Optimal Estimates for Lower and Upper Bounds of Approximation Errors in the p-version of the Finite Element Method in Two Dimensions

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Dedicated to Prof. O. Widlund on the occasion of his 60th birthday

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Abstract. This paper will analyze the lower and upper error bounds of the finite element solution of the p-version for linear elliptic problems in polygonal domains. The optimal rate of convergence is rigorously proved based on the sharp estimates of lower and upper bounds of the approximation error.

1 Introduction

The p-version, the h-version and h–p-version are the three basic approaches of the finite element method. The p-version achieves accuracy by increasing the element degree p on a fixed mesh. It was proved in 1980 that the p-version finite element solution converges in the energy norm at least as fast as the h-version and that it converges twice as fast as the h-version if the singularity of r^γ-type in solution occurs near vertices of nonsmooth domains, where r denotes the distance to the vertices. The convergence rate, given in ([**?**]), is of order $O(p^{-2\gamma+\epsilon})$ with $\epsilon > 0$, arbitrary. The result was sharpened in 1986 by removing ϵ (see [**?**, **?**]). Since then, the p-version of the finite element method has developed rapidly in all aspects such as approximation, solution of the resulting linear algebraic system, and applications to mechanics and engineering. There are several commercial and research codes based on the p and h–p version, including MSC/PROBE, Applied Structure, PolyFEM, Stresscheck, PHLEX and STRIPE.

Although significant progress has been made in the past two decades, several important issues of the p-version in two dimensions remain to be addressed. First, the lower bound of approximation error and the optimal rate of convergence for the p-version finite element method has to be established. Second, the inverse approximation theorems for the p-version, which provide knowledge on the regularity of solutions of the problems under consideration based on information gathered by computation, is of great theoretical and practical interest. Third, a reliable and accurate a-posteriori error estimator for the p-version is not available,

It is known that the usual Sobolev spaces are not adequate for problems on nonsmooth domains. The usual Besov space may be a very good vehicle for the h-version approximation, but it is obviously not adequate for the p-version approximation for the problems on nonsmooth domains. Hence it seems that usual Sobolev and Besov spaces are not effective mathematical tools to address the important issues mentioned above. The performance of p-version in one dimension was adequately analyzed in [?] and a precise asymptotic expression of the approximation error of the p-version was given there. Unfortunately, the arguments in [?] can not be carried over to the higher-dimensional setting.

After two decades of effort, we have recently found that the best mathematical tool for the p-version in two and three dimensions is the weighted Besov spaces with Jacobi weights (see [?]). These spaces allow us to effectively characterize the singularity of the solution of problems on nonsmooth domains, to precisely estimate the lower and upper bounds of the approximation error, and to establish optimal convergence of the p-version. The direct and inverse approximation theorems in the framework of weighted Besov spaces perfectly reflect the relations between the approximation and the singularity of the solutions. All of these results can lead to reliable a-posteriori error estimation and adaptive strategy for the p-version.

This paper will concentrate on the estimate on lower and upper bounds of the p-version approximation error in two dimensions. The rest of the paper is organized as follows : In Section 2 we introduce the weighted Sobolev space $H^{k,\beta}(Q)$ and Besov space $B^{s,\beta}(Q)$ over square domain $Q = (-1,1)^2$ for functions of r^γ-type, and the lower and upper bounds of the p-version approximation error are proved based on the space $B^{s,\beta}(Q)$. In order to adequately analyze the lower and upper bound of the p-version approximation for functions of $r^\gamma \log^\nu r$-type with integer $\nu > 0$, we modify the weighted Besov space $B^{s,\beta}(Q)$ to $B^{s,\beta}_\nu(Q)$. It is interesting to note that the results for integer γ and non-integer γ are different. For example, if γ is an integer and $\nu = 1$, the factor $\log p$ will not appear in both lower and upper bound of the error. Furthermore, we generalize the weighted Besov spaces and approximation results to an n-dimensional setting. The lower and upper bound of errors of the p-version finite element solutions for elliptic boundary value problems and the optimal rate of convergence are proved in Section 3.

2 Approximation Properties in the n-dimensional Cube

In this section we shall investigate the approximation properties in the n-dimensional cube for functions in weighted Sobolev spaces and weighted Besov spaces. For the sake of exposition we shall give rigorous proofs in 2 dimensions, then generalize the result to the n-dimensional case.

Throughout this section, we denote the n-dimensional cube by $I^n = (-1,1)^n$; in particular, we write $I = I^1 = (-1,1)$ and $Q = I^2 = (-1,1)^2$.

2.1 Sobolev and Besov spaces with Jacobi weights

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be two n-tuples with integers $\alpha_i \geq 0$ and real numbers $\beta_i > -1$, $1 \leq i \leq n$. A weight function associated with α and β is defined as

$$W_{\alpha,\beta}(x) = \prod_{i=1}^{n} (1 - {x_i}^2)^{\alpha_i + \beta_i} \tag{2.1}$$

and the spaces $H^{k,\beta}(I^n)$ with integer $k \geq 0$ is defined as a closure of C^∞-functions furnished with a weighted norm:

$$\begin{aligned} \|u\|_{H^{k,\beta}(I^n)} &= \left\{ \sum_{|\alpha|=0}^{k} \int_{I^n} |D^\alpha u|^2 W_{\alpha\beta}(x) dx \right\}^{1/2} \\ &= \left\{ \sum_{|\alpha|=0}^{k} \int_{I^n} |D^\alpha u|^2 \prod_{i=1}^{n} (1 - {x_i}^2)^{\alpha_i + \beta_i} dx \right\}^{1/2}, \end{aligned} \tag{2.2}$$

where $D^\alpha u = u_{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}}$. By $|u|_{H^{k,\beta}(Q)}$ we denote the semi-norm involving the highest derivatives of u.

For integers ℓ and k, $\ell < k$, and $\theta \in (0,1)$ we introduce an interpolation space

$$B^{s,\beta}(I^n) = \left(H^{\ell,\beta}(I^n), H^{k,\beta}(I^n)\right)_{\theta,\infty} \tag{2.3a}$$

with $s = (1-\theta)\ell + \theta k$. This space is referred to as a weighted Besov space with the norm

$$\|u\|_{B^{s,\beta}(I^n)} = \sup_{t>0} \; t^{-\theta} K(t,u), \tag{2.3b}$$

where

$$K(t,u) = \inf_{u=v+w} \left(\|v\|_{H^{\ell,\beta}(I^n)} + t\|w\|_{H^{k,\beta}(I^n)} \right). \tag{2.3c}$$

Note that the Jacobi polynomials $p_m^{(\beta,\beta)}(t)$ and its derivatives $\frac{d^\alpha}{dt^\alpha} p_m^{(\beta,\beta)}(t)$ are orthogonal with the weight $W_{\alpha\beta}(t) = (1-t^2)^{\alpha+\beta}$. Hence we refer to the weight function in (2.1) as the Jacobi weight and refer to $\beta = (\beta_1, \beta_2, \ldots \beta_n)$ as the exponent of the Jacobi weight, or the Jacobi exponent. The spaces $H^{k,\beta}(I^n)$ and $B^{s,\beta}(I^n)$ are referred to as the weighted Sobolev space and Besov space with the Jacobi weight, respectively.

By $P_p(I^n)$ we denote the collection of polynomials of (separate) degree $\leq p$. We have the following approximation properties for functions in $H^{k,\beta}(I^n)$ and $B^{s,\beta}(I^n)$

Theorem 2.1. *Let $u \in H^{k,\beta}(I^n)$, $k \geq 0$, and let u_p be the projection of u on $P_p(I^n)$ in $H^{0,\beta}(I^n)$. Then for $\ell \leq k$*

$$|u - u_p|_{H^{\ell,\beta}(I^n)} \leq cp^{-(k-\ell)}.|u|_{H^{k,\beta}(I^n)}. \tag{2.4}$$

Proof. For $u \in H^{k,\beta}(I^n)$, $k \geq 0$, there is an expansion by the Jacobi polynomials

$$u = \sum_{0 \leq m_1 < \infty} \cdots \sum_{0 \leq m_n < \infty} a_{m_1,m_2 \ldots m_n} \prod_{i=1}^{n} p_{m_i}^{(\beta_i,\beta_i)}(x_i).$$

The properties of the Jacobi polynomials (see [11]):

$$\frac{d}{dt} p_m^{(\beta,\beta)}(t) = \frac{1}{2}(m + \beta + 1) p_{m-1}^{(\beta+1,\beta+1)}(t),$$

and

$$\int_{-1}^{1} \frac{d^k}{dt^k} p_m^{(\beta,\beta)}(t) \frac{d^k}{dt^k} p_\ell^{(\beta,\beta)}(t)(1 - t^2)^{k+\beta}\, dt$$
$$= \begin{cases} 0 & \text{if } m \neq \ell \\ \dfrac{2^{2(\beta+1)} \left[(m + 2\beta + 1) \cdots (m + 2\beta + k)\right]^2 \Gamma(m + 2\beta + 1)^2}{(m - k)!(2m + 2\beta + 1)\Gamma(m + k + 2\beta + 1)} & \text{if } m = \ell \end{cases}$$

lead directly to (2.4). For the details, we refer to [?]. □

Since $B^{s,\beta}(I^n)$ is an exact interpolation space of exponent θ, defined by the usual K-method, the standard arguments of interpolation spaces yields the approximability of functions in $B^{s,\beta}(I^n)$.

Theorem 2.2. *Let $u \in B^{s,\beta}(I^n)$, $s > 0$, and let u_p be the projection of u on $P_p(I^n)$ in $H^{0,\beta}(I^n)$. Then for $\ell < s$*

$$\|u - u_p\|_{H^{\ell,\beta}(I^n)} \leq cp^{-(s-\ell)} \|u\|_{B^{s,\beta}(I^n)}. \tag{2.5}$$

2.2 Upper Bound of Approximation Error for Functions of r^γ-type

Let $A_1 = (-1, -1)$ be the vertex of Q and Γ_1 be the edge $\{x = (x_1, -1) | x_1 \in I\}$ and let (r, θ) denote the polar coordinates with respect to the vertex A_1 and the edge Γ_1.

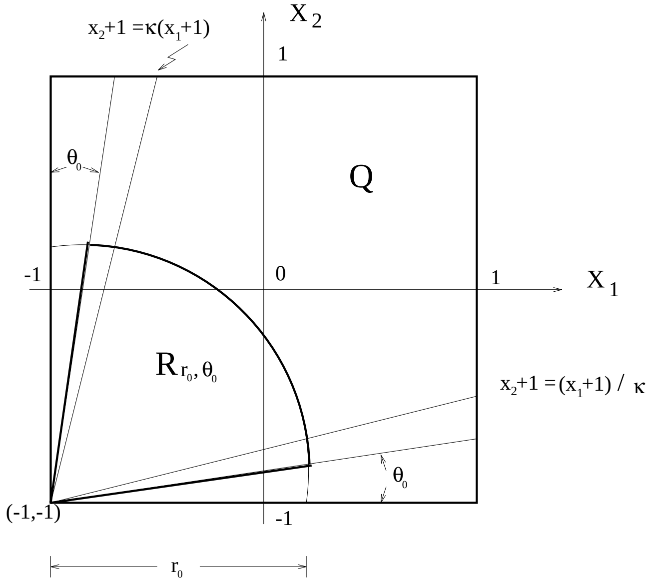

Fig. 2.1 Square Domain Ω and sub region R_{r_0,θ_0}

We consider the approximation to the function on Q

$$u(x) = r^\gamma \chi(r) \Phi(\theta), \tag{2.6}$$

where $\gamma > 0$, $\chi(r)$ and $\Phi(r)$ are C^∞ functions such that for $0 < r_0 < 2$ and $0 < \theta_0 < \pi/2$,

$$\begin{matrix} \chi(r) = 1 \quad \text{for } 0 < r < r_0/2 \\ \chi(r) = 0 \quad \text{for } r > r_0 \end{matrix} , \tag{2.7a}$$

and

$$\Phi(\theta) = 0 \quad \text{for } \theta \bar{\in} (\theta_0, \pi/2 - \theta_0). \tag{2.7b}$$

Obviously, $\operatorname{Supp} u \subset R_0 = R_{r_0,\theta_0}$, where

$$R_{r_0,\theta_0} = \{x \in Q | 0 < r < r_0, \theta \in (\theta_0, \pi/2 - \theta_0)\} \tag{2.8}$$

, which is shown in Fig. 2.1.

In order to effectively approximate the function given in (2.6) we have to characterize the function in terms of the weighted Sobolev and Besov spaces introduced in Section 2.1.

