Solution of Viscoelastic Scattering Problems in Linear Acoustics Using hp Boundary/Finite Element Method

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Abstract

The interaction of acoustic waves with submerged structures remains one of the most difficult and challenging problems in underwater acoustics. Many techniques such as coupled Boundary Element (BE)/Finite Element (FE) or coupled Infinite Element (IE)/Finite Element approximations have evolved. In the present work, we focus on the steady-state formulation only, and study a general coupled \( hp \)-adaptive BE/FE method. A particular emphasis is placed on an a-posteriori error estimation for the viscoelastic scattering problems.

The highlights of the proposed methodology are as follows:

- The exterior Helmholtz equation and the Sommerfeld radiation condition are replaced with an equivalent Burton-Miller (BM) boundary integral equation on the surface of the scatterer.
- The BM equation is coupled to the steady-state form of viscoelasticity equations modeling the behavior of the structure.
- The viscoelasticity equations are approximated using \( hp \)-adaptive FE isoparametric discretizations with order of approximation \( p \geq 5 \) in order to avoid the "locking" phenomenon.
- A compatible \( hp \) superparametric discretization is used to approximate the BM integral equation.
- Both the FE and BE approximations are based on a weak form of the equations, and the Galerkin method, allowing for a full convergence analysis.
- An a-posteriori error estimate for the coupled problem of a residual type is derived, allowing for estimating the error in pressure on the wet surface of the scatterer.
- An adaptive scheme, an extension of the Texas Three Step Adaptive Strategy is used to manipulate the mesh size \( h \) and the order of approximation \( p \) so as to approximately minimize the number of degrees of freedom required to produce a solution with a specified accuracy. The use of this \( hp \)-scheme may exhibit exponential convergence rates.

Several numerical experiments illustrate the methodology. These include detailed convergence studies for the problem of scattering of a plane acoustic wave on a viscoelastic sphere, and adaptive solutions of viscoelastic scattering problems for a series of MOCK0 models.

1 Introduction

A starting point for the numerical work presented in this paper has been the recent contribution by Demkowicz and Oden [9] on the application of \( hp \)-adaptive BE/FE approximations to the solution of the rigid and

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elastic scattering problems. In particular, the a-posteriori error estimate based on the $L^2$-residual corresponding to the boundary integral equation, proposed in [14] and investigated subsequently [12, 13, 16, 17] was shown to work satisfactory for the rigid scattering but not for the coupled, elastic scattering problems. Consequently, the issue of designing optimal $hp$-meshes for the elastic scattering was never resolved.

As pointed out in [9], the main reason for the failure of the a-posteriori error estimate for the coupled problems, has been the dominating FE discretization error, an order of magnitude larger than its corresponding BE counterpart. It was precisely this observation which had motivated us to study the possibility of using the superparametric boundary elements, with a higher order approximation for geometry, matching the FE discretization for the structure (with high $p$ to avoid locking), and a lower order discretization to approximate the pressure. In that way we might not only be able to save a significant number of degrees of freedom in the final BE system of equations, but also force the BE and FE discretization errors to be of the same order, allowing thus for a successful a-posteriori error estimation.

Indeed, the subsequently presented results will show that the better resolution of geometry improves significantly the quality of the a-posteriori error estimate for the rigid scattering, and makes it possible for the elastic scattering problems. The error estimate is then used as a basis for an $hp$-adaptive strategy, generalizing the Texas Three Step method proposed in [9, 22].

The validation of the numerical simulations is done in the context of spherical shells. The exact 3-D solutions for the forced vibration of a viscoelastic hollow sphere, and for the acoustic scattering of a plane wave on a viscoelastic hollow sphere problems are used. In this aspect, the work is a continuation of [2, 3, 4].

The plan of the paper is as follows. In Section 2, we review mathematical formulation of the scattering problem [9]. The corresponding approximation [9] is described in Section 3. We discuss the forced vibration problem of a viscoelastic hollow sphere in Section 4. Finally, Section 5 and Section 6 are devoted to the study of the acoustic scattering from rigid, elastic, and viscoelastic scatterers.

### 2 Mathematical Formulation of the Scattering Problem

#### 2.1 Classical and Boundary Integral Formulations

**Formulation of the Problems**

Although the eventual problem of interest is the acoustic scattering from a damped structure submerged in water, it is convenient to start with simpler, related problems involving vibrations of the fluid or structure separately.

In all three cases to be discussed, $\Omega$ denotes a bounded domain occupied by the structure, with boundary consisting of the external ("wet") part $\Gamma$ and the internal ("dry") part $\Gamma_0$ (see Fig. 1), and $\Omega^e$ denotes the exterior domain occupied by the fluid.
Figure 1: Acoustic scattering of a plane wave from a viscoelastic body.

Model Scattering Problem

Given an incident pressure field $p^{inc}$, prescribed in the whole of $\mathbb{R}^3$, we wish to determine a (complex-valued) total pressure
\[ p = p^{inc} + p^s, \text{ in } \Omega^e \] satisfying the following

- the Helmholtz equation in $\Omega^e$
  \[ \nabla^2 p + k^2 p = 0 \] (2)

- the Sommerfeld radiation condition at $\infty$
  \[ |\frac{\partial p^s}{\partial r} - ikp^s| = O\left(\frac{1}{r^2}\right) \] (3)

and

- an impedance boundary condition on $\Gamma$
  \[ \frac{\partial p}{\partial n} + i\omega\rho_f c_p = 0 \] (4)

where the following notation has been used (see (1)-(4)) : $\omega$ - the frequency, $k = \omega/c_f$ - the wave number, with $c_f$ - the sound speed in water, $\rho_f$ - the density of water, $n$ - the outward, normal unit vector to boundary
\( \Gamma, r \) - the distance from the origin, \( i \) - the imaginary unit, and \( \varepsilon \) - the impedance. In the rigid scattering case \( \varepsilon = 0 \) and (4) reduces to the Neumann boundary condition, in the case of a void scatterer \( \varepsilon = \infty \) and the Dirichlet boundary condition \( p = 0 \) is used. A general, local response of the scatterer is modeled using the Robin (Cauchy) boundary condition (4).

**Forced Vibrations of a Viscoelastic Body in Vacuum**

Identifying the displacement vector \( u \) defined on \( \Omega \) as the unknown, we have to satisfy

- the frequency-domain form of the viscoelasticity equations in \( \Omega \)
  \[
  \text{div}(E^* : \varepsilon(u)) + \rho_s \omega^2 u = 0
  \]  
  \[(5)\]

- traction boundary condition on the exterior surface \( \Gamma \)
  \[
  \sigma_n(u) = -p \\
  t_r(u) = 0
  \]  
  \[(6)\]

- homogeneous traction boundary condition on the interior surface \( \Gamma_0 \)
  \[
  t(u) = 0
  \]  
  \[(7)\]

Here \( E^* \) denotes the frequency-dependent tensor of viscoelastic moduli, \( \rho_s \) is the density of solid, \( t(u) \) stands for the stress vector corresponding to displacement \( u \)

\[
  t_i(u) = E^*_{ijkl} u_{kl} n_j
  \]  
  \[(8)\]

with the corresponding normal and tangential components \( \sigma_n, t_r \), respectively:

\[
\begin{align*}
  \sigma_n(u) &= t_i(u) n_i = E^*_{ijkl} u_{kl} n_j n_i \\
  t_r(u) &= t(u) - \sigma_n(u) n
\end{align*}
  \]  
  \[(9, 10)\]

**Coupled Problem - Acoustic Scattering from a Viscoelastic Body**

The final, coupled problem consists of determining both fields, the total pressure \( p = p(x), x \in \Omega^e \) and the displacement \( u = u(x), x \in \Omega \). The equations to be satisfied include the Helmholtz equation (2), the viscoelasticity equations (5), the Sommerfeld radiation condition (3), the traction-free boundary conditions (7) on interior surface \( \Gamma_0 \), and the compatibility conditions on the exterior surface \( \Gamma \) expressing equality of stresses,

\[
\begin{align*}
  \sigma_n(u) &= -p \\
  t_r(u) &= 0
\end{align*}
  \]  
  \[(11)\]

and the normal accelerations

\[
\rho_f \omega^2 u_n = \frac{\partial p}{\partial n}
  \]  
  \[(12)\]
Burton-Miller Boundary Integral Formulation

