to guarantee optimal convergence in the graph norm, since the discrete counterpart of (6.14) is usually not satisfied uniformly with respect to the mesh-size. As a consequence, when approximating this type of equation supplemented with non-smooth data, the approximate solution exhibits spurious node-to-node oscillations. The two-level subgrid viscosity technique developed in Guermond [22, 24, 23] is one possible cure to this problem.

6.4.2. *The discrete setting.* Let us introduce three finite dimensional spaces X_h, X_H, X_h^H such that

$$V \supset X_h = X_H \oplus X_h^H. \tag{6.15}$$

We assume that X_h, X_H have suitable interpolation properties; that is, there is a dense subspace $W \subset V$ together with a linear interpolation operator $I_H \in \mathcal{L}(W; X_H)$ and two constants $k > 0$, $c > 0$ such that

$$\forall H,\ \forall v \in W, \quad \|v - I_H v\|_L + H\|v - I_H v\|_V \leq cH^{k+1}\|v\|_W. \tag{6.16}$$

One may view X_h as a fine scale space, X_H a coarse scale space and X_h^H a subgrid scale space where basis functions are highly fluctuating.

Denoting by h and H the mesh-sizes on which X_h and X_H are built respectively, we assume that H and h are of the same order; i.e., $c_1 h \leq H \leq c_2 h$. In practice we shall always use $H = 2h$. As a result, X_h being finite dimensional, we assume that there exists $c_i > 0$, independent of h and H, s.t.

$$\forall v_h \in X_h, \qquad \|v_h\|_V \leq c_i H^{-1} \|v_h\|_L. \tag{6.17}$$

Note that this hypothesis indirectly implies that A is a first order differential operator and justifies the assumption $c_1 h \leq H \leq c_2 h$.

We define $P_H : X_h \longrightarrow X_H$ as being the projection of X_h onto X_H that is parallel to X_h^H. We assume that P_H is stable in the norm of L uniformly with respect to H and h. For all v_h in X_h we denote

$$v_H = P_H v_h \qquad \text{and} \qquad v_h^H = (1 - P_H) v_h. \tag{6.18}$$

We introduce $b_h \in \mathcal{L}(X_h^H, X_h^H)$ s.t. for all $(v_h^H, w_h^H) \in X_h^H \times X_h^H$,

$$b_h(v_h^H, w_h^H) = H \int_\Omega \nabla v_h^H \cdot \nabla w_h^H \, \mathrm{d}x. \tag{6.19}$$

Note that b_h is a dissipative bilinear form. It is a subgrid viscosity that acts only on the subgrid scales. This property has to be put in parallel with the subgrid model (6.5) advocated in Hughes, Mazzei, and Jansen [27] for which the theory discussed herein may be regarded as a partial justification.

Now we introduce the main hypothesis of this section, *i.e.* we assume that a discrete version of (6.14) is satisfied. More precisely, we assume that there are $c_1 > 0$ and $c_2 \geq 0$, independent of (H, h), such that

$$\forall v_h \in X_h, \quad \sup_{\phi_h \in X_h} \frac{(Av_H, \phi_h)_L}{\|\phi_h\|_L} \geq c_1 \|v_H\|_V - c_2 \|v_h\|_L. \tag{6.20}$$

Let us assume that $u_0 \in W$ so that u_0 can be approximated by $I_H u_0$. Then, the discrete problem we consider is

$$\begin{cases} \text{Find } u_h \in \mathcal{C}^1([0, +\infty[; X_h) \text{ s.t.} \\ (d_t u_h, v_h)_L + (Au_h, v_h)_L + b_h(u_h^H, v_h^H) = (f, v_h)_L, \qquad \forall v_h \in X_h, \\ u_{h|t=0} = I_H u_0. \end{cases} \tag{6.21}$$

This problem has a unique solution, for it is a system of linear ordinary differential equations. The major convergence result of this section is the following.

Theorem 6.1. *Under hypotheses (6.16), (6.17), (6.18), and (6.19), (6.20), if u is in $\mathcal{C}^2([0,T];W)$, then u_h satisfies the following error estimates.*

$$\|u - u_h\|_{\mathcal{C}^0([0,T];L)} \leq c_1 H^{k+1/2}, \tag{6.22}$$

$$\left[\frac{1}{T}\int_0^T \|u - u_h\|_V^2\right]^{1/2} \leq c_2 H^k, \tag{6.23}$$

where the constants c_1 and c_2 are bounded from above as follows.

$$c_1 \leq c\,[H + T\,(1+T)]^{1/2}\,\|u\|_{\mathcal{C}^2([0,T];W)}, \qquad c_2 \leq c\,[1+T]\,\|u\|_{\mathcal{C}^2([0,T];W)}.$$

Note that the norms used in the error estimates are the same as those of the stability estimates (6.13). The estimate (6.23) is optimal in the graph norm. The estimate (6.22) is the same as that obtained by the Discontinuous Galerkin technique (see for instance Johnson–Pitkäranta [29]).