Theorem 2.3. *Let u be the function given in (2.6). Then $u \in H^{1+[2\gamma],\beta}(Q)$ and $u \in B^{1+2\gamma,\beta}(Q)$ with $\beta = (-1/2, -1/2)$. Hereafter $[a]$ denotes the biggest integer $< a$.*

Proof. It is trivial to verify that $u \in H^{1+[2\gamma],\beta}(Q)$ by definition. To show $u \in B^{1+2\gamma,\beta}(Q)$, we need to write $u = v + w$ with $v = \varphi_\delta(r)u$ and $w = (1 - \varphi_\delta(r))u$ where $\varphi_\delta(r)$ is a C^∞-function such that $\varphi_\delta(r) = 1$ for $0 < r < \delta/2$ and $\varphi_\delta(r) = 0$ for $r > \delta$. Obviously $v \in H^{\ell,\beta}(Q)$ for any $0 \leq \ell < 1 + [2\gamma]$ and $w \in H^{k,\beta}(Q)$ for any $k > 1 + 2\gamma$. Furthermore, it is seen that

$$\|v\|_{H^{\ell,\beta}(Q)} \leq C\delta^{\gamma+\frac{1}{2}(1-\ell)}$$

and

$$\|w\|_{H^{k,\beta}(Q)} \leq C\delta^{\gamma+\frac{1}{2}(1-k)},$$

which implies that

$$K(t,u) \leq C\left(\delta^{\gamma+\frac{1}{2}(1-\ell)} + t\delta^{\gamma+\frac{1}{2}(1-k)}\right).$$

Selecting $\delta = t^{\frac{2}{k-\ell}}$ and $\theta = \frac{1+2\gamma-\ell}{k-\ell}$, we have for $0 < t < 1$,

$$\sup_{0<t<1} t^{-\theta} K(t,u) \leq C.$$

For $t > 1$ it always holds that

$$\sup_{t>1} t^{-\theta} K(t,u) \leq \|u\|_{H^{\ell,\beta}(Q)} \leq \|u\|_{H^{1+[2\gamma],\beta}(Q)},$$

which completes the proof. $\square$

Remark 2.1. In one dimension, we consider the function of x^γ-type with $\gamma > 1/2$

$$u(x) = x^\gamma \chi(x).$$

It can easily be shown that $u'(x) \in B^{2\gamma-1,\beta}(I)$ with $\beta = 0$, which is the interpretation of Theorem 2.3 for $n = 1$ with $\beta = -1$.

The theorem above, together with Theorem 2.2, leads to the next approximation theorem.

Theorem 2.4. *Let u be given in (2.6). Then*

$$\inf_{\varphi \in P_p(Q)} \|u - \varphi\|_{H^1(R_{r_0,\theta_0})} \leq C p^{-2\gamma}, \tag{2.9}$$

where $C > 0$ is independent of p.

Proof. Due to Theorems 2.2 and 2.3, it suffices to show that with $\beta = (-1/2, -1/2)$

$$\|u - \varphi\|_{H^1(R_{r_0,\theta_0})} \leq C \|u - \varphi\|_{H^{1,\beta}(Q)}. \tag{2.10}$$

Note that for $x \in R_{r_0,\theta_0} \subset R^2$, there hold for $i, j = 1, 2$

$$\begin{aligned} 1 - r_0 \leq (1 - x_i) \leq 2, \\ 1/\kappa_0 \leq \frac{1 + x_i}{1 + x_j} \leq \kappa_0, \end{aligned} \tag{2.11}$$

where κ_0 is a constant depending on θ_0, which implies for $|\alpha| = 1$

$$\begin{aligned} \int_{R_{r_0,\theta_0}} |D^\alpha(u - \varphi)|^2 dx &\leq C \int_{R_{r_0,\theta_0}} |D^\alpha(u - \varphi)|^2 \prod_{i=1}^{2} (1 - x_i^2)^{\alpha_i - 1/2} dx \\ &\leq C |u - \varphi|^2_{H^{1,\beta}(Q)} \end{aligned}$$

and

$$\int_{R_{r_0,\theta_0}} |u-\varphi|^2 dx \le \int_Q |u-\varphi|^2 \prod_{i=1}^{2}(1-x_i^2)^{-1/2} dx$$
$$\le \|u-\varphi\|^2_{H^{0,\beta}(Q)}.$$

Then (2.10) follows immediately. □

2.3 Upper Bound of Approximation Error for Functions of $r^\gamma \log^\nu r$-type

Let

$$u(x) = r^\gamma \log^\nu r\ \chi(r)\Phi(\theta), \tag{2.12}$$

where $\chi(r)$ and $\Phi(\theta)$ are the same as in Section 2.2, $\gamma > 0$, integer $\nu \ge 0$. It can be proved that $u \in H^{1+[2\gamma],\beta}(Q)$ and $u \in B^{1+2\gamma-\epsilon,\beta}(Q)$, with $\beta = (-1/2,-1/2)$ and an arbitrary $\epsilon > 0$. It will result in a loss of $O(p^{-\epsilon})$ in approximation. To avoid the loss, we have to modify the weighted Besov space $B^{s,\beta}(Q)$. For $\nu > 0$, $\ell < k$, $0 < \theta < 1$, define

$$B^{s,\beta}_\nu(Q) = \left(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\right)_{\theta,\infty,\nu} \tag{2.13a}$$

with the norm

$$\|u\|_{B^{s,\beta}_\nu(Q)} = \sup_{t>0} \frac{t^{-\theta}}{(1+|\log t|)^\nu} K(t,u), \tag{2.13b}$$

where $K(t,u)$ is defined in (2.3c).

The factor $(1+|\log t|)^{-\nu}$ in (2.13b) is supposed to balance the factor $\log^\nu r$ in the functions given in (2.12). Hence the space $B^{s,\beta}_\nu(Q)$ can characterize the functions of $r^\gamma \log^\nu r$-type more precisely than the space $B^{s,\beta}(Q)$.

Theorem 2.5. *Let u be given in (2.12) with $\gamma > 0$ and $\nu \geq 0$ Then $u \in B_{\nu}^{1+2\gamma,\beta}(Q)$ if γ is not an integer, or γ is an integer and $\nu = 0$, and $u \in B_{\nu-1}^{1+2\gamma,\beta}(Q)$ if γ is an integer and $\nu \geq 1$, with $\beta = (-1/2, -1/2)$.*

Proof. Let $\varphi_\delta(r)$ be a cut-off C^∞ function as before, and $u = v + w$ with $v = \varphi_\delta(r)u$ and $w = (1 - \varphi_\delta(r))u$. By a straightforward calculation we have for $\ell < 1 + 2\gamma < k$

$$\|v\|_{H^{\ell,\beta}(Q)} \leq C\delta^{\gamma+\frac{1}{2}(1-\ell)}|log\delta|^\nu$$

and

$$\|w\|_{H^{k,\beta}(Q)} \leq C\delta^{\gamma+\frac{1}{2}(1-k)}|\log\delta|^\nu.$$

By selecting $\delta = t^{\frac{2}{k-\ell}}$ for $0 < t < 1$ and $\theta = \dfrac{2\gamma + 1 - \ell}{k - \ell}$, we have

$$\sup_{0<t<1} \frac{t^{-\theta}K(t,u)}{(1 + |\log t|)^\nu} \leq C$$

and

$$\sup_{t>1} \frac{t^{-\theta}K(t,u)}{(1 + |\log t|)^\nu} \leq \|u\|_{H^{\ell,\beta}(Q)} \leq C,$$

which implies $u \in B_{\nu}^{1+2\gamma,\beta}(Q)$.

If γ is an integer > 0, we write $u = v + w$, with

$$w = r^\gamma \log^\nu(r + \delta)\chi(r)\Phi(\theta)$$

and

$$v = r^\gamma \left(\log^\nu r - \log^\nu(r + \delta)\right)\chi(r)\Phi(\theta).$$

By a straightforward calculation, we have for $k > 2r + 1$

$$\| w \|_{H^{k,\beta}(\theta)} \leq C\delta^{\gamma+(1-k)/2} \mid \log\delta \mid^{\nu-1}$$

and

$$\| v \|_{H^{0,\beta}(\theta)} \leq C\delta^{\gamma+1/2} \mid \log \delta \mid^{\nu-1},$$

which implies

$$\kappa(t,u) \leq c \mid \log \delta \mid^{\nu-1} \left(\delta^{\gamma+1/2} + t\delta^{\gamma+(1-k)/2}\right)$$

and by selecting $\delta = t^{\frac{2}{k}}$ and $\theta = (2\gamma+1)/k$ we get

$$\sup_{t>0} \frac{t^{-\theta}K(t,u)}{(1+|\log t|)^{\nu-1}} \leq C,$$

which implies $u \in B^{1+2\gamma,\beta}_{\nu-1}(Q)$. For details of the above argument we refer to [**?**]. $\square$

Theorem 2.6. *Let* $u \in B^{s,\beta}_{\nu}(Q)$, $s > 0$, $\nu \geq 0$. *Then there exists a polynimial* $\varphi \in P_p(Q)$ *such that for integer* $\ell \leq s$

$$\|u-\varphi\|_{H^{\ell,\beta}(Q)} \leq Cp^{-(s-\ell)}\left(1+\log p\right)^{\nu} \|u\|_{B^{s,\beta}_{\nu}(Q)} \tag{2.14}$$

where C *is dependent of* p.

Proof. Due to Definition (2.13), there are $v \in H^{\ell,\beta}(Q)$ and $w \in H^{k,\beta}(Q)$ such that for all $t > 0$

$$\|v\|_{H^{\ell,\beta}(Q)} + t\|w\|_{H^{k,\beta}(Q)} \leq Ct^{\theta}\left(1+|\log t|\right)^{\nu} \|u\|_{B^{s,\beta}_{\nu}(Q)} \tag{2.15}$$

where $\ell < s < k$, $\theta = \dfrac{s-\ell}{k-\ell}$. By Theorem 2.1 the polynomial w_p, which is the $H^{0,\beta}$-projection of w on $P_p(Q)$, satisfies

$$\|w-w_p\|_{H^{\ell,\beta}(Q)} \leq Cp^{-(k-\ell)}\|w\|_{H^{k,\beta}(Q)}.$$

Therefore

$$\begin{aligned} \|u-w_p\|_{H^{\ell,\beta}(Q)} &\leq \|v\|_{H^{\ell,\beta}(Q)} + \|w-w_p\|_{H^{\ell,\beta}(Q)} \\ &\leq \|v\|_{H^{\ell,\beta}(Q)} + Cp^{-(k-\ell)}\|w\|_{H^{k,\beta}(Q)} \\ &\leq C\left(\|v\|_{H^{\ell,\beta}(Q)} + p^{-(k-\ell)}\|w\|_{H^{k,\beta}(Q)}\right). \end{aligned} \tag{2.16}$$

Letting $t = p^{-(k-\ell)}$ and combining (2.15) and (2.16), we have

$$\begin{aligned}\|u-\varphi\|_{H^{\ell,\beta}(Q)} &\le C\left(\|v\|_{H^{\ell,\beta}(Q)} + t\|w\|_{H^{k,\beta}(Q)}\right)\\ &\le Ct^{\theta}\left(1+\log t\right)^{\nu}\|u\|_{B_{\nu}^{s,\beta}(Q)}\\ &\le Cp^{-(s-\ell)}\left(1+\log p\right)^{\nu}\|u\|_{B_{\nu}^{s,\beta}(Q)},\end{aligned}$$

which leads to (2.14). □

Remark 2.2. It is worth noting that the space $B_{\nu}^{s,\beta}(Q)$, $\nu > 0$, is not an exact interpolation space of exponent θ defined by usual K-method. We proved Theorem 2.6 by using the definition (2.13) of $B_{\nu}^{s,\beta}(I^n)$ and Theorem 2.5, but not by using the usual arguments of exact interpolation spaces as we argued for Theorem 2.2 in [?]. The space $B_{\nu}^{s,\beta}(Q)$ is actually a uniform interpolation space. For various important properties of the space $B_{\nu}^{s,\beta}(Q)$, such as reiteration and equivalent norms by the K-method and the J-method, we refer to [?].

For the functions in $B_{\nu}^{s,\beta}(Q)$ there is a sharp estimate on the upper bound of approximation error, which is parallel to Theorem 2.4.

Theorem 2.7. *Let u be given in (2.12). Then*

$$\inf_{\varphi\in P_p(Q)}\|u-\varphi\|_{H^1(R_{r_0,\theta_0})} \le Cp^{-2\gamma}\left(1+\log p\right)^{\nu} \tag{2.17a}$$

if γ is not an integer, or γ is an integer and $\nu = 0$; and

$$\inf_{\varphi\in P_p(Q)}\|u-\varphi\|_{H^1(R_{r_0,\theta_0})} \le Cp^{-2\gamma}\left(1+\log p\right)^{\nu-1} \tag{2.17b}$$

if γ is an integer and $\nu > 0$, where C is independent of p.