Following [9], we replace the Helmholtz equation (2) and the Sommerfeld radiation condition (3) with the equivalent Burton-Miller boundary integral equation

$$\left\{ \frac{1}{2} p - C p + A \frac{\partial p}{\partial n} \right\} + \frac{i}{k} \left\{ \frac{1}{2} \frac{\partial p}{\partial n} + B \frac{\partial p}{\partial n} + D p \right\} = p^{\text{inc}} + \frac{i}{k} \frac{\partial p^{\text{inc}}}{\partial n}$$

(13)

where the boundary integral operators are defined as follows:

- the single layer potential
  $$A p(x) = \int_{\Gamma} \Phi(x, y) p(y) dS(y)$$

(14)

- the double layer potential
  $$C p(x) = \int_{\Gamma} \frac{\partial \Phi}{\partial n(y)} (x, y) p(y) dS(y)$$

(15)

- the adjoint of the double layer potential
  $$B p(x) = \int_{\Gamma} \frac{\partial \Phi}{\partial n(x)} (x, y) p(y) dS(y)$$

(16)

- the hypersingular operator
  $$D p(x) = \int_{\Gamma} \frac{\partial^2 \Phi}{\partial n(x) \partial n(y)} (x, y) p(y) dS(y)$$

(17)

with fundamental solution to the Helmholtz equation

$$\Phi(x, y) = \Phi(r) = \frac{1}{4\pi} \frac{e^{ikr}}{r}, \quad r = |x - y|$$

(18)

and its derivatives

$$\frac{\partial \Phi}{\partial n(x)} = \Phi'(r) \frac{\partial r}{\partial n(x)} , \quad \frac{\partial \Phi}{\partial n(y)} = \Phi'(r) \frac{\partial r}{\partial n(y)} \quad \frac{\partial^2 \Phi}{\partial n(x) \partial n(y)} = \Phi''(r) \frac{\partial r}{\partial n(x)} \frac{\partial r}{\partial n(y)} + \Phi'(r) \frac{\partial^2 r}{\partial n(x) \partial n(y)}$$

(19)

While the definition of the single layer potential involves the usual Lebesgue integral (kernel $\Phi(x, y)$ is only weakly singular), integrals in the double layer potential and its adjoint are to be understood in the Cauchy Principal Value (CPV) sense, and the hypersingular integral is defined using the notion of the Hadamard finite part integral. For smooth domains, $\partial r/\partial n(y) = O(|x - y|)$, and the CPV integrals reduce to the Lebesgue integrals as well.

Using impedance condition (4), we can eliminate $\partial p/\partial n$, reducing the model scattering problem to the Burton-Miller integral equation with total pressure $p$ on boundary $\Gamma$ as the only unknown. Once the pressure on the boundary is known, derivative $\partial p/\partial n$ on $\Gamma$ is determined from the impedance condition, and the usual Helmholtz representation formula is used to determine values of the solution in the exterior domain.

Similarly, for the coupled problem, compatibility condition (12) is used to replace the normal derivative $\partial p/\partial n$ in the Burton-Miller equation with normal displacement $u_n$ and, consequently, the whole problem reduces to viscoelasticity equations (5) coupled with the integral equation with total pressure $p$ on boundary $\Gamma$ and displacement vector $u$ in domain $\Omega$ being the unknowns.
2.2 Variational Formulations

Burton-Miller Integral Equation for the Acoustic Fluid

In order to get the variational form, we multiply equation (13) by a test function \( q \), integrate one more time over boundary \( \Gamma \), and integrate the hypersingular term by parts moving one derivative to the test function. For technical details, see [8, 16].

We end up with the identity,

\[
d(p, q) + c(\frac{\partial p}{\partial n}, q) = \ell(q)
\]

for any admissible \( q \). Here the sesquilinear and antilinear forms are defined as follows:

\[
d(p, q) = \left\{ \frac{1}{2} \int_{\Gamma} p(x)q(x) dS_x - \int_{\Gamma} \int \frac{\partial \Phi}{\partial y}(x, y)p(y)q(x) dSydS_x \right\}
\]

\[
+ \frac{1}{2} \int_{\Gamma} \int \Phi(x, y) \text{rot}_y p(y) \text{rot}_x q(x) dSydS_x
\]

\[
- k^2 \int_{\Gamma} \int \Phi(x, y)n_x \cdot n_y p(y)q(x) dSydS_x \right\}
\]

\[
c(r, q) = \rho_f \omega^2 \left\{ \int_{\Gamma} \int \Phi(x, y)r(y)q(x) dSydS_x \right\}
\]

\[
+ \frac{1}{2} \int_{\Gamma} \int r(x)q(x) dS_x
\]

\[
+ \int_{\Gamma} \int \frac{\partial \Phi}{\partial n(x)}(x, y)r(y)q(x) dSydS_x \right\}
\]

\[
\ell(q) = \int_{\Gamma} p^{inc}(x)q(x) dS_x + \frac{i}{k} \int_{\Gamma} \frac{\partial p^{inc}}{\partial n(x)}(x)q(x) dS_x
\]

with \( \text{rot}_x p = \nabla p \cdot n(x) \). Again, using impedance condition (4), we reduce (20) to the standard variational formulation

\[
\begin{cases}
\text{Find } p \in H^{1/2}(\Gamma) \text{ such that } \\
D(p, q) = \ell(q) \quad \forall q \in H^{1/2}(\Gamma)
\end{cases}
\]

where \( H^{1/2}(\Gamma) \) is the Sobolev space of order 1/2 for functions defined on the boundary, and

\[
D(p, q) = d(p, q) - i\omega \rho_f c(p, q)
\]

Forced Vibrations of a Viscoelastic Body in Vacuum

Following the common procedure, we obtain the formulation

\[
\begin{cases}
\text{Find } u \in H^1(\Omega) \text{ such that } \\
a(u, v) = -b(p, v) \quad \forall v \in H^1(\Omega)
\end{cases}
\]

where \( H^1(\Omega) = (H^1(\Omega))^3 \) is the Sobolev space of order 1 and

\[
a(u, v) = \int_{\Omega} E_{ijkl} u_{k,l} v_{i,j} dx - \omega^2 \int_{\Omega} \rho_s u_k v_k dx
\]

\[
b(p, v) = \int_{\Gamma} pv_n dS
\]

with the normal component of the test function \( v_n = v_i n_i \).
Coupled Problem - Acoustic Scattering from a Viscoelastic Body

Using notation (21) and (25) we have

\[ \begin{align*}
\text{Find } & \ u \in H^1(\Omega), p \in H^\frac{1}{2}(\Gamma) \text{ such that } \\
a(u, v) + b(p, v) &= 0 \quad \forall v \in H^1(\Omega) \\
\sum_{I=1}^{3} c_I(u_I, q) + d(p, q) &= \ell(q) \quad \forall q \in H^\frac{1}{2}(\Gamma)
\end{align*} \tag{26} \]

where, to account for possible discontinuities in outward normal \( n \) and therefore normal displacement \( u_n = u_I n_I \) as well, we have introduced three new sesquilinear forms defined as

\[ c_I(u_I, q) = \rho_f \omega^2 c(u_I n_I, q) \quad (\text{no summation over } I) \tag{27} \]

