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Analysis of a Coupled Finite-Infinite Element Method for Exterior Helmholtz Problems

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Dedicated to Prof. Wolfgang Wendland on occasion of his 60th birthday

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Abstract

This analysis of convergence of a coupled FEM-IEM is based on our previous work on the FEM and the IEM for exterior Helmholtz problems. The key idea is to represent both the exact and the numerical solution by the DtN operators that they induce on the coupling hypersurface in the exterior of an obstacle. The investigation of convergence can then be related to a spectral analysis of these DtN operators. We give a general outline of our method and then proceed to a detailed investigation of the case in which the coupling surface is a sphere. Our main goal is to explore the convergence mechanism. In this context, we show well-posedness of both the continuous and the discrete models. As a corollary, we obtain a rather elementary proof of existence/ uniqueness for the exterior problem (on the continuous level). We further show that the discrete inf-sup constants have a positive lower bound that does not depend on N (number of DOF of the IEM). The proofs are based on lemmas on the spectra of the continuous and the discrete DtN operators, where the spectral characterization of the DtN operator is given as a conjecture from numerical experiments. In our convergence analysis, we show algebraic (in terms of N) convergence of arbitrary order and generalize this result in a proof of exponential convergence. The theoretical convergence results are illustrated with numerical evaluations.

1 Introduction

Numerical methods for the reduced wave equation in unbounded domains are, in general, semianalytic. In boundary element methods, the analytical solution in the exterior is included into the numerical model via the integral solution representation. If finite element methods are used for the numerical solution of exterior problems, analytical information can be efficiently included employing the solution obtained from separation of variables in a reduced exterior domain with separable geometry. In this coupled approach, the FEM is used to discretize the domain between a scatterer of general shape and a separable hypersurface. The exterior of this coupling surface can be numerically resolved by means of infinite elements or absorbing boundary conditions . The potential advantages of this approach (in terms of computational efficiency compared to BEM) had been pointed out by Bayliss et al. [2] as early as in 1980. In recent years, finite element methods have been successfully used for large-scale computations in numerical acoustics – see, e.g., [4].

In this paper, we study the convergence of a numerical method coupling finite and infinite elements (FEM-IEM) with the focus on the IEM error. To our knowledge, this is the first convergence analysis for coupled FEM-IEM. We draw on a recent convergence analysis for the IEM in separable exterior domain given by the first author and Gerdes in [6]. The specific convergence behavior of the FEM for Helmholtz problems in bounded regions has been recently analyzed in a series of papers by the second author and Babuška [12, 13, 9].

In our analysis, we disregard the FEM error and focus on the influence of the IEM approximation on the convergence of the coupled solution. The key idea is to replace (for analytical purposes only) the exact and IEM solutions in the exterior of the coupling surface with a Dirichlet-to-Neumann (DtN) map on the coupling surface. DtN maps have been previously used in acoustic computations for the construction of absorbing boundary conditions, cf. [10] and references therein. In a general convergence theorem (Theorem 1), we relate approximation of the exact solution by the IEM solution to the difference of the DtN operators associated with either solution. Then we proceed to the investigation of a particular case, namely, the spherical coupling boundary. This is the most simple (though sometimes not the most practical) coupling surface. We discuss in detail the assumptions of the Convergence Theorem and analyze the DtN operators. More precisely, we prove existence/ uniqueness and stability results of both the exact and the approximate models. The proof of existence-uniqueness of the exact coupled problem (Theorem 3) is based on the spectral analysis of the DtN operator. We remark that this proof is easily extended to an elementary *proof of* existence/ uniqueness of the original exterior problem.

The spectral characterization of the approximate DtN for the IEM is obtained in the form of a conjecture from numerical evaluations. We show discrete stability (Theorem 4) by proving the discrete Babuška-Brezzi condition, with all discrete inf-sup constants bounded from below by a positive number. The general convergence theorem is then specified for the spherical case, yielding algebraic convergence rates $N^{-\alpha}$, where N is the number of DOF for the IEM-approximation, with arbitrarily high rate α. This result is naturally generalized to a statement on exponential convergence (Theorem 5). The section on the spherical case is concluded with some results from the numerical evaluation of the upper bound for convergence shown in the proposition on algebraic convergence. Conclusions of this investigation are summarized in section 4.

2 General Setting and Convergence Theorem

2.1 Model problem

We consider the following scattering problem. Let $\Omega \subset \mathbb{R}^3$ be a finite domain with sufficiently smooth boundary Γ and denote $\Omega^+ = \mathbb{R}^3 - \Omega$. The problem of rigid scattering is written as

$$\begin{aligned} -\triangle u - k^2 u &= 0 && \text{in} \quad \Omega^+ \\ \frac{\partial u}{\partial n} &= g(s) && \text{on} \quad \Gamma \\ \lim_{R\to\infty} R\left|\frac{\partial u}{\partial R} - iku\right| &= 0 \end{aligned} \tag{2.1}$$

where k is the wave number, $g(s)$ is the boundary data on Γ computed from a given incident pressure, and the Sommerfeld radiation condition is prescribed in the far field. To cast the problem into a weak form, we multiply the Helmholtz equation by the complex conjugate of a test function v and apply Green's theorem of partial integration. A suitable functional-analytical setting for this procedure is presented in [6], after Leis [16]. Namely, we assume that $g \in H^{-1/2}(\Gamma), u \in H^1_w(\Omega^+)$ and $v \in H^1_{w^*}(\Omega^+)$, where $H^{-1/2}(\Gamma)$ is the fractional Sobolev space defined in the usual way, and the weighted Sobolev spaces $H^1_w(\Omega^+)$ and $H^1_{w^*}(\Omega^+)$ are defined by

$$H^1_{\varpi}(\Omega^+) = \{u : \|u\|_{1,\varpi} < \infty\} \tag{2.2}$$

with the norm $\|u\|_{1,\varpi}$ corresponding to the weighted inner product

$$(u,v)_{1,\varpi} = \int_{\Omega^+} \varpi\,(\nabla u \cdot \nabla \overline{v} + u\,\overline{v})\, dV + \int_{\Omega^+} \left(\frac{\partial u}{\partial r} - iku\right) \overline{\left(\frac{\partial v}{\partial r} - ikv\right)}\, dV \tag{2.3}$$

where $\varpi = w = |x|^{-2}$ or $\varpi = w^* = |x|^2$, respectively. Functions in these trial and test spaces automatically satisfy the Sommerfeld condition (following the approach of Leis, it has been included into the definition of the function spaces) and the weak formulation of the scattering problem becomes:

$$\begin{cases} \text{Find } u \in H^1_w(\Omega^+) : \\ \qquad b(u,v) = \langle g, v \rangle_\Gamma\,, \qquad \forall v \in H^1_{w^*}(\Omega^+) \end{cases} \tag{2.4}$$

with

$$b(u,v) := (\nabla u, \nabla v)_{\Omega^+} - k^2 (u,v)_{\Omega^+} \tag{2.5}$$

We use the notation

$$(u,v)_\Omega = \int_\Omega u\,\overline{v}\;dV\,; \qquad\qquad \langle u,v \rangle_\Gamma = \int_\Gamma u\,\overline{v}\;dS$$

for the L^2 inner product on domain Ω and for the duality pairing on $H^{1/2}(\Gamma)$, respectively. It is well known that both the strong and the weak formulations of the scattering problem at hand have a unique solution.

2.2 Solution by FEM-IEM

To solve problem (2.4) numerically by coupled FEM-IEM, one introduces a smooth artificial boundary $\Gamma_a \subset \Omega^+$, enclosing Ω. Let Ω_a denote the annular domain between Γ and Γ_a, and let Ω_a^+ denote the reduced exterior domain, as depicted in Fig. 1. Subdomain Ω_a is triangulated and a standard subspace $S^p_h(\Omega_a) \subset H^1(\Omega_a)$, based on conforming h-p finite elements, is defined. This naturally induces a finite element partition and a finite element space $S^p_h(\Gamma_a)$ on the artificial boundary. Defining finite-dimensional IE test and trial spaces $S^{pN}_{h,w}(\Omega_a^+) \subset H^1_w(\Omega_a^+)$ and $S^{pN}_{h.w^*}(\Omega_a^+) \subset H^1_{w^*}(\Omega_a^+)$ on Ω_a^+ such that each function v in these

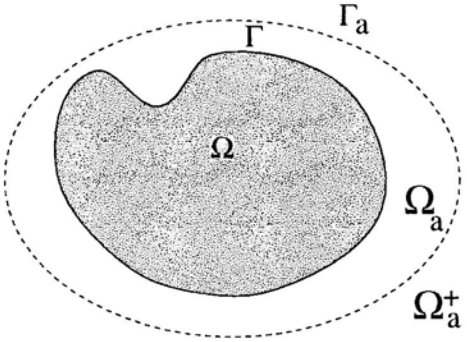


Figure 1: Scatterer and artificial boundary

spaces reduces on Γ_a to a function $v_h \in S^p_h(\Gamma_a)$. one obtains the test and the trial spaces of the coupled FEM-IEM as the sets of functions

$$\begin{aligned} V^p_{hN} &= \left\{ v \in H^1_w(\Omega^+) \,;\, v|_{\Omega_a} \in S^p_h(\Omega_a) \wedge v|_{\Omega^+_a} \in S^{pN}_{h,w}(\Omega^+_a) \right\} && (2.6) \\ V^{p*}_{hN} &= \left\{ v \in H^1_{w^*}(\Omega^+) \,;\, v|_{\Omega_a} \in S^p_h(\Omega_a) \wedge v|_{\Omega^+_a} \in S^{pN}_{h,w^*}(\Omega^+_a) \right\} . && (2.7) \end{aligned}$$