Proof. Due to Theorem 2.5, $u \in B_{\nu}^{1+2\gamma,\beta}(Q)$. Applying (2.10) and (2.14) for $\ell = 1$, we have a $\varphi \in P_p(Q)$ such that

$$\begin{aligned}\|u-\varphi\|_{H^1(R_{r_0,\theta_0})} &\le C\|u-\varphi\|_{H^{1,\beta}(R_{r_0,\theta_0})}\\ &\le Cp^{-(s-\ell)}\left(1+\log p\right)^{\nu}\|u\|_{B_{\nu}^{1+2\gamma,\beta}(Q)}\end{aligned}$$

which yields (2.17a). Similarly (2.17b) stands for integer γ and $\nu > 0$. ☐

2.4 Lower Bound of Approximation Error for Functions of $r^\gamma \log^\nu r$-type

To prove the lower bounds of approximation error for functions of $r^\gamma \log^\nu r$-type with $\nu > 0$ or with non-integer γ and $\nu = 0$ in n-dimensions, we need an asymptotic expression of approximation error in one dimension. For functions of x^γ-type in one dimension it has been explored very well in [?]. We shall establish the asymptotic expression of approximation error for the function of $x^\gamma \log^\nu x$-type with $\nu > 0$.

Let $H^{k,\beta}(I)$, $\beta > -1$ be the weighted Sobolev space on I with Jacobi weight defined in Section 2.1, and let $P_p(I)$ denote the set of polynomials of degree $\leq p$ on I.

Lemma 2.1. *Let $u \in H^{1,\beta}(I)$ with $\beta = 0$, and let $u_p \in P_p(I)$ be the projection of u on $P_p(I)$ in $L^2(I)$. Then*

$$|u - u_p|_{H^{1,\beta}(I)} \geq p\|u - u_p\|_{L^2(I)}. \tag{2.18}$$

Proof. Let $u = \sum_{j=0}^{\infty} a_j L_j(x)$, where $L_j(x)$ is the Legendre polynomial of degree j. Then $u_p = \sum_{j=0}^{p} a_j L_j(x)$, and

$$\|u - \varphi_p\|_{L^2(I)} = \sum_{j=p+1}^{\infty} |a_j| \frac{2}{2j+1}.$$

Since $u'(x) = \sum_{j=1}^{\infty} a_j L_j'(x) = \sum_{j=1}^{\infty} a_j \frac{j+1}{2} p_{j-1}^{(1,1)}(x)$ where $p_{j-1}^{(1,1)}(x)$ is the Jacobi polynomial of degree $(j-1)$, and $u_p'(x) = \sum_{j=1}^{p} a_j \frac{j+1}{2} p_{j-1}^{(1,1)}(x)$. Due to the orthogonal properties of the Jacobi polynomial (see [?])

$$\int_{-1}^{1} p_i^{(1,1)}(x)p_j^{(1,1)}(x)(1-x^2)dx = \begin{cases} 0 & \text{if } i \neq j \\ \dfrac{8(j+1)}{(2j+3)(j+2)} & \text{if } i = j \end{cases},$$

we have

$$\begin{aligned}\int_{-1}^{1} |u' - u_p'|(1-x^2)dx &= \sum_{j=p+1}^{\infty} |a_j|^2 \left(\frac{j+1}{2}\right)^2 \frac{8j}{(2j+1)(j+1)} \\ &= \sum_{j=p+1}^{\infty} |a_j|^2 \frac{2}{2j+1} j(j+1) \\ &\geq p^2 \sum_{j=p+1}^{\infty} |a_j|^2 \frac{2}{2j+1} \\ &= p^2 \|u - \varphi\|_{L^2(I)}^2,\end{aligned}$$

which yields (2.18). □

Theorem 2.8. *Let* $u = (x+1)^\gamma$, $\gamma > -1/2$, *and let* $\varphi_\gamma^0(x)$ *be the projection of* u *on* $P_p(I)$, $p \geq \max(1,\gamma)$ *in* $L^2(I)$. *Then, for* $\nu \geq 0$, $\varphi_\gamma^\nu(x) = \frac{d^\nu}{d\gamma^\nu}\varphi_\gamma^0(x)$ *is the projection of the function* $u_\nu = (1+x)^\gamma \ln^\nu(1+x)$ *in* $L^2(I)$, *and*

$$\|u_\nu - \varphi_\gamma^\nu\|_{L^2(I)} = p^{-(2\gamma+1)} E_\nu(\gamma,p)\left(1 + O(\frac{1}{p})\right) \tag{2.19}$$

with

$$E_\nu(\gamma,p)^2 = \frac{1}{4\alpha+2}\left(\sum_{s=0}^{\nu} C_{\nu-s}(\gamma)\ln^s(1+p)\right)^2, \tag{2.20}$$

where $C_s(\gamma)$, $0 \leq s \leq \nu$, *are analytic for* $\gamma > -1$.

Proof. Since $u \in L^2(I)$, we have the expansion:

$$u(x) = \sum_{i=0}^{\infty} a_i(\gamma)L_i(x)$$

where $L_i(x)$ is the Legendre polynomial of degree i, and according to [?]

$$a_i(\gamma) = (-1)^{i-1} C_0(\gamma) \frac{\Gamma(i-\gamma)}{\Gamma(i+\gamma+2)}(i+2) \tag{2.21}$$

with

$$C_0(\gamma) = \frac{2^{\gamma+1}\Gamma(1+\gamma)^2 \sin \pi\gamma}{\pi} \tag{2.22}$$

Let $\varphi_\gamma^0(x) = \sum_{i=0}^{p} a_i(\gamma) L_i(x)$; then it was proved in [?, Theorem 5] that

$$\|u - \varphi_\gamma^0(x)\|_{L^2(I)} = \sum_{i=p+1}^{\infty} a_i(\gamma)^2 \frac{2}{2i+1} = E_0(\gamma, p) p^{-(2\gamma+1)} \left(1 + O(\frac{1}{p})\right)$$

with

$$E_0(\gamma, p)^2 = \frac{C_0^2(\gamma)}{(4\alpha+2)},$$

where $O(\frac{1}{p})$ depends on γ. This is (2.20) for $\nu = 0$.

By differentiation with respect to γ, we have

$$u_1 = \frac{d}{d\gamma}(1+x)^\gamma = (1+x)^\gamma \ln(1+x).$$

Then $u_1 \in L^2(I)$, and has the expansion:

$$u_1 = \sum_{i=1}^{\infty} b_i(\gamma) L_i(x)$$

with $b_i(\gamma) = a_i'(\gamma)$ for $\gamma > -1$. Also

$$\varphi_\gamma^1(x) = \frac{d}{d\gamma}\varphi_\gamma^0(x) = \sum_{i=0}^{p} a_i'(\gamma) L_i(x) = \sum_{i=0}^{p} b_i(\gamma) L_i(x)$$

which is the projection of u_1 on $P_p(I)$ in $L^2(I)$, and

$$\|u_1 - \varphi_\gamma^1\|_{L^2(I)}^2 = \sum_{i=p+1}^{\infty} a_i'(\gamma)^2 \frac{2}{2i+1}. \tag{2.23}$$

It follows from (2.21) that

$$a_i'(\gamma) = (-1)^{i-1}(i+1/2))\Big\{C_0'(\gamma)\frac{\Gamma(i-\gamma)}{\Gamma(i+\gamma+2)} + C_0(\gamma)\Big(\frac{\Gamma'(i-\gamma)}{\Gamma(i+\gamma+2)} + \frac{\Gamma(i-\gamma)\Gamma'(i+\gamma+2)}{\Gamma(i+\gamma+2)^2}\Big)\Big\}. \tag{2.24}$$

Due to formula 6.3.8 of [?]

$$\Gamma'(z) = \frac{d}{dz}\left(\ln\Gamma(z)\right)\Gamma(z) = \psi(z)\Gamma(z) \tag{2.25a}$$

with

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{250z^6} + \dots \tag{2.25b}$$

Therefore for $z > 1$

$$\psi(z) = \ln z + O(\frac{1}{z})$$

and

$$\Gamma'(z) \le \Gamma(z)\left(\ln z + O(\frac{1}{z})\right).$$

Due to Stiring's formula [?] and (2.24)-(2.25), we obtain

$$\begin{aligned} a_i'(\gamma) &= \frac{C_0'(\gamma)}{i^{2\gamma+1}}\left(1 + O(\frac{1}{i})\right) + \frac{C_0(\gamma)\ln i}{i^{2\gamma+1}}\left(1 + O(\frac{1}{i})\right) \\ &= \frac{C_0'(\gamma) + C_0(\gamma)\ln i}{i^{2\gamma+1}}\left(1 + O(\frac{1}{i})\right) \end{aligned}$$

and

$$\sum_{i=p+1}^{\infty}|a_i'(\gamma)|^2\frac{2}{2i+1} = \sum_{i=p+1}^{\infty}\frac{C_0'(\gamma)^2 + C_0^2(\gamma)(\ln i)^2 + 2C_0(\gamma)C_1(\gamma)\ln i}{i^{4\gamma+3}}\left(1 + O(\frac{1}{i})\right),$$

which yields

$$\|u_1 - \varphi_\gamma^1\|_{L^2(I)}^2 = E_1(\gamma,p)p^{-(2\gamma+1)}\left(1 + O(\frac{1}{p})\right)$$

with

$$\begin{aligned}
E_1^2(\gamma,p) &= \frac{C_0^2(\gamma)}{(4\gamma+2)}\ln^2(1+p) \\
&+ \frac{2\left(C_0^2(\gamma)+C_0(\gamma)C_0'(\gamma)(4\alpha+2)\right)}{(4\gamma+2)^2}\ln(1+p) \\
&+ \frac{2C_0^2(\gamma)+2C_0(\gamma)C_0'(\gamma)(4\gamma+2)+C_0'(\gamma)^2(4\gamma+2)^2}{(4\gamma+2)^3} \\
&= \frac{1}{4\gamma+2}\sum_{s=0}^{2} C_{2-s}^2(\gamma)\ln^s(p+1),
\end{aligned}$$

where $C_{2-s}(\gamma), s = 0,1,2$ are analytic functions for $\gamma > -1$, which is (2.24) for $\nu = 1$.

For $\nu > 1$ we can derive analogously

$$a_i^{(\nu)}(\gamma) = \frac{C_0(\gamma)\ln^\nu i + C_1(\gamma)\ln^{\nu-1} i + \ldots C_\nu(\gamma)}{i^{2\gamma+1}}\left(1+O(\frac{1}{i})\right),$$

which leads directly to

$$\|u_\nu(x)-\varphi_\gamma^\nu(x)\|_{L^2(I)} = E_\nu(\gamma,p)p^{-(2\gamma+1)}\left(1+O(\frac{1}{p})\right)$$

with

$$E_\nu(\gamma,p)^2 = \frac{1}{(4\gamma+2)}\left(\sum_{s=0}^{\nu} C_{\nu-s}(\gamma)\ln^s(1+p)\right)^2,$$

where $C_{\nu-s}(\gamma)$, $s = 0,1,2,\ldots\nu$ are analytic for $\gamma > -1$. Thus we complete the proof of the lemma for $\nu \geq 0$. □

Corollary 2.1. *If γ is not an integer, there holds with $C_0(\gamma) \neq 0$*

$$\|u_\nu(x)-\varphi_\gamma^\nu(x)\|_{L^2(I)} = C_0(\gamma)p^{-(2\gamma+1)}\ln^\nu(1+p)\left(1+O(\frac{1}{\ln(1+p)})\right). \tag{2.26}$$

If γ is an integer and $\nu = 0$, $C_0(\gamma) = E_0(\gamma,p) = 0$, and there is no approximation error in this case.

If γ is an integer and $\nu > 0$, $C_0(\gamma) = 0$ and $C_1(\gamma) \neq 0$, and there holds

$$\|u_\nu(x)-\varphi_\gamma^\nu(x)\|_{L^2(I)} = C_1(\gamma)p^{-(2\gamma+1)}\ln^{\nu-1}(1+p)\left(1+O(\frac{1}{\ln(1+p)})\right). \tag{2.27}$$

Remark 2.3. (2.25) implies that $\sum_{i=0}^{\infty}\left|\frac{d^\nu}{d\gamma^\nu}a_i(\gamma)\right|^2$ converges uniformly with respect to parameter $\gamma \in [\gamma_0, A]$ for any $\gamma_0 > -1$ and $A < \infty$. Hence, for $\gamma > -1/2$ and integer $\nu \geq 0$, the following always holds

$$\frac{d^\nu}{d\gamma^\nu}(1+x)^\gamma = (1+x)^\gamma \ln^\nu(1+x) = \sum_{i=1}^{\infty}\frac{d^\nu}{d\gamma^\nu}a_i(\gamma)L_i(x).$$

Theorem 2.9. *Let $u = r^\gamma\Phi(\theta)\chi(r)$, with non-integer $\gamma > 0$, where $\chi(r)$ and $\Phi(\theta)$ are smooth functions defined as in (2.6). Then*

$$\inf_{\varphi\in P_p(Q)} \|u-\varphi\|_{H^1(Q)} \geq Cp^{-2\gamma}, \tag{2.28}$$

where $C > 0$ is independent of p.