3 Galerkin approximation

Based upon the variational formulation, the usual Galerkin approximation is applied. Given a set of basis functions on surface \( \Gamma \),

\[ e_i(x), \quad i = 1, \ldots, N \tag{28} \]

and a set of basis functions in domain \( \Omega \),

\[ \phi_j(x), \quad j = 1, \ldots, M \tag{29} \]

we introduce approximations

\[ p^h(x) = \sum_{i=1}^{N} p_i e_i(x), q^h(x) = \sum_{i=1}^{N} q_i e_i(x) \tag{30} \]

\[ p_i, q_k \in C, \quad i, k = 1, \ldots, N \]

\[ u^h(x) = \sum_{I=1}^{3} u_I(x) i_I = \sum_{I=1}^{3} \sum_{j=1}^{M} u^I_j \phi_j(x) i_I = \sum_{j=1}^{M} (\sum_{I=1}^{3} u^I_j i_I) \phi_j(x) \]

\[ v^h(x) = \sum_{J=1}^{3} v_J(x) i_I = \sum_{J=1}^{3} \sum_{l=1}^{M} v^J_l \phi_l(x) i_J = \sum_{l=1}^{M} (\sum_{J=1}^{3} v^J_l i_J) \phi_l(x) \tag{31} \]

\[ u^I_j, v^J_l \in C, \quad I, J = 1, 2, 3, \quad j, l = 1, \ldots, M \]

where \( i_I, I = 1, 2, 3 \), denote unit vectors of a global, Cartesian system of coordinates. Upon replacing \( p \) and \( q \), \( u \) and \( v \) in the variational formulation with their approximations (28)-(31), we obtain the following systems of linear equations
Model scattering problem

\[ \sum_{i=1}^{N} D_{ik} p_i = l_k , \quad k = 1, \ldots, N \]  

(32)

where

\[ D_{ik} = D(e_i, e_k), \quad i, k = 1, \ldots, N \]  

(33)

\[ l_k = l(e_k), \quad k = 1, \ldots, N \]

Forced Vibrations

\[ \sum_{j=1}^{3} \sum_{j=1}^{M} a_{jI}^J u_j^I = b_I^J , \quad J = 1, 2, 3, \quad l = 1, \ldots, M \]  

(34)

where

\[ a_{jI}^J = a(\phi_j, \phi_I), \quad I, J = 1, 2, 3, \quad j, l = 1, \ldots, M \]  

(35)

\[ b_I^J = -b(p, \phi_I), \quad J = 1, 2, 3, \quad l = 1, \ldots, M \]  

(36)

 Coupled Problem

\[ \sum_{j=1}^{3} \sum_{j=1}^{M} a_{jI}^J u_j^I + \sum_{i=1}^{N} D_{ik} p_i = 0 , \quad J = 1, 2, 3, \quad l = 1, \ldots, M \]  

(37)

\[ \sum_{j=1}^{M} c_{jk}^J u_j^I + \sum_{i=1}^{N} d_{ik} p_i = \ell_k , \quad k = 1, \ldots, N \]  

(38)

where \( a_{jI}^J, \ell_k \) are defined as before and

\[ b_I^J = b(e_i, \phi_I), \quad J = 1, 2, 3, \quad i = 1, \ldots, N, \quad l = 1, \ldots, M \]  

(39)

\[ c_{jk}^J = c_I(\phi_j, e_I), \quad J = 1, 2, 3, \quad j = 1, \ldots, M, \quad k = 1, \ldots, N \]  

(40)

\[ d_{ik} = d(e_i, e_k), \quad i, k = 1, \ldots, N \]  

(41)

3.1 \( h_p \)-Adaptive Approximation

The Galerkin basis functions are constructed using \( h_p \)-approximations based on the notion of constrained approximations [15]. An arbitrary, unstructured initial triangular grid may be used on the wet surface \( \Gamma \). The grid extends into the body in the form of layers of prismatic solid elements. In this way, the full compatibility of the surface and body meshes is enforced.

Master elements. The elements are constructed using the classical idea of isoparametric or superparametric approximations. A master triangle, shown in Fig. 2, consists of three vertex nodes, three mid-side nodes and a middle node. Each of the mid-side nodes may have an arbitrary order of approximation \( p \) (in
practice $p = 1, \ldots, 9$). Consequently, the element (vector) order of approximation $\mathbf{p}_K = (p_1^K, p_2^K, p_3^K, p_4^K)$ characterizes the approximation on each element $K$. The corresponding shape functions include standard linear shape functions for the vertex nodes and Lagrange-like functions for the mid-side nodes, and standard Lagrange shape functions for the middle node (see [7] for a precise definition).

Similarly, the master prismatic element, shown in Fig. 2, consists of six vertex nodes and fifteen higher-order nodes: nine mid-edge, two mid-base, three mid-side, and one middle node. The corresponding shape functions are tensor products of the 2-D triangle shape functions and 1-D shape functions (see [6, 11]). Consequently, the mid-side nodes and the middle node have two corresponding orders of approximations: a horizontal $p$, and vertical $q$. For that reason, we frequently talk about $hpq$-approximation.

**Geometry representation.** The structure is viewed as a union of disjoint blocks. Topologically, each of the blocks is a triangular or rectangular prism, and it is specified as an image of the reference triangular or rectangular prism through a particular parameterization. The concept is illustrated in Fig. 2. In practice, the parameterizations may be explicit or implicit, with an additional assumption on compatibility of parameterization being enforced. For details concerning practical implementation, we refer to [5].

**Initial mesh generation.** The method is based on the idea of an algebraic mesh generator and $hp$-interpolation. The concept is illustrated in Fig. 3. Given, for each reference figure, a number $m$ of divisions in “horizontal” and “vertical” directions (compatible for neighboring elements, the initial mesh is always regular), the reference blocks are covered with uniform, regular grids consisting of elements $\bar{K}$. By constructing a composition of the standard affine map $\eta$ transforming master element $\hat{K}$ onto element $\hat{K}$ and the (restriction of) the block parameterization $x_b$, we construct a map from master element $\hat{K}$ onto a curvilinear element $K$ identified as the image of element $\hat{K}$ under the particular parameterization $x_b$

$$K = T(\hat{K}) = x_b(\hat{K}), \quad T = x_b \circ \eta \quad (42)$$

In principle, this map could be used directly to define the curvilinear element, i.e. in the element calculations. In practice, we choose instead to approximate it with polynomials using the idea of isoparametric approximation. More precisely, given a particular order of approximation of element $K$ (may vary for different nodes), we replace transformation $T$ with its $hp$-interpolation. The idea of the $hp$-interpolation follows from the convergence theory of $hp$ approximations [1] and has been introduced in [6, 21]. Roughly speaking, the $hp$-interpolation combines the classical interpolation for vertex nodes with local $H^1_0$-projections for higher-order nodes. Given a sufficiently regular function, the corresponding $hp$-interpolant exhibits the same orders of convergence (in terms of both $h$ and $p$) as the corresponding global $H^1_0$-projection (solution to the Laplace equation with Dirichlet boundary conditions imposed using the $H^1_0$ projection on the boundary).

**h-adaptivity.** Elements may be refined vertically into four sons ($h4$-refinements) or horizontally into two sons ($h2$-refinement), resulting in the necessity of introducing the constrained nodes and the constrained approximation. As usual, we prefer to avoid the propagation of the constraints in the sense that the constrained degrees of freedom (dof) are expressible through unconstrained dof only. This is the case for so-called one-irregular meshes [23]. Unfortunately, with the possibility of anisotropic refinements, dealing with so-called double constrained nodes turns out to be unavoidable. For details concerning the corresponding
algorithm, generalizing the classical procedure for one-irregular meshes, we refer to [12].

Data structure. Implementation of constrained approximation. All necessary details can be found in [6]. Contrary to the earlier implementation in [15], all data for constrained nodes is stored explicitly, allowing one for a very concise and simple implementation of the constrained approximation, based on the idea of the modified element [6].