With this notation, the numerical solution by coupled FEM-IEM is now $\hat{u} \in V_{hN}$ such that

$$b(\hat{u}, \hat{v}) = \langle g, \hat{v} \rangle_\Gamma, \qquad \forall \hat{v} \in V^*_{hN} \,. \tag{2.8}$$

In this investigation, we focus on the convergence of the IE approximation. We will thus disregard the FE error, enlarging the approximation spaces to

$$\begin{aligned} V_N &= \left\{ v \in H^1_w(\Omega^+) \,;\, v|_{\Omega_a} \in H^1(\Omega_a) \wedge v|_{\Omega^+_a} \in H^1_{w,N}(\Omega^+_a) \right\} && (2.9) \\ V^*_N &= \left\{ v \in H^1_{w^*}(\Omega^+) \,;\, v|_{\Omega_a} \in H^1(\Omega_a) \wedge v|_{\Omega^+_a} \in H^1_{w^*,N}(\Omega^+_a) \right\} . && (2.10) \end{aligned}$$

where $H^1_{w.N}(\Omega^+_a)$ is a subspace of functions that reduce to H^1 functions on Ω_a and lie in the span of a N-dimensional IE basis in Ω^+_a (see also next section). More precisely, space $H^1_{\varphi.N}(\Omega^+_a)$ is defined as follows,

$$H^1_{\varphi.N} := \left\{ \sum_{j=1}^{N} v_j e_j \;:\; v_j \in H^{1/2}(\Gamma_a),\; j = 1, \ldots, N \right\} \subset H^1_\varphi$$

with appropriately selected basis functions $e_j \in H^1_\varphi$ – for more details, see the spherical case in the next section. We then look for a solution of (2.8) where V_{hN}, V^*_{hN} are replaced by V_N, V^*_N, respectively. Since there is not FEM partition given on the coupling surface, the exterior domain is in fact approximated by a single "infinite element" in radial direction.

2.3 Domain decomposition and DtN condition

We now apply the natural domain decomposition

$$\begin{aligned} b(u,v) &:= b_1(u,v) + b_2(u,v) \\ &= \Big((\nabla u, \nabla v)_{\Omega_a} - k^2 (u,v)_{\Omega_a} \Big) + \Big((\nabla u, \nabla v)_{\Omega^+_a} - k^2 (u,v)_{\Omega^+_a} \Big) . \end{aligned}$$

On the exterior domain Ω^+_a, we solve, weakly for all $v \in H^1_{w^*}(\Omega^+_a)$, the exterior Dirichlet problem: Given $u^- \in H^{1/2}(\Gamma_a)$, find a radiating solution $u \in H^1_w(\Omega^+_a)$ of Helmholtz' equation with the Dirichlet condition

$$u|_{\Gamma_a} = u^-$$

where u^- is the trace of $u|_{\Omega_a}$ on Γ_a. We then use the generalized Greens formula [19, p.598] to define the *Dirichlet-to-Neumann* operator

$$G : H^{1/2}(\Gamma_a) \to H^{-1/2}(\Gamma_a) \tag{2.11}$$

via

$$\langle Gu, v\rangle_{\Gamma_a} := b_2(u,v) - (\mathcal{H}u, v)_{L^2(\Omega_a^+)}$$

for all $v \in H^1_{w^*}(\Omega_a^+)$, where u is a radiating solution in Ω_a^+, v is identified with its trace on the boundary Γ_a and $\mathcal{H} = -\triangle - k^2$ is the Helmholtz differential operator. Since $\mathcal{H}u \equiv 0$ in Ω_a^+, the second term on the right hand side vanishes and we have

$$b_2(u,v) = \langle Gu, v\rangle_{\Gamma_a}\,.$$

Hence

$$b(u,v) = b_1(u,v) + \langle Gu, v\rangle_{\Gamma_a} = \langle g, v\rangle_\Gamma \tag{2.12}$$

holds for all $v \in H^1(\Omega_a)$, whereby the original problem (2.4) is reduced to a finite domain.

Similarly, solving the exterior Dirichlet problem for trial functions $u^N \in H^1_{w.N}(\Omega_a^+)$ over test space $H^1_{w^*,N}(\Omega_a^+)$, the IEM leads to an approximate Dirichlet-to Neumann map

$$G_N : H^{1/2}(\Gamma_a) \to H^{-1/2}(\Gamma_a)\,.$$

via $\langle G_N u, v\rangle_{\Gamma_a} := b_2(u,v)$ for all $v \in H^1_{w^*.N}(\Omega_a^+)$, and we have the equality

$$b_1(u^N, v) + \langle G_N u, v\rangle_{\Gamma_a} = \langle g, v\rangle_\Gamma$$

for all $v \in V_N^*$. Thus we seek the IEM solution from the the variational problem:

$$\left\{\begin{array}{l} \text{Find } u^N \in H^1(\Omega_a) : \\ \\ \qquad b_N(u^N, v) = \langle g, v\rangle_\Gamma\,, \qquad \forall v \in H^1(\Omega_a) \end{array}\right. \tag{2.13}$$

with

$$b_N(u^N, v) := (\nabla u^N, \nabla v)_{\Omega_a} - k^2(u^N, v)_{\Omega_a} + \langle G_N u^N, v\rangle_{\Gamma_a}\,. \tag{2.14}$$

The exact solution of (2.4) is obtained equivalently from the reduced problem with exact DtN map G, viz.

$$\left\{\begin{array}{l} \text{Find } u \in H^1(\Omega_a) : \\ \\ \qquad b(u, v) = \langle g, v\rangle_\Gamma\,, \qquad \forall v \in H^1(\Omega_a) \end{array}\right. \tag{2.15}$$

with

$$b(u,v) = (\nabla u, \nabla v)_{\Omega_a} - k^2(u,v)_{\Omega_a} + \langle Gu, v\rangle_{\Gamma_a}\,.$$

2.4 Convergence Theorem

We now will formulate a sufficient condition for convergence of the IEM. The key idea of our approach is to relate the exact and IEM solutions to their DtN operators on the artificial boundary Γ_a. As usual, the convergence rate depends on the regularity of the solution. We therefore assume that trace $\hat{u}$ lies in the subspace $H^\alpha(\Gamma_a) \subset H^{1/2}(\Gamma_a)$ with $\alpha \geq 1/2$ (for a more detailed discussion of regularity, see below). The operators G and G_N, applied to u, then lie in the subspace $\mathcal{L}(H^\alpha, H^{-1/2}) \subset \mathcal{L}(H^{1/2}, H^{-1/2})$.

Theorem 1 *Suppose, for given data g, that the variational problem (2.15) has a unique solution $u \in H^1(\Omega_a)$ with trace $\hat{u} \in H^\alpha(\Gamma_a)$, $\alpha \geq 1/2$. Assume further that the discrete problem (2.13) satisfies the inf-sup condition with constant $\gamma_N > 0$.*

Then the IE error $u - u^N$ is bounded as

$$\|u - u^N\|_1 \leq C(\alpha, \Gamma_a, u)\, \gamma_N^{-1} \, \|G - G_N\|_{\mathcal{L}(H^\alpha, H^{-1/2})} .$$

Hence the IEM solution u^N converges to u for $N \to \infty$ if

$$\gamma_N^{-1} \, \|G - G_N\|_{\mathcal{L}(H^\alpha, H^{-1/2})} \to 0$$

for $N \to \infty$.

Proof : Consider

$$\begin{aligned} |b_N(u - u^N, v)| &= |b_N(u, v) - b(u, v) + b(u, v) - b_N(u^N, v)| \\ &= |b(u, v) - b_N(u, v)| \\ &= |\langle (G - G_N)u, v\rangle| . \end{aligned}$$

Then, by the discrete inf-sup condition,

$$\gamma_N \|u - u^N\|_1 \leq \sup_{v \in H^1(\Omega_a)} \frac{|\langle (G - G_N)u, v\rangle|}{\|v\|_1} \leq T \|G - G_N\|_{\mathcal{L}(H^\alpha, H^{-1/2})} \|\hat{u}\|_{H^\alpha(\Gamma_a)}$$

where T is the trace constant relating $\|v\|_1$ and $\|\hat{v}\|_{1/2}$. Hence

$$\|u - u^N\|_1 \leq T \gamma_N^{-1} \|G - G_N\|_{\mathcal{L}(H^\alpha, H^{-1/2})} \, \|\hat{u}\|_{H^\alpha(\Gamma_a)}$$

yielding the statement with $C = T \, \|\hat{u}\|_{H^\alpha(\Gamma_a)}$. $\lhd$

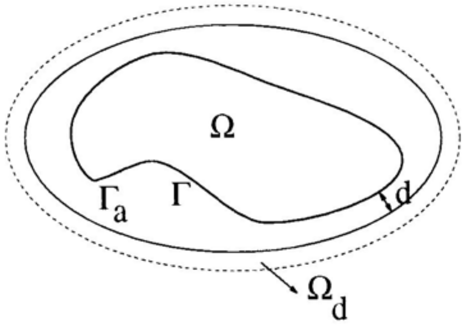


Figure 2: Domain Ω_d

2.5 A remark on regularity

The constant in the bound shown in Theorem 1 depends on the regularity of u and on the size of the annular domain Ω_a. The exact solution u is analytic everywhere in Ω^+ and, in particular, on Γ_a. This follows by analytic continuation from the well-known Wilcox expansion theorem for radiating solutions of the exterior Helmholtz equation. Hence, in principle, one could use any α in the upper bound. Let us sketch briefly the relation of norm $\|u\|_\alpha$ to the wave number and the distance between wet surface Γ and coupling surface Γ_a. Let d be the minimal distance between Γ and Γ_a, and consider an auxiliary domain Ω_d that is obtained if one enlarges Ω_a everywhere by distance d – see Fig. 2. Let B_d be the open sphere with radius d around some point $x \in \Gamma_a$. Then u can be written in the standard integral representation

$$u(x) = \int_{\partial B} u(t)\frac{\partial G(x,t)}{\partial n}dt - \int_{\partial B} \frac{\partial u(t)}{\partial n} G(x,t)dt$$

using the free space Green function

$$G(x,t) = \frac{e^{ikR}}{4\pi R}, \qquad R = |x-t|\,.$$

Taking the derivatives in x leads to the estimate

$$|D^s u(x)| \leq \left\| D^s \frac{\partial G}{\partial n} \right\|_{H^{-1/2}(\partial B)} \|u\|_{H^{1/2}(\partial B)} + \left\| \frac{\partial u}{\partial n} \right\|_{H^{-1/2}(\partial B)} \|D^s G\|_{H^{1/2}(\partial B)}\,.$$