Proof. We assume that $\Phi(\theta) \not\equiv 0$ and that $\chi(r) \equiv 1$ for $0 < r < 1$. There is an interval $[\theta_1, \theta_2]$ on which $|\Phi(\theta)| \geq \Phi_0 > 0$. For any $\varphi(x) \in P_p(Q)$, $\varphi(r,\theta)$ is a polynomial of r with θ as a parameter. Therefore

$$\begin{aligned}\int_Q \left|\frac{\partial}{\partial r}(u-\varphi)\right|^2 dx &\geq \int_{\theta_1}^{\theta_2}\int_0^1 \left|\frac{\partial}{\partial r}(r^\gamma\Phi(\theta)-\varphi)\right|^2 rdrd\theta \\ &= \int_{\theta_1}^{\theta_2} |\Phi(\theta)|^2 \left(\int_0^1 \left|\frac{\partial}{\partial r}(r^\gamma - \Phi^{-1}(\theta)\varphi)\right|^2 rdr\right) d\theta.\end{aligned}$$

Since $|\Phi(\theta)| \geq \Phi_0 > 0$ on $[\theta_1,\theta_2]$, $\Phi^{-1}(\theta)\varphi(r,\theta)$ is a polynomial of r, which is well-defined on $[0,1]$. Therefore

$$\begin{aligned}\int_0^1 \left|\frac{\partial}{\partial r}\left(r^\gamma - \Phi^{-1}(\theta)\varphi\right)\right|^2 rdr &\geq \inf_{w\in P_p(I_1)}\int_0^1\left|\frac{\partial}{\partial r}\left(r^\gamma - w\right)\right|^2 rdr \\ &\geq \inf_{w\in P_p(I_1)}\int_0^1\left|\frac{\partial}{\partial r}\left(r^\gamma - w\right)\right|^2 r(1-r)dr \\ &= \inf_{w\in P_p(I_1)} |r^\gamma - w|^2_{H^{1,\beta}(I)},\end{aligned}$$

where $\beta = (0,0)$, $I_1 = (0,1)$. By Lemma 2.1

$$|r^\gamma - w|^2_{H^{1,\beta}(I_1)} \geq Cp\,|r^\gamma - w|_{L^2(I_1)}.$$

Since $\gamma \neq$ integer, by Theorem 5 of [?] we have

$$|r^\gamma - w|_{L^2(I_1)} = |C_0(\gamma)| p^{-(2\gamma+1)} \left(1 + O(\frac{1}{p})\right).$$

Summarizing the above, we have

$$\int_Q \left| \frac{\partial}{\partial r}(u - \varphi) \right|^2 dx \geq C\Phi_0^2(\theta_2 - \theta_1) p^{-2\gamma},$$

which implies (2.28). □

Theorem 2.10. *Let* $u = r^\gamma \log^\nu r \chi(r)\Phi(\theta)$ *with* $\Phi(\theta)$ *and* $\chi(r)$ *being* C^∞ *functions as in the previous lemma. Then*

$$\inf_{\varphi \in P_p(Q)} \|u - \varphi\|_{H^1(Q)} \geq C p^{-2\gamma}(1 + \log p)^{\nu^*}, \tag{2.29}$$

where $C > 0$ *is independent of* p, $\nu^* = \nu - 1$ *if* γ *is an integer and* $\nu > 0$, *and* $\nu^* = \nu$. *otherwise.*

Proof. We can proceed as in the proof of Theorem 2.9 except for using Theorem 2.8 and Corollary 2.1 instead of Theorem 5 of [?]. □

2.5 Lower Bound of Approximation Error for Functions of r^γ-type with Integer γ

We now consider the lower bound of the approximation error for functions of r^γ-type with an integer γ. Obviously, the arguments in the proof of Theorem 2.9–2.10, which are based on one-dimensional analysis, are unable to give us a precise estimate on lower bounds for n-dimensional cases with an integer γ. Namely, it will give a zero lower bound, but it is not sharp. Hence we have to develop a totally different approach to derive a sharp estimate on a lower bound for n dimensions, $n \geq 2$.

Lemma 2.2. *Let $F(t)$ be a non-increasing function on $[0,\infty)$, and $\lim\limits_{t\to\infty} F(t) = 0$. Then, there is a function $G(t)$ on $(0,\infty)$ with the following properties:*

(P1) $G(t) \geq F(t)$ *for* $t \in (0,\infty)$,

(P2) $G(t)$ *is non-increasing,*

(P3) $\lim\limits_{t\to\infty} G(t) = 0$,

(P4) $\dfrac{G(t^k)}{G(t)} > \dfrac{1}{2}$ *for* $t \in (1,\infty)$ *and integer* $k \geq 1$.

Proof. Let $\{p_m\}_{m=1}^{\infty}$ and $\{p_{m_\ell}\}_{\ell=1}^{\infty}$ be a sequence and its subsequence with $p_m = 2^{mk}$ and $p_{m_\ell} = 2^{k^\ell}$ where $m_\ell = k^{\ell-1}, \ell = 1, 2, \ldots$. Define

$$G(t) = F(t) \qquad \text{for } t \in (0, p_{m_1}],$$

$$G(t) = G(p_{m_\ell}) \qquad \text{for } p_{m_\ell} \leq t < p_{m_{\ell+1}},$$

and

$$G(p_{m_{\ell+1}}) = \max\{F(p_{m_{\ell+1}}), \frac{1}{2}G(p_{m_\ell})\}.$$

Obviously, (P1) and (P2) hold. Since $G(t)$ is non-increasing, (P3) holds if there is a sequence $\{x_m\}_{m=1}^{\infty}$ such that $\lim\limits_{m\to\infty} x_m = \infty$ and $\lim\limits_{m\to\infty} G(x_m) = 0$. Let $p_{m_{\ell'}}$ be such that $G(p_{m_{\ell'}}) = F(p_{m_{\ell'}})$. If these $p_{m_{\ell'}}$ form an infinite sequence, we have

$$\lim_{\ell'\to\infty} p_{m_{\ell'}} = \infty \qquad \text{and} \qquad \lim_{\ell'\to\infty} G(p_{m_{\ell'}}) = \lim_{\ell'\to\infty} F(p_{m_{\ell'}}) = 0$$

If $\{p_{m'_\ell}\}_{\ell'}$ contains only finite number of terms, then there exists $\ell_0 > 0$ such that $G(p_{m_0}) = (\frac{1}{2})^{\ell-\ell_0} G(p_{m_{\ell_0}})$ for $\ell \geq \ell_0$, which immediately leads to

$$\lim_{\ell\to\infty} p_{m_\ell} = \infty \qquad \text{and} \qquad \lim_{\ell\to\infty} G(p_{m_\ell}) = 0.$$

which implies (P3).

Due to the definition of $G(t)$ we have for $p_{m_\ell} \leq t < p_{m_{\ell+1}}$

$$\frac{G(t^k)}{G(t)} = \frac{G(p^k_{m_\ell})}{G(p_{m_\ell})} = \frac{G(p_{m_{\ell+1}})}{G(p_{m_\ell})} \geq \frac{1}{2}.$$

Hence, (P4) holds. □

Remark 2.4. Monotonicity of $F(t)$ is not necessary for constructing $G(t)$ in Lemma 2.2. Since $\lim\limits_{t\to\infty} F(t) = 0$ there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and $F(t_{n+1}) < F(t_n)$, $n = 1, 2, \ldots$. Define F

$$\tilde{F}(t) = \frac{t - t_n}{t_{n+1} - t_n}\left(f(t_{n+1}) - f(t_n)\right) + f(t_n), \qquad t_n \leq t \leq t_{n+1}, \qquad n = 1, 2, \ldots$$

$$\tilde{F}(t) = F(t_1), \qquad 0 < t \leq t_1.$$

Then $G(t)$ can be constructed based on $\tilde{F}(t)$ such that $G(t) \geq \tilde{F}(t)$, $\tilde{F}(t_n) = F(t_n)$, and (P2)–(P4) hold.

We now introduce a weighted Besov space associated with the function G satisfying (P1)–(P4) :

$$B^{s,\beta}_G(Q) = \left(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\right)_{\theta,\infty,G}$$

with norm

$$\|u\|_{B^{s,\beta}_G(Q)} = \sup_{t>0} \frac{t^{-\theta}K(t,u)}{G(1/t)},$$

where $0 < \theta < 1$, $s = (1-\theta)\ell + \theta k$.

Lemma 2.3. *If for all $p \geq 1$ there holds*

$$\inf_{\varphi \in P_p(Q)} \|u - \varphi\|_{L^2(Q)} \leq C p^{-s} G(p)$$

where $G(t)$ satisfies (P1)–(P4), then $u \in B^{s,\beta'}_G(Q)$ with $\beta' = (0,0)$.

Proof. We write $u = v + w$ with $w = \varphi_{p_\ell} \in P_{p_m}(Q)$, where $p_m = 2^{mk}$ with integer $k > s$. Then

$$\begin{aligned}
\|v\|_{H^{0,\beta'}(Q)} &= \|v\|_{L^2(Q)} = \|u - \varphi_{p_m}\|_{L^2(Q)} \le A p_m^{-s} G(p_m), \\
\|w\|_{H^{k,\beta'}(Q)} &= \|\varphi_{p_1} + \sum_{j=2}^{m} (\varphi_{p_j} - \varphi_{p_{j-1}})\|_{H^{k,\beta'}(Q)} \\
&\le \|\varphi_{p_1}\|_{H^{k,\beta'}(Q)} + \sum \|\varphi_{p_j} - \varphi_{p_{j-1}}\|_{H^{k,\beta'}(Q)},
\end{aligned}$$

by inverse inequality [?, Lemma 5.1],

$$\begin{aligned}
&\le C\Big\{ p_1^k \|\varphi_{p_1}\|_{L^2(Q)} + \sum_{j=2}^{m} p_j^k \|\varphi_{p_j} - \varphi_{p_{j-1}}\|_{L^2(Q)} \Big\} \\
&\le C\Big\{ p_1^k \left(\|u\|_{L^2(Q)} + \|u - \varphi_{p_1}\|_{L^2(Q)} \right) \\
&\qquad + \sum_{j=2}^{m} p_j^k \|u - \varphi_{p_j}\|_{L^2(Q)} + \|u - \varphi_{p_{j-1}}\|_{L^2(Q)} \Big\} \\
&\le C\Big\{ \|u\|_{L^2(Q)} + p_1^{k-s} G(p_1) + \sum_{j=2}^{m} \left(p_j^{k-s} G(p_j) + p_j^k p_{j-1}^{-s} G(p_{j-1}) \right) \Big\} \\
&\le C\Big\{ \|u\|_{L^2(Q)} + \sum_{j=1}^{m} p_j^{k-s} G(p_j) \Big\} \\
&\le C\Big\{ \|u\|_{L^2(Q)} + p_m^{k-s} G(p_m) \sum_{j=1}^{m} \left(\frac{p_j}{p_m} \right)^{k-s} \frac{G(p_j)}{G(p_m)} \Big\}.
\end{aligned}$$

Here we used the facts that $p_{j-1} = 2^{-k} p_j$, $G(p_{j-1}) = G(p_j)$ if $2^{k^\ell} \le p_{j-1}$, $p_j < 2^{k^{\ell+1}}$ and $G(p_{j-1}) < 2G(p_j)$ if $p_{j-1} < 2^{k^{\ell+1}} \le p_j$.

Due to the definition of $G(t)$ and (P4), we have for $k > s + 1$

$$\sum_{j=1}^{m} \left(\frac{p_j}{p_m} \right)^{k-s} \frac{G(p_j)}{G(p_m)} \le \sum_{j=1}^{m} 2^{(j-m)(k-s)} 2^{m-j} = \sum_{j=1}^{m} 2^{(j-m)(k-s-1)},$$

which implies for $k > s + 1$

$$\|w\|_{H^{k,\beta'}(Q)} \le C\Big\{ \|u\|_{L^2(Q)} + p_m^{k-s} G(p_m) \Big\}.$$

Therefore, for $0 < t < 1$

$$\begin{aligned} K(t,u) &\le \|v\|_{H^{0,\beta'}(Q)} + t\|w\|_{H^{k,\beta'}(Q)} \\ &\le Ap_m^{-s} G(p_m) + Ct\left(\|u\|_{L^2(Q)} + p_m^{k-s} G(p_m)\right) \\ &\le C\left(A + \|u\|_{L^2(Q)}\right)\left(p_m^{-s} G(p_m) + tp_m^{k-s} G(p_m)\right). \end{aligned}$$

Here we used the fact that $p_m^{k-s} G(p_m) > 2^{m(k-s-1)} G(p_1) > C$. Selecting $t = p_m^{-k}$ and $\theta = s/k$ we have

$$\frac{t^{-\theta} K(t,u)}{G(1/t)} \le C\left(A + \|u\|_{L^2(Q)}\right) \frac{G(t^{-1/k})}{G(t^{-1})},$$

by (P4),

$$\le 2C\left(A + \|u\|_{L^2(Q)}\right).$$

Fot $t > 1$, it is trivial to show that

$$\frac{t^{-\theta} K(t,u)}{G(t^{-1})} \le C\|u\|_{L^2(Q)}.$$

Thus the proof of the lemma is completed. □

Let $S = (0,1) \times (0,1)$, and let T_N be the rectangular radical mesh with nodal points $(x_{1,i}, x_{2,j})$,

$$x_{1,i} = \left(\frac{i}{N}\right)^{\sigma}, \qquad x_{2,j} = \left(\frac{j}{N}\right)^{\sigma} \qquad i,j = 0,1,2,\dots N \tag{2.30}$$

where $\sigma > 0$. By $h_i^{(x_1)} = x_{1,i} - x_{1,i-1}$ and $h_j^{(x_2)} = x_{2,j} - x_{2,j-1}$ we denote the length of edges of element $\tau_{ij} = (x_{1,i-1}, x_{1,i}) \times (x_{2,j-1}, x_{2,j})$, $i,j = 1,2,\dots N$.

