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Dedicated to Professor Ivo Babuška on the occasion of his seventith birthday

Abstract

Domain decomposition method for the *h-p* version of the finite element method in two and three dimensions are discussed. Using the framework of additive Schwarz method, various iterative methods are described, with their condition numbers estimated. Further, to reduce the cost for solving the sub-problems on element interfaces, different inexact interface solvers are proposed. The effects on the overall condition number, as well as their efficient implementation, are analyzed.

Key Words: domain decomposition, additive Schwarz method, condition number, *h-p* version.
AMS(MOS) Subject Classification: 65F10, 65N30, 65N55

1 Introduction

Domain decomposition method provides a powerful iterative and parallel solution technique for the large-scale linear systems arising from the finite element approximations. The basic idea behind the domain decomposition method is to decompose the approximation space in the finite element method into a number of subspaces, each of them corresponds to a specific set of geometric objects (subdomains or substructures), and then to correct the intermediate solution in every step of iteration on these subspaces separately. These corrections are equivalent to solve the subproblems with unknowns associated with the subdomains or substructures, which are much smaller in size than that of the original one and much easier to solve. Among many others, two primary concerns for the study of the domain decomposition method are: (1) the condition number of the iteration operator, which determines the total number of iteration to achieve a convergence criterion, and (2) the cost to solve the subproblems on the subspaces, which contribute directly to the overall cost of the iterative solutions. Clearly, a desirable domain decomposition method should resulted in small condition numbers for the iteration operators and inexpensive solvers for the subproblems. See, e.g.,

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for the conventional finite element method (i.e., the h-version) has been studied extensively over the last decade, see, e.g., [5, 6, 7, 11, 15, 16, 26, 34, 35]. By properly define the space decomposition and the subproblems on subspaces, a typical domain decomposition method usually results in a condition number which grows only logarithmically with the number of unknowns in subdomains if the subdomains do not overlap with each other, or in a condition number which is bounded from above by a constant if the subdomains overlap "generously" with each other. It has become an efficient and effective alternative to classical direct solvers. Domain decomposition method for the p-version and spectral element methods has also been developed considerably, see, e.g., [2, 27, 28, 31, 32]. Similar ideas as in the h-version can also be used in the case of p-version, though there are essential differences between the properties of the approximation spaces of the two methods. A properly defined domain decomposition method typically results in a condition number growing logarithmically with the degree of polynomial used in the approximations.

The h-p version of the finite element method is a new approach for the discretization of the elliptic boundary value problems in engineering practices. It is designed to adjust both the element sizes and polynomial degrees according to the smoothness of the problems to optimize the approximations. It is shown under the frame of countably normed spaces that this new approach can achieve the exponential rate of convergence, if the computational mesh and polynomial degrees are properly designed, see [4, 18, 19, 20, 21]. It is the most effective approach for solving the problems in nonsmooth domains in computational mechanics and engineering. Direct solution methods are the classical techniques for solving the linear systems resulted in the h-p version. With the advances of modern parallel computer technology, more interests have also been attracted to the iterative solution techniques for the h-p version. In recent years, many authors studied the domain decomposition method for the h-p version approximations, see, e.g., [1, 22, 23, 24, 29, 30]. Though the analysis for the two dimensional problems are relatively inclusive in these paper, there are only limited results for the three dimensional problems and numerical experiments. More investigations are required in the future for the design, analysis, and implementation of the domain decomposition method for the h-p version.

In this paper we will discuss the domain decomposition method using the frame of additive Schwarz method for the h-p version of the finite element method for two and three dimensional elliptic problems. Our basic idea is to distinguish the linear modes (the h-version components) from the high order modes (the p-version components) in the h-p version approximation, and to treat the h-version components as in the h-version finite element method and the p-version components as in the p-version finite element method. The resulted algorithm can be implemented in parallel on the subdomain level for the h-version components and on the element level for the p-version components. The analysis will be focused on the decomposition of the approximation spaces and the condition number of the iteration operators. To reduce the cost of the solution of subproblems on element interfaces, various inexact solvers, together with their fast solution technique, are described and analyzed.

This paper is organized as follows. In Section 2 we introduce the h-p version of the finite element method for the model problem. In Section 3 we describe an abstract additive Schwarz method. In Section 4 and Section 5 we study separately the two and three dimensional problems. In Section 6 we discuss the issue of the inexact solvers on element interfaces.

2 The h-p Version Approximation

2.1 The Model Problem and Two-level Mesh

Let Ω be a polyhedral domain in $\mathbf{R}^d$, $d = 2, 3$. $L^2(\Omega)$, $H^1(\Omega)$, and $H_0^1(\Omega)$ are the usual Sobolev spaces. Given $f \in L^2(\Omega)$, consider the following Poisson equation with homogeneous Dirichlet condition:

$$\begin{cases} -\Delta u &= f, \quad \text{in } \Omega, \\ \quad u &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Define a bilinear form $a_\Omega(\cdot,\cdot)$: $H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbf{R}^1$ as

$$a_\Omega(u, v) = \int_\Omega \nabla u \cdot \nabla v \; d\Omega.$$

Then the weak formulation of (2.1) is to find $u \in H_0^1(\Omega)$ such that

$$a_\Omega(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(\Omega)$.

Now we consider the approximation of (2.2) by the h-p version of the finite element method. First we partition Ω into a two-level mesh:

(1) **Coarse Mesh.** We divide Ω into a regular family of triangles and quadrilaters (in $\mathbf{R}^2$), or tetrahedrals, pentahedrals, and hexahedrals (in $\mathbf{R}^3$), i.e., $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$. See [12]. We refer to the Ω_i's as *subdomains*, and use H_i to denote the diameter of Ω_i. Also we will refer to the mesh constituted by the subdomains in Ω as the *coarse mesh*: $\Omega_H = \{\Omega_i, 1 \le i \le N\}$.

(2) **Fine Mesh.** Each subdomain Ω_i is further partitioned into a number of smaller triangles and quadrilaters (in $\mathbf{R}^2$), or smaller hexahedrals (in $\mathbf{R}^3$), such that $\bar{\Omega}_i = \cup_{m=1}^{N_i} \bar{K}_m^{(i)}$. We require that and $K_m^{(i)}, 1 \le m \le N_i$, are all shape regular and of the same order of diameters, i.e., the partition of Ω_i is quasi-uniform. We refer to the $K_m^{(i)}$'s as *elements*, and use h_i to denote the characteristic diameter of the elements in Ω_i. We refer to the mesh constituted by all the elements in Ω as the *fine mesh*: $\Omega_h = \{K_m^{(i)}, 1 \le m \le N_i, 1 \le i \le N\}$.

To distinguish between the geometric objects in the coarse mesh and those in the fine mesh, we refer to the vertices of a subdomain as *vertices*, while those of an element as *nodes*, and refer to the edges of a subdomain as *edges*, while those of an element as *sides*. Further, we will usually use Γ and γ (with or without subscripts) to denote an edge of the subdomains and a side of the elements, and use F and f (with or without subscripts) to denote a face of the subdomains and a face of the elements, respectively. For convenience, we will omit the subscripts denoting the subdomain or elements in various quantities below, when no confusion occurs.

2.2 Shape Functions in Reference Element

• **Degree of Freedoms:** Let $\hat{K}$ be the reference element in $\mathbf{R}^2$ or in $\mathbf{R}^3$. We use $\hat{\gamma}$ to denote one of its sides, and use $\hat{f}$ to denote one of its faces if $d = 3$. For each side and face, a polynomial degree $p_{\hat{\gamma}}$ or $p_{\hat{f}}$ is assigned to construct the approximation subspace, and a polynomial degree $p_{\hat{0}}$ is also assigned for the interior of $\hat{K}$. We define a multiple index $\hat{\mathcal{P}}$ to denote all these polynomial degrees associated with $\hat{K}$.