The norms of u can be estimated as

$$\begin{aligned} \|u\|_{H^{1/2}(\partial B)} \leq C\|u\|_{H^1(B)} &\leq C\|u\|_{H^1(\Omega_d)} \\ \left\| \frac{\partial u}{\partial n} \right\|_{H^{-1/2}(\partial B)} &\leq C(k^2+1)\|u\|_{H^1(\Omega_d)} \end{aligned}$$

where we used Green's formula of partial integration on B to get the second estimate. It is easy to see that the norms of the Green function can be bounded by $\|G\|_{H^{s+1/2}(\partial B)}$. Using

$$\left(\frac{e^{ikz}}{z}\right)^{(s)}(d) = (ik - \frac{1}{d})^s,$$

and assuming that $d \ll k$, we have

$$\left|\frac{d^s G}{dR^s}\right|_{R=d} \simeq \frac{k^s}{d^s}.$$

from which we conclude that there exists some C, not depending on k, d, such that

$$\|G\|_{H^s(\partial B)} \leq C\,\frac{k^s}{d^s}.$$

Interpolation between H^s and H^{s+1} leads to

$$\|G\|_{H^{s+\frac{1}{2}}(\partial B)} \leq C\,\left(\frac{k}{d}\right)^{s+\frac{1}{2}}.$$

and thus

$$|D^s u(x)| \leq C(s)\,\frac{k^{s+\frac{5}{2}}}{d^{s+\frac{1}{2}}}\,\|u\|_{H^1(\Omega_d)}$$

for $x \in \Gamma_a$. By interpolation, the reasoning can be extended to Sobolev norms $\|\cdot\|_{H^\alpha}$ for real α. Thus the calculations above indicate an upper bound of the form

$$\|u\|_{H^\alpha(\Gamma_a)} \leq C(\Omega, \alpha)\,\frac{k^{\alpha+\frac{5}{2}}}{d^{\alpha+\frac{1}{2}}}\,\|u\|_{H^1(\Omega_d)}. \tag{2.16}$$

This rather crude upper bound may be suited to demonstrate the principal behavior of the IEM for small d and large k. Let us comment on the assumptions from a practical point of view. $d \ll k$ should be read in nondimensional coordinates which are scaled to some measure of the scatterer Ω. In practice, it is always attempted to fit the scatterer as closely as possible with the coupling surface in order to keep the cost of FE computations low. The size of the FEM domain is sometimes adapted to the wavenumber k by letting d be some fraction of the wavelength $\lambda = 2\pi/k$, leading to small distance d for large wave number k. Our estimate, together with the convergence theorem, indicates that the norms $\|u\|_\alpha$ may grow significantly for small d, adversely influencing the convergence rate of the IEM.

3 Spherical case

We now consider the special case that the artificial boundary Γ_a is a spherical surface S with radius a. Otherwise, we keep the notation of the previous section as depicted in Fig. 1.

Remark 1: A spherical domain is usually not the best choice for computational analysis of elongated objects. Firstly, one wants to minimize the size of the computational domain in order to minimize the number of finite elements. Secondly, due to the pollution effect of the FEM [12, 13, 14], the FEM error increases both with the frequency *and* the domain size. It is therefore recommended to fit the scatterer more closely with the artificial boundary, using, e.g., prolate spheroidal coordinates for elongated obstacles [4].

3.1 Spectral expansion of the exact DtN operator

The DtN-operator is found as follows: Assume $u(a,\theta,\phi)$ is given on the sphere $r = a$ and expand it into spherical harmonics as

$$u(a,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_{mn} Y_{mn}(\theta,\phi)$$

where

$$Y_{mn}(\theta,\phi) = \alpha_{nm} P_n^{|m|}(\cos\theta) e^{im\phi}$$

with normalization factor

$$\alpha_{nm} = \sqrt{\frac{(2n+1)(n+|m|)!}{4\pi(n-|m|)!}}\,.$$

The coefficients are

$$u_{mn} = a^2 \int_{\mathcal{S}} u(a,\theta',\phi') \overline{Y_{mn}}(\theta',\phi')\, dS'$$

where $\mathcal{S}$ is the unit sphere. In the exterior domain Ω_a^+, solution $u(r,\phi,\theta)$ can be found by separation of variables as

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} h_n(kr) Y_n(\theta,\phi)$$

with

$$Y_n(\theta,\phi) = \sum_{m=-n}^{n} c_{mn} Y_{mn}(\theta,\phi)\,.$$

The constants c_{mn} are found from the boundary conditions. In the present case, one simply gets, by setting $r = a$ and using orthonormality of the spherical harmonics – cf. [10],

$$c_{mn} = \frac{u_{mn}}{a^2 h_n(ka)}\,.$$

Thus

$$u(r,\theta,\phi)=\sum_{n=0}^{\infty}\frac{h_n(kr)}{h_n(ka)}\sum_{m=-n}^{n}Y_{mn}(\theta,\phi)\int_S u(a,\theta',\phi')\overline{Y_{mn}}(\theta',\phi')\,dS'\,.$$

Differentiating in r and setting $r=a$ finally leads to

$$\frac{\partial u}{\partial r}(a,\theta,\phi)=\sum_{n=0}^{\infty}k\frac{h_n'(ka)}{h_n(ka)}\sum_{m=-n}^{n}Y_{mn}(\theta,\phi)\int_S u(a,\theta',\phi')\overline{Y_{mn}}(\theta',\phi')\,dS'\,. \tag{3.17}$$

The Dirichlet-to-Neumann operator is now simply obtained by setting

$$Gu=\left.\frac{\partial u}{\partial n}\right|_{\Gamma_a}=-\left.\frac{\partial u}{\partial r}\right|_{r=a}$$

(note that $\mathbf{n}$ is the exterior normal of the exterior region Ω_a^+, pointing in the negative radial direction).

The DtN condition is an exact (nonlocal) absorbing condition on the artificial boundary and thus reduces the exterior problem to a boundary value problem on the finite domain Ω_a: Find solution u of

$$\begin{aligned} -\triangle u-k^2u &= 0 && \text{in} && \Omega^+\\ \frac{\partial u}{\partial n} &= g(s) && \text{on} && \Gamma\\ \frac{\partial u}{\partial r} &= -Gu && \text{on} && S\,. \end{aligned} \tag{3.18}$$

The corresponding variational weak form is

$$b(u,v)=(\nabla u,\nabla v)_{\Omega_a}-k^2(u,v)_{\Omega_a}+\langle Gu,v\rangle_S=\langle g,v\rangle_\Gamma,\qquad \forall v\in H^1(\Omega_a)\,. \tag{3.19}$$

Expanding testfunction v on S, we get

$$\langle Gu,v\rangle=\sum_{n=0}^{\infty}\lambda_n\sum_{m=-n}^{n}u_{mn}\bar{v}_{mn} \tag{3.20}$$

with

$$\lambda_n:=-k\frac{h_n'(ka)}{h_n(ka)}\,. \tag{3.21}$$

Lemma 1 *The eigenvalues λ_n have the following properties:*

(1) The imaginary parts of all λ_n are negative and

$$Im\,\lambda_n\to -0\qquad as\; n\to\infty\,. \tag{3.22}$$

(2) The real parts of all λ_n are positive and bounded from below as

$$\frac{1}{a} \leq Re\, \lambda_n\,. \tag{3.23}$$

(3) The real parts of the eigenvalues are bounded as

$$\frac{n+1}{a} - k \leq Re\, \lambda_n \leq \frac{n+1}{a} + k\,. \tag{3.24}$$

Hence for large n, we have the asymptotic relation

$$\lambda_n \simeq \frac{n+1}{a}\,. \tag{3.25}$$

Proof : By definition of the spherical Hankel functions,

$$\lambda_n := -k\frac{j_n'(ka) + iy_n'(ka)}{j_n(ka) + iy_n(ka)}$$

where j_n, y_n are the spherical Bessel functions of the first and second kind, respectively. Hence, with $ka = x$,

$$\text{Im}\ \lambda_n = -\frac{k}{j_n^2 + y_n^2}\begin{vmatrix} j_n'(x) & y_n'(x) \\ j_n(x) & y_n(x) \end{vmatrix}\,.$$

By [1, 10.1.6], the Wronskian is $-x^2$, hence

$$\text{Im}\ \lambda_n = -\frac{k}{|h_n(x)|^2 x^2}\,.$$

Hence Im $\lambda_n < 0$ for all n. Further, by [1, 10.1.27],

$$|h_n(x)|^2 = x^{-2} S_n(x) \tag{3.26}$$

with

$$S_n(x) = \sum_{k=0}^{n} \frac{(2n-k)![2(n-k)]!}{k![(n-k)!]^2}(2x)^{2k-2n}\,. \tag{3.27}$$

We see that x^2 in the expression for Im λ cancels out with that in (3.26). Rewriting the members of the sum above as

$$s_{nk} = \frac{(2n-k)!}{n!} \cdot \frac{[2(n-k)]!}{(n-k)!} \cdot \frac{n!}{k!(n-k)!}(2x)^{2k-2n}$$

we immediately see that

$$s_{nk} \geq \begin{pmatrix} n \\ k \end{pmatrix} (2x)^{2k-2n}$$

for all n, k. Hence

$$S_n \geq \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} 1^k \left(\frac{1}{(2x)^2}\right)^{n-k} = \left(1 + \frac{1}{(2x)^2}\right)^n$$

and therefore the imaginary part vanishes asymptotically.