Let $P_2(\tau_{ij})$ (resp. $P_2(S)$) be set of polynomials of (separate) degree ≤ 2 over τ_{ij} (resp. S), and let $P_2(T_N) = \{\varphi(x) \,|\, \varphi(x)\Big|_{\tau_{ij}} \in P_2(\tau_{ij})\}$.

Lemma 2.4. *Let $u \in H^k(S) \cap C^0(\bar{S}), k \geq 1$. Then for $1 \leq \ell, m \leq k$*

$$\|u - \Pi_2 u\|_{L^2(S)} \leq C \left(\left\| \frac{\partial^m u}{\partial x_1^m} \right\|_{L^2(S)} + \left\| \frac{\partial^\ell u}{\partial x_2^\ell} \right\|_{L^2(S)} \right) \tag{2.31}$$

where $\Pi_2 u$ is the projection of u on $P_2(S)$ in $L^2(S)$.

Proof. The standard arguments of the projection in the framework of the Legendre expansion lead directly to the assertion of the lemma. □

Lemma 2.5. *Let T_N be radical mesh defined by (2.30) with $\sigma \geq 3$. Assume that $u = 0$ at $(1/2, 1/2)$ and*

$$\int_S \left(\left(\frac{\partial^3 u}{\partial x_1} \right)^2 x_1^3 + \left(\frac{\partial^3 u}{\partial x_2} \right)^3 x_2^3 \right) dx = |u|^2_{H^{3*}(S)} < \infty.$$

Then

$$\begin{aligned} \|u - \Pi_2 u\|^2_{L^2(S)} &= \sum_{i,j=1}^{N} \|u - \Pi_2 u\|^2_{L^2(\tau_{ij})} \\ &\leq C N^{-6} |u|^2_{H^{3*}(S)}, \end{aligned} \tag{2.32}$$

where Π_2 is a projection operator in $L^2(S)$ (resp. $L^2(\tau_{ij})$) on $P_2(S)$ (resp. $P_2(\tau_{ij})$).

Proof. On elements $\tau_{ij}, i, j > 1$, by Lemma 2.4 we have

$$\|u - \Pi_2 u\|^2_{L^2(\tau_{ij})} \leq C \Big\{ \left| h_i^{(x_1)} \right|^6 \left\| \frac{\partial^3 u}{\partial x_1^3} \right\|^2_{L^2(\tau_{ij})} + \left| h_j^{(x_2)} \right|^6 \left\| \frac{\partial^3 u}{\partial x_2^3} \right\|^2_{L^2(\tau_{ij})} \Big\},$$

by (2.30),

$$\leq C N^{-6} |u|^2_{H^{3*}(\tau_{ij})}. \tag{2.33}$$

We next consider the approximation on $\tau_{1,j}, j > 1$. By Schwarz inequality we get

$$\begin{aligned} \left| \frac{\partial^2 u}{\partial x_1^2} \right| &= \left| \int_{1/2}^{x_1} \frac{\partial^3 u(t, x_2)}{\partial x_1^3} dt \right| \\ &\leq \left(\int_{1/2}^{x_{1,1}} \left| \frac{\partial^3 u}{\partial x_1^3} \right| x_1^3 dx_1 \right)^{1/2} \left(\int_{1/2}^{x_1} t^{-3} dt \right)^{1/2} \\ &\leq C \left(\int_{1/2}^{x_{1,1}} \left| \frac{\partial^3 u}{\partial x_1^3} \right| x_1^3 dx_1 \right)^{1/2} \frac{1}{x_1}, \end{aligned}$$

which implies

$$\begin{aligned}\left|\frac{\partial u}{\partial x_1}\right| &\le \left|\int_{1/2}^{x_1} \frac{\partial^2 u(t,x_2)}{\partial x_1^2} dt\right| \\ &\le \left(\int_{1/2}^{x_{1,1}} \left|\frac{\partial^2 u(t,x_2)}{\partial x_1^2}\right|^2 x_1^{5/4} dx_1\right)^{1/2} \left(\int_{1/2}^{x_1} t^{-5/4} dt\right)^{1/2} \\ &\le C\left(\int_{1/2}^{x_{1,1}} \left|\frac{\partial^3 u(t,x_2)}{\partial x_1^3}\right|^2 x_1^3 dx_1\right)^{1/2} x_1^{-1/8}.\end{aligned}$$

Therefore we have

$$\int_{\tau_{1j}} \left|\frac{\partial u}{\partial x_1}\right|^2 dx \le C\left(h_1^{(x_1)}\right)^{3/4} \int_{\tau_{1/2,j}} \left|\frac{\partial^3 u}{\partial x_1^3}\right|^2 x_1^3 dx,$$

where $\tau_{1/2,j} = (0,1/2) \times (x_{2,j-1}, x_{2,j})$.

For $j > 1$ we have

$$\int_{\tau_{1j}} \left|\frac{\partial^3 u}{\partial x_2^3}\right|^2 dx \le \frac{C}{x_{2,j-1}^3} \int_{\tau_{1j}} \left|\frac{\partial^3 u}{\partial x_2^3}\right|^2 x_2^3 dx.$$

Applying Lemma 2.4 with $m = 1$ and $\ell = 3$ we get

$$\begin{aligned}\int_{\tau_{1j}} |u - \Pi_2 u|^2 dx \le C\Big\{ &\left(h_1^{(x_1)}\right)^2 \int_{\tau_{1/2,j}} \left|\frac{\partial^3 u}{\partial x_1^3}\right|^2 x_1^3 dx \\ &+ \frac{\left|h_j^{(x_2)}\right|^6}{x_{2,j-1}^3} \int_{\tau_{1j}} \left|\frac{\partial^3 u}{\partial x_2^3}\right|^2 x_2^3 dx\Big\},\end{aligned}$$

by (2.30),

$$\le C\left(N^{-7}|u|_{H^{3*}(S)}^2 + N^{-6}|u|_{H^{3*}(\tau_{1j})}^2\right). \tag{2.34a}$$

In the same way, we have for $i > 1$

$$\int_{\tau_{i1}} |u - \Pi_2 u|^2 dx \le CN^{-6}\left(|u|_{H^{3*}(S)} + |u|_{H^{3*}(\tau_{i1})}\right). \tag{2.34b}$$

Similarly we have by applying Lemma 2.4 with $m = \ell = 1$

$$\begin{aligned}\int_{\tau_{11}} |u - \Pi_2 u|^2 dx &\le C\Big\{\left|h_1^{(x_1)}\right|^2 \int_{\tau_{i1}} \left|\frac{\partial^3 u}{\partial x_1^3}\right|^2 x_1^3 dx \\ &\qquad + \left|h_1^{(x_2)}\right|^2 \int_{\tau_{i1}} \left|\frac{\partial^3 u}{\partial x_2^3}\right|^2 x_2^3 dx\Big\} \\ &\le CN^{-6}|u|^2_{H^{3*}(S)}. \end{aligned} \tag{2.35}$$

Then (2.32) follows from the combination of (2.33)∼(2.35). □

Lemma 2.6. *Let* $u \in B_G^{2,\beta'}(Q)$ *with* $\beta' = (0,0)$, *and let* $\tilde{T}_N$ *be a radical mesh on* Q *with* $\sigma \ge 3$ *which is a union of four radical meshes on unit squares* $S_i, i = 1,2,3,4$ *with* $S_1 = (-1,0)^2, S_2 = (-1,0) \times (0,1), S_3 = (0,1) \times (-1,0)$, *and* $S_4 = (0,1)^2$. *Then*

$$\inf_{\varphi \in P_2(\tilde{T}_N)} \|u - \varphi\|_{L^2(Q)} \le CN^{-2} G(N^3) \tag{2.36}$$

with $C > 0$ *independent of* N.

Proof. Due to the definition of $B_G^{2,\beta'}(Q)$, there exist $v \in H^{0,\beta'}(Q)$ and $w \in H^{3,\beta'}(Q)$ such that for $t > 0$

$$\|v\|_{H^{0,\beta'}(Q)} + t\|w\|_{H^{3,\beta'}(Q)} \le Ct^{2/3} G(1/t) \|u\|_{B_G^{2,\beta'}(Q)}. \tag{2.37}$$

Let $\chi(t)$ be a C^∞ function such that $\chi(t) = 1$ for $0 < t < 1/2$ and $\chi(t) = 0$ for $t > 1$. We compose $w = \sum_{\ell=1}^4 w_\ell$ where

$$\begin{aligned} w_1 &= w\chi(x_1)\chi(x_2), \\ w_2 &= w\chi(x_1)(1 - \chi(x_2)), \\ w_3 &= w(1 - \chi(x_1))\chi(x_2), \\ w_4 &= w(1 - \chi(x_1))(1 - \chi(x_2)). \end{aligned}$$

Note that supp. $w_1 \subset Q_1 = (-1,0) \times (-1,0)$, supp. $w_2 \subset Q_2 = (-1,0) \times (-\frac{1}{2},1)$, supp. $w_3 \subset Q_3 = (-\frac{1}{2},1) \times (-1,0)$, and supp. $w_4 \subset Q_4 = (-\frac{1}{2},1) \times (-\frac{1}{2},1)$. Since $H^{3,\beta'}(Q) \subset C^0(\bar{Q})$ (see [?, ?]) we have by Lemma 2.6

$$\begin{aligned}
\|w_1 - \Pi_2 w_1\|^2_{L^2(Q)} &= \|w_1 - \Pi_2 w_1\|^2_{L^2(Q_1)} \\
&\leq CN^{-6} \int_{Q_1} \left(\left| \frac{\partial^3 w_1}{\partial x_1^3} \right|^2 (1+x_1)^3 + \left| \frac{\partial^3 w_1}{\partial x_2^3} \right|^2 (1+x_2)^3 \right) dx \\
&\leq CN^{-6} \int_{Q_1} \left(\left| \frac{\partial^3 w_1}{\partial x_1^3} \right|^2 (1-x_1^2)^3 + \left| \frac{\partial^3 w_1}{\partial x_2^3} \right|^2 (1-x_2^2)^3 \right) dx \\
&\leq CN^{-6} |w_1|^2_{H^{3,\beta'}(Q)} \\
&\leq CN^{-6} |w|^2_{H^{3,\beta'}(Q)},
\end{aligned}$$

where $\Pi_2 w_1$ is the projection of w_1 on $P_2(Q_1)$. Similarly, the following hold

$$\begin{aligned}
\|w_2 - \Pi_2 w_2\|^2_{L^2(Q)} &= \|w_2 - \Pi_2 w_2\|^2_{L^2(Q_1)} + \|w_2 - \Pi_2 w_2\|^2_{L^2(Q_2)}, \\
\|w_2 - \Pi_2 w_2\|^2_{L^2(Q_2)} &\leq CN^{-6} \int_{Q_2} \left(\left| \frac{\partial^3 w_2}{\partial x_1^3} \right|^2 (1+x_1)^3 + \left| \frac{\partial^3 w_2}{\partial x_2^3} \right|^2 (1+x_2)^3 \right) dx \\
&\leq CN^{-6} |w|^2_{H^{3,\beta'}(Q)},
\end{aligned}$$

and

$$\begin{aligned}
\|w_2 - \Pi_2 w_2\|^2_{L^2(Q_1)} &\leq CN^{-6} \int_{-1/2}^{0} \int_{-1}^{0} \left(\left| \frac{\partial^3 w_2}{\partial x_1^3} \right|^2 (1+x_1)^3 + \left| \frac{\partial^3 w_2}{\partial x_2^3} \right|^2 (1+x_2)^3 \right) dx \\
&\leq CN^{-6} \int_{-1/2}^{0} \int_{1}^{0} \left(\left| \frac{\partial^3 w_2}{\partial x_1^3} \right|^2 (1-x_1^2)^3 + \left| \frac{\partial^3 w_2}{\partial x_2^3} \right|^2 (1-x_2^2)^3 \right) dx \\
&\leq CN^{-6} |w_2|^2_{H^{3,\beta'}(Q)} \\
&\leq CN^{-6} |w|^2_{H^{3,\beta'}(Q)}.
\end{aligned}$$

The functions w_3 and w_4 are approximated by the projection on $P_2(Q_l), 1 \leq l \leq 4$, and similar estimates hold.