4 FE Modeling of Forced Vibrations of a Viscoelastic Sphere in Vacuum

4.1 Locking

In principle, we would like to model the entire vibrating structure, including its thin-walled parts, using the 3-D viscoelasticity formulation only. This allows for a logically consistent approximation, and the use of \( hp \)-adaptive approximations [9]. Recent investigations [9, 25], however, have shown that the classical viscoelasticity formulation is sensitive to the thickness \( t \) of the structure and degenerates as \( t \) approaches zero. The phenomenon is especially articulated for lower order approximations and, in the presence of appropriate boundary and load conditions, may lead to completely wrong, stiff solutions; hence the name locking is frequently used. The whole problem can be related to the classical spectral analysis. Decomposing the exact solution into its spectral components (at least for the elastic case)

\[
\mathbf{u} = \sum_{i=1}^{\infty} u_i \mathbf{e}_i
\]

where \( \mathbf{e}_i \) - are the eigenvectors of the elasticity operators, we can represent the corresponding approximate solution in the same form

\[
\mathbf{u}_h = \sum_{i=1}^{\infty} u_{h_i} \mathbf{e}_{h_i}
\]

where \( \mathbf{e}_{h_i} \) denote the approximate eigenvectors. We may argue that the quality of approximation \( u_h \) will depend upon two factors
• a number of (essentially) non-zero terms in (43)
• the quality of the numerical resolution of the involved eigenvectors and eigenvalues (the spectral degrees of freedom $u_i$ depend upon the corresponding load and eigenvalues $\lambda_i$ of the elasticity operator)

If some of the involved eigenfunctions and eigenvalues are underresolved, the approximate solution will essentially differ from the exact one. The resolution of the eigenvalues and/or eigenvectors turns out to be sensitive to the thickness of the structure and the order of approximation $p$, with a decreasing sensitivity for growing $p$ (in [9, 18, 20, 25] $p \geq 5$ for modeling shells has been suggested).

4.2 Numerical Experiments I

We summarize here results of several numerical experiments on the performance of the proposed approximation. All presented results were obtained on a DEC 3000 workstation, using double precision. Unless otherwise specified, the 3-D exact solution [3, 4] was used in the study and all tests involving the spherical shell were run with the following data:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water density</td>
<td>$\rho_f = 1000$ kg/m$^3$</td>
</tr>
<tr>
<td>Steel density</td>
<td>$\rho_s = 7800$ kg/m$^3$</td>
</tr>
<tr>
<td>Sound speed in water</td>
<td>$c_f = 1524$ m/sec</td>
</tr>
<tr>
<td>Elastic Young’s modulus</td>
<td>$E = 2.1 \times 10^{11}$ N/m$^2$</td>
</tr>
<tr>
<td>Fictitious Young’s modulus</td>
<td>$E^* = 2.1 \times 10^{11} \times (1 - \eta i) N/m^2$</td>
</tr>
<tr>
<td>Poisson Ratio</td>
<td>$\nu = 0.3$</td>
</tr>
<tr>
<td>Fictitious Poisson Ratio</td>
<td>$\nu^* = 0.3$</td>
</tr>
<tr>
<td>Midsurface radius</td>
<td>$a = 1$ m</td>
</tr>
<tr>
<td>Thickness of the shell</td>
<td>$t = 0.01$ m</td>
</tr>
</tbody>
</table>

The sphere is partitioned into eight equal octants with each of the octants parametrized using the area coordinates described in [5]. For all the details concerning parametrization and mesh generation we refer to [5, 6]. All meshes are characterized by three parameters:

• the number of subdivisions of one spherical octant, $m = 1, 2, 3, \ldots$, ($h = 1/m$),

• the lateral order of approximation, $p = 1, 2, \ldots$,

• the order of approximation across the thickness (applies to shell approximation only), $q = 1, 2, \ldots$

Convergence Rates

All of the following results are for wave number $ka = 2$. Figures 4-9 present $h$- and $p$-convergence rates for the forcing term coinciding with the third, fifth and eighth eigenfunctions of the shell, while Figures 10-15
correspond to the viscoelastic hollow sphere with loss factor $\eta = 5\%$. In order to avoid the inappropriate limit phenomenon (see e.g. [24]), in all considered examples, the order of approximation across thickness was set to $q = 2$. The error in the normal displacement on the outer surface is measured in terms of the $L^2$-norm, and compared with the best approximation error. The following observations can be made:

- the actual error and the best approximation error display identical convergence rates, but do not converge to each other (notice that the $L^2$-norm is not the energy norm for the problem),
- damping has no significant impact neither on the $h$- nor on the $p$-convergence curves at this frequency (the effect of damping becomes visible only for wave numbers close to resonant frequencies).

Next, we investigate the $h$-convergence rates for the forcing term coinciding with the exact pressure from the viscoelastic scattering problem with $\eta = 5\%$, shown in Fig. 16. The rates of convergence are consistent with the theoretical values $p + \frac{1}{2}$ for $p$th order of approximation.

**Sensitivity to Thickness, Locking**

We study next the sensitivity of the approximation properties of the formulation with respect to the thickness of the shell. For a particular mesh of $8 \times 36$ ($m=6$) elements of order $p = 2$, and $ka = 2$, Table 1 displays the corresponding variation of the error for the third, fifth and eighth modes as a function of thickness ratio $t/a = .08, .04, .02, .01, .005, .0025$. As expected, the third mode, dominated by the membrane behavior, is practically insensitive to the thickness. As the mode number increases, and the relative contribution of the bending energy to the total energy grows, the results deteriorate with the decreasing thickness. The corresponding approximate eigenvalue converges to the exact one slower, resulting in a “stiffer” behavior of
Figure 6: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 0\%$, fifth mode loading. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Order of approximation $p = 2$, number of subdivisions $m = 1, 2, \ldots, 7$.

Figure 7: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 0\%$, fifth mode loading. Experimental $p$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Number of subdivisions $m = 1$, order of approximation $p = 2, 3, \ldots, 8$.

Figure 8: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 0\%$, eighth mode loading. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Order of approximation $p = 2$, number of subdivisions $m = 1, 2, \ldots, 7$.

Figure 9: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 0\%$, eighth mode loading. Experimental $p$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Number of subdivisions $m = 1$, order of approximation $p = 2, 3, \ldots, 8$. 
Figure 10: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 5\%$, third mode loading. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Order of approximation $p = 2$, number of subdivisions $m = 1, 2, \ldots, 7$.

Figure 11: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 5\%$, third mode loading. Experimental $p$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Number of subdivisions $m = 1$, order of approximation $p = 2, 3, \ldots, 8$.

Figure 12: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 5\%$, fifth mode loading. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Order of approximation $p = 2$, number of subdivisions $m = 1, 2, \ldots, 7$.

Figure 13: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 5\%$, fifth mode loading. Experimental $p$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Number of subdivisions $m = 1$, order of approximation $p = 2, 3, \ldots, 8$. 
Figure 14: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 5\%$, eighth mode loading. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Order of approximation $p = 2$, number of subdivisions $m = 1, 2, \ldots, 7$.

Figure 15: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 5\%$, eighth mode loading. Experimental $p$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm. Number of subdivisions $m = 1$, order of approximation $p = 2, 3, \ldots, 8$.

Figure 16: Forced vibrations of a viscoelastic spherical shell, $ka = 2$ and $\eta = 5\%$, exact pressure (loading). Experimental $h$-convergence rates for the actual error, measured in $L^2$-norm. Order of approximation $p = 3, 4, 5$, number of subdivisions $m = 1, 2, \ldots, 5$. 
Table 1: Forced vibrations of a spherical shell, \( ka = 2 \) and \( \eta = 0\% \). Dependence of the error upon the thickness ratio \( t/a \) for a mesh of quadratic elements. Values shown are \( \|e_h\|/\|u\| \) where \( \| \cdot \| \) is the \( L^2 \)-norm.

<table>
<thead>
<tr>
<th>( t/a )</th>
<th>0.08</th>
<th>0.04</th>
<th>0.02</th>
<th>0.01</th>
<th>.005</th>
<th>.0025</th>
</tr>
</thead>
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<td>mode 3</td>
<td>1.35</td>
<td>1.32</td>
<td>1.31</td>
<td>1.31</td>
<td>1.31</td>
<td>1.31</td>
</tr>
<tr>
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<td>8.14</td>
</tr>
<tr>
<td>mode 8</td>
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<td>33.02</td>
<td>38.80</td>
<td>40.77</td>
<td>41.31</td>
<td>41.45</td>
</tr>
</tbody>
</table>

Table 2: Forced vibrations of a spherical shell, \( ka = 2 \) and \( \eta = 3\% \). Dependence of the error upon the thickness ratio \( t/a \) for a mesh of quadratic elements. Values shown are \( \|e_h\|/\|u\| \) where \( \| \cdot \| \) is the \( L^2 \)-norm.