• **Shape Functions:** Let p be a positive integer. If $\hat{K}$ is a triangle or a hexahedral, we define $\mathcal{P}_p(\hat{K})$ to be the set of polynomial of total degree $\le p$ on $\hat{K}$; if $\hat{K}$ is a quadrilateral or hexahedral, we define $\mathcal{P}_p(\hat{K})$ to be the set of polynomial of separate degree $\leq p$ on $\hat{K}$; We define the following sets of shape functions (or modes):

1). The set $\Psi^{[N]}(\hat{K})$ of *nodal* shape functions. It is composed of the linear or bilinear functions which have the value one at one node of $\hat{K}$, and zero at all others;

2). The set $\Psi^{[\hat{\gamma}]}_{\hat{\mathcal{P}}}(\hat{K})$ of *side* shape functions. If $\hat{\gamma}$ is a side of $\hat{K}$, then a side shape function associated with $\hat{\gamma}$ vanishes on all the sides of $\hat{K}$ except $\hat{\gamma}$, and on all the faces not touching γ (if $d = 3$). $\Psi^{[\hat{\gamma}]}_{\hat{\mathcal{P}}}(\hat{K})$ is composed of all side shape functions in $\mathcal{P}_{p_{\hat{\gamma}}}(\hat{K})$ associated with $\hat{\gamma}$;

3). The set $\Psi^{[\hat{f}]}_{\hat{\mathcal{P}}}(\hat{K})$ of *face* shape functions (if $d = 3$). If $\hat{f}$ is a face of $\hat{K}$, then a face shape function associated with $\hat{f}$ vanishes on $\partial\hat{K}\backslash\hat{f}$. $\Psi^{[\hat{f}]}_{\hat{\mathcal{P}}}(\hat{K})$ is composed of all face shape functions in $\mathcal{P}_{p_{\hat{f}}}(\hat{K})$ associated with $\hat{f}$;

4). The set $\Psi^{[\hat{b}]}_{\hat{\mathcal{P}}}(\hat{K})$ of *internal* shape (or *bubble*) functions. These shape functions are in $\mathcal{P}_{p_{\hat{0}}}(\hat{K})$ and vanish on $\partial\hat{K}$.

For brevity, we also use $\Psi^{[N]}(\hat{K})$, $\Psi^{[\hat{\gamma}]}_{\hat{\mathcal{P}}}(\hat{K})$, $\Psi^{[\hat{f}]}_{\hat{\mathcal{P}}}(\hat{K})$, and $\Psi^{[\hat{b}]}_{\hat{\mathcal{P}}}(\hat{K})$ to denote the spaces spanned by the corresponding set of shape functions. There are many different ways to create the shape functions, see, e.g., [3, 8, 31, 32, 33].

2.3 Approximation Subspace

Consider an element K. Assume that for each side, face, and the interior of K, a polynomial degree p^K_γ, p^K_f, and p^K_0 is assigned, respectively. We define $\mathcal{P}^K$ as the multiple index composed of all these polynomial degrees associated with the element K. Let F_K be the affine mapping from the reference element $\hat{K}$ onto K, and let $\hat{\gamma}$ and $\hat{f}$ be the imagines of γ and f under F_K^{-1}, respectively.

We define correspondingly the function spaces associated with the nodes, sides, faces, and the interior of element K as follows:

$$\begin{cases} \Psi^{[N]}(K) & = \{v \circ F_K^{-1} \mid \forall v \in \Psi^{[N]}(\hat{K})\}, \\ \Psi^{[\gamma]}(K) & = \{v \circ F_K^{-1} \mid \forall v \in \Psi^{[\hat{\gamma}]}_{\mathcal{P}^K}(\hat{K})\}, \\ \Psi^{[f]}(K) & = \{v \circ F_K^{-1} \mid \forall v \in \Psi^{[\hat{f}]}_{\mathcal{P}^K}(\hat{K})\}, \\ \Psi^{[b]}(K) & = \{v \circ F_K^{-1} \mid \forall v \in \Psi^{[\hat{b}]}_{\mathcal{P}^K}(\hat{K})\}. \end{cases}$$

Now the approximation subspace in the element K is defined as

$$\Psi(K) = \Psi^{[N]}(K) \oplus \cup_\gamma \Psi^{[\gamma]}(K) \oplus \cup_f \Psi^{[f]}(K) \oplus \Psi^{[b]}(K).$$

The approximation subspace for the h-p version of FEM in Ω is

$$\Psi(\Omega) = \{v \mid v|_K \in \Psi(K)\} \cap H^1_0(\Omega).$$

The h-p version finite element method for solving (2.2) is to find $u \in \Psi(\Omega)$ such that

$$a_\Omega(u, v) = (f, v), \qquad \forall v \in \Psi(\Omega). \tag{2.3}$$

3 Iterative Solution Method

We describe in this section an iterative and parallel method, the so called *additive Schwarz method (ASM)*, for solving a general symmetric positive definite system like (2.3). Let $\mathcal{V}$ be a Hilbertian space with inner product $(\cdot,\cdot)$. $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbf{R}$, is symmetric positive definite bilinear form. We want to find $u \in \mathcal{V}$ such that

$$a(u,v) = (f,v), \qquad \forall v \in \mathcal{V}. \tag{3.1}$$

Assume that

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1 + \cdots + \mathcal{V}_n$$

and that an inner product $b_k(\cdot,\cdot)$ is defined for each $\mathcal{V}_k$. Define the linear mapping $T_k : \mathcal{V} \to \mathcal{V}_k$ as for $w \in \mathcal{V}$, $T_k w \in \mathcal{V}_k$ satisfying

$$b_k(T_k w, v) = a(w,v), \quad \forall v \in \mathcal{V}_k. \tag{3.2}$$

Let $T = T_0 + T_1 + \cdots + T_n$, and let $g = \sum_{k=0}^{n} g_k$, with $g_k \in \mathcal{V}_k$ defined by

$$b_k(g_k, v) = (f,v), \qquad \forall v \in \mathcal{V}_k.$$

Now we solve instead of (3.1) but the following problem: to find $u^* \in \mathcal{V}$ such that

$$Tu^* = g. \tag{3.3}$$

If T is invertible, then the solution u^* of (3.3) is identical to the solution u of (3.1).

Consider the iterative solution of (3.3), e.g.,

$$u^{(m+1)} = u^{(m)} + \alpha(g - Tu^{(m)}), \quad m = 0,1,2,\cdots,$$

where α is an iteration parameter. In each step of the iteration we have to evaluate the action of T on the intermediate solution $u^{(m)}$. It can be accomplished by solving $n+1$ subproblems as follows:

$$b_k(T_k u^{(m)}, v) = a(u^{(m)}, v), \quad \forall v \in \mathcal{V}_k. \tag{3.4}$$

Clearly, in order to achieve a convergence criterion, the number of iterations depends on the condition number of T. Therefore, to define an efficient additive Schwarz method, the following two aspects are of the primary concern: (1) the condition number of T, and (2) the cost of the solution of the subproblems (3.4). In the remaining of this paper, our discussion will mainly focus on these two issues for the linear system (2.3) arising from the *h-p* version finite element method.

4 Two dimensional problems

4.1 Non-overlapping case

To define an iterative method using the framework of the ASM, we need only to define the decomposition of the space $\Psi(\Omega)$ and the bilinear forms $b_k(\cdot,\cdot)$ associated with each subspaces.