Letting $\Pi_2 w = \sum_{\ell=1}^{4} \Pi_2 w_\ell$, we have

$$\begin{aligned}\|w - \Pi_2 w\|^2_{L^2(Q)} &= \sum_{\ell=1}^{4} \|w_\ell - \Pi_2 w_\ell\|^2_{L^2(Q_\ell)} \\ &\leq CN^{-6}\|w\|^2_{H^{3,\beta'}(Q)},\end{aligned}$$

which implies

$$\begin{aligned}\|u - \Pi_2 w\|^2_{L^2(Q)} &\leq \|v\|_{L^2(Q)} + \|w - \Pi_2 w\|^2_{L^2(Q)} \\ &\leq \|v\|_{L^2(Q)} + CN^{-3}\|w\|^2_{H^{3,\beta'}(Q)},\end{aligned}$$

by selecting $t = N^{-3}$,

$$\leq C\left(\|v\|_{L^2(Q)} + t\|w\|^2_{H^{3,\beta}(Q)}\right),$$

by (2.37),

$$\begin{aligned}&\leq Ct^{2/3}G(1/t)\|u\|_{B^{2,\beta'}_G(Q)} \\ &= CN^{-2}G(N^3)\|u\|_{B^{2,\beta'}_G(Q)},\end{aligned}$$

which leads to (2.36). □

Lemma 2.7. *Let $u = r\chi(r)\Phi(\theta)$, where $r = (x_1^2 + x_2^2)^{1/2}$, $\chi(r)$ and $\Phi(\theta)$ are C^∞ functions such that $\chi(r) = 1$ for $r < \delta/2 < 1/2$ and $\chi(r) = 0$ for $r > \delta$, and $\Phi(\theta) \not\equiv 0$ on $(0, \pi/2)$. Then there exist $\theta_1, \theta_2 \in (0, \pi/2)$ such that*

$$\inf_{\varphi \in P_2(T_N)} \|u_{x_1} - \varphi\|_{L^2(S_0)} \geq CN^{-2}, \tag{2.38}$$

where T_N is a radical mesh with $\sigma = 4$, $S_0 = \{x \in S \mid r < \delta \text{ and } \theta_1 < \theta < \theta_2\}$, and (r, θ) are polar coordinates.

Proof. By straightforward calculation we have for $|\alpha| = 4$

$$|D^\alpha u_{x_1}| \leq C\frac{1}{r^4},$$

and for $|\alpha| \leq 3$ and $|x| < \delta/2$

$$D^\alpha u_{x_1} = \frac{1}{r^3}\Psi_\alpha(\theta),$$

where $\Psi_\alpha(\theta)$ is analytic and not identically zero. Hence there exists an interval $[\theta_1, \theta_2]$ such that for $\theta \in [\theta_1, \theta_2]$ and all α with $|\alpha| = 3$

$$|\Psi_\alpha(\theta)| \geq C_0 > 0,$$

which implies that there holds in S_0 for $|\alpha| = 3$

$$|D^\alpha u_{x_1}| \geq C_0/r^3, \tag{2.39a}$$

and for $|\alpha| = 4$,

$$|D^\alpha u_{x_1}| \leq C\frac{1}{r}\left(\left|\frac{\partial^3 u_{x_1}}{\partial x^3}\right| + \left|\frac{\partial^3 u_{x_1}}{\partial y^3}\right|\right). \tag{2.39b}$$

Let M_{ij} be a linear mapping of Q onto the elements $\tau_{ij} \subset Q_{\delta/2,\kappa}$, and by U_{ij} we denote the mapped functions

$$U_{ij}(\xi, \eta) = u_{x_1}(M_{ij}(\xi, \eta)).$$

By scaling arguments, (2.39) leads to, for $|\alpha| = 4$,

$$|D^\alpha U_{ij}| \leq C\frac{|h_i^{(x_1)}|^{\alpha_1}\,|h_j^{(x_2)}|^{\alpha_2}}{(|h_i^{(x_1)}|^3 + |h_j^{(x_2)}|^3)\sqrt{x_{1,i-1}^2 + x_{2,j-1}^2}}\left(\left|\frac{\partial^3 U_{ij}}{\partial \xi^3}\right| + \left|\frac{\partial^3 U_{ij}}{\partial \eta^3}\right|\right) \tag{2.40}$$

Note that there holds for some $c_m > 0, m = 1, 2,$

$$\begin{aligned} &\frac{1}{\kappa}x_{1,i-1} < x_{2,j-1} < \kappa x_{1,i-1}, \\ &c_1 h_i^{(x_2)} \leq h_j^{(x_2)} \leq c_2 h_i^{(x_1)}. \end{aligned} \tag{2.41}$$

which, together with (2.30) and (2.40), leads to

$$\begin{aligned}|D^\alpha U_{ij}| &\leq C\frac{h_i^{(x_1)}}{x_{i-1}}\left(\left|\frac{\partial^3 U_{ij}}{\partial\xi^3}\right| + \left|\frac{\partial^3 U_{ij}}{\partial\eta^3}\right|\right)\\ &\leq \frac{C}{i}\left(\left|\frac{\partial^3 U_{ij}}{\partial\xi^3}\right| + \left|\frac{\partial^3 U_{ij}}{\partial\eta^3}\right|\right)\\ &\leq C\left(\left|\frac{\partial^3 U_{ij}}{\partial\xi^3}\right| + \left|\frac{\partial^3 U_{ij}}{\partial\eta^3}\right|\right) \qquad (2.42)\end{aligned}$$

where C is independent of i, γ and N. We now assume and will prove later that there exists $C > 0$ such that

$$\|U_{ij} - \Pi_2 U_{ij}\|^2_{L^2(S)} \geq C|U_{ij}|^2_{H^{3*}(S)}, \qquad (2.43)$$

where $\Pi_2 U_{ij}$ is the projection of U_{ij} on the $P_2(S)$ and $|U_{ij}|^2_{H^{3*}(S)} = \left|\frac{\partial^3 U_{ij}}{\partial\xi^3}\right|^2 + \left|\frac{\partial^3 U_{ij}}{\partial\eta^3}\right|^2$, which immediately leads to

$$\|u_{x_1} - \Pi_2 u_{x_1}\|_{L^2(\tau_{ij})} \geq C\left(\left|h_i^{(x_1)}\right|^3 \left\|\frac{\partial^3 u_{x_1}}{\partial x_1^3}\right\|_{L^2(\tau_{ij})} + \left|h_j^{(x_2)}\right|^3 \left\|\frac{\partial^3 u_{x_1}}{\partial x_2^3}\right\|_{L^2(\tau_{ij})}\right)$$

by (2.30),(2.39) and (2.41)

$$\begin{aligned}&\geq C\left|h_i^{(x_1)}\right|^4 x_{i-1}^{-3}\\ &\geq CN^{-4}.\end{aligned}$$

This yields

$$\begin{aligned}\inf_{\varphi\in P_2(T_N)} \|u-\varphi\|^2_{L^2(S_0)} &\geq C\sum_{i=1}^{[\delta N]}\sum_{j=[K_1 i]+1}^{[K_2 i]} CN^{-4}\\ &\geq CN^{-2},\end{aligned}$$

where $K_\ell = \tan\theta_\ell$.

Now it remains to prove (2.43). If it is not true, there exists a sequence $\{U_m\}_{m=1}^\infty$ among those $U_{ij}, 1 \leq i, j \leq N, N = 1, 2, \dots$ such that

$$\lim_{m\to\infty} \|U_m - \Pi_2 U_m\|_{L^2(Q)} = 0, \qquad (2.44)$$

$$|U_m|^2_{H^{3*}(Q)} = \left|\frac{\partial^3 U_m}{\partial \xi^3}\right|^2_{L^2(Q)} + \left|\frac{\partial^3 U_m}{\partial \eta^3}\right|^2_{L^2(Q)} = 1 \tag{2.45}$$

Furthermore, we have by (2.42) and (2.45)

$$|D^4 U_m|^2 = \sum_{|\alpha|=4} |D^\alpha U_m|^2 \leq C. \tag{2.46}$$

We introduce an equivalent norm on $H^4(Q)$ (see [?])

$$\left\{ |U|^2_{H^4(Q)} + |U|^2_{H^{3*}(Q)} + \|\Pi_2 u\|^2_{L^2(Q)} \right\}^{1/2},$$

which, together with (2.44)∼(2.46), implies that $\{U_m\}_{m=1}^\infty$ is a bounded sequence in $H^4(Q)$. By compactness theorem (see e.g. [?]) there is a subsequence, denoted by $\{U_m\}_{m=1}^\infty$ again, such that

$$\lim_{m\to\infty} U_m = \bar{U} \quad \text{in} \quad H^3(Q).$$

Consequently

$$\|\bar{U} - \Pi_2 \bar{U}\|_{L^2(Q)} = 0$$

and

$$\bar{U}_{H^{3*}(Q)} = 1,$$

but these two contradict each other. Thus we complete the proof of the lemma. □

We now conclude with our main result of this subsection.

Theorem 2.11. *Let* $u = r^\gamma \chi(r)\Phi(\theta)$ *with integer* γ *where* $\chi(r)$ *and* $\Phi(\theta)$ *are* C^∞ *function satisfying (2.7) as before. Then, if* $r^\gamma \Phi(\theta)$ *is not a polynomial, there exists a constant* C *independent of* p *such that*

$$\inf_{\varphi \in P_p(Q)} \|u - \varphi\|_{H^1(Q)} \geq C p^{-2\gamma}. \tag{2.47}$$

Proof. For the sake of simplicity we shall prove the case for $\gamma = 1$. The proof for integer $\gamma > 1$ is similar to what follows. Suppose (2.48) is false, then there exists some $F(p)$ such that $\lim_{p\to\infty} F(p) = 0$, and

$$\inf_{\varphi \in P_p(Q)} \|u - \varphi\|_{H^1(Q)} \leq Cp^{-2}F(p).$$

Due to Lemma 2.2 and Remark 2.3 we can construct a function $G(t)$ satisfying (P1)–(P4), and introduce a weighted Besov space $B_G^{k,\beta'}(Q)$ with $\beta' = (0,0)$. Then $D^\alpha u \in B_G^{2,\beta'}(Q)$ for $|\alpha| \leq 1$. It is proved in Lemma 2.6 that for $|\alpha| \leq 1$

$$\inf_{\varphi \in P_p(Q)} \|D^\alpha u - \varphi\|_{L^2(Q)} \leq CN^{-2}G(N^3), \tag{2.48}$$

where $\lim_{N\to\infty} G(N^3) = 0$. On the other hand, Lemma 2.8 indicates that there exists a constant $\tilde{C}$ independent of N such that

$$\inf_{\varphi \in P_p(Q)} \|u_{x_1} - \varphi\|_{L^2(Q)} \geq \tilde{C}N^{-2}, \tag{2.49}$$

which contradicts (2.48). Hence the theorem is proved for $\gamma = 1$.

For integer $\gamma > 1$ we can analogously show that $D^\alpha u \in B_G^{2\gamma,\beta'}(Q)$ with $\beta' = (0,0)$, and for $|\alpha| = 1$

$$\inf_{\varphi \in P_p(Q)} \|D^\alpha u - \varphi\|_{L^2(Q)} \leq Cp^{-2\gamma}G(p^{2\gamma+1}), \tag{2.50}$$

and

$$\inf_{\varphi \in P_p(Q)} \|D^\alpha u - \varphi\|_{L^2(Q)} \geq Cp^{-2\gamma}. \tag{2.51}$$

The contradiction between (2.50) and (2.51) leads to (2.47) immediately for all integer γ. $\square$

Remark 2.5. If $u = r^\gamma$ with even integer γ then there is no approximation error if a polynomial of degree $p \geq \gamma$ is used. In general, the estimate (2.47) holds if $u = r^\gamma \Phi(\theta)$ is not a polynomial on Q.

2.6 Lower and Upper Bounds of Approximation Error in the n-dimensional Cube $(-1,1)^n$

We now consider the approximation to functions of $r^\gamma \log^\nu r$-type in the n-dimensional cube I^n. Let

$$u = r^\gamma \log^\nu r \chi(r)\Phi(z) \tag{2.52}$$

with $\gamma > 0$ and $\nu \geq 0$, where $r = \left(\sum_{i=1}^n (x_i+1)^2\right)^{1/2}, z = (z_1, z_2, \ldots, z_{n-1})$ is the variable on the unit sphere $S = \{x \,|\, r = 1\}$ centered at the vertex $(-1,-1,\cdots-1)$. $\chi(r)$ and $\Phi(z)$ are C^∞ functions such that $\chi(r)$ satisfies (2.7a) and

$$\Phi(z) = 0 \quad \text{for} \quad z \bar{\in} S_{\kappa_0}, \tag{2.53}$$

where $S_{\kappa_0} \subset S_{I^n} = S \cap I^n$ such that the the (angular) distance between S_{κ_0} and the x_i-axis is larger than κ_0, $1 \leq i \leq n$. Then

$$\operatorname{Supp} u \subset R_0 = R_{r_0,\kappa_0} = \{x \in I^n \,|\, r < r_0, z \in S_{\kappa_0}\}, \tag{2.54}$$

and (2.11) holds for $1 \leq i,j \leq n$ with constant κ_0 depending on θ_0.