The present theory can be extended to the coercive case; *i.e.*, when the differential operator is of the form $A+\varepsilon D$, where A is a first order differential operator and D is a coercive second order differential operator. From the mathematical point of view, the coercivity of D implies that the evolution equation is parabolic. If ε is $\mathcal{O}(1)$, the Galerkin technique is optimal, but if ε is small, the coercivity is not strong enough to guarantee that the Galerkin approximation is satisfactory. It is shown in [23] that by using the same two-level framework as above and by perturbing the Galerkin technique with the same bilinear form b_h as above, Theorem 6.1 still holds. The remarkable result here is that the estimates are uniform with respect to ε, and optimal convergence is obtained in the graph norm of A. If we think of A as being a transport operator and D being a Laplacian, then convergence on the

gradient of the solution in the streamwise direction is guaranteed whereas only L^2 convergence is guaranteed in the crosswind direction.

6.4.3. *Examples and extension to the nonlinear case.* Different discrete functional framework satisfying the hypotheses above are described in [22, 24, 23]. Let us describe the simplest one, *i.e.* the $\mathbb{P}_1$ framework, for it is probably used implicitly by many authors in LES. For the sake of simplicity we assume that Ω is a polyhedron in $\mathbb{R}^d$ ($d = 2$ or 3) and $\mathcal{T}_H = \cup\{K_H\}$ is a regular triangulation of Ω composed of affine simplices. Let us assume that V is composed of vector-valued functions in $\mathbb{R}^m$. Let us define first X_H by

$$X_H = \{v_H \in H^1(\Omega)^m;\ v_{H|K_H} \in \mathbb{P}_1(K_H)^m,\ \forall K_H \in \mathcal{T}_H\}. \tag{6.24}$$

In 2D, from each triangle $K_H \in \mathcal{T}_H$, we create four new triangles by connecting the middles of the three edges of K_H. In 3D, from each tetrahedron, we create eight new tetrahedra by proceeding as follows: on each face we connect the middles of the edges, and choosing arbitrarily two non-intersecting edges we connect the middles of these two edges. Let $h = H/2$ and $\mathcal{T}_h$ denote the resulting new triangulation. For each macro-simplex K_H, we define $\mathbb{P}$ as being the space of functions that are continuous on K_H, vanish at the vertices of K_H, and are piecewise $\mathbb{P}_1$ on each sub-simplex of K_H. We define

$$X_h^H = \{v_h^H \in H^1(\Omega)^m \mid v_{h|K_H}^H \in \mathbb{P}^m,\ \forall K_H \in \mathcal{T}_H\}. \tag{6.25}$$

By setting $X_h = X_H \oplus X_h^H$, it is clear that we can characterize X_h by

$$X_h = \{v_h \in H^1(\Omega)^m \mid v_{h|K_h} \in \mathbb{P}_1(K_h)^m,\ \forall K_h \in \mathcal{T}_h\}. \tag{6.26}$$

The couple (X_H, X_h) is referred to as the two-level $\mathbb{P}_1$ setting.

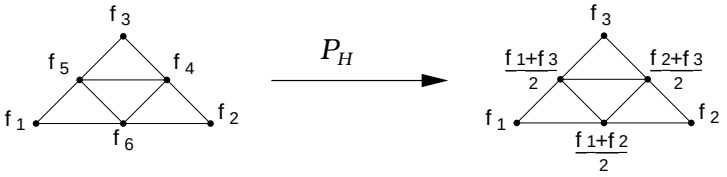

FIGURE 4. Definition of P_H for the two-level $\mathbb{P}_1$ setting.

In Figure 4 we show a schematic representation of the action of the filter $P_H : X_h \longrightarrow X_H$ on a macro-element K_H of $\mathcal{T}_H$.

Numerical tests presented in [22, 24, 23] reveal that the present technique performs as predicted in the theory. However, when it is tested on rough solutions, *i.e.* discontinuous solutions or solutions exhibiting shocks, localized spurious oscillations are still present in the vicinity of discontinuities. These residual oscillations are due to the Gibbs phenomenon, which is well-known to those working on nonlinear conservation laws. It is the manifestation of a well-known theorem in analysis that states that truncated Fourier series of a given function does not converge uniformly to the function in question unless the function is very smooth (continuity is not enough), see Rudin [52,

p. 97–98] for more details. A simple trick to eliminate this undesired oscillations consists of adding strong dissipation in the region where the solution is rough. Of course, one does not know *a priori* where the solution is rough, but one may expect that in this region the quantity $\nabla u_h^H = \nabla(u_h - P_H u_h)$ is of the same order as ∇u_h. Hence, we are led to introduce the following shock-capturing nonlinear form:

$$c_h(u_h; u_h, v_h) = c_{sc} H \int_\Omega \frac{|\nabla u_h^H|}{|\nabla u_h|} \nabla u_h \cdot \nabla v_h \, \mathrm{d}x. \tag{6.27}$$

Then, (6.21) must be modified by replacing $b_h(u_h^H, v_h^H)$ by $b_h(u_h^H, v_h^H) + c_h(u_h^H, u_h, v_h)$. Unfortunately, no theory is available for supporting the presence of c_h, though it has been shown to be extremely efficient in applications. Note that in regions where $|\nabla u_h^H|$ is of order $|\nabla u_h|$, the added viscosity is $\mathcal{O}(H)$ as it should be. Note also the similarity between this term and (6.1) and (6.2) where for dimensional reasons $H|\nabla u_h^H|/|\nabla u_h|$ is replaced by $H^2|\nabla u_h^H|$; hence, we can also interpret (6.3) as a shock-capturing operator.