<table>
<thead>
<tr>
<th>( t/a )</th>
<th>0.08</th>
<th>0.04</th>
<th>0.02</th>
<th>0.01</th>
<th>.005</th>
<th>.0025</th>
</tr>
</thead>
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<tr>
<td>mode 3</td>
<td>1.35</td>
<td>1.32</td>
<td>1.31</td>
<td>1.31</td>
<td>1.31</td>
<td>1.31</td>
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<td>33.02</td>
<td>38.79</td>
<td>40.74</td>
<td>41.30</td>
<td>41.44</td>
</tr>
</tbody>
</table>

the shell (in that sense the approximation locks). If, for a particular forcing term, the contribution of higher order modes to the exact solution is large, the error grows with decreasing thickness. The same comments apply to the viscoelastic cases, \( \eta = 3\% \) and \( \eta = 5\% \), as well (see Table 2 and Table 3).

Finally, we study the sensitivity of the approximation properties of the formulation to the thickness, for elements of higher order, particularly for \( p = 5 \). Table 4 summarizes the same experiment on a mesh with \( p = 5, m = 2 \) and, as before, \( q = 2 \). Suprisingly, while the resolution of the higher mode is less sensitive to the thickness, the lower modes exhibit much greater variation with the thickness than those corresponding to the mesh of quadratic elements. The results for the viscoelastic cases, \( \eta = 3\% \) and \( \eta = 5\% \), (see Table 5 and Table 6) are indistinguishable from the elastic case.

5 Rigid Scattering Problems

5.1 The Residual-Based A-Posteriori Error Estimation

The scattering problems fall into the standard setting for abstract variational problems. Given two Hilbert spaces \( V \) and \( H \), with \( V \) continuously and compactly imbedded in \( H \), we consider the standard abstract
Table 4: Forced vibrations of a spherical shell, $ka = 2$ and $\eta = 0\%$. Dependence of the error upon the thickness ratio $t/a$ for a mesh of quintic elements. Values shown are $\|e_h\|/\|u\|$ where $\|\cdot\|$ is the $L^2$-norm.

<table>
<thead>
<tr>
<th>$t/a$</th>
<th>0.08</th>
<th>0.04</th>
<th>0.02</th>
<th>0.01</th>
<th>0.005</th>
<th>0.0025</th>
</tr>
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<td>4.71</td>
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<tr>
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<td>14.91</td>
<td>15.06</td>
<td>15.29</td>
<td>15.43</td>
</tr>
</tbody>
</table>

Table 5: Forced vibrations of a spherical shell, $ka = 2$ and $\eta = 3\%$. Dependence of the error upon the thickness ratio $t/a$ for a mesh of quintic elements. Values shown are $\|e_h\|/\|u\|$ where $\|\cdot\|$ is the $L^2$-norm.

<table>
<thead>
<tr>
<th>$t/a$</th>
<th>0.08</th>
<th>0.04</th>
<th>0.02</th>
<th>0.01</th>
<th>0.005</th>
<th>0.0025</th>
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<td>1.91</td>
<td>2.52</td>
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<tr>
<td>mode 5</td>
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<td>5.03</td>
<td>5.14</td>
</tr>
<tr>
<td>mode 8</td>
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<td>14.90</td>
<td>15.06</td>
<td>15.28</td>
<td>15.43</td>
</tr>
</tbody>
</table>

Table 6: Forced vibrations of a spherical shell, $ka = 2$ and $\eta = 5\%$. Dependence of the error upon the thickness ratio $t/a$ for a mesh of quintic elements. Values shown are $\|e_h\|/\|u\|$ where $\|\cdot\|$ is the $L^2$-norm.

<table>
<thead>
<tr>
<th>$t/a$</th>
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</thead>
<tbody>
<tr>
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<td>2.52</td>
<td>2.88</td>
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<td>4.70</td>
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<td>5.14</td>
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<tr>
<td>mode 8</td>
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<td>13.37</td>
<td>14.90</td>
<td>15.05</td>
<td>15.28</td>
<td>15.43</td>
</tr>
</tbody>
</table>
where $A, C : V \to V'$ are the operators corresponding to sesquilinear forms $a$ and $c$, respectively,

$$< Au, v >= a(u, v), \quad < Cu, v >= c(u, v), \quad u, v \in V$$  \hfill (50)

Subtracting $Au_h + Cu_h$ from (49), we obtain

$$A e_h + C e_h = r_h$$  \hfill (51)

Here, $r_h$ denotes the residual

$$r_h = I - Au_h - Cu_h$$  \hfill (52)

Applying $A^{-1}$ to both sides of (52), we have

$$e_h + Ke_h = A^{-1}r_h$$  \hfill (53)

where $K = A^{-1}C$. With operator $K$ being compact, the term $Ke_h$ converges to zero faster than the error itself and therefore we expect that in the limit

$$\lim_{h \to 0} \frac{\|A^{-1}r_h\|}{\|e_h\|} = 1$$  \hfill (54)

The idea of the residual a-posteriori error estimation is applied to the Helmholtz formulation

$$p - 2Cp = 2p^{inc}$$  \hfill (55)

where $C$ is the double-layer potential. Consequently, operator $A$ reduces to the identity, and the energy norm to the $L^2$-norm on the scatterer.

5.2 Numerical Experiments II

Acoustic Scattering from a Rigid Sphere

We begin with a convergence study for the isoparametric and superparametric boundary elements for the rigid scattering problem. The exact solution can be found, e.g., in [3]. Unless otherwise specified, the data used in this study was the same as in Section 4.2. The numerical results are given in Tables 7-10. Table 7 and Table 8 summarize the $h$-convergence experiments with the residual a-posteriori error estimator for the rigid scattering problems using the isoparametric and the superparametric elements ($p_{BE} = 2$ and $p_{GE} = 3$, here $p_{BE} = 2$ stands for the order of horizontal approximation for boundary elements and $p_{GE} = 3$ stands for the order of horizontal approximation for the geometry) with $ka = 2$, respectively. Table 9 and Table 10 summarize the $p$-convergence experiments with the residual a-posteriori error estimator for the rigid scattering problems, using the isoparametric and the superparametric elements, respectively. To visualize the presented results, the corresponding plots are displayed in Figures 17-20. All errors are measured in the $L^2$-norm and are expressed in percent of the total $L^2$-norm of the solution. Comparing Fig. 17 and Fig. 18, we may conclude that the use of the superparametric elements (compare the effectivity indices for $m = 1$) makes the a-posteriori error estimate fully reliable even for very coarse meshes. The same conclusion applies to the $p$-convergence rates (see Fig. 19-Fig. 20).
Acoustic Scattering from a Rigid MOCK0 Model

As a final example in this section, we present a solution to the rigid scattering problem using the superparametric BE approximation, and the $L^2$-residual error estimate combined with the Texas Three Step Strategy described in [22, 9]. The model consists of a plane wave interacting with a rigid cylindrical shell with conical and spherical end caps, as shown in Fig. 21. For precise geometric data we refer to [5]. The incident wave propagates along the main axis of the cylinder with frequency $ka = 1.09$, where $2a$ is the diameter of the main cylinder. The Texas Three Step Adaptive Strategy [9, 22] is used to obtain the optimal meshes, starting with a quasiuniform mesh of superparametric quadratic elements $(p_{BE} = 2$ and $p_{GE} = 3)$ and the following control data:

- the anticipated rate of convergence (squared) $\beta = 5.0$
- intermediate target error for the pressure $\eta_i = 0.05$
- final target error for the pressure $\eta_f = 0.03$

Note that, in the $p$-refined stages, the approximation for the geometry without refinement stays one order higher, consistently with the initial mesh data, while isoparametric boundary elements are used for the higher-order approximation, $p_{BE} \geq 3$. The resulting, final $hp$-BE grids for the geometry and for the pressure together with the corresponding contours of the magnitude of the scattered pressure are shown in Figures 21-23, respectively. The final calculated residual equals 0.6 percent of the total $L^2$-norm of the solution.