• **Decomposition of space:** First, we define $\Psi_H^*(\Omega)$ to be the linear finite element space based on the coarse mesh Ω_H, and $\Psi_h(\Omega)$ to be the linear finite element space based on the fine mesh Ω_h. As pointed out in [34], a subspace of $\Psi(\Omega)$ associated with the coarse mesh will play the role of overall communication of information in the iterative method, and is essential to the efficiency of the algorithm. $\Psi^*_H(\Omega)$ would be a suitable choice if it were a subspace of $\Psi(\Omega)$. However, this is generally not true. To replace $\Psi^*_H(\Omega)$, we define a subspace of $\Psi(\Omega)$ as follows:

$$\Psi_H(\Omega) = \Pi_h \Psi^*_H(\Omega), \tag{4.1}$$

where $\Pi_h : C(\Omega) \to \Psi_h(\Omega)$, is the piecewise linear interpolation operator on the nodes of the fine mesh. Clearly, $\Psi_H(\Omega)$ has the same dimensions as $\Psi^*_H(\Omega)$, and will be used for the overall communication.

Next for each subdomain Ω_i, we define a space spanned by the nodal basis functions associated with the nodes inside Ω_i, i.e.,

$$\Psi_{\Omega_i}(\Omega) = \{v \in \Psi_h(\Omega) \mid v = 0 \text{ outside } \Omega_i\}.$$

Let Γ be a edge shared by the subdomains Ω_i and Ω_j. We define a space associated with the edge Γ by the partial orthogonalization on the subdomain level as follows:

$$\Psi_\Gamma(\Omega) = \{v \in \Psi_h(\Omega) \mid \quad \begin{array}{l} v = 0 \text{ in } \Omega_\ell, \quad \forall \ell \neq i, j, \text{ and} \\ a(v, \phi) = 0, \forall \phi \in \Psi_{\Omega_i}(\Omega) \cup \Psi_{\Omega_j}(\Omega)\}, \end{array}$$

Next we consider the subspaces related to the elements. For each element K, we define

$$\Psi_K(\Omega) = \{v \in \Psi(\Omega) \mid v = 0 \text{ outside } K\}.$$

Let γ be a side shared by two elements K_m and K_n. We define a subspace associated with the side γ by the partial orthogonalization on the element level:

$$\Psi_\gamma(\Omega) = \{v \in \Psi(\Omega) \mid \quad \begin{array}{l} v = 0 \text{ in } K_\ell, \quad \forall \ell \neq m, n, \text{ and} \\ a(v, \phi) = 0, \forall \phi \in \Psi_{K_m}(\Omega) \cup \Psi_{K_n}(\Omega)\}. \end{array}$$

Now the decomposition of $\Psi(\Omega)$ is defined as

$$\Psi(\Omega) = \Psi_H(\Omega) \oplus \cup_\Gamma \Psi_\Gamma(\Omega) \oplus \cup_{\Omega_i} \Psi_{\Omega_i}(\Omega) \oplus \cup_\gamma \Psi_\gamma(\Omega) \oplus \cup_K \Psi_K(\Omega). \tag{4.2}$$

• **Bilinear forms:** We take here all the bilinear forms $b_k(\cdot,\cdot)$ in the ASM as $a_\Omega(\cdot,\cdot)$. Then the iterative operator in the ASM is defined by

$$T = T_H + \sum_\Gamma T_\Gamma + \sum_{\Omega_i} T_{\Omega_i} + \sum_\gamma T_\gamma + \sum_K T_K. \tag{4.3}$$

where T_H, T_Γ, T_{Ω_i}, T_γ, and T_K are the projection operators, with respect to $a_\Omega(\cdot,\cdot)$, onto the subspaces in the right hand side of (4.2), respectively.

Theorem 4.1 *([23, Theorem 4.1]) Assume that $p_0^K \geq \max_\gamma p_\gamma^K$ holds for all the elements. Let $p_i = \max_{K \in \Omega_i} p_0^K$. Then for the ASM with T defined in (4.3),*

$$c \min_i (1 + \ln \frac{H_i p_i}{h_i})^{-2} a_\Omega(u,u) \leq a_\Omega(Tu, u) \leq c' a_\Omega(u,u), \forall u \in \Psi(\Omega),$$

and thus $\kappa(T) \leq c \max_i (1 + \ln \frac{H_i p_i}{h_i})^2$

4.2 Overlapping case

• **Decomposition of space:** We construct in this subsection another ASM for (2.3), which overlaps the h-version components (i.e., linear nodal functions) "generously" and does not overlap the p-version components (i.e., high order side and internal functions). We define in the same way as in the previous subsection the subspaces $\Psi_h(\Omega)$, $\Psi_H(\Omega)$, $\Psi_\gamma(\Omega)$, and $\Psi_K(\Omega)$. Let Q be one of the vertices of the subdomains, and let S_Q be the union of all the subdomains touching Q. Define

$$\Psi_Q(\Omega) = \{v \in \Psi_h(\Omega) \mid v = 0 \text{ on } \Omega \setminus S\}.$$

The sum of all $\Psi_Q(\Omega)$'s is the space $\Psi_h(\Omega)$. Therefore, it is easy to see that

$$\Psi(\Omega) = \Psi_H(\Omega) + \sum_Q \Psi_Q(\Omega) + \sum_\gamma \Psi_\gamma(\Omega) + \sum_K \Psi_K(\Omega). \tag{4.4}$$

Note that in the above decomposition of $\Psi(\Omega)$, the subspaces $\Psi_H(\Omega)$ and $\Psi_Q(\Omega)$ (representing the h-version components) overlap with each other.

• **Bilinear form:** We take all the bilinear forms $b_k(\cdot,\cdot)$ in the ASM to be $a_\Omega(\cdot,\cdot)$. Then the iterative operator in the ASM is defined by

$$T = T_H + \sum_Q T_Q + \sum_\gamma T_\gamma + \sum_K T_K. \tag{4.5}$$

where T_H, T_Q, T_γ, and T_K are the projection operators onto the subspaces in the decomposition (4.4), respectively.

Theorem 4.2 *([23, Theorem 3.2]) Assume that $p_0^K \geq \max_\gamma p_\gamma^K$ holds for all the elements. Let $p = \max_K p_0^K$. Then for the ASM with T defined in (4.5),*

$$c(1+\ln p)^{-2} a_\Omega(u,u) \leq a_\Omega(Tu,u) \leq c' a_\Omega(u,u), \quad \forall u \in \Psi(\Omega),$$

and thus $\kappa(T) \leq c(1+\ln p)^2$.

5 Three dimensional problem

• **Decomposition of space:** We define the subspaces $\Psi_h(\Omega)$ and $\Psi_H(\Omega)$ in the same way as in Sec.4.1 in two dimensions. Next we introduce several subspaces related to the subdomains. Let $\mathcal{W}$ denote the union of all the vertices and edges of the subdomains, and let $\mathcal{W}_h$ be the set of all the element nodes lying on $\mathcal{W}$. $\mathcal{W}$ is usually called the wire basket of the subdomains. Define

$$\Psi_\mathcal{W}(\Omega) = span\{\psi_q(x,y,z),\ \forall q \in \mathcal{W}_h\}.$$

where $\psi_q(x,y,z)$ is the usual nodal basis function associated with the node q of the elements. Note that the support of any function in $\Psi_h(\Omega)$ is contained in the union of all the elements touching the wire basket $\mathcal{W}$, and the values of a function in $\Psi_h(\Omega)$ at $\mathcal{W}_h$ determine uniquely this function. For each subdomain Ω_i, define

$$\Psi_{\Omega_i}(\Omega) = \{v \in \Psi_h(\Omega) \mid v = 0 \text{ outside } \Omega_i\}.$$