We proceed with the investigation of the real part

$$\text{Re } \lambda_n = -k \frac{j_n'(x) j_n(x) + y_n'(x) y_n(x)}{|h_n(x)|^2} . \tag{3.28}$$

By direct computation,

$$\text{Re } \lambda_o = \frac{1}{a} .$$

For $n \geq 1$, we replace j_n' and y_n' using [1, 10.1.21],

$$f_n' = f_{n-1} - \frac{n+1}{x} f_n ,$$

to get

$$k^{-1} \text{Re } \lambda_n = \frac{n+1}{x} - r_n(x)$$

with

$$r_n(x) := \frac{j_{n-1}(x) j_n(x) + y_{n-1}(x) y_n(x)}{|h_n(x)|^2} .$$

We prove that $|r_n| < 1$. Indeed, by [1, 10.1.26],

$$r_n(x) = \frac{\frac{\pi}{2x} M_{n-\frac{1}{2}}(x) M_{n+\frac{1}{2}}(x) (\cos \theta_{n-\frac{1}{2}} \cos \theta_{n+\frac{1}{2}} + \sin \theta_{n-\frac{1}{2}} \sin \theta_{n+\frac{1}{2}})}{\frac{\pi}{2x} M_{n+\frac{1}{2}}^2}$$

where $M_{n+1/2}(x) = \sqrt{2x/\pi}\, |h_n(x)|$. Hence

$$r_n^2(x) \leq \frac{S_{n-1}(x)}{S_n(x)}$$

where we have used again (3.26). Transforming the summation indices as $l = n - 1 - k$ and $l = n - k$, respectively, we get

$$\begin{aligned} S_{n-1}(x) &= \sum_{l=0}^{n-1} \frac{(n-1+l)!(2l)!}{(n-1-l)!(l!)^2} (2x)^{-2l} \\ S_n(x) &= \sum_{l=0}^{n} \frac{(n+l)!(2l)!}{(n-l)!(l!)^2} (2x)^{-2l} \end{aligned}$$

We can now write

$$S_n(x) = \sum_{l=0}^{n-1} a_l \frac{n+l}{n-l}(2x)^{-2l} + s_n(2x)^{-2n}$$

where a_l are the coefficients of $S_{n-1}(x)$, to see that

$$S_n(x) \geq S_{n-1}(x) + s_n(2x)^{-2n}$$

yielding

$$|r_n(x)| \leq 1.$$

This proves claim (3). Property (4) now follows directly from (1) and (3).

To show the lower bound (3.23) in (2), observe that (3.28) is equivalently written as

$$\text{Re } \lambda_n = -k \frac{(|h_n(x)|^2)'}{2|h_n(x)|^2} = \frac{|h_n(x)|'}{|h_n(x)|}.$$

Inserting $|h_n(x)| = x^{-1}\sqrt{S_n(x)}$ we get

$$\text{Re } \lambda_n = k \frac{1 - x \frac{S_n'(x)}{2S_n(x)}}{x}.$$

By direct computation, it is easily checked that $-xS_n'(x)/S_n(x) \geq 0$. With this, $x = ka$ yields the lower bound. The lemma is proved. ◁

3.2 Spectral expansion of the DtN operator for the IEM

Let $u^N \in H^1(\Omega_a)$ be the solution of (2.13) in the spherical case, i.e., u^N solves weakly the BVP

$$\begin{aligned} -\triangle u - k^2 u &= 0 && \text{in} \quad \Omega^+ \\ \frac{\partial u}{\partial n} &= g(s) && \text{on} \quad \Gamma \\ \frac{\partial u}{\partial r} &= -G_N u && \text{on} \quad S. \end{aligned} \tag{3.29}$$

Expanding the trial and test functions into spherical harmonics we get again

$$\langle G_N u, v\rangle = \sum_{n=0}^{\infty} \lambda_n^N \sum_{m=-n}^{n} u_{mn}\bar{v}_{mn}. \tag{3.30}$$

The eigenvalues λ_n^N are found from the approximate (using IEM) solution of the the exterior Dirichlet problem: Given $\hat{u}$ on S, find $u \in H^1_{wN}(\Omega_a^+)$ such that $u = \hat{u}$ on S and

$$b_2(u,v) = (\nabla u, \nabla v)_{\Omega_a^+} - k^2(u,v)_{\Omega_a^+} = 0 \tag{3.31}$$

holds for all $v \in H^1_{w^*N}(\Omega_a^+)$ satisfying $v = 0$ on S. Accordingly, we seek u^N in the form

$$u^N(r,\theta,\phi) = \Phi_o(KR)\,\hat{u} + \sum_{j=1}^{N} \Phi_j(KR)\, w^j(\theta,\phi)$$

where $R = r/a$, $K = ka$ are nondimensional parameters, Φ_j are the shape functions of the IEM, and w^j are unknown functions in angular coordinates. Functions w^j can be expanded into spherical harmonics,

$$w^j(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} w^j_{mn} Y_{mn}(\theta,\phi)\,.$$

We define

$$\begin{aligned} \Phi_o(KR) &= \varphi_1(KR) \\ \Phi_j(KR) &= \varphi_{j+1}(KR) - \varphi_1(KR), \qquad j = 1,\ldots,N \end{aligned}$$

where

$$\varphi_l(KR) = \frac{e^{iK(R-1)}}{R^l}, \qquad l = 1,\ldots$$

are scaled members of the Wilcox expansion for radiating solutions, cf. [6]. Obviously, $\{\Phi_j\}_{j=1}^N$ form a basis of the trial space $V_{N,o} = \{u \in H^1_{wN}(\Omega_a^+) \,|\, u|_{R=1} = 0\}$. Similarly, we define

$$\Psi_j(KR) = \psi_{j+1}(KR) - \psi_1(KR), \qquad j = 1,\ldots,N$$

with

$$\psi_l(KR) = \frac{e^{iK(R-1)}}{R^{l+2}}, \qquad l = 1,\ldots$$

as a basis of the test space $V^*_{N,o} = \{v \in H^1_{w^*N}(\Omega_a^+) \,|\, v|_{R=1} = 0\}$. Now, testing

$$u = \Phi_o(KR) \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \hat{u}_{m'n'} Y_{m'n'}(\theta,\phi) + \sum_{j=1}^{N} \Phi_j(KR) \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} w^j_{m'n'} Y_{m'n'}(\theta,\phi)$$

in eq (3.31) with

$$v = \Psi_i(KR) v^i_{mn} Y_{mn}(\theta,\phi)$$

for some fixed i, m, n, we get

$$\sum_{j=1}^{N}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} a\int_{\mathcal{S}}\int_{1}^{\infty}\Big(R^2\overline{\Psi}_i'\Phi_j'\overline{Y}_{mn}Y_{m'n'}+\overline{\Psi}_i\Phi_j\nabla_S\overline{Y}_{mn}\nabla_S Y_{m'n'}$$
$$-K^2R^2\overline{\Psi}_i\Phi_j\overline{Y}_{mn}Y_{m'n'}\Big)\,w_{m'n'}\,dR\,dS=$$
$$-\sum_{j=1}^{N}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} a\int_{\mathcal{S}}\int_{1}^{\infty}\Big(R^2\overline{\Psi}_i'\Phi_o\overline{Y}_{mn}Y_{m'n'}+\overline{\Psi}_i\Phi_o\nabla_S\overline{Y}_{mn}\nabla_S Y_{m'n'}$$
$$-K^2R^2\overline{\Psi}_i\Phi_o\overline{Y}_{mn}Y_{m'n'}\Big)\,\hat{u}_{m'n'}\,dR\,dS\,.$$

We used the identity $\nabla f = \partial f/\partial r\,\mathbf{e}_r + \nabla_S f$ for the gradient in spherical coordinates; $\nabla_S f$ being the gradient on the unit sphere. Applying

$$\int_{\mathcal{S}}\nabla_S\overline{Y}_{mn}\nabla_S Y_{m'n'} = -\int_{\mathcal{S}}\triangle_S\overline{Y}_{mn}Y_{m'n'} = n(n+1)\int_{\mathcal{S}}\overline{Y}_{mn}Y_{m'n'}$$

and using the orthonormality of the spherical harmonics, we arrive at the series of linear systems

$$\sum_{j=1}^{N} a_{ij}(n)w_{mn}^{j} = -a_{io}\hat{u}_{mn};\qquad i=1,\ldots,N$$

for $n=0,1,\ldots;\ m=-n,\ldots,n$. Observe that coefficients a_{ij} depend only on n. Thus, letting $\mathbf{u}_n=\{u_n^1,\ldots,u_n^N\}$ be the solution of

$$\mathbf{A}_n\mathbf{u}_n=\mathbf{b}_n$$

with coefficients $a_{ij}(n)$ as above and $b_i(n)=-a_{io}(n)$, we have

$$w_{mn}^{j}=u_n^{j}\hat{u}_{mn}$$

(no summation over n). Reordering the sums, we get

$$u^N(r,\theta,\phi)=\sum_{n=0}^{\infty}h_n^N(kr)\sum_{m=-n}^{n}\hat{u}_{mn}Y_{mn}(\theta,\phi)$$

with

$$h_n^N(kr):=\sum_{j=0}^{N}u_n^j\Phi_j(kr) \tag{3.32}$$

where formally $u_o^j:=1$. Thus, the discrete DtN condition is

$$G_N u = -\frac{\partial u}{\partial r}\bigg|_{r=a} = -\sum_{n=0}^{\infty}k\,(h_n^N)'(ka)\sum_{m=-n}^{n}Y_{mn}(\theta,\phi)\int_{\mathcal{S}}u(a,\theta',\phi')\overline{Y_{mn}}(\theta',\phi')\,dS' \tag{3.33}$$

hence the IE-eigenvalues are

$$\lambda_n^N = -k(h_n^N)'(ka)\,. \tag{3.34}$$

Remark 2: Observing that the IEM basis is chosen such that $h_n^N(ka) \equiv 1$ for all n, N, we can equivalently write

$$\lambda_n^N = -k\frac{(h_n^N)'(ka)}{h_n^N(ka)}$$

to get formal similarity with the exact eigenvalues (3.21).