6 Elastic Scattering and Viscoelastic Scattering Problems

6.1 An A-Posteriori Error Estimation for the Coupled Problem

As for the rigid scattering problem, we use the Helmholtz part of the Burton-Miller operator only, starting with the following formulation of the coupled problem.

$$\frac{1}{2} p - Cp + \rho \omega^2 Au_n = p^{inc}$$

$$Lu = -p$$

where $L$ is the viscoelasticity operator, prescribing for a given displacement field the corresponding normal stress component on surface $\Gamma$, and $A$ and $C$ are the single layer and double layer potentials, respectively. Introducing the Green operator $G$ corresponding to (57)

$$u_n = Gp.$$  
(58)

and substituting (58) into (56), we arrive at the equation

$$\frac{1}{2} p - Cp + \rho \omega^2 AGp = p^{inc}$$

(59)

Define now the residual for the approximate solution $p_h$ as the following

$$r_h = p^{inc} - \left\{ \frac{1}{2} p_h - C p_h + \rho \omega^2 AG_h p_h \right\}$$

(60)
Notice that in the definition of the residual, the Green operator $G$ has been replaced with its approximate counterpart corresponding to the approximate elasticity or viscoelasticity problem. Subtracting (60) from (59), we obtain

$$\frac{1}{2}(p - p_h) = C(p - p_h) - \rho_f \omega^2 A(Gp - G_h p_h) + r_h$$

$$= C(p - p_h) - \rho_f \omega^2 A G(p - p_h) + \rho_f \omega^2 A(G_h - G) p_h + r_h$$

(61)

Following the idea from [9], we use the fact that the operators $C$ and $AG$ defined on $L^2(\Gamma)$ into itself are compact, and therefore the corresponding terms should converge faster to zero than the term on the left hand side. Consequently, asymptotically,

$$\| p - p_h \|_{L^2(\Gamma)} \equiv 2 \| \rho_f \omega^2 A(G_h - G) p_h + r_h \|_{L^2(\Gamma)}$$

(62)

In order to make the term involving the Green function small, when compared with the residual, we introduce now a better approximation of the Green operator $G_h^{\text{better}}$ with the corresponding residual

$$r_h^{\text{better}} = p^{\text{inc}} - \left\{ \frac{1}{2} p_h - C p_h + \rho_f \omega^2 A G_h^{\text{better}} p_h \right\}$$

(63)

Rewriting (62), we obtain

$$\| p - p_h \|_{L^2(\Gamma)} \equiv 2 \| \rho_f \omega^2 A(G_h^{\text{better}} - G) p_h + r_h^{\text{better}} \|_{L^2(\Gamma)}$$

(64)

For $G_h^{\text{better}}$ sufficiently close to $G$, we have

$$\| p - p_h \|_{L^2(\Gamma)} \approx 2 \| r_h^{\text{better}} \|_{L^2(\Gamma)}$$

(65)

In practice, we use the isoparametric finite elements of higher order to model the structure and the superparametric boundary elements to obtain the better approximate Green operator $G_h^{\text{better}}$.

6.2 Numerical Experiments III

Numerical Verification of the A-Posteriori Error Estimate for the Coupled Problem

The problem of scattering of a plane wave on the viscoelastic hollow sphere is used one more time to validate the proposed a-posteriori error estimate and examine the effect of the superparametric BE approximations. Tables 11-14 summarize the results for both the isoparametric and the superparametric boundary elements. The corresponding graphs are plotted in Figures 24-27. For wave number $ka$ equal to 1, the superparametric elements $p_{\text{me}} = 2$ combined with $p_{\text{OE}} = 4$, $p_{\text{OE}} = 5$, $p_{\text{AS}} = 6$ deliver consistently reliable estimates (for $m \geq 2$, effectivity index $\geq 0.9$, see Tables 12-14), while the isoparametric boundary elements do not (see Table 11). The results for the viscoelastic scattering problems with $\eta = 5\%$ are similar to those for the elastic scattering problems (see Tables 15-18 and Figures 28-31).
We next investigate the error estimate in context of p-convergence for the viscoelastic problem with \( \eta = 5\% \). Table 19 and Table 20 summarize the p-convergence results for \( p_{OE} = p_{BE} + 2 \) and \( p_{OE} = p_{BE} + 3 \). The results suggest that in context of the hp-adaptivity, \( p_{OE} = p_{BE} + 3 \) may be a better choice than \( p_{OE} = p_{BE} + 2 \).

Finally, we verify the error estimate for a higher wave number, \( k = 2 \). Table 21 summarizes the h-convergence results for the model problem, using the superparametric boundary element with \( p_{OE} = 4/p_{BE} = 2 \) coupled with order of approximation \( p = 4 \) for the shell. Comparing the effectivity indices, we see that the estimate works only for sufficiently refined meshes, starting with \( m = 4 \). In other words, the estimate for the pressure is reliable only once the shell has been sufficiently well resolved. This suggests that the error estimate for the pressure must be combined with an estimate for the velocity and, if necessary, some mesh refinements to guarantee the resolution of the shell. These observations have led us to an hp-mesh refinement strategy discussed next.

6.3 An hp Adaptive Strategy

The principal idea of the two-step strategy to estimate the error is as follows. Beginning with a coarse mesh, we first solve the coupled problem and determine the approximate pressure. Next we decouple the equations and consider only the viscoelasticity problem with the approximate pressure from the first step as a forcing term. We perform a uniform \( h^4 \)-refinement or uniform \( p \)-refinement of the entire body mesh and compare the corresponding normal displacements on the wet surface for both meshes. If the difference is greater than a prescribed tolerance, then we restart the whole procedure with the new, \( h^4 \)-refined or \( p \)-refined mesh. This, rather crude, error estimation procedure for the forced vibrations problems is justified by two facts:

1. The cost of the solution of the forced vibrations problem constitutes only a small portion of the cost of the corresponding boundary integral equation. Hence, at least in this context, it makes little sense to use more sophisticated techniques.

2. Experiments with the force vibrations of the spherical shell have shown that a local mesh refinement may not always necessarily lead to a smaller error in normal displacement on the wet surface. Contrary to the local refinements, a global refinement decreases the error always.

Obviously, there is always a room for a better \( hp \)-refinement strategy for the forced vibrations problem. Only once the resolution of the shell is accepted, the error for the pressure is estimated and local \( h^4 \) or \( p \)-refinements follow using the Texas Three Step Strategy. For the details concerning the evaluation of the optimal distribution of order of approximation \( p \) or number of subdivisions of an element we refer to [22, 9]. The final algorithm looks as follows

Given:

- \( \beta > 0 \),
- a target error \( \eta_v \) for the forced vibration problem,
• an intermediate target error \( \eta_i \) for the pressure,
• a final target error \( \eta_f \) for the pressure,
• an initial mesh of elements \( K \) of order \( p_0 \).

Solution and a Two-Step A-Posteriori Error Estimate Procedure

**Step 1**

1.1. Solve the coupled problem on the current mesh.
1.2. Perform a temporary \( h^4 \)-global or \( p \)-global refinement on the body mesh, and solve the force vibrations problem with the forcing term equal to the approximate pressure from step 1.1.
1.3. Compare the normal displacement for the original and refined meshes in \( L^2 \) norm on the outer surface. If the difference exceeds the target error \( \eta_v \), keep the global refinement and return to step 1.1, otherwise proceed to step 2.

**Step 2:**

2.1. Evaluate the element \( L^2 \)-residuals \( e_k \) for the Helmholtz boundary integral equation for the coupled problem.

One should notice that very often in the performed experiments the second, \( h^4 \)-refinement step is skipped, as global \( h^4 \)-refinements, necessary for a successful error estimation, drive the error down below the required tolerance \( \eta_i \).