Let F be a face shared by the subdomains Ω_i and Ω_j. Define by the partial orthogonalization on the subdomain level

$$\Psi_F(\Omega) = \{v \in \Psi_h(\Omega) \mid \quad v = 0 \text{ in } \Omega_\ell, \quad \forall \ell \neq i, j, \text{ and} \\ a(v, \phi) = 0, \forall \phi \in \Psi_{\Omega_i^0}(\Omega) \cup \Psi_{\Omega_j^0}(\Omega)\}.$$

In order to describe the p-version components in the decomposition of $\Psi(\Omega)$, we introduce the shape functions based on the Gauss-Lobatto-Legendre (GLL) interpolation on the reference element. Let $\hat{K} = (-1, 1)^3$ be the reference element, and let p be a positive integer. $L_p(t)$ is the Legendre polynomial of degree p. Denote by $-1 = t_0^{(p)} < t_1^{(p)} < \cdots < t_p^{(p)} = 1$ the GLL points, i.e., the zeros of $(1 - t^2)L_p'(t)$. Let $\hat{\ell}_j^{(p)}(t), 0 \leq j \leq p$, denote the Lagrange interpolation polynomials such that $\hat{\ell}_j^{(p)}(t_{j'}^{(p)}) = \delta_{jj'}$. We define

$$\hat{\phi}_{\hat{q}} = \hat{\ell}_0^{p_{\hat{\gamma}_1}}(\xi)\hat{\ell}_0^{p_{\hat{\gamma}_2}}(\eta)\hat{\ell}_0^{p_{\hat{\gamma}_3}}(\zeta)$$

if $\hat{q}$ is a node of $\hat{K}$ which joins the sides $\hat{\gamma}_1, \hat{\gamma}_2$ and $\hat{\gamma}_3$, and

$$\hat{\phi}_j^{[\hat{\gamma}]} = \hat{\ell}_j^{p_{\hat{\gamma}}}(\xi)\hat{\ell}_0^{p_{\hat{f}_1}}(\eta)\hat{\ell}_0^{p_{\hat{f}_2}}(\zeta), 1 \leq j \leq p_{\hat{\gamma}_1} - 1$$

if $\hat{\gamma}$ is a side of $\hat{K}$ which lies in the ξ direction and joins the faces $\hat{f}_1$ and $\hat{f}_2$.

Now we consider the subspaces associated with the elements. Let w be the union of all the nodes and sides (i.e., the wire basket) of the elements in Ω, and let w_p be the set of all the GLL points lying on w. Note that for each element side, there is a polynomial degree associated with it, and thus the number and positions of the GLL points vary from side to side according to the polynomial degrees. Let

$$\Psi_{\hat{p}}^{[\mathsf{W}]}(\hat{K}) = span\{\hat{\phi}_{\hat{q}_1}, \hat{\phi}_{\hat{q}_2}, \cdots, \hat{\phi}_{\hat{q}_8}\} + \sum_{\hat{\gamma}} span\{\hat{\phi}_1^{[\hat{\gamma}]}, \hat{\phi}_2^{[\hat{\gamma}]}, \cdots, \hat{\phi}_{p_{\hat{\gamma}}-1}^{[\hat{\gamma}]}\}$$

Define

$$\Psi_{\mathsf{W}}(\Omega) = \{v \in \Psi(\Omega) \mid v|_K = \hat{v} \circ F_K^{-1} \text{ and } \hat{v} \in \Psi_{pK}^{[\mathsf{W}]}(\hat{K}), \ \forall K\}.$$

It is easy to see that any function in $\Psi_{\mathsf{W}}(\Omega)$ is determine uniquely by its values on w_p.

For each element K, define

$$\Psi_K(\Omega) = \{v \in \Psi(\Omega) \mid v = 0 \text{ outside } K\}.$$

For any element face f shared by two elements K_m and K_n, define by the orthogonalization on the element level

$$\Psi_f(\Omega) = \{v \in \Psi(\Omega) \mid \quad v = 0 \text{ in } K_\ell, \quad \forall \ell \neq m, n, \text{ and} \\ a(v, \phi) = 0, \forall \phi \in \Psi_{K_m}(\Omega) \cup \Psi_{K_n}(\Omega)\}.$$

$\Psi(\Omega)$ can be decomposed as

$$\Psi(\Omega) = \Psi_H(\Omega) + \quad \Psi_{\mathcal{W}}(\Omega) + \cup_F \Psi_F(\Omega) + \cup_{\Omega_i} \Psi_{\Omega_i}(\Omega) \\ + \Psi_{\mathsf{W}}(\Omega) + \cup_f \Psi_f(\Omega) + \cup_K \Psi_K(\Omega). \tag{5.1}$$

• **Bilinear Forms:** We take the bilinear forms $b_k(\cdot, \cdot)$ in ASM to be $a_\Omega(\cdot, \cdot)$ for all the subspaces in (5.1), except only the two, $\Psi_{\mathcal{W}}(\Omega)$ and $\Psi_{\mathsf{W}}(\Omega)$, associated with the wire basket of the subdomains and the wire basket of elements. For these two subspaces, we define

(1) For all $u, v \in \Psi_{\mathcal{W}}(\Omega)$

$$b_{\mathcal{W}}(u,v) = \sum_i (h_i \sum_{q \in \mathcal{W}_h \cap \partial\Omega_i} u(q)v(q)),$$

where h_i is the characteristic diameter of the elements in Ω_i.

(2) For all $u, v \in \Psi_{\mathsf{W}}(\Omega)$,

$$b_{\mathsf{W}}(u,v) = \sum_K (h_K \sum_{q \in \mathsf{W}_p \cap \partial K} u(q)v(q)w_q),$$

where

$$w_q = \frac{2}{p_{\gamma_1}(p_{\gamma_1}+1)} + \frac{2}{p_{\gamma_2}(p_{\gamma_2}+1)} + \frac{2}{p_{\gamma_3}(p_{\gamma_3}+1)}$$

if q is a node of K shared by its three edges γ_1, γ_2, and γ_3, and

$$w_q = \frac{2}{p_\gamma(p_\gamma+1)[L_{p_\gamma}(t_j^{(p_\gamma)})]^2}, \qquad 1 \le j \le p_\gamma - 1,$$

if q is the j-th GLL point of a side γ of K.

The operator T in the ASM is then defined as

$$T = T_H + T_{\mathcal{W}} + \sum_F T_F + \sum_{\Omega_i} T_{\Omega_i} + T_{\mathsf{W}} + \sum_f T_f + \sum_K T_K. \tag{5.2}$$

where T_H, T_F, T_{Ω_i}, T_f, and T_K are the projection operators, with respect to $a_\Omega(\cdot,\cdot)$, onto the corresponding subspaces (5.1), and $T_{\mathcal{W}}$ and T_{W} are defined as in (3.2) with the above bilinear forms $b_{\mathcal{W}}$ and b_{W}.

Theorem 5.1 *([24, Theorem 4.1]) Assume that all elements are hexahedral, for which* $\max_f p_f^K \le p_0^K \le const. \min_{\gamma,f}\{p_\gamma^K, p_f^K\}$. *Let* $p_i = \max_{K \subset \Omega_i} p_0^K$. *Then for the ASM with* T *defined by (5.2),*

$$c \min_i (1 + \ln \frac{H_i p_i}{h_i})^{-2} a_\Omega(u,u) \le a_\Omega(Tu,u) \le c' a_\Omega(u,u), \quad \forall u \in \Psi(\Omega)$$

and thus $\kappa(T) \le c \max_i (1 + \ln \frac{H_i p_i}{h_i})^2$.