Let us now characterize the discrete eigenvalues. Obviously , for all N,

$$\lambda_n^N = \lambda_n, \qquad n = 0, 1, \ldots, N\,. \tag{3.35}$$

Indeed, the first N basis functions lie in the Galerkin subspace hence the exact eigenvalues are reproduced.[1] Further, from computational experiments we get the following

Lemma 2 *(Conjecture!) The discrete eigenvalues have the properties:*

(1) The imaginary parts of all λ_n^N are negative and bounded with

$$Im\,\lambda_n^N \to -k \qquad as\, n \to \infty\,. \tag{3.36}$$

(2) The real parts of all λ_n^N form for each N an increasing sequence, converging to the limit

$$\lambda_*^N = \lim_{n\to\infty} Re\,\lambda_n^N = \frac{N^2 + 4N + 1}{2a}\,. \tag{3.37}$$

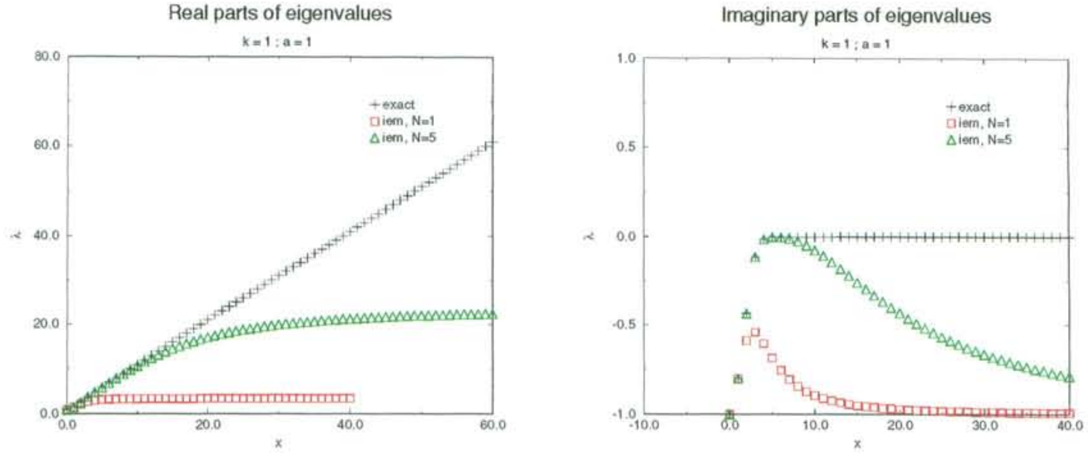
(3) The real parts are positive and bounded from below as

$$\frac{1}{a} \leq Re\,\lambda_n^N\,.$$

This lemma remains so far a conjecture. For illustration of numerous computational experiments, see the following pictures. We compare the eigenvalues of the exact DtN operator to those of the approximate operator for different N. Fig. 3 shows the case $k = a = 1$. The influence of parameters k, a is exemplified in Figures 4,5.

---

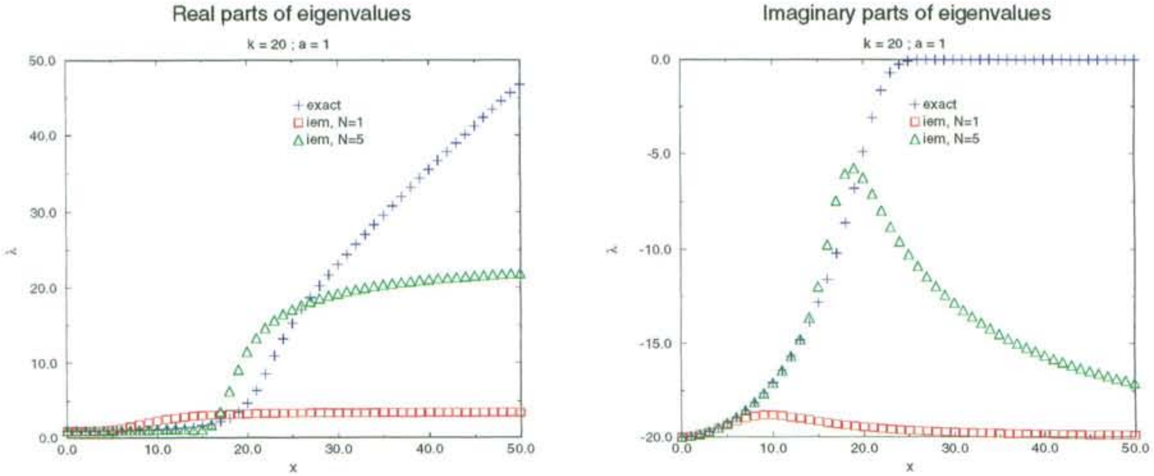
[1] Here we tacitly assume that all modal Galerkin problems have a unique solution, compare section 3.4.



(a) Real parts

(b) Imaginary parts

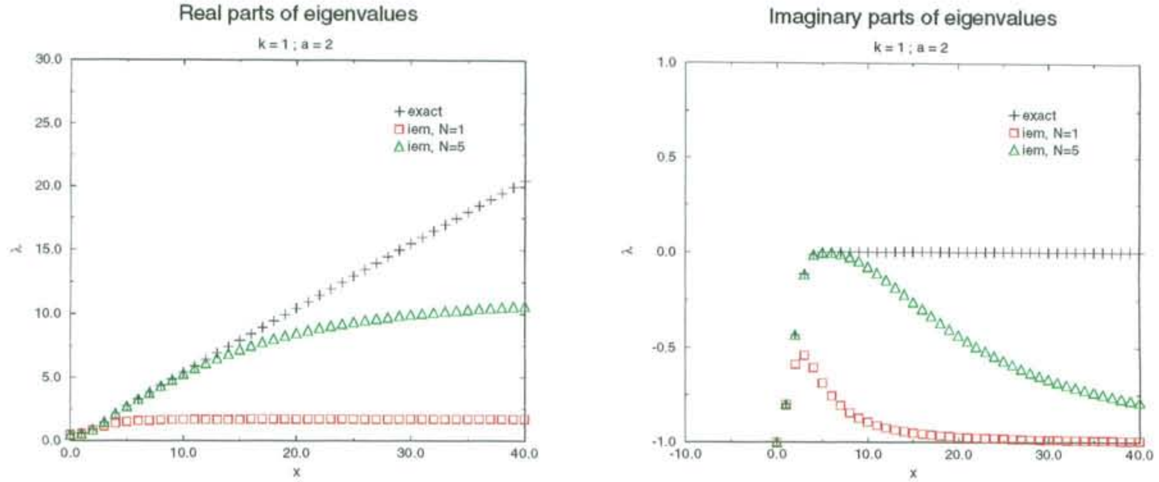
Figure 3: Eigenvalues for $k = a = 1$



(a) Real parts

(b) Imaginary parts

Figure 4: Influence of wavenumber ($k = 20$)



(a) Real parts

(b) Imaginary parts

Figure 5: Influence of radius ($a = 2$)

3.3 Existence and uniqueness of the reduced problems

We first investigate well-posedness of the reduced continuous problem (3.18). Start with uniqueness. Consider the weak form (3.19) and assume that there exists $u \in H^1(\Omega_a)$ such that $b(u, v) = 0$ for all $v \in H^1(\Omega_a)$. Then, in particular,

$$0 = b(u,u) = b_1(u,u) + \sum_{n=0}^{\infty} \lambda_n |u_{mn}|^2$$

hence

$$b_1(u,u) + \sum_{n=0}^{\infty} \operatorname{Re} \lambda_n |u_{mn}|^2 = -i \sum_{n=0}^{\infty} \operatorname{Im} \lambda_n |u_{mn}|^2 .$$

According to Lemma 1, $\operatorname{Im} \lambda_n < 0$ for all n. Hence the term on the right hand side can be a real number only if $u_{mn} \equiv 0$ for all m, n. Thus $u \equiv 0$ on S from which we conclude $u \equiv 0$ in Ω_a, for instance, by a scaling argument, see [17, p.117][2].