**6.4 Numerical Experiments IV**

We conclude our presentation with a number of examples illustrating the described method.

**Scattering of a Plane Wave on the Viscoelastic Hollow Sphere, \( \eta = 5\% \), \( k\alpha = 1 \).**

The following control parameters have been used [9]:

- the anticipated rate of convergence (squared) \( \beta = 5.0 \)
- final target error for the forced vibrations \( \eta_v = .01 \)
- intermediate target error for the pressure \( \eta_i = .01 \)
- final target error for the pressure \( \eta_t = .0055 \)

The optimal mesh is displayed in Fig. 32. Numerical solutions for the original and final mesh are compared with the exact ones in Fig. 33 and Fig. 34, respectively. The final calculated residual equals 0.7% of \( L^2 \)-norm of the total pressure.
Scattering of a plane wave on the viscoelastic hollow sphere, \( \eta = 5\% \), \( ka = 2 \).

The optimal mesh is displayed in Fig. 35. Numerical solutions for the original and final mesh are compared with the exact one in Fig. 36 and Fig. 37, respectively. The following control parameters have been used:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>the anticipated rate of convergence (squared)</td>
<td>( \beta = 5.0 )</td>
</tr>
<tr>
<td>final target error for the forced vibrations</td>
<td>( \eta_v = 0.012 )</td>
</tr>
<tr>
<td>intermediate target error for the pressure</td>
<td>( \eta_i = 0.01 )</td>
</tr>
<tr>
<td>final target error for the pressure</td>
<td>( \eta_t = 0.01 )</td>
</tr>
</tbody>
</table>

The final calculated residual equals 0.4\% of \( L^2 \)-norm of the total pressure.

**Acoustic Scattering from a Viscoelastic MOCK0 Model**

With the same geometric data as in the previous section and with superparametric quadratic elements \( p_{ae} = 5 \) and \( p_{be} = 2 \), the full coupled problem is solved. The adaptive strategy is driven by the \( L^2 \)-residual corresponding to the Helmholtz integral equation and the following control parameters:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>the anticipated rate of convergence (squared)</td>
<td>( \beta = 5.0 )</td>
</tr>
<tr>
<td>final target error for the forced vibrations</td>
<td>( \eta_v = 0.02 )</td>
</tr>
<tr>
<td>intermediate target error for the pressure</td>
<td>( \eta_i = 0.011 )</td>
</tr>
<tr>
<td>final target error for the pressure</td>
<td>( \eta_t = 0.01 )</td>
</tr>
</tbody>
</table>

The shell was modeled with a single layer of prismatic elements. The resulting, final \( hp-BE \) grids for the geometry and for the pressure together with the corresponding contours of the magnitude of the scattered pressure are shown in Figures 38-40, respectively. The final calculated residual equals 3 percent of the total \( L^2 \)-norm of the solution.

**Acoustic Scattering from a Viscoelastic MOCK0 Model. An Effect of Stiffeners**

As a final illustrative example, we present a result of a study on the effect of stiffening rings shown in Fig. 41. The \( hp \)-adaptive solution for the shell without rings is shown in Fig. 42, while the solution for the shell with rings is plotted in Fig. 43. In both cases, the shell is modeled with a single layer of prismatic elements with the fifth order elements used to model the rings. The stiffening effect of the rings is visible.

**Acknowledgement:** The work has been supported by the Office of Naval Research under Contract N00014-89-J-1451.

**References**


Table 7: Scattering from a rigid sphere, $k\alpha = 2$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($p_{GE} = 2$, $p_{BE} = 2$, isoparametric).

<table>
<thead>
<tr>
<th>$m$</th>
<th>best appr. error</th>
<th>appr. error</th>
<th>residual</th>
<th>eff. index</th>
</tr>
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<td>12.80</td>
<td>20.21</td>
<td>10.69</td>
<td>0.53</td>
</tr>
<tr>
<td>2</td>
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<td>2.73</td>
<td>2.49</td>
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<td>0.41</td>
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</table>

Figure 17: Acoustic scattering from a rigid sphere, $k\alpha = 2$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2, \ldots, 6$. 
Table 8: Scattering from a rigid sphere, \( k\alpha = 2 \). Comparison of the \( L^2 \)-error with \( L^2 \)-residual for \( h \)-refined meshes (\( p_{GE} = 3, p_{BE} = 2 \), superparametric).

<table>
<thead>
<tr>
<th>( m )</th>
<th>best appr. error</th>
<th>appr. error</th>
<th>residual</th>
<th>eff. index</th>
</tr>
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<tbody>
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<td>1.06</td>
<td>1.03</td>
</tr>
<tr>
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<td>0.50</td>
<td>0.52</td>
<td>1.04</td>
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<td>0.27</td>
<td>0.28</td>
<td>0.30</td>
<td>1.06</td>
</tr>
</tbody>
</table>

Figure 18: Acoustic scattering from a rigid sphere, \( k\alpha = 2 \). Experimental \( h \)-convergence rates for the actual and best approximation errors, measured in \( L^2 \)-norm, compared with the \( L^2 \)-residual. Order of approximation \( p_{BE} = 2 \), number of subdivisions \( m = 1, 2, \ldots, 6 \).
Table 9: Scattering from a rigid sphere, $ka = 2$. Comparison of the $L^2$-error with $L^2$-residual for $p$-refined meshes ($p_{GE} = p_{BE}$, isoparametric).

<table>
<thead>
<tr>
<th>$p_{BE}$</th>
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<th>appr. error</th>
<th>residual</th>
<th>eff. index</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>12.80</td>
<td>20.21</td>
<td>10.69</td>
<td>0.53</td>
</tr>
<tr>
<td>3</td>
<td>3.37</td>
<td>4.05</td>
<td>3.70</td>
<td>0.91</td>
</tr>
<tr>
<td>4</td>
<td>1.28</td>
<td>1.45</td>
<td>1.40</td>
<td>0.97</td>
</tr>
<tr>
<td>5</td>
<td>0.35</td>
<td>0.45</td>
<td>0.48</td>
<td>1.07</td>
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</table>

Figure 19: Acoustic scattering from a rigid sphere, $ka = 2$. Experimental $p$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Number of subdivisions $m = 1$, order of approximation $p_{BE} = 2, \ldots, 5$. 
Table 10: Scattering from a rigid sphere, $ka = 2$. Comparison of the $L^2$-error with $L^2$-residual for $p$-refined meshes ($p_{GE} = p_{BE} + 1$, superparametric).

<table>
<thead>
<tr>
<th>$p_{BE}$</th>
<th>best appr. error</th>
<th>appr. error</th>
<th>residual</th>
<th>eff. index</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>11.14</td>
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<td>12.16</td>
<td>1.02</td>
</tr>
<tr>
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<td>3.81</td>
<td>3.99</td>
<td>4.05</td>
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<tr>
<td>5</td>
<td>0.36</td>
<td>0.55</td>
<td>0.59</td>
<td>1.08</td>
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</table>

Figure 20: Acoustic scattering from a rigid sphere, $ka = 2$. Experimental $p$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Number of subdivisions $m = 1$, order of approximation $p_{BE} = 2, \ldots, 5$. 
Figure 21: Acoustic scattering from a rigid MOCK0 model, $ka = 1.09$. Final $hp$-mesh for the geometry.

Figure 22: Acoustic scattering from a rigid MOCK0 model, $ka = 1.09$. Final $hp$-mesh for the pressure.

Figure 23: Acoustic scattering from a rigid MOCK0 model, $ka = 1.09$. Final contours of the magnitude of the scattered pressure.
Table 11: Scattering from an elastic hollow sphere, $ka = 1$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($p_{GE} = 2, p_{BE} = 2$, isoparametric).

<table>
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<tr>
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<th>residual</th>
<th>eff. index</th>
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<td>3.11</td>
<td>0.51</td>
<td>0.16</td>
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<tr>
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<td>0.27</td>
<td>0.94</td>
<td>0.28</td>
<td>0.30</td>
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<td>5</td>
<td>0.15</td>
<td>0.35</td>
<td>0.19</td>
<td>0.54</td>
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</table>

Figure 24: Acoustic scattering from an elastic spherical shell, $ka = 1$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2, \ldots, 5$. 