6 Inexact solver on element interfaces

Consider the implementation of the additive Schwarz method defined in the previous two sections. In each step of the iteration, we have to find the solution $T_k v \in \mathcal{V}_k$ of the system

$$b_k(T_k v, \phi) = a(v, \phi), \quad \forall \phi \in \mathcal{V}_k. \tag{6.1}$$

They correspond to a number of sub-problems associated with element nodes on the wire basket, element interfaces, and interiors of elements. It is easy to see that the sub-problem associated with the element nodes on the wire basket is a global problem in Ω with much less unknowns, and the sub-problem associated with the interior of an element simply corresponds to a homogeneous Dirichlet problem in the element, which can be solved in straightforward ways. For the sub-problems associated with the element interfaces, they can be solved in one of the two fashions: (1) Form the

Schur complement blocks with respect to each interface, and then solve a linear system of equations with unknowns on the (open) interface; (2) Solve a linear system of equations with unknowns in the union of the two elements sharing the interface. By both methods, the computation of these sub-problems on element interfaces will account for a considerable percentage of the total computational work. To reduce the cost for solving these interface problems, we can choose a new bilinear form $\tilde{b}_k(\cdot,\cdot)$ in such a way that (6.1) can be easily solved, e.g., choose $\tilde{b}_k(\cdot,\cdot)$ such that (6.1) corresponds to a linear system with diagonal coefficient matrix. Such a choice will result in a new operator $\tilde{T}$ in ASM, and a condition number $\kappa(\tilde{T})$ different from $\kappa(T)$. By suitably choosing $\tilde{b}_k(\cdot,\cdot)$, we expect to obtain the inexact solvers for the sub-problems on element interfaces, whose solution can be computed at very low cost, meanwhile the condition number $\kappa(\tilde{T})$ does not increase substantially compared with $\kappa(T)$. In this way the more efficient additive Schwarz method is resulted.

Lemma 6.1 *Let T be the operator in the ASM using the bilinear form $b_k(\cdot,\cdot)$, $\tilde{T}$ be that using the bilinear forms $\tilde{b}_k(\cdot,\cdot)$. Assume for $k = 0, 1, \cdots, n$*

$$\alpha_k b_k(u,u) \le \tilde{b}_k(u,u) \le \beta_k b_k(u,u), \quad \forall u \in \mathcal{V}_k. \tag{6.2}$$

Then we have

$$\kappa(\tilde{T}) \le \frac{\max(\beta_k)}{\min(\alpha_k)}\,\kappa(T). \tag{6.3}$$

Proof Note that T and $\tilde{T}$ are symmetric positive definite bilinear forms on $\mathcal{V}\times\mathcal{V}$ with respect to $a(\cdot,\cdot)$, and for $u \in \mathcal{V}$,

$$a(\tilde{T}_k u, u) = \tilde{b}_k(\tilde{T}_k u, \tilde{T}_k u) \le \beta_k\, b_k(T_k u, T_k u) = \beta_k a(T_k u, u),$$

it follows that

$$\begin{aligned}
\lambda_{max}(\tilde{T}) &= \sup_{u\in\mathcal{V}} \frac{a(\tilde{T}u,u)}{a(u,u)} = \sup_{u\in\mathcal{V}} \frac{\sum a(\tilde{T}_k u,u)}{a(u,u)} \\
&\le \max\beta_k \sup_{u\in\mathcal{V}} \frac{\sum a(T_k u,u)}{a(u,u)} = \max\beta_k \sup_{u\in\mathcal{V}} \frac{a(Tu,u)}{a(u,u)} \\
&= \max\beta_k\, \lambda_{max}(T).
\end{aligned}$$

Similarly

$$\lambda_{min}(\tilde{T}) \ge \min\alpha_k\, \lambda_{min}(T).$$

Then the conclusion of the lemma follows from the above two inequalities. □

When we apply the above lemma to the ASM using inexact solvers on element interfaces, $\tilde{b}_k(\cdot,\cdot) = b_k(\cdot,\cdot) = a_\Omega(\cdot,\cdot)$ for all the subspaces in the decomposition (4.2) and (4.4) except $\Psi_\gamma(\Omega)$ if $d = 2$, and for all the subspaces in the decomposition (5.1) except $\Psi_f(\Omega)$ if $d = 3$. On the subspaces $\Psi_\gamma(\Omega)$ and $\Psi_f(\Omega)$, $b_k(\cdot,\cdot) = a_\Omega(\cdot,\cdot)$ will be replaced by $\tilde{b}_\gamma(\cdot,\cdot)$ or $\tilde{b}_f(\cdot,\cdot)$ defined below. Then the factor appeared on the right hand side of (6.3) is $\dfrac{\max_\gamma(1,\beta_\gamma)}{\min_\gamma(1,\alpha_\gamma)}$ if $d = 2$, or $\dfrac{\max_f(1,\beta_f)}{\min_f(1,\alpha_f)}$ if $d = 3$, where $\alpha_\gamma, \beta_\gamma, \alpha_f$, and β_f are the constants such that:

$$\begin{cases} \alpha_\gamma a_\Omega(u,u) \le \tilde{b}_\gamma(u,u) \le \beta_\gamma a_\Omega(u,u), & \forall u \in \Psi_\gamma(\Omega), \quad \text{for } d = 2, \\ \alpha_f a_\Omega(u,u) \le \tilde{b}_f(u,u) \le \beta_f a_\Omega(u,u), & \forall u \in \Psi_f(\Omega), \quad \text{for } d = 3. \end{cases} \tag{6.4}$$

By the assumption that all the elements are shape regular, the above inequalities can be translated into relations in the reference element by regular mappings, and the constants α_γ, β_γ, α_f, and β_f are of the same order with respect to the polynomial degree p_γ or p_f for all the elements. Therefore, we need only to study the above inequalities in the reference element. Once a bilinear form $\tilde{b}_{\hat{\gamma}}(\hat{u}, \hat{v})$ or $\tilde{b}_{\hat{f}}(\hat{u}, \hat{v})$ on the reference element is defined, we can define

$$\begin{cases} \tilde{b}_\gamma(u,v) = \tilde{b}_{\hat{\gamma}_m}(u \circ F_{K_m}, v \circ F_{K_m}) + \tilde{b}_{\hat{\gamma}_n}(u \circ F_{K_n}, v \circ F_{K_n}), \quad \forall u \in \Psi_\gamma(\Omega), & \text{for } d = 2, \\ \tilde{b}_f(u,v) = h_{K_m}\tilde{b}_{\hat{f}_m}(\hat{u} \circ F_{K_m}, v \circ F_{K_m}) + h_{K_n}\tilde{b}_{\hat{f}_n}(u \circ F_{K_n}, v \circ F_{K_n}), \quad \forall u \in \Psi_f(\Omega), & \text{for } d = 3, \end{cases}$$

where K_m and K_n are the elements sharing γ or f, F_{K_m} and F_{K_n} are the affine mapping from $\hat{K}$ onto K_m and K_n, and $\hat{\gamma}_m, \hat{\gamma}_n, \hat{f}_m$, and $\hat{f}_n$ are the imagines of γ or f under $F_{K_m}^{-1}$ or $F_{K_n}^{-1}$. By a scaling argument, the inequality (6.4) holds on both the reference and physical elements with the constants independent of h. Therefore, in the remaining part of this section, we will only describe the inexact solvers on a side or a face of the reference element. For convenience, we will omit the superscript " ˆ " which indicates the reference element and the functions on it.