To show existence, we split $b(u, v) = a(u, v) - c(u, v)$, defining

$$\begin{aligned} a(u,v) &= (\nabla u, \nabla v) + \sum_{n=0}^{\infty} \operatorname{Re} \lambda_n \sum_{m=-n}^{n} u_{mn} \overline{v}_{mn} , \\ c(u,v) &= c_1(u,v) + c_2(u,v) \end{aligned}$$

---

[2] Alternatively, one can use the analyticity of the solution.

with

$$\begin{aligned} c_1(u,v) &:= k^2(u,v) \qquad u,v \in L^2(\Omega_a)\,, \\ c_2(u,v) &:= -i \sum_{n=0}^{\infty} \operatorname{Im} \lambda_n \sum_{m=-n}^{n} u_{mn}\overline{v}_{mn} \qquad u,v \in L^2(S)\,. \end{aligned}$$

By Lemma 1, $a(u,v)$ is a continuous and coercive sesquilinear form on $V \times V$, where $V = H^1(\Omega_a)$, and hence it defines an isomorphism from V into its dual V'. Further, $c_1(u,v)$ is obviously continuous on $L^2(\Omega_a) \times L^2(\Omega_a)$. The form $c_2(u,v)$ is continuous on $L^2(S) \times L^2(S)$ by property (1) of Lemma 1. Due to the compact imbeddings of $H^1(\Omega_a)$ into $L^2(\Omega_a)$ and $H^{1/2}(S)$ into $L^2(S)$, form $c(u,v)$ defines a *compact* operator C from V into V'. The variational problem can then be rewritten as

$$(A+C)u = F, \qquad u \in V,\ F \in V'$$

or, after applying A^{-1} to both sides of the equation,

$$(I+K)u = f, \qquad u, f \in V,$$

where $f = a^{-1}F$ and $K = A^{-1}C$ is a compact operator. By the Fredholm alternative for operators of the second kind, see, e.g., [19], this equation has either a nontrivial solution for $f = 0$ or there exists a solution for all f. Since uniqueness has been shown existence follows.

We have proved

Theorem 2 *The reduced variational problem*

$$b(u,v) = (\nabla u, \nabla v)_{\Omega_a} - k^2(u,v)_{\Omega_a} + \langle Gu, v\rangle_S = \langle g, v\rangle_\Gamma, \qquad \forall v \in H^1(\Omega_a),$$

where G is the the DtN operator on a sphere S, has for all $g \in L^2(\Gamma)$ a unique solution $u \in H^1(\Omega_a)$.

Remark 3: By analytic continuation onto Ω_a^e, the above argument leads to an elementary proof of existence and uniqueness for the original exterior problem (2.4).

The same argument holds with minor modifications for existence and uniqueness of the reduced IEM problem (2.13) – provided Lemma 2 on the discrete eigenvalues holds and the eigenvalues behave as shown in the numerical evaluation (positive real parts, negative bounded imaginary parts).

3.4 Stability analysis

We will show that the inf-sup constants γ_N for the reduced IEM problems (2.13) have a uniform (independent of N) positive lower bound. Let us first show

Lemma 3 *Assume that Re* $\lambda_n^N \geq 0$. *Then*

$$\liminf_{N\to\infty} \quad \gamma_N \;\to\; 0\ .$$

Proof : Assume to the contrary that there exists a subsequence γ_{N_l} converging to zero, where

$$\gamma_{N_l} = \inf_{\|u\|=1} \sup_{\|v\|=1} \ |b_{N_l}(u,v)|$$

with the norm of $H^1(\Omega_a)$ redefined for convenience as

$$\|u\|^2 = \int_\Omega |\nabla u|^2 dx + \text{Re}\ \lambda_o \int_S |u_o|^2 dS$$

where u_o denotes the zeroth spherical component of trace $\hat{u}$ on S. Then there exists a corresponding sequence of unit vectors u_l, $\|u_l\| = 1$, such that

$$\sup_{\|v\|=1} |b_{N_l}(u_l,v)| \to 0 \qquad \text{as} \quad l \to \infty\,. \tag{3.38}$$

At the cost of replacing the sequence with a subsequence, we can assume that sequence u_l converges *weakly* in $H^1(\Omega_a)$ to a limit u, $\|u\| \leq 1$ [19, p.462]. Note that weak convergence in $H^1(\Omega_a)$ implies strong convergence in $L^2(\Omega_a)$. We shall show that u would be a nontrivial solution to the *continuous* problem with a zero right hand side which is a contradiction to uniqueness.

Step 1: Let $v \in H^1(\Omega_a)$ be a test function with a *finite* spectral representation on the coupling sphere, i.e., $v_n = 0$, $n \geq n_o$ for some n_o. Then $b(u,v) \equiv 0$. Indeed,

$$|b(u,v)| \leq |b(u-u_l,v)| + |b_{N_l}(u_l,v)| + |b(u_l,v) - b_{N_l}(u_l,v)|\,. \tag{3.39}$$

The first term converges to zero by the definition of weak convergence, the second by (3.38) and the last term vanishes for $N_l \geq n_o$ since the approximate eigenvalues $\lambda_n^{N_l}$ reproduce the exact λ_n for $N_l \leq n$ and thus

$$b(u_l,v) - b_{N_l}(u_l,v) = \sum_{n=N_l}^{n_o} (\lambda_n - \lambda_n^{N_l}) u_{ln} v_n$$

is zero for sufficiently large l. The left hand side in (3.39) does not depend on l, hence it must be equal to zero since it is nonnegative and can be bounded by an arbitrarily small number.

Step 2: Let $v \in H^1(\Omega_a)$ be arbitrary, and let $\hat{v} \in H^{1/2}(S)$ denote its trace with the corresponding spectral representation

$$\hat{v} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} v_{mn} Y_{mn} \,.$$

Set

$$v_{n_o} = \sum_{n=0}^{n_o} \sum_{m=-n}^{n} v_{mn} Y_{mn} \,,$$

then $\|\hat{v} - \hat{v}_{n_o}\|$ can be made arbitrarily small for sufficiently large n_o. Denoting by v_{n_o} the best approximation of v in $H^1(\Omega_a)$, coinciding with $\hat{v}_{n_o}$ on S, we have

$$\|v_{n_o} - v\|_{H^1(\Omega_a)} \to 0 \qquad \text{as } n_o \to \infty \,.$$

Thus

$$b(u,v) \leq |b(u, v_{n_o})| + |b(u, v - v_{n_o})| \to 0$$

Indeed, the first term on the right hand side vanishes asymptotically by step 1, and the second term can be bounded, by the continuity of form b, as $|b(u, v - v_{n_o})| \leq M\|u\|\|v - v_{n_o}\|$ where M is the continuity constant.

Step 3: We show that u cannot be zero. Indeed,

$$\begin{aligned}
\operatorname{Re} b(u_l, u_l) &= \int_\Omega |\nabla u_l|^2 \, dx - k^2 \int_\Omega |u_l|^2 \, dx + \int_S \sum_{n=0}^{\infty} \operatorname{Re} \lambda_n^{N_l} |u_{ln}|^2 \, dS \\
&\geq \|u_l\|^2 - k^2 \int_\Omega |u_l|^2 \, dx \\
&= 1 - k^2 \int_\Omega |u_l|^2 \, dx \,.
\end{aligned}$$

By (3.38), the left hand side converges to zero. Hence in the limit $l \to \infty$

$$0 \geq 1 - k^2 \|u\|_{L^2(\Omega_a)}$$

where we employed the strong convergence in L^2 norm. Thus

$$\|u\|_L^2(\Omega_a) \geq \frac{1}{k^2}$$

hence $u \neq 0$. We have shown the contradiction to uniqueness of the continuous problem, proving the lemma. $\lhd$

By the lemma, the inf-sup constant γ_N can be zero at most for a finite number of Ns. To preclude that situation, we further assume (in accordance with the conjecture on the approximate eigenvalues) that Im $\lambda_n^N < 0$. Then uniqueness of solutions for the approximate problems can be shown similarly to the contiuous case, yielding the final result

Theorem 3 *Assume that Re* $\lambda_n^N \geq 0$ *and Im* $\lambda_n^N < 0$ *for all* n, N. *Then*

$$\inf_N \gamma_N > 0 \, .$$

The discrete inf-sup constants are strongly bounded away from zero.

3.5 Convergence analysis

We are now in a position to apply Theorem 1 of section 2. The operator norm $\|G - G_N\|$ can be computed from the eigenvalues in the expansions of the exact and IE solution as follows. Defining, for real $\alpha \geq 0$, on the unit sphere the Sobolev spaces $H^\alpha(\mathcal{S})$ with the norm

$$\|u\|_\alpha = \left(\sum_{n=0}^{\infty} (1+n^2)^\alpha \sum_{m=-n}^{n} |u_{mn}|^2 \right)^{1/2}$$

we have

$$\begin{aligned} \|G - G_N\|_{\mathcal{L}(H^\alpha, H^{-1/2})} &= \sup_{\|u\|_\alpha = 1} \left(\sup_{\|v\|_{1/2}=1} | \langle (G - G_N) u, v \rangle | \right) \\ &= \sup_{\|u\|_\alpha = 1} \sup_{\|v\|_{1/2}=1} \left| \sum_{n=0}^{\infty} \frac{(\lambda_n - \lambda_n^N)}{(1+n^2)^{\frac{1+2\alpha}{4}}} \sum_{m=-n}^{n} (1+n^2)^{\alpha/2} u_{mn} (1+n^2)^{1/4} \bar{v}_{mn} \right| \\ &\leq \sup_{n \geq 0} \left| \frac{(\lambda_n - \lambda_n^N)}{(1+n^2)^{\frac{1+2\alpha}{4}}} \right| \end{aligned} \tag{3.40}$$

by Cauchy-Schwarz inequality. We thus specify Theorem 1 for the spherical case as follows:

Theorem 4 *With the assumptions of Theorem 1, in the case that coupling surface* Γ_a *is a sphere* S, *the error of the IEM solution can be estimated as*

$$\|u - u_N\|_{H^1(\Omega_a)} \leq T \gamma_N^{-1} \frac{C}{N^{\alpha - 3/2}} \|u\|_{H^\alpha(\Gamma_a)}, \qquad \alpha > \frac{3}{2} \, . \tag{3.41}$$