We have the following results for the n-dimensional setting which are analogous to Theorem 2.5 $\sim$ 2.7 and Theorem 2.9 $\sim$ 2.11.

Theorem 2.12. *Let u be given as in (2.52). Then $u \in B_\nu^{s,\beta}(I^n)$ with $\beta = \left(-\frac{1}{n}, -\frac{1}{n}, \cdots - \frac{1}{n}\right)$ and $s = 2\gamma + n - 1$.*

Theorem 2.13. *Let u be given as in (2.52). Then, if $r^\gamma \log^\nu r\Phi(z)$ is not a polynomial,*

$$C_1 p^{-(2\gamma+n-2)}(1+\log p)^{\nu^*} \leq \inf_{\varphi \in P_p(I^n)} \|u-\varphi\|_{H^1(R_{r_0,\theta_0})} \leq C_2 p^{-(2\gamma+n-2)}(1+\log p)^{\nu^*}, \tag{2.55}$$

where C_1 and C_2 are constants independent of p, $\nu^* = \nu - 1$ if γ is an integer and $\nu > 0$, and $\nu^* = \nu$, otherwise .

3 Approximation Error of the p-version Finite Element Solution in Two Dimensions

In this section we shall analyze the approximation error of the p-version finite element solutions for elliptic problems on polygonal domains in two-dimensions in the framework of weighted Besov spaces. The lower and upper bounds of the approximation error will be rigorously proven.

3.1 The p-version Finite Element Solution in Two Dimensions

Let Ω be a polygonal domain shown in Fig. 3.1, with vertices $A_i, 1 \leq i \leq m$, $(A_{m+1} = A_1)$, and (open) edge $\Gamma_i = A_iA_{i+1}$. By ω_i we denote the internal angle between Γ_i and Γ_{i+1}. Let $\mathcal{D}$ be a subset of $\mathcal{M} = \{0, 1, \ldots, M\}$, and $\mathcal{N} = \mathcal{M} \setminus \mathcal{D}$. We refer to $\Gamma_D = \cup_{i\in\mathcal{D}}\bar{\Gamma}_i$ as the Dirichlet boundary and $\Gamma_N = \cup_{i\in\mathcal{N}}\Gamma_i$ as the Neumann boundary.

Consider a boundary value problem:

$$\begin{cases} -\Delta u + u = f \quad \text{in } \Omega \\ u|_{\Gamma_D} = 0 \\ \left.\frac{\partial u}{\partial n}\right|_{\Gamma_N} = g. \end{cases} \tag{3.1}$$

By $H^k(\Omega), k \geq 0$, integer, we denote the usual Sobolev space and $H^1_D(\Omega)) = \{u \in$

$H^1(\Omega)\,|\,u|_{\Gamma_D} = 0\}$. The variational form is to seek $u(x) \in H^1_D(\Omega)$ such that

$$B(u, v) = F(v) \qquad \forall v \in H^1_D(\Omega) \tag{3.2a}$$

where B is the bilinear form on $H^1_D(\Omega) \times H^1_D(\Omega)$:

$$B(u, v) = \int_\Omega (\nabla u \cdot \nabla v + uv)dx, \tag{3.2b}$$

and F is the linear functional on $H^1(\Omega)$:

$$F(v) = \int_\Omega fvdr + \int_{\Gamma_N} gvds. \tag{3.2c}$$

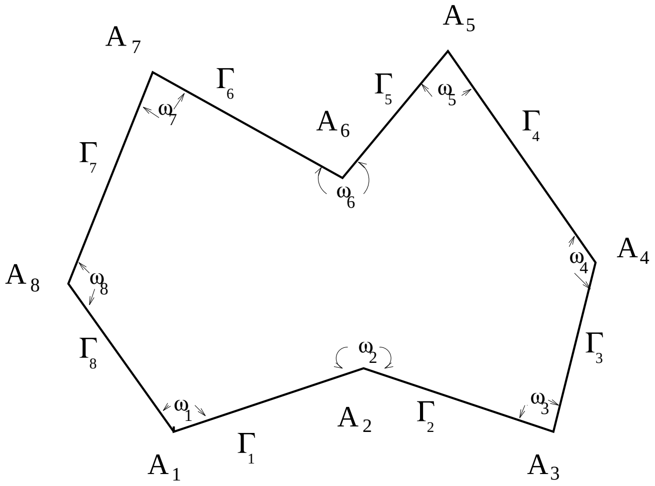

Fig. 3.1 Polygonal domain Ω

Let $\mathcal{T} = \{\Delta\}$ be a family of quasi-uniform meshes $\Delta = \{\Omega_i\}_i^n$ with shape regular elements $\Omega_i's$ which are (open) triangles and quadrilaterals. We assume that $\bar{\Omega} = \cup_{i=1}^n \bar{\Omega}_i$ and $\bar{\Omega}_i \cap \bar{\Omega}_j, i \neq j$ is an entire side, a vertex in common, or empty.

By M_i we denote an affine mapping of $S = (-1, 1)^2$ onto Ω_i. The finite element space is defined as

$$S^p(\Omega; \Delta) = \{\varphi \in H^1(\Omega) | \varphi \circ M_i \text{ is a polynomial of degree } p \text{ on } S\}$$

and

$$S^p_D(\Omega; \Delta) = S^p(\Omega; \Delta) \cap H^1_D(\Omega).$$

The finite element formulation is to seek $u_p \in S^p_D(\Omega;\Delta)$ such that

$$B(u_p, v) = F(v), \qquad \forall v \in S^p_D(\Omega;\Delta). \tag{3.3}$$

Due to the coercivity of B, it follows easily that

$$\|u - u_p\|_{H^1(\Omega)} \le C \inf_{v \in S^p_D(\Omega;\Delta)} \|u - v\|_{H^1(\Omega)}. \tag{3.4}$$

Theorem 3.1. *Let $u \in H^k(\Omega), k > 1$ be the solution of the problem (3.2), and let $u_p \in S^p(\Omega;\Delta)$ be the finite element solution of (3.3). Then there holds for $p \ge k-1$*

$$\|u - u_p\|_{H^1(\Omega)} \le C p^{-(k-1)} \|u\|_{H^k(\Omega)}. \tag{3.5}$$

Proof. First, we introduce a linear mapping M which maps Ω onto $Q' \subset Q = (-1,1)^2$, and then we extend the function $\tilde{u} = u \circ M$ on Q with preserving the H^k-norm, denoted again by $\tilde{u}$. Obviously $\tilde{u} \in H^{k,\beta}(Q)$ with any $\beta > 0$. Then by Theorem 2.1 there is a polynomial $\tilde{\varphi} \in P_p(Q)$ such that

$$\begin{aligned} |\tilde{u} - \tilde{\varphi}|_{H^{1,\beta}(Q)} &\le C p^{-(k-1)} |\tilde{u}|_{H^{k,\beta}(Q)} \\ &\le C p^{-(k-1)} \|\tilde{u}\|_{H^{k,\beta}(Q)} \\ &\le C p^{-(k-1)} \|u\|_{H^k(Q)}. \end{aligned} \tag{3.6}$$

Note that $\varphi = \tilde{\varphi} \circ M^{-1} \in S^p(\Omega;\Delta)$ and

$$|u - \varphi|_{H^1(\Omega)} \le C \|\tilde{u} - \tilde{\varphi}\|_{H^{1,\beta}(Q)},$$

which,together with (3.4) and (3.6), leads to (3.5) if $|\Gamma_D| = 0$.

For $|\Gamma_D| \ne 0$ we refer to [?] for the techniques to adjust the Dirichlet boundary conditions on elements Ω_i such that $|\partial\Omega_i \cap \Gamma_D| \ne 0$. We also refer to [?] for the arguments of partition of unity for homogeneous Dirichlet boundary conditions. □

3.2 The Lower and Upper Bounds of Approximation Errors for Elliptic Problem with Singular Solution in Polygonal Domains

It is well known that the solutions of problem (2.1) in polygonal domains are singular in neighborhoods of vertices A_i's, governing the performance of the finite element solutions.

Let $S_{\delta_i} = \{x \in \Omega \,|\, \mathrm{dist}(x, \Lambda_i) < \delta_i\}$ be a neighborhood of the vertex A_i with $\delta_i \in (0,1)$ shown in Fig. 3.2. δ_i is selected such that $S_{\delta_i} \cap S_{\delta_j} = \emptyset$ for $i = j$. $\Omega_0 = \Omega \setminus \cup_{i \in M} S_{\delta_i/2}$ contains no vertices of Ω, and $\Omega_0 \cap S_{\delta_i} \neq \emptyset$ for $i \in \mathcal{M}$. Ω_0 is called the regular part of the domain Ω.

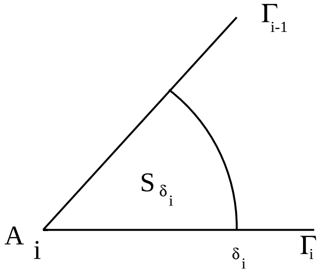

Fig. 3.2 A neighborhood of the vertex A_i

It is known that if $f \in H^{k-2}(\Omega), g \in H^{k-3/2}(\Omega), k \geq 2$ then $u \in H^k(\Omega_0)$, and in each neighborhood S_{δ_i} $u(x)$ has an asymptotic expansion in terms of singular functions of $r^\gamma \log^\nu r$-type:

$$u = \sum_{\substack{m \geq 1 \\ 0 < \gamma_m^{[i]} \leq k-1}} r_i^{\gamma_m^{[i]}} (\log r_i)^{\nu_m^{[i]}} \Phi_m^{[i]}(\theta_i) \chi^{[i]}(r_i) + u_0^{[i]} \tag{3.7}$$

where (r_i, θ_i) are polar coordinates with the origin located at the vertex A_i, and $u_0^{[i]} \in H^k(S_{\delta_i})$ is the smooth part of u, $\nu_m^{[i]}, \gamma_m^{[i]} \geq 0$, $\nu_m^{[i]}$ are integers. We assume that $\nu_m^{[i]} > \nu_{m+1}^{[i]} \geq 0$ and $\gamma_m^{[i]} \leq \gamma_{m+1}^{[i]}$, $\chi^{[i]}(r_i)$ and $\Phi^{[i]}(\theta_i)$ are C^∞-functions such that $\chi^{[i]}(r_i) = 1$ for $0 < r_i < \delta_i/2$, $\chi^{[i]}(r_i) = 0$ for $r > \delta_i$. Let γ, ν and i_0 be such that

$$\gamma = \min_i \gamma_1^{[i]} = \gamma_1^{[i_0]}, \qquad \nu = \max_{\gamma_1^{[i]}=\gamma} \nu_1^{[i]} = \nu_1^{[i_0]}. \tag{3.8}$$

We will apply the estimates of error based on the weighted Besov space to the problem (3.1).

Theorem 3.2. *Let u be the solution of the problem (3.1) with $f \in H^{k-2}(\Omega)$ and $g \in H^{k-3/2}(\Gamma)$, $k \geq \max\{2, 2\gamma + 1\}$ and let $u_p \in S_D^p(\Omega; \Delta), p \geq k - 1$ be the finite element solution. Then*

$$\|u - u_p\| \leq Cp^{-2\gamma}(1 + \log p)^{\nu^*} \tag{3.9}$$

with a constant C independent of p, with γ and ν given in (3.8), and

$$\nu^* = \begin{cases} \nu & \text{if } \gamma \text{ is not an integer} \\ \nu - 1 & \text{if } \gamma \text{ is an integer and } \nu \geq 1 \\ 0 & \text{if } \nu = 0 \end{cases} \tag{3.10}$$

Remark 3.1. The estimate (3.9) was given in [?], where special techniques were used for the corner singularity and adjustment between elements was made for C^0-continuity while retaining the optimal rate of the approximation error on each element. Here we will use a different approach, namely using the weighted Besov space to obtain the error estimate and using partition of unity for the C^0-continuity.

Proof. of Theorem 3.2 Due to (3.4), it suffices to construct a series of polynomial $\varphi_p \in S_D^p(\Omega; \Delta), p = k - 1, k, \ldots,$ such that for each p

$$\|u - \varphi_p\|_{H^1(\Omega)} \leq Cp^{-2\gamma}(1 + \log p)^{\nu^*} \tag{3.11}$$

By $T_\ell, 1 \leq \ell \leq L$, we denote vertices of the mesh Δ and by Q_ℓ we denote a union of elements which have T_ℓ as a vertex, i.e. $Q_\ell = \cup_{T_\ell \in \bar{\Omega}_j} \bar{\Omega}_j$. T_ℓ and Q_ℓ are referred to as a node and a patch centered at T_ℓ, respectively. Let $\phi_\ell \in S^1(\Omega; \Delta)$ such that $\phi_\ell(T_\ell) = 1$, $\phi_\ell(T_j) = 0$ for $j \neq \ell$. Note that supp. $\Phi_\ell = Q_\ell$ and that $\sum_{\ell=1}^L \phi_\ell \equiv 1$. Thus $\{\phi_\ell\}_{\ell=1}^L$ is a partition of unity.