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Table 12: Scattering from an elastic hollow sphere, $ka = 1$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($p_{GE} = 4, p_{BE} = 2$, superparametric).

<table>
<thead>
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<th>residual</th>
<th>eff. index</th>
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<td>2.09</td>
<td>0.96</td>
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<tr>
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<td>0.70</td>
<td>0.78</td>
<td>0.70</td>
<td>0.90</td>
</tr>
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</table>

Figure 25: Acoustic scattering from an elastic spherical shell, $ka = 1$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2, 3$. 
Table 13: Scattering from an elastic hollow sphere, $ka = 1$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($p_{GE} = 5, p_{BE} = 2$, superparametric).

<table>
<thead>
<tr>
<th>$m$</th>
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<th>appr. error</th>
<th>residual</th>
<th>eff. index</th>
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<tr>
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<td>2.06</td>
<td>2.25</td>
<td>2.03</td>
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</table>

Figure 26: Acoustic scattering from an elastic spherical shell, $ka = 1$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2$. 
Table 14: Scattering from an elastic hollow sphere, $ka = 1$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($p_{GE} = 6, p_{BE} = 2$, superparametric).

<table>
<thead>
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<th>$m$</th>
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<th>residual</th>
<th>eff. index</th>
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<td>0.73</td>
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<tr>
<td>2</td>
<td>2.06</td>
<td>2.17</td>
<td>1.99</td>
<td>0.92</td>
</tr>
</tbody>
</table>

Figure 27: Acoustic scattering from an elastic spherical shell, $ka = 1$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2$. 
Table 15: Scattering from a viscoelastic hollow sphere, $ka = 1$ and $\eta = 5\%$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($p_{GE} = 2, p_{BE} = 2$, isoparametric).

<table>
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<th>residual</th>
<th>eff. index</th>
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<td>0.17</td>
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<td>0.90</td>
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</table>

Figure 28: Acoustic scattering from a viscoelastic spherical shell, $ka = 1$ and $\eta = 5\%$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2, \ldots, 5$. 
Table 16: Scattering from a viscoelastic hollow sphere, $ka = 1$ and $\eta = 5\%$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($p_{GE} = 4, p_{BE} = 2$, superparametric).

<table>
<thead>
<tr>
<th>$m$</th>
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<th>residual</th>
<th>eff. index</th>
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<td>2.08</td>
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<td>0.77</td>
<td>0.70</td>
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Figure 29: Acoustic scattering from a viscoelastic spherical shell, $ka = 1$ and $\eta = 5\%$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2, 3$. 

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Table 17: Scattering from a viscoelastic hollow sphere, $ka = 1$ and $\eta = 5\%$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($PGE = 5, PBE = 2$, superparametric).

<table>
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<th>m</th>
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<th>residual</th>
<th>eff. index</th>
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<td>1</td>
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<td>12.10</td>
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<td>0.73</td>
</tr>
<tr>
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<td>2.05</td>
<td>2.23</td>
<td>2.03</td>
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Figure 30: Acoustic scattering from a viscoelastic spherical shell, $ka = 1$ and $\eta = 5\%$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2$. 
Table 18: Scattering from a viscoelastic hollow sphere, $ka = 1$ and $\eta = 5\%$. Comparison of the $L^2$-error with $L^2$-residual for $h$-refined meshes ($p_{GE} = 6, p_{BE} = 2$, superparametric).

<table>
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<th>$m$</th>
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<th>residual</th>
<th>eff. index</th>
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<td>2.05</td>
<td>2.17</td>
<td>1.98</td>
<td>0.92</td>
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</table>

Figure 31: Acoustic scattering from a viscoelastic spherical shell, $ka = 1$ and $\eta = 5\%$. Experimental $h$-convergence rates for the actual and best approximation errors, measured in $L^2$-norm, compared with the $L^2$-residual. Order of approximation $p_{BE} = 2$, number of subdivisions $m = 1, 2$. 

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Figure 32: Adaptive solution of the problem of scattering of a plane wave on a viscoelastic sphere, \( ka = 1 \). The final optimal mesh from the viewpoint (1,1,1).

Figure 33: Adaptive solution of the problem of scattering of a plane wave on a viscoelastic sphere, \( ka = 1 \). Comparison of the numerical and exact solutions on the initial mesh.

<table>
<thead>
<tr>
<th>( p_{BE} )</th>
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<th>appr. error</th>
<th>residual</th>
<th>eff. index</th>
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<td>0.67</td>
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<td>0.73</td>
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<td>0.86</td>
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<td>0.63</td>
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Table 19: Scattering from a viscoelastic hollow sphere, \( ka = 1 \) and \( \eta = 5\% \). Comparison of the \( L^2 \)-error with \( L^2 \)-residual for \( p \)-refined meshes (\( p_{GE} = p_{BE} + 2 \), superparametric).

<table>
<thead>
<tr>
<th>( p_{BE} )</th>
<th>best appr. error</th>
<th>appr. error</th>
<th>residual</th>
<th>eff. index</th>
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<tbody>
<tr>
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<td>8.61</td>
<td>12.10</td>
<td>8.83</td>
<td>0.73</td>
</tr>
<tr>
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<td>1.85</td>
<td>1.94</td>
<td>1.87</td>
<td>0.97</td>
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<td>0.42</td>
<td>0.52</td>
<td>0.47</td>
<td>0.91</td>
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</table>

Table 20: Scattering from a viscoelastic hollow sphere, \( ka = 1 \) and \( \eta = 5\% \). Comparison of the \( L^2 \)-error with \( L^2 \)-residual for \( p \)-refined meshes (\( p_{GE} = p_{BE} + 3 \), superparametric).

<table>
<thead>
<tr>
<th>( m )</th>
<th>best appr. error</th>
<th>appr. error</th>
<th>residual</th>
<th>eff. index</th>
</tr>
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</tr>
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<td>2</td>
<td>3.62</td>
<td>12.43</td>
<td>4.97</td>
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<td>0.42</td>
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</table>

Table 21: Scattering of a plane wave on a viscoelastic hollow sphere with \( ka = 2 \). Comparison of the \( L^2 \)-error with \( L^2 \)-residual for \( h \)-refined meshes (\( p_{GE} = 4, p_{BE} = 2 \), superparametric).
Figure 34: Adaptive solution of the problem of scattering of a plane wave on a viscoelastic sphere, \( ka = 1 \). Comparison of the numerical and exact solutions of on the final mesh.

Figure 35: Adaptive solution of the problem of scattering of a plane wave on a viscoelastic sphere, \( ka = 2 \). The final optimal mesh from the view point (1,1,1).

Figure 36: Adaptive solution of the problem of scattering of a plane wave on a viscoelastic sphere, \( ka = 2 \). Comparison of the numerical and exact solutions of on the initial mesh.

Figure 37: Adaptive solution of the problem of scattering of a plane wave on a viscoelastic sphere, \( ka = 2 \). Comparison of the numerical and exact solutions of on the final mesh.
Figure 38: Acoustic scattering from a viscoelastic MOCK0 model, $ka = 1.09$. Final $hp$-mesh for the geometry.

Figure 39: Acoustic scattering from a viscoelastic MOCK0 model, $ka = 1.09$. Final $hp$-mesh for the pressure.

Figure 40: Acoustic scattering from a viscoelastic MOCK0 model, $ka = 1.09$. Final contours of the magnitude of the scattered pressure.

Figure 41: Acoustic scattering from a MOCK0 model. Details of the geometric representation.
Figure 42: Acoustic scattering from a viscoelastic MOCK0 model without rings, $ka = 1.09$. Final contours of the magnitude of the scattered pressure and a profile of pressure along the main section.

Figure 43: Acoustic scattering from a viscoelastic MOCK0 model with rings, $ka = 1.09$. Final contours of the magnitude of the scattered pressure and a profile of pressure along the main section.