6.1 Inexact solvers on element sides in $\mathbf{R}^2$

We study in this section several inexact solvers on the element sides for two dimensional problems. They result in the linear systems with diagonal coefficient matrices, and thus very easy to compute. Let K be the reference element, and let γ be one of its sides. We identify γ with $I = \{x \mid -1 < x < 1\}$. Denote by $\{\phi_j(x)\}_{j=1}^{p-1}$ a set of the basis functions of $\mathcal{P}_p^0(I)$. $\Psi^{[\gamma]}(K)$ is the set of side functions associated with γ and orthogonal to all the internal modes in K (see Sec.2). Define $\mathcal{H}\phi_j \in \Psi^{[\gamma]}(K)$ be the (unique) discrete harmonic extension of ϕ_j from γ to K. Then $\{\mathcal{H}\phi_j\}_{j=1}^{p_\gamma-1}$ is a set of basis functions of $\Psi^{[\gamma]}(K)$. For $u \in \Psi^{[\gamma]}(\Omega)$,

$$u = \sum_{j=1}^{p_\gamma-1} u_j(\mathcal{H}\phi_j)(x,y). \tag{6.5}$$

Basis functions (1):

$$\phi_j(x) = \sqrt{j+\frac{1}{2}}\int_{-1}^{x} L_j(t)dt, \tag{6.6}$$

where $L_j(t)$ is the Legendre polynomial of degree j. This set of basis functions is widely used in the p-version finite element method for its hierarchical structure and relatively small condition number of the stiffness matrix (see [3, 33]).

Theorem 6.1 *For the basis functions defined in (6.6), let $u, v \in \Psi^{[\gamma]}(K)$ be expressed as in (6.5). Define*

$$\tilde{b}_\gamma(u,v) = \sum_{j=1}^{p_\gamma-1} \frac{1}{j} u_j v_j.$$

Then

$$\frac{c}{p_\gamma \ln p_\gamma} a_K(u,u) \le \tilde{b}_\gamma(u,u) \le c a_K(u,u)$$

and thus

$$\kappa(\tilde{T}) \le c \max_\gamma\{p_\gamma \ln p_\gamma\}\, \kappa(T).$$

Proof See the proof of Theorem 1 of [10]. □

Basis functions (2):

$$\phi_j(x) = \ell_j(x), \tag{6.7}$$

where $\ell_j(x)$ is the basis functions of the Lagrange interpolation based on the GLL points (of order p_γ). This set of basis function is widely used in the spectral and spectral element methods (see [8]). In this case all the integrations in the bilinear forms are replaced by numerical integrations with the GLL quadrature formulas.

Theorem 6.2 *For the basis functions defined in (6.7), let $u, v \in \Psi^{[\gamma]}(K)$ be expressed as in (6.5). Define*

$$\tilde{b}_\gamma(u, v) = \sum_{j=1}^{p_\gamma - 1} u_j v_j.$$

Then

$$\frac{c}{p_\gamma \ln p_\gamma} a_K(u, u) \leq \tilde{b}_\gamma(u, u) \leq c a_K(u, u)$$

and thus

$$\kappa(\tilde{T}) \leq c \max_\gamma \{p_\gamma \ln p_\gamma\} \, \kappa(T).$$

Proof See the proof of Theorem 2 of [10]. □

Basis functions (3):

$$\phi_j(x) = \begin{cases} T_{j+1}(x) - T_0(x), & \text{if } j \text{ odd}, \\ T_{j+1}(x) - T_1(x), & \text{if } j \text{ even}, \end{cases} \tag{6.8}$$

where $T_j(x) = \cos(j \cos^{-1} x)$ is the Chebyshev polynomial of the first kind of degree j. This set of basis functions is frequently used in the spectral and spectral element methods with the fast Fourier transformation (FFT) between the function values and its expansion coefficients, see [8, 13].

Theorem 6.3 *For the basis functions defined in (6.8), let $u, v \in \Psi^{[\gamma]}(K)$ be expressed as in (6.5). Define*

$$\tilde{b}_\gamma(u, v) = \sum_{j=1}^{p_\gamma - 1} j u_j v_j$$

Then

$$c a_K(u, u) \leq \tilde{b}_\gamma(u, u) \leq c (\ln p_\gamma)^3 a_K(u, u)$$

and thus

$$\kappa(\tilde{T}) \leq c \max_\gamma \{(\ln p_\gamma)^3\} \, \kappa(T).$$

Proof See the proof of Theorem 3.3 in [9] □

6.2 Inexact solvers on Element Faces in $\mathbf{R}^3$

(1) Diagonal inexact solvers on element faces

Let $K = (-1,1)^3$ be the reference element. Denote by f one of its faces. Let $I_x = \{-1 < x < 1\}$ and $I_y = \{-1 < y < 1\}$. We identify f with $I_x \times I_y$. Let $\{\phi_i(x)\}_{i=1}^{p-1}$ be a set of basis functions of $\mathcal{P}_p^0(\mathcal{I})$. Then $\{\phi_{ij}(x,y) = \phi_i(x)\phi_j(y)\}_{i,j=1}^{p-1}$ is a set of basis functions for the space $\mathcal{P}_p^0(f) = \mathcal{P}_p^0(I_x) \otimes \mathcal{P}_p^0(I_y)$. Denote by $\mathcal{H}\phi_{ij}$ the unique discrete harmonic extension of ϕ_{ij} from f onto K. Then $\{\mathcal{H}\phi_{ij}\}_{i,j=1}^{p-1}$ is a set of basis functions of the space $\Psi^{[f]}(K)$, which consists of all the polynomials of (separate) degree $\leq p$, vanishing on $\partial K \setminus f$, and orthogonal to the internal modes of K. For $u \in \Psi^{[f]}(K)$,

$$u = \sum_{i,j=1}^{p_f-1} u_{ij}(\mathcal{H}\phi_{ij})(x,y,z). \tag{6.9}$$

Next, we discuss the inexact face solvers for various basis functions.

Theorem 6.4 *([9, Theorem 4.1]) Let the basis functions be chosen as in (6.6), and let $u, v \in \Psi^{[f]}(K)$ be expressed as in (6.9). Define*

$$\tilde{b}_f(u,v) = \sum_{i,j=1}^{p_f-1} (\frac{1}{i^2 j} + \frac{1}{ij^2}) u_{ij} v_{ij}.$$

Then we have

$$\frac{c}{(p_f)^3 \ln p_f} a_K(u,u) \leq \tilde{b}_\gamma(u,u) \leq c a_K(u,u)$$

and thus

$$\kappa(\tilde{T}) \leq c \max_f \{(p_f)^3 \ln p_f)\} \, \kappa(T).$$

Theorem 6.5 *([9, Theorem 4.2]) Let the basis functions be chosen as in (6.7). Let $w_i, 1 \leq i \leq p_f - 1$, be the GLL quadrature weights, and let $u, v \in \Psi^{[f]}(K)$ be expressed as in (6.9). Define*

$$\tilde{b}_f(u,v) = \sum_{i,j=1}^{p_f-1} (w_i + w_j) u_{ij} v_{ij}.$$

Then we have

$$\frac{c}{p_f \ln p_f} a_K(u,u) \leq \tilde{b}_\gamma(u,u) \leq c a_K(u,u)$$

and thus

$$\kappa(\tilde{T}) \leq c \max_f \{p_f \ln p_f\} \, \kappa(T).$$

Theorem 6.6 *([9, Theorem 4.3]) Let the basis functions be chosen as in (6.8), and let $u, v \in \Psi^{[f]}(K)$ be expressed as in (6.9). Define*

$$\tilde{b}_f(u,v) = \sum_{i,j=1}^{p_f-1} (i+j) u_{ij} v_{ij}.$$

Then we have

$$\frac{c}{p_f} a_K(u,u) \leq \tilde{b}_f(u,u) \leq c p_f (\ln p_f)^3 a_K(u,u)$$

and thus

$$\kappa(\tilde{T}) \leq c \max_f \{(p_f)^2 (\ln p_f)^3\} \, \kappa(T).$$

(2) Inexact solver based on $|\cdot|_{H_{00}^{1/2}(F)}$-norm and its fast implementation

It has been demonstrated in the previous subsection that the condition numbers $\kappa(\tilde{T})$ using the diagonal inexact face solvers involve at least a factor p. We discuss in this subsection a non-diagonal face solver based on the bilinear form $\tilde{b}_f(\cdot,\cdot)$ which is induced by the $H_{00}^{1/2}$-norm on the element faces. It results in a smaller condition number $\kappa(\tilde{T})$ than those using the diagonal face solvers. Moreover, it admits a fast algorithm to compute the subproblems on the element faces.