We recall in passing that if in (6.27), $|\nabla u_h^H|$ is replaced by the residual of the equation, then it can be proved for scalar conservation laws that this term yields an L^∞-estimate on the approximate unknown (*i.e.* a maximum principle) that guarantees convergence to the entropy solution (see the series of papers by Johnson, Szepessy *et al.* [30, 31, 59])

We now provide a spectral interpretation of the respective effects of b_h and c_h. The bilinear form b_h is a viscosity term that takes care of the non-coercive (*i.e.* hyperbolic) character of the equation, whereas c_h is a shock capturing term that takes care of the Gibbs phenomenon induced by discontinuities. The bilinear form b_h suppresses wide-spreading (*i.e. unlocalized*) node-to-node oscillations induced by the lack of coercivity. As a result, this term attenuates only the high wave-number modes, possibly producing a dip on the tail of the spectrum of the solution as observed in [6] when Galerkin Least Square is used alone to simulate the Kolmogorov cascade. On the other hand, Gibbs oscillations induced by discontinuities are very localized around the discontinuities, which means that their spectral range is wide and centered in the intermediate wave-number modes. As a result, c_h acts on intermediate wave-number modes. These crude observations may explain why stabilization and shock capturing must be used jointly to reproduce the $k^{-5/3}$ cascade in LES as observed in [6]. In conclusion, the selection of the values of the constants scaling b_h and c_h is an issue that seems to be important in practice. Some tentative mathematical answers such as in [4] could be proposed to evaluate optimal constants, but also a pragmatic option could be to utilize one of the strategies suggested by Lilly in [43].

7. Concluding remarks

Our motivation and goal in writing the present review were to provide a mathematically sound answer to the question: what should be a good LES model?

We have shown that filtering, although widely accepted as a paradigm in the LES literature, yields a paradox. Indeed, after a result from Germano, it can be shown that there exist filters such that exact closure is possible, *i.e.*, the subgrid tensor appearing in the filtered Navier–Stokes equations can be expressed exactly in terms of the filtered velocity only. However the whole procedure does not yield any gain since the two solution sets are isomorphic. Roughly speaking, this means that the number of degree of freedoms to represent a solution of the Navier–Stokes equations should be identical to that necessary to represent a solution of the filtered equations. This result shed doubts on the relevance of filtering the whole Navier-Stokes equations as usually done in the literature.

Mathematical analysis has lead us to advance two criteria that good LES models should satisfy:

(i) They should be regularization techniques that transform the (possibly ill-posed) Navier-Stokes equations into a wellposed set of PDE's.
(ii) They should select physically relevant solutions of the Navier-Stokes equations (*i.e.* "dissipative solutions" *à la* Duchon-Robert or "suitable weak solutions" *à la* Cafarelli, Khon, Nirenberg, and Scheffer)

Among the admissible techniques are those that regularize the nonlinear advection term by partial filtering (the Leray regularization and the NS-α model). It seems that these techniques are the most appropriate justifications to date for all the filtering methodologies currently employed in LES. Other admissible techniques are those which consist in adding a vanishing nonlinear viscosity to the momentum equation (*i.e.*, the Smagorinsky and Ladyženskaja models). Contrary to what is frequently claimed in the literature, these regularization techniques do not require any kind of filtering to be justified. Concerning the second criterion, regularization techniques *à la* Leray seem to be good candidates. That is, Navier–Stokes solutions obtained as a limit of a sequence (or subsequence) of regularized solutions satisfy the local energy inequality (3.15), whereas limits of Galerkin approximations of the non-regularized Navier–Stokes equations may not.

We have also studied vanishing eddy viscosity methods based on spectral approximation. Contrary to what is sometimes claimed, we have shown that these methods are radically different from the nonlinear regularizations *à la* Smagorinsky, for the latter yields unique solutions whereas uniqueness is an open question for the former. Hence, spectral eddy viscosity methods do not comply with the first criterion we have proposed. Comparing Kraichnan-like eddy viscosity techniques and Tadmor's spectral viscosity, it appears that the former pollutes the spectral accuracy whereas the latter is optimal in this respect.

Finally we have made an attempt to mathematically justify some classes of two-scale LES approximation techniques. We have shown that these numerical methods are related to a subgrid stabilization technique that solves non-coercive PDE's.

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[2]TICAM, The University of Texas at Austin, TX 78712, USA
Current address: [1]LIMSI (CNRS-UPR 3152), BP 133, 91403, Orsay, France
E-mail address: guermond@limsi.fr
E-mail address: oden@ticam.utexas.edu, serge@ticam.utexas.edu