Proof : We will evaluate the upper bound of the norm $\|G - G_N\|$ as given in (3.40). First observe that $\sup_{n \geq 0}$ is in fact the $\sup_{n \geq N+1}$ since the first N eigenvalues are reproduced exactly. Then

$$\sup_{n \geq 0} \left| \frac{(\lambda_n - \lambda_n^N)}{(1+n^2)^{\frac{1+2\alpha}{4}}} \right| \leq C \sup_{n \geq N+1} \frac{|\lambda_n - \lambda_n^N|}{n^{1/2+\alpha}} \leq C \sup_{n \geq N+1} \frac{1}{n^{\alpha - 3/2}} \sup_{n \geq N+1} \frac{|\lambda_n - \lambda_n^N|}{n^2}$$

for some C. The first supremum on the right is obviously achieved for the lowest possible $n = N + 1$. From Lemmas 1 and 2 it is easy to see that the second supremum is bounded. Thus

$$\|G - G_N\| \leq \frac{C}{N^{\alpha - 3/2}}$$

and the statement follows from Theorem 1. $\lhd$

Remark 4: The convergence estimate above is probably suboptimal. The numerical evalutation will show that convergence is assured for $\alpha > 1/2$. This value of α is also obtained from the following assessment in the asymptotic range. Replacing $\lambda_n \simeq n$ and $\lambda_n^N \simeq N^2$ asymptotically and evaluating

$$\left(\frac{n - N^2}{n^\beta} \right)'_n = 0$$

$(\beta = \alpha + 1/2)$ we see that an extremal value is achieved at

$$n = N^2 \, \frac{\beta}{\beta - 1} \, .$$

Thus the supremum is approximately

$$\sup_n \approx \frac{1}{\alpha - 1/2} \, \frac{1}{N^{2\alpha - 1}}$$

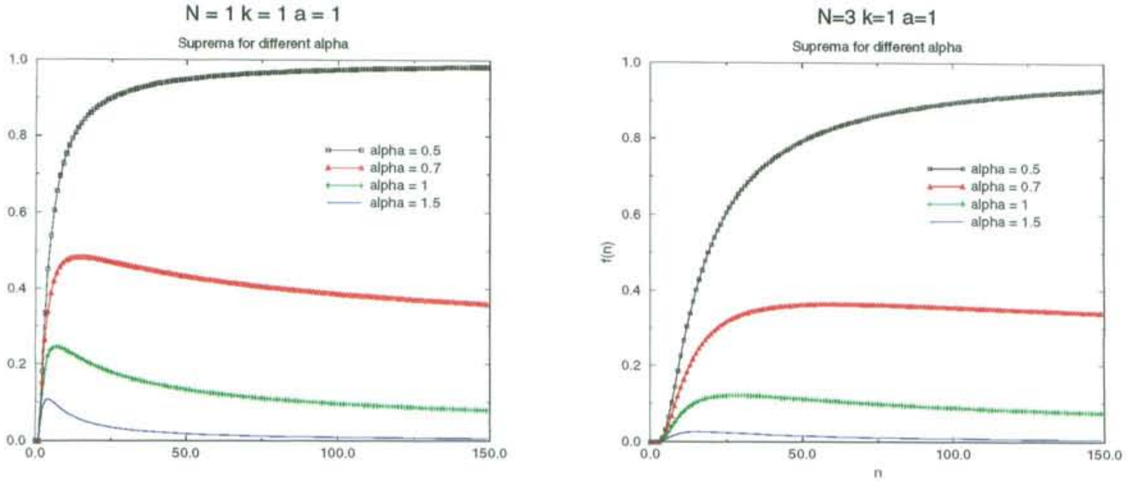
hence the IEM converges for $\alpha > 1/2$. We also see that the constant in front of the rate is likely to grow as $\alpha \rightarrow 1/2 + 0$.

Remark 5: Since the exact solution of the homogeneous Helmholtz equation is analytic in the whole exterior we can use in principle any regularity on Γ_a and thus achieve an arbitrary rate of convergence. However, we saw that the norm $\|u\|_{H^\alpha(\Gamma_a)}$ grows with α hence for each fixed N there is some optimal α for error estimation. So far, practical experience (with rather smooth and regular problems, cf. [6, 7, 8, 9, 20]) shows that sufficient accuracy is obtained using only a few DOF in the IEM region.

3.6 Numerical evaluation

We first evaluate the function

$$f(n, N, \alpha) = \frac{(\lambda_n - \lambda_n^N)}{(1 + n^2)^{\frac{1+2\alpha}{4}}}$$



(a) $N = 1$

(b) $N = 3$

Figure 6: Function f for different n and α

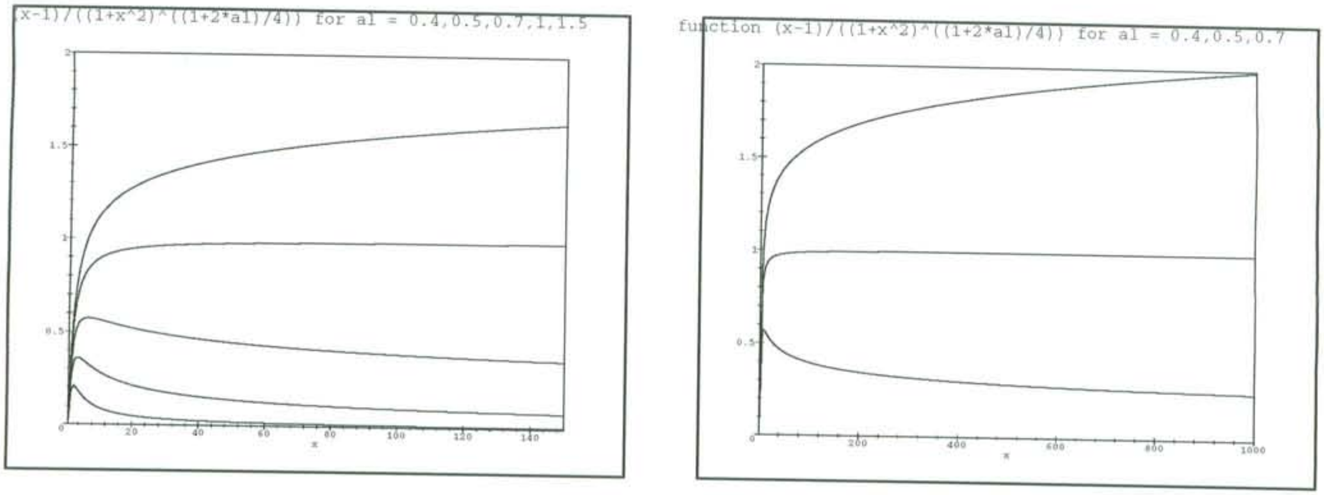
to indicate the existence and position of the supremum for different N and α. Fig. 6 shows that, as predicted, function f has a supremum for $\alpha \geq 1/2$. The position of the suprema is shifted to the right if N grows. We compare the figures to plots where we have replaced λ and λ_n by their asymptotic patterns. In Fig. 7 we show first the same region as above and then the extension to higher n (due to an instability in the evaluation of the Bessel functions, we could not increase n arbitrarily in the computation of λ_n). The plots show that indeed $\alpha \geq 0.5$ is sufficient for the existence of a supremum and hence the expected convergence rate of the IEM is $N^{2\alpha-1}$. The influence of parameters a and k is shown in Fig. 8. As expected, radius a acts as a scaling factor whereas wave number k shifts the position of the supremum but does not change the asymptotic behavior of $f(n)$. Finally, we check the convergence rate by evaluating $\sup_n f(n, N, \alpha)$ for different α. We have shown rigorously a rate of

$$f(n) \sim N^{\alpha-3/2}$$

and we indicated that a finer estimate should lead to

$$f(n) \sim N^{2\alpha-1}.$$

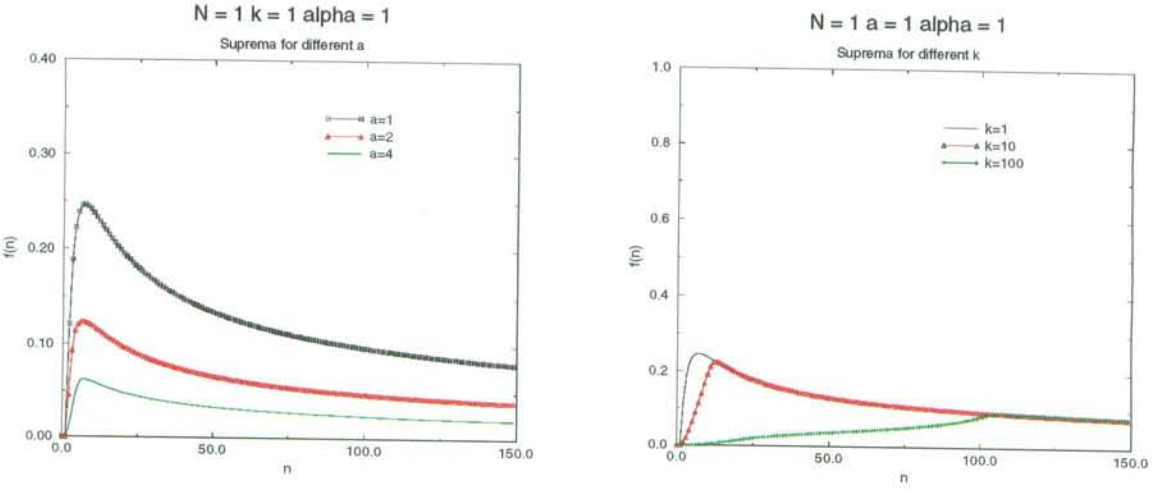
We evaluate the rates for $\alpha = 1, 2.5, 0.5$ – see Table 1. The last column shows the rates as measured in the computations, cf. Fig. 3.6.



(a) $n = 1, \ldots, 150$

(b) $n = 1, \ldots, 1000$

Figure 7: Evaluation of asymptotics



(a) Influence of a

(b) Influence of k

Figure 8: Influence of parameters a and k.