Let $H_{00}^{1/2}(F)$ be the interpolation space which is half way between $H_0^1(F)$ and $L^2(F)$, An equivalent norm for this space is (see Theorem 13.1 of [25])

$$\|v\|_{H_{00}^{1/2}(F)} \asymp (\|v\|^2_{L^2(I_x,H_{00}^{1/2}(I_y))} + \|v\|^2_{L^2(I_y,H_{00}^{1/2}(I_x))})^{1/2}, \tag{6.10}$$

where

$$\|v\|^2_{L^2(I_x,H_{00}^{1/2}(I_y))} = \int_{I_x} \|v(x,\cdot)\|^2_{H_{00}^{1/2}(I_y)}\, dx,$$

and $\|v\|_{L^2(I_y,H_{00}^{1/2}(I_x))}$ is defined similarly. By the trace theorem and the discrete harmonic extension theorem (see [31, 32]), we have

$$c\|v\|^2_{H_{00}^{1/2}(F)} \le a_{\hat{K}}(\mathcal{H}v,\mathcal{H}v) \le c'\|v\|^2_{H_{00}^{1/2}(F)}, \quad \forall v \in \mathcal{P}_p^0(F). \tag{6.11}$$

Let $\{\phi_j\}_{j=1}^{p-1}$ be a set of basis functions of $\mathcal{P}_p^0(I)$. Define

$$M = (m_{ij}), \quad Q = (q_{ij}),$$

where $m_{ij} = (\phi_i, \phi_j)$ and $q_{ij} = (\phi_i', \phi_j')$, with $(\cdot,\cdot)$ denoting the L^2-inner product in I. They are the mass and stiffness matrices in one dimension, and are symmetric positive definite. For $v = \sum_{j=1}^{p-1} v_j\phi_j$, define $\mathbf{v} = (v_1, v_2, \cdots, v_{p-1})$. Clearly

$$\|v\|^2_{L^2(I)} = \mathbf{v}^T M\mathbf{v}, \qquad \|v\|^2_{H^1(I)} = \mathbf{v}^T Q\mathbf{v}.$$

A relation between the $H_{00}^{1/2}(I)$-norm of v and a bilinear form of its coefficients $\mathbf{v}$ can be described as follows:

Lemma 6.2 *([9, Lemma 4.3]) Let*

$$J = M^{1/2}(M^{-1/2}QM^{-1/2})^{1/2}M^{1/2}.$$

Then for all $v \in \mathcal{P}_p^0(I)$

$$\frac{c}{p}\mathbf{v}^T J\mathbf{v} \le \|v\|^2_{H_{00}^{1/2}(I)} \le c'\ \mathbf{v}^T J\mathbf{v}. \tag{6.12}$$

Now we are in a position to describe an inexact face solver in $\mathbf{R}^3$ by using J. Note that the basis functions in elements and its faces are tensor products of one dimensional ones. We introduce a "dot product" of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbf{R}^{n\times n}$ as follows:

$$A \cdot B = \sum_{i,j=1}^{n} a_{ij}b_{ij}.$$

Theorem 6.7 *Let u and v in $\Psi^{[f]}(K)$ be expressed as in (6.9), and let U and V be the $(p_f-1)\times(p_f-1)$ matrices with entries u_{ij} and v_{ij}. Define*

$$\tilde{b}_f^{(0)}(u,v) = (MU)\cdot(JV) + (MV)\cdot(JM). \tag{6.13}$$

Then we have

$$c a_K(u,u) \le \tilde{b}_f^{(0)}(u,u) \le c p_f a_K(u,u)$$

and thus

$$\kappa(\tilde{T}) \le c \max_f\{p_f\}\, \kappa(T).$$

Proof By (6.10) and Lemma 6.2

$$c\|u\|^2_{H^{1/2}_{00}(F)} \le \tilde{b}_f^{(0)}(u,u) \le c p_f \|u\|^2_{H^{1/2}_{00}(F)},$$

which, together with (6.11), implies the conclusion of the theorem. □

Although the condition number bound for $\tilde{T}$ using $\tilde{b}_f^{(0)}$ seems no better than for those using $\tilde{b}_f$ defined in part (1) of this section, numerical experiments indicate that the former is much better that the laters. Indeed for all the three kinds of basis functions, $\kappa(\tilde{T})/\kappa(T)$ are close to 1 for moderate p ($p \le 12$), see Sec.5 in [9]. Now we consider how to implement the ASM using $\tilde{b}_f^{(0)}$ efficiently. Clearly, in each step of the iteration, we have to find the solutions of the sub-problems on element faces as (6.1). Define $U = (U_{ij})$ and $R = (R_{ij})$ be the matrices composed of the expansion coefficients of the unknown function u and the given function r (corresponding to the residual in the iteration), respectively. Then equation (6.1) with $\tilde{b}_k(\cdot,\cdot)$ choosing as $\tilde{b}_k^{(0)}(\cdot,\cdot)$ is equivalent to

$$MUJ + JUM = R. \tag{6.14}$$

Substitute $J = M^{1/2}(M^{-1/2}QM^{-1/2})^{1/2}M^{1/2}$ into the above equation, and define

$$\tilde{U} = M^{1/2}UM^{1/2}, \qquad \tilde{R} = M^{-1/2}RM^{-1/2}.$$

Then (6.14) is changed into

$$\tilde{U}(M^{-1/2}QM^{-1/2})^{1/2} + (M^{-1/2}QM^{-1/2})^{1/2}\tilde{U} = \tilde{R}. \tag{6.15}$$

Let G be the unitary matrix composed of all the eigenvectors of $M^{-1/2}QM^{-1/2}$, then

$$M^{-1/2}QM^{-1/2} = G\Lambda G', \tag{6.16}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_{p-1})$ with λ_i's being the eigenvalues of $M^{-1/2}AM^{-1/2}$. Substitute (6.16) into (6.15) and define

$$\tilde{\tilde{U}} = G'\tilde{U}G, \qquad \tilde{\tilde{R}} = G'\tilde{R}G = G'M^{-1/2}RM^{-1/2}G.$$

Then (6.15) can be changed further into

$$\tilde{\tilde{U}}\Lambda^{1/2} + \Lambda^{1/2}\tilde{\tilde{U}} = \tilde{\tilde{R}},$$

whose solution is readily obtained by

$$(\tilde{\tilde{U}})_{ij} = \frac{1}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}(\tilde{\tilde{R}})_{ij}, \qquad \forall 1 \le i,j \le p-1.$$

Finally, U can be recovered from $\tilde{\tilde{U}}$ as follows

$$U = M^{-1/2} G \tilde{\tilde{U}} G' M^{-1/2} = E \tilde{\tilde{U}} E',$$

where $E = M^{-1/2} G$.

In summary, the solution of (6.14) can be accomplished in the following two steps:

(1) **Preprocessing:** Calculate $H = M^{-1/2}, B = HAH$; solve all the eigenvalues and the orthogonal eigenvectors of B such that $BG = G\Lambda$; and calculate $E = HG$.