Let $\varphi_p = \sum_{\ell=1}^L \phi_\ell \varphi_{p-1}^{[\ell]}$ with $\varphi_{p-1}^{[\ell]} \in P_{p-1}(Q_\ell)$, where $P_{p-1}(Q_\ell)$ is a set of polynomial of degree p on Q_ℓ. Then, $\varphi_p \in S^p(\Omega; \Delta)$ and

$$u - \varphi_p = \sum_{\ell=1}^{L} \phi_\ell (u - \varphi_{p-1}^{[\ell]}).$$

We now need to construct a polynomial $\varphi_{p-1}^{[\ell]} \in P_{p-1}(Q_\ell)$ in each patch Q_ℓ such that $\varphi_{p-1}^{[\ell]}$ satisfies the homogeneous Dirichlet boundary condition on Γ_D, and

$$\|u - \varphi_{p-1}^{[\ell]}\|_{H^1(Q)} \leq C p^{-(k-1)} \|u\|_{H^k(Q)} \tag{3.12}$$

if $\bar{Q}_\ell$ contains no vertices of Ω, or

$$\|u - \varphi_{p-1}^{[\ell]}\|_{H^1(Q)} \leq C p^{-2\gamma} (1 + \log p)^{\nu^*} \tag{3.13}$$

if $\bar{Q}_\ell$ contains a vertex of Ω.

For the sake of simplicity, we assume that $|\Gamma_D| = 0$. We refer to [?] for technical detail of arguments for the Dirichlet boundary condition and the mixed boundary condition.

We shall construct $\varphi_{p-1}^{[\ell]}$ on each Q_ℓ for two different cases:

(A) Q_ℓ contains no vertex;

(B) Q_ℓ contains a vertex.

Case (A): Since Q_ℓ contains no vertices of Ω, $Q_\ell \subset \Omega_0$ and $u \in H^k(\Omega_0)$. We introduce a linear mapping M_ℓ which maps $\tilde{Q}_\ell$ onto $Q_\ell \subset Q = (-1,1)^2$, and $\tilde{u} =$

$u \circ M_\ell^{-1}$. $\tilde{u}$ can be extended to whole Q preserving the H^k norm. By standard argument of the p-version on Q (e.g. see [**?**, Lemma 4.1]) there exists a polynomial $\tilde{\varphi}_{p-1} \in P_{p-1}(Q)$ such that

$$\|\tilde{u} - \tilde{\varphi}_{\ell-1}\|_{H^1(Q)} \leq Cp^{-(k-1)}\|\tilde{u}\|_{H^k(Q)}. \tag{3.14}$$

Let $\varphi_{p-1}^{[\ell]} = \tilde{\varphi}_{p-1} \circ M_\ell$. Then $\varphi_{p-1}^{[\ell]} \in P_{p-1}(Q_\ell)$, and (3.12) follows from (3.14).

Case (B): We shall analyze the case that the center T_ℓ is the vertex A_i of Ω because the treatments for those with T_ℓ located on the boundary or in interior are the same with what follows.

We assume that $T_\ell = A_i$ is the origin and that there are several elements Ω_t, $1 \leq t \leq m$ around the vertex A_i located on the line $\theta = \theta_t$, $0 \leq t \leq m$, where $\theta_0 = 0, \theta_m = \omega_i$. In Q_ℓ the solution has an asymptotic expansion (3.7), namely $u = v + u_0$, where $u_0 \in H^k(Q_\ell)$ is the smooth part, and

$$v = \sum_{\substack{m \geq 1 \\ \gamma_m \leq k-1}} r^{\gamma_m}(\log r_m)^{\nu_m}\Phi_m(\theta)\chi(r)$$

is the singular part, where we omitted the super-index [i].

Since $u_0 \in H^k(Q_\ell)$, using the argument in Case (A), we have a polynomial $\varphi_{p-1}^{[0]} \in P_{p-1}(Q_\ell)$ (or $S^{p-1}(\Omega;\Delta)$) such that

$$\|u_0 - \varphi_{p-1}^{[0]}\|_{H^1(Q_\ell)} \leq Cp^{-(k-1)}\|u_0\|_{H^k(Q_\ell)}. \tag{3.15}$$

We shall next construct a polynomial $\varphi_{p-1}^{[\nu]} \in P_{p-1}(Q_\ell)$ for two cases: (B1) the internal angle $\omega_i < \pi$; (B2) the internal angle $\omega_i \geq \pi$.

In case (B1), there is a $\sigma > 0$ such that $\omega_i + 2\sigma < \pi$. We extend $\Phi(\theta)$ to interval $[-\sigma, \omega_i + \sigma]$; then v is extended to Q_ℓ^* which contains Q_ℓ and is between the lines $\theta = -\sigma$ and $\theta = \omega_i + \sigma$. We now introduce an affine mapping M which maps

Q_ℓ^* into $\tilde{Q}_\ell^* \subset R_0 = R_{r_0,\theta_0} \subset Q = (-1,1)^2$, where R_{r_0,θ_0} is defined in (2.8). Then $\tilde{v} = v \circ M^{-1} \in B_\nu^{s,\beta}(Q)$, with $s = 1 + 2\gamma$ and $\beta = (-1/2, -1/2)$. By Theorem 2.7, there exists a polynomial $\tilde{\varphi}_{p-1} \in P_{p-1}(Q)$ such that

$$\|\tilde{v} - \tilde{\varphi}_{p-1}\|_{H^1(R_0)} \leq C p^{-2\gamma_1} (1 + \log p)^{\nu_1^*}. \tag{3.16}$$

Letting $\varphi_{p-1}^{[v]} = \tilde{\varphi}_{p-1} \circ M \in P_{p-1}(Q_\ell^*)$, we have

$$\begin{aligned} \|v - \varphi_{p-1}^{[v]}\|_{H^1(Q_\ell)} &\leq \|v - \varphi_{p-1}^{[v]}\|_{H^1(Q_\ell^*)} \\ &\leq C \|\tilde{v} - \tilde{\varphi}_{p-1}\|_{H^1(R_0)}, \end{aligned}$$

by (3.16),

$$\leq C p^{-2\gamma_1} (1 + \log p)^{\nu_1^*}. \tag{3.17}$$

Noting that $k \geq 2\gamma + 1$, and combining (3.14) and (3.16), we have (3.13) for case (B1).

In case (B2), we extend $\Phi(\theta)$ to $[-\sigma, \omega_i + \sigma]$, with $\sigma > 0$, such that $\theta_1 + \sigma < \pi$ and $(\theta_m - \theta_{m-1}) + \sigma < \pi$. Let $\theta_{-1} = -\sigma$ and $\theta_{m+1} = \omega_i + \sigma$, and let Ω_0 and Ω_{m+1} denote the two additional elements, which are between lines $\theta = \theta_{-1}$ and $\theta = \theta_0$, and the lines $\theta = \theta_m$ and $\theta = \theta_{m+1}$, respectively.

Let $S_t = (\theta_{t-1} + \sigma_t, \theta_{t+1} - \sigma_t)$, with $\sigma_t 1 \leq t \leq m$ properly selected, such that $\{S_t\}_{t=0}^m$ will be a cover of $[0, \omega_i]$. Let $\{\psi_t\}_{t=0}^m$ be a partition of unity subordinated to the cover $\{S_t\}_{t=0}^m$. Set $v_t = \psi_t v$, $\operatorname{supp.} v_t \subset S_t \subset (\bar{\Omega}_t \cup \bar{\Omega}_{t+1})$. We assume here that $(\theta_{t+1} - \theta_{t-1}) < \pi$ for all $0 \leq t \leq m$. For each v_t it becomes case (B1); namely, there exists a polynomial $\varphi^{[t]} \in P_{p-1}(\bar{\Omega}_t \cup \bar{\Omega}_{t+1})$, such that

$$\|v_t - \varphi_{p-1}^{[t]}\|_{H^1(\bar{\Omega}_t \cup \bar{\Omega}_{t+1})} \leq C p^{-2\gamma_1} (1 + \log p)^{\nu_1^*}.$$

We extend $\varphi^{[t]}$ by zero outside of S_t, and set $\varphi^{[v]}_{p-1} = \sum_{t=0}^{m} \varphi^{[t]}$. Then $\varphi^{[v]}_{p-1} \in S^{p-1}(\Omega; \Delta)$, and

$$\begin{aligned} \|v - \varphi^{[v]}_{p-1}\|_{H^1(Q_t)} &\leq \sum_{t=0}^{m} \|v_t - \varphi^{[t]}_{p-1}\|_{H^1(\bar{\Omega}_t \cup \bar{\Omega}_{t+1})} \\ &\leq Cp^{-2\gamma_1}(1 + \log p)^{\nu_1^*}, \end{aligned}$$

which together with (3.15) leads to (3.13).

If $(\theta_{t_1} - \theta_{t-1}) \geq \pi$ for some t, we introduce an additional linear mapping M_t which maps the line $\theta = \theta_t$ onto itself and maps the line $\theta = \theta_{t-1}$ onto $\theta = \theta'_{t-1}$ such that $\theta_t - \theta'_{t-1} < \pi - (\theta_{t+1} - \theta_t)$. Then $(\theta_{t+1} - \theta'_{t-1}) < \pi$, the case is converted to the one which we have analyzed above. For details of this technique, we refer to [?]. ☐

We shall next prove a lower bound theorem.

Theorem 3.3. *Let u be the solution of the problem (3.1) with $f \in H^{k-2}(\Omega)$ and $g \in H^{k-3/2}(\Gamma_N)$, $k = \max\{2, 1 + 2\gamma\}$, and let $u_p \in S^p_D(\Omega; \Delta)$ be the finite element solution of the corresponding problem (3.3). Then*

$$\|u - u_p\| \geq Cp^{-2\gamma}(1 + \log p)^{\nu^*}, \tag{3.18}$$

where C is independent of p, ν^* is given in (3.10), γ and ν are given by (3.8).

Proof. We assume that Ω_1 is the element containing the vertex A_{i_0} where the strongest singularity occurs. It suffices to prove

$$\|u - u_p\|_{H^1(\Omega_1)} \geq Cp^{-2\gamma}(1 + \log p)^{\nu^*} \tag{3.19}$$

By M_1 we denote a linear mapping which maps Ω_1 onto $R \subset Q = (-1, 1)^2$ and A_i to the vertex $(-1, 1)$. We may assume $R \subset R_{r_0,\theta_0} \subset Q$ with some $r_0 \in (0, 1)$ and $\theta_0 \in (0, \frac{\pi}{2})$, with R_{r_0,θ_0} as defined in (2.8). Let $\tilde{u} = u \circ M_1$ and $\tilde{u}_p = u_p \circ M_1$. Then

$\tilde{u}_p \in P_p(Q)$. For non-integer γ and any $\nu \geq 0$, and integer γ with $\nu > 0$, we have by Theorem 2.9–2.10

$$\|\tilde{u} - \tilde{u}_p\|_{H^1(Q)} \geq Cp^{-2\gamma}(1 + \log p)^{\nu^*}, \tag{3.20a}$$

and for integer γ with $\nu = 0$ by Theorem 2.11 we have

$$\|\tilde{u} - \tilde{u}_p\|_{H^1(Q)} \geq Cp^{-2\gamma}, \tag{3.20b}$$

where C is independent of p. Because M_1 is a linear mapping (3.18) follows immediately from (3.20). $\square$

Combining the estimate of the upper and lower bounds of errors of the p-version in the H^1-norm derived above, we now conclude with the optimal error estimate for the elliptic problem on polygonal domains.

Theorem 3.4. *Let u and u_p be the solution of the problem (3.1) on polygonal domain Ω and the finite element solution of the p-version in $S^p_D(\Omega; \Delta)$, respectively. Then there are constants C_1 and C_2 independent of p such that*

$$C_1 p^{-2\gamma} (1 + \log p)^{\nu^*} \leq \|u - u_p\|_{H^1(\Omega)} \leq C_2 p^{-2\gamma} (1 + \log p)^{\nu^*},$$

where γ and ν are given in (3.8), which represent the strongest singularity of the solution of the problem (3.1). ν^* is given in (3.10).

Remark 3.2. We have rigorously proved the optimal convergence of the p-version of the finite element method for two dimensions. The mathematical tools and techniques can be generalized to three dimensional problems.

Remark 3.3. Whether C_1 and C_2 are asymptotically the same remains to be answered yet. Nevertheless, the same order on the the upper and lower bound of errors allows us to develop a-posteriori error estimators by extrapolation of computational solutions, which will be reasonably reliable in practice if the difference between C_1 and C_2 is not very large.

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