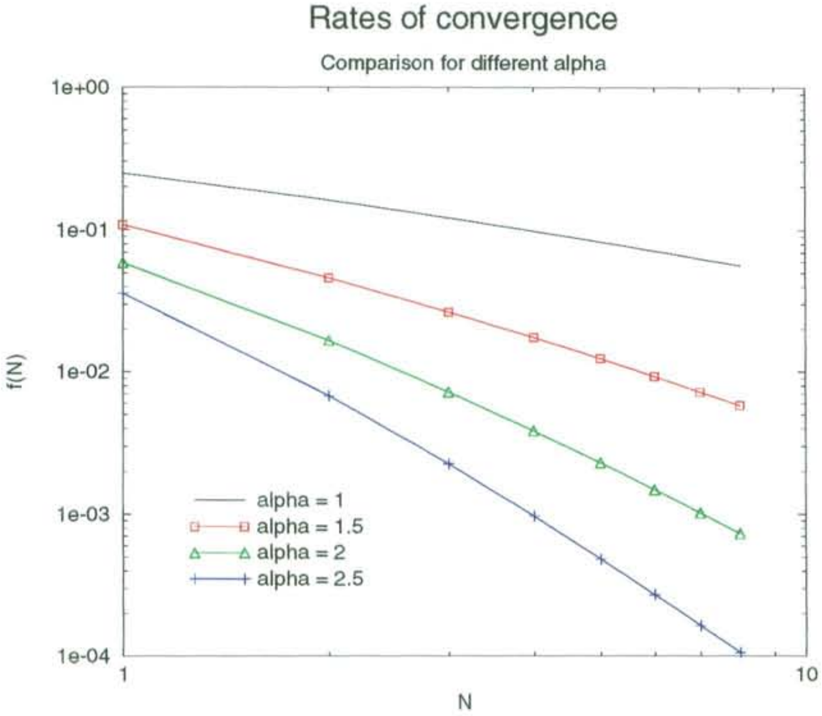


Figure 9: Rates of convergence in upper bound established by supremum

α	$\alpha - 3/2$	$2\alpha - 1$	rate
1	-	1	0.7938
1.5	-	2	1.588
2	0.5	3	2.393
2.5	1	4	3.194

Table 1: Sample values of α and convergence rates

We observe that the rates are close to the optimal rates as obtained in Remark 7. We reiterate that these rates are not actual rates from a FEM-IEM computation but rather the rates predicted by our Convergence Theorem. We will compare our theoretical results with computational results in a future publication.

3.7 Exponential convergence

Analyticity of solution u implies that the trace $\hat{u}$ on the coupling sphere S lies in $H^{\alpha}(S)$ for arbitrarily large α. Consequently, the algebraic rate of convergence shown above is unlimited and one expects an exponential convergence rate. We will now show that this is indeed the case.

We start from

$$\begin{aligned}\gamma_N \|u-u_N\|_1 &\le \sup_{\|v\|_1=1} |\langle (G-G_N)u, v\rangle| \\ &\le \sup_{\|v\|_1=1} \left|\sum_{n=0}^{\infty}(\lambda_n - \lambda_n^N) \sum_{m=-n}^{n} u_{mn}\bar{v}_{mn}\right| \\ &\le \sup_{\|v\|_1=1} \left(\sum_{n=N+1}^{\infty} \frac{|\lambda_n - \lambda_n^N|}{(1+n^2)^{1/2}} \sum_{m=-n}^{n} |u_{mn}|^2\right) \|v\|_{H^{1/2}}(S) \\ &\le T \sum_{n=N+1}^{\infty} \frac{|\lambda_n - \lambda_n^N|}{(1+n^2)^{1/2}} \sum_{m=-n}^{n} |u_{mn}|^2\end{aligned}$$

where T is the trace constant. Then, by the properties of the eigenvalues (assuming that Lemma 2 holds), we have the bound

$$\gamma_N \|u-u_N\|_1 \le C\, T \sum_{n=N+1}^{\infty} (1+n^2)^{1/2} \sum_{m=-n}^{n} |u_{mn}|^2 . \tag{3.42}$$

We will show an exponential bound of the form $Ce^{-\kappa N}$ for the right hand side. Denote

$$a_n = (1+n^2)^{1/2} c_n$$

where

$$c_n = \sum_{m=-n}^{n} |u_{mn}|^2 .$$

We will show that the c_n satisfy an exponential bound. Then also the a_n satisfy such bound since a sequence $a_n = b_n c_n$, where the b_n are algebraically growing and the c_n are exponentially decreasing, is exponentially decreasing. The exponential bound for the residual $r_n = \sum_{n=N+1}^{\infty} a_n$ is a consequence of the following

Lemma 4 *Consider a series $\sum_{n=0}^{\infty} a_n$ with $a_n \ge 0$. Then*

$$\sum_{n=N+1}^{\infty} a_n \le Ce^{-\kappa N}$$

holds for some $C > 0$, $\kappa > 0$, if and only if there exists some $D > 0$ such that

$$a_n \le De^{-\kappa N}, \qquad \forall n$$

for the same $\kappa > 0$.

Proof : ($\Rightarrow$):

$$a_n \leq r_{n-1} \leq Ce^{\kappa}e^{-\kappa n}\,.$$

($\Leftarrow$):

$$\begin{aligned} r_N = \sum_{n=N+1}^{\infty} a_n \leq D \sum_{n=N+1}^{\infty} e^{-\kappa n} &= De^{-\kappa(N+1)}\frac{1}{1-e^{-\kappa}} \\ &= \frac{De^{-\kappa}}{1-e^{-\kappa}}e^{-\kappa N}\,. \end{aligned}$$

$\triangleleft$

It remains to establish the exponential bound for the c_n. Recall that the exact solution can be written in the exterior $r > r_o$ of any sphere $r = r_o$, enclosing the obstacle, as

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} U_m n Y_{mn}(\theta,\phi)$$

where

$$U_{mn} = \frac{u_{mn}}{a^2 h_n(ka)}\,.$$

The solution representation is absolutely convergent in the L^2 norm, implying the bound

$$|h_n(kr)|^2 \sum_{m=-n}^{n} |U_{mn}|^2 \leq C\,.$$

Consequently,

$$\begin{aligned} \sum_{m=-n}^{n} |u_{mn}|^2 &\leq \frac{|h_n(ka)|^2}{a^2|h_n(kr)|^2}\,|h_n(kr)|^2 \sum_{m=-n}^{n} |U_{mn}|^2 \\ &\leq \frac{C}{a^2}\frac{|h_n(ka)|^2}{a^2|h_n(kr)|^2} \end{aligned}$$

Using the relations for the asymptotic (in n) behavior of the Bessel functions, [1, 9.3.1, 10.1.1], it is easy to show that

$$|h_n(z)|^2 \sim \frac{2}{(2n+1)z}\left(\frac{ez}{2n+1}\right)^{-(2n+1)}$$

and therefore

$$\frac{|h_n(ka)|^2}{a^2 h_n(kr)|^2} \sim \frac{kr}{ka}\left(\frac{eka}{ekr}\right)^{-(2n+1)} = \left(\frac{a}{r}\right)^{-2n} = e^{-2\kappa n}$$

with

$$\kappa = \ln\frac{a}{r}\,.$$

We have shown

Theorem 5 *If the coupling surface Γ_a is a sphere, then the approximate solution u^N of (2.13) converges exponentially to the exact solution u of (2.15) as $N \to \infty$. More precisely,*

$$\|u - u_N\|_1 \le C\, e^{-\kappa N}$$

for some positive constants C, κ.

Remark 6: Note that $\kappa = ln(a/r)$ with $r_o < r < a$ where r_o is the radius of a smallest sphere enclosing the obstacle. The further away the coupling sphere $r = a$ the larger the ratio a/r_o. Hence the rate of convergence is expected to grow with the distance between the obstacle and the coupling surface.

4 Conclusions

We analyse a coupled finite-infinite element method, focusing on the convergence of the IEM. To our knowledge, this is the first numerical analysis of FEM-IEM coupling. We relate approximability of the exact solution to the distance between two operators in the norm of the linear space $\mathcal{L}(H^\alpha(\Gamma_a), H^{-1/2}(\Gamma_a)$ where Γ_a is an auxiliary coupling surface in the exterior. In the general setting of section 2 it is not assumed that the exterior Ω_a^+ of Γ_a is separable in some coordinate system. In our convergence theorem we essentially prove the standard result that approximability (here: measured in operator norm) and (discrete) stability yield convergence. The operator norm depends on the regularity of the exact solution. In our discussion of this topic we show that – though one has theoretically infinite regularity on any Γ_a, arbitrarily close to wet surface Γ, the norms $\|u\|_{H^\alpha(\Gamma_a)}$ are likely to grow inversely proportional with the distance and proportionally with wave number k.

In our investigation of the special case that Γ_a is a sphere, we characterize the DtN operators by their countable spectra. The spectral characterization of the IEM-DtN operator is formulated as a conjecture from numerical experiments. Based on the spectral analysis, we show existence-uniqueness of the continuous and discrete formulation. As a corollary, we obtain an elementary proof of existence-uniqueness for the original exterior problem. We then proceed to the proof of a discrete Babuška-Brezzi condition, showing that the inf-sup constants γ_N are bounded from below, indepentently of N, by some positive number. The aforementioned results are used to specify the convergence thorem for the spherical case. We first show that convergence rates in terms of $N^{-\alpha}$ can be achieved for any α since the exact solution is analytic in the exterior domain. (N is the number of DOF in the IEM). The rate in the error bound is multiplied by a constant that grows with α (but does not depend on N). The observation that algebraic rate of any rate is theoretically possible leads to the statement of exponential convergence which concludes our analysis of the spherical case. Finally, we evaluate numerically the upper bound for approximability in terms of N. The rates of the upper bound are close to the predicted ones. Our methodology can be applied to the analysis of a wide range of coupling or absorbing boundary conditions for the FEA of exterior problems.

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