(2) **In iteration:** For any R, the solution U to (6.14) is computed by

(i) Calculate $\tilde{\tilde{R}} = E'RE$;

(ii) Calculate $(\tilde{\tilde{U}})_{ij} = \dfrac{1}{\sqrt{\lambda_i} + \sqrt{\lambda_j}} (\tilde{\tilde{R}})_{ij}, \qquad \forall 1 \leq i, j \leq p-1$;

(iii) Calculate $U = E \tilde{\tilde{U}} E'$;

By using the above algorithm, the total number of arithmetic operations is only $4(p_f - 1)^3 + (p_f - 1)^2$ for each element face, which is the same order as the number of unknowns in one element in the h-p version finite element method in three dimensions. Therefore the above inexact face solver is very cheap for implementation. The computation of the eigenpairs of the matrices M and B in the preprocessing step is minor, since they are matrices only of order $p-1$. Besides, this step is needed only once for all elements and all iterations.

References

[1] Ainsworth, M., A preconditioner based on domain decomposition of *h-p* finite element approximation on quasi-uniform meshes, *SIAM J. Numer. Anal.*, 33(1996), 1358-1376.

[2] Babuška, I., Craig, A., Mandel, J. and Pitkäranta, J., Efficient preconditioning for the *p* version finite element method in two dimensions, *SIAM J. Numer. Anal.,* 28(1991), 624-661.

[3] Babuška, I.,Griebel, M. and Pitkäranta, J., The problem of selecting the shape functions for a *p*-type finite element, *Int'n J. Numer. Methods. Engrg.*, 28(1989), 1891-1908.

[4] Babuška, I. and Guo, B. Q., The *h-p* version of the finite element method for domains with curved boundaries, *SIAM J. Numer. Anal.*, 25(1988), 837-861.

[5] Bjørstad, P. E. and Widlund, O. B., Iterative methods for the solution of elliptic problems on regions partitioned into substructures, *SIAM J. Numer. Anal.*, 23(1986), 1097-1120.

[6] Bramble, J., Pasciak, J. and Schatz, A., The construction of preconditioners for elliptic problems by substructuring, I, *Math. of Comput.*, 175(1986), 103-134.

[7] Bramble, J., Pasciak, J. and Schatz, A., The construction of preconditioners for elliptic problems by substructuring, IV, *Math. of Comput.*, 53(1989), 1-24.

[8] Canuto, C., Hussaini, M.Y., Quarteroni, A., and Zang, T.A., Spectral Methods in Fluid Dynamics, Springer-verlag, Berlin, 1987.

[9] Cao, W.M. and Guo, B.Q., Preconditioning on element interfaces for *p*-version finite element method and spectral element method, submitted, 1996.

[10] Casarin, M., Diagonal side preconditioning for the p-version and spectral element method, *Tech. Report 704,* Department of Computer Science, Courant Institute of Mathematical Science, New York University, 1995.

[11] Chan, T.F. and Resasco, D.C., Analysis of domain decomposition preconditioners on irregular regions, *CAM Report 87-05*, UCLA, 1987.

[12] Ciarlet, P.G. The Finite Element Method for Elliptic Problems, North-Holland, 1978.

[13] Don,W.S. and Gottlieb, D., The Chebyshev-Legendre method: implementing Legendre methods on Chebyshev points, *SIAM J. Numer. Anal.,* 31(1994), 1519-1534.

[14] Dryja, M. and Widlund, O. B., An additive variant of the Schwarz method alternating method for the case of many subregions, *Tech. Report 339,* Department of Computer Science, Courant Institute of Mathematical Science, New York University, 1987.

[15] Dryja, M. and Widlund, O. B., Some domain decomposition algorithms for elliptic problems, in *Iterative Methods for Large Scale Linear Systems,* eds. L. Hayes and D. Kincaid, Academic Press, San Diego, CA, 1989, 273-291.

[16] Dryja, M. and Widlund, O. B., Towards a unified theory of domain decomposition algorithms for elliptic problems, in proceedings of *The Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, eds. T. F. Chan, R. Glowinski, J. Périaux, and O. B. Widlund, SIAM Philadelphia, PA, (1990).

[17] Dryja, M., Smith, B., and Widlund, O.B., Schwarz analysis of iterative substructuring algorithms for problems in three dimensions, *SIAM J. Numer. Anal.* 31(1994), 1662-1694.

[18] Guo, B. Q. and Babuška, I., The *h-p* version of the finite element method, part 1: the basic approximation results, *Comput. Mech.,* 1(1986), 21-41.

[19] Guo, B. Q. and Babuška, I., The *h-p* version of the finite element method, part 2: General results and applications, *Comput. Mech.,* 1(1986), 203-220.

[20] Guo, B. and Babuška, I., Regularity of the solution of elliptic problems on nonsmooth domains in $\mathbf{R}^3$, part 1: countably normed spaces on polyhedral domains, *Tech Note BN 1181*, Institute for Physical Science and Technology, Univ. of Maryland, College Park, 1995, to appear in *Proceedings of Royal Society of Edinburg.*

[21] Guo, B. and Babuška, I., Regularity of the solution of elliptic problems on nonsmooth domains in $\mathbf{R}^3$, part 2: Regularity in neighborhoods of edges, *Tech Note BN 1182*, Institute for Physical Science and Technology, Univ. of Maryland, College Park, 1995, to appear in *Proceedings Royal Society of Edinburg.*

[22] Guo, B. Q. and Cao, W. M., A preconditioner for the *h-p* version of the finite element method in two dimensions, *Numer. Math.* 75 (1996), 59-77.

[23] Guo, B. Q. and Cao, W. M., Additive Schwarz method for the *h-p* version of the finite element method in two dimensions, to appear in *SIAM J. Sci. Comput.*.

[24] Guo, B. Q. and Cao, W. M., Additive Schwarz method for the *h-p* version of the finite element method in three dimensions, to appear in *SIAM J. Numer. Anal.*.

[25] Lions, J.L., and Magenes, E., Non-Homogeneous Boundary Value Problems and Applications, Vol.1, Springer-Verlag, Heidelberg, 1972.

[26] Lions, P. L., On the Schwarz alternating method (I), in proceedings of "*The First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, eds. R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, SIAM Philadelphia, PA, (1988).

[27] Mandel, J., Iterative solvers by substructuring for the *p*-version finite element method, *Comput. Methods Appl. Mech. Engrg.*, 80(1990), 117-128.

[28] Mandel, J., Two-level domain decomposition preconditioning for the *p*-version finite element method in three dimensions, *Int. J. Numer. Meth. Engrg*, **29**(1990), pp1095-1108.

[29] Oden, J. T., Patra, A. and Feng, Y.-S., Domain decomposition for adaptive *hp* finite element methods, in *Contemporary Mathematics*, AMS, Vol. 180, 1994, 295-301.

[30] Oden, J. T., Patra, A. and Feng, Y.-S., Parallel Domain decomposition solvers for adaptive *h-p* finite element methods, *TICAM Report 94-11*, 1994.

[31] Pavarino, L.F. and Widlund, O.B., A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions, *SIAM J. Numer. Anal.,* 33(1996), 1303-1335.

[32] Pavarino, L.F. and Widlund, O.B., Iterative substructuring methods for spectral elements: problems in three dimensions based on numerical quadrature, *Tech. Report 663,* Department of Computer Science, Courant Institute of Mathematical Science, New York University, 1994. To appear in *Computers Math. Applic.*.

[33] Szabó, B. and Babuška, I., Finite Element Analysis, Wiley, New York, 1990.

[34] Widlund, O. B., Iterative substructuring methods: Algorithms and theory for elliptic problems in the plane, in proceedings of "*The First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, eds. R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, SIAM Philadelphia, PA, (1988).

[35] Widlund, O. B., Optimal iterative refinement methods, in proceedings of *The Second International Symposium on Domain Decomposition Methods for Partial Differential Equations*, eds. T. F. Chan, R. Glowinski, J. Périaux, and O. B. Widlund, SIAM Philadelphia, PA, (1989).

[36] Xu, J., Iterative methods by space decomposition and subspace correction, *SIAM Rev.*, 34(1992), 581-613.