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Abstract: In part I of this investigation, we proved that the standard a-posteriori estimates, based only on local computations, may severely underestimate the exact error for the classes of wave-numbers and the types of meshes employed in engineering analyses. We showed that this is due to the fact that the local estimators do not measure the pollution effect inherent to the FE-solutions of Helmholtz' equation with large wavenumber. Here, we construct a-posteriori estimates of the pollution error. We demonstrate that these estimates are reliable and can be used to correct the standard a-posteriori error estimates in any patch of elements of interest.

1 Introduction

In part I of this investigation, we studied the reliability two popular a-posteriori error estimators for the Helmholtz equation $\triangle u + k^2 u = 0$. Problems that are governed by this equation arise in many applications, for instance in acoustics and electromagnetics. The solutions for Helmholtz' equation with large wavenumber $k$ are highly oscillatory, hence the finite element mesh has to be very fine in order to resolve the oscillations. Beyond that, the resolution rule that is usually applied in practice for piecewise linear approximation fails as the wavenumber increases – see [1, 9, 11]. Apriori error estimates that exactly quantify this effect have been shown in [11, 13]. These preasymptotic estimates established a link between the observations in numerical experiments and well-known estimates from previous mathematical investigations of FEM convergence for Helmholtz' equation. It was shown that the previous estimates hold only asymptotically whereas the error is polluted in the range of practical computations. The notion of numerical pollution for Helmholtz' equation was first introduced in [3]. For numerical investigations of the pollution effect in higher dimensions see [2] and [10]. In [3], it is shown that pollution cannot be, in general, 'circumvented' by the application of stabilized FEM. The more feasible approach so far is to accelerate convergence by higher-order approximation with $h$-$p$-FEM [12] of analytical trial functions as in the Partition of Unity method [16, 17]. An overview of higher-order methods for Helmholtz equation is given in [14]. However, though a lot of research effort is devoted to higher-order FEM, most of the codes applied in the practical analysis of acoustic and electromagnetic problems use linear elements [9, 15]. We showed in part I that the Babuška-Miller residual estimator and the ZZ-estimator by Zienkiewcz and Zhu are not reliable for FEM-solutions of Helmholtz' equation with high wavenumber unless the mesh is very fine. The effectivity index is close to one only if the meshsize $h$ is in the asymptotic range characterized by $k^2 h \ll 1$. On meshes used in practice, the error indicators from both methods significantly underestimate the true error and may thus lead to wrong judgement of the solution quality. We showed that the decrease in accuracy of the estimators for large wavenumber is due to an increasing phase shift of the finite element solution at large wavenumber. The pollution introduced by the phase shift cannot be “seen” by the traditional local estimators.

In this paper, we propose a methodology of a-posteriori estimation of the pollution effect. For problems of linear elasticity, this topic has has been extensively studied in [4]-[8]. We start here from the results and methodologies developed in these references. However, the basic ideas have to be adapted to the highly oscillatory character of the solutions in the present case.

The paper is organized as follows. In section 2, we refer the basic ideas for the a-posteriori estimation of the pollution error developed in [4]-[8]. We show that these estimators are asymptotically exact also for FE-solutions of Helmholtz' equation. However, both a theoretical estimate and numerical evaluation (also section 2) indicate that the estimators are highly oscillatory themselves and that their quality is only slightly better than the local estimators analyzed in part 1. This leads
to the idea of computing estimators of the locally smoothened pollution error. The outline of this idea concludes section 2. We test this new approach both analytically and numerically. As the analytical tool, we derive in section 3 formulae for the asymptotic behavior of the pollution index. The asymptotic expressions do not depend on meshsize $h$; hence they are suited to investigate the dependence of the pollution estimator on wavenumber $k$. In section 4, we evaluate the pollution estimator and compare its effectivity to the local estimators of part 1. An algorithmic presentation of the proposed methodology and the main results of its evaluation conclude this paper (section 5).

2 Basic ideas and analysis

In this section, we outline and analyse the basic ideas for the estimation of the pollution error. We use the notation and the model problem introduced in part I, considering the variational problem: Find $u \in V = H^1_0(0,1)$ such that

$$B(u,v) = \int_0^1 \frac{du}{dx} \frac{dv}{dx} dx - k^2 \int_0^1 uv dx - iku(1)v(1) = \int_0^1 f v dx$$ (2.1)

holds for all $v \in V$. Denoting by $I_j$ a finite element with number $j$, we denote by $B_{I_j}$ the restriction of the form $B$ to $I_j$.

2.1 A posteriori estimation of the pollution error by Green’s function approach

Reviewing the methodology for estimation of the pollution error proposed in [4]-[8] we see that the key point is to use the Green’s function $G(x,s) \in V = H^1_0(I)$ as a test-function. The Green’s function lies in the test-space as a function of $s$ for arbitrarily fixed $x$ or, vice versa, as a function of $x$ for fixed $s$. Furthermore, by definition, $B(w(\cdot),\tilde{G}(x,\cdot)) = w(x)$ for $0 \leq x \leq 1$; $w \in V$. Taking $w = e_h = u - u_h$ we have, in particular,

$$e_h(x) = B(e_h(s),\tilde{G}(x,s)) = B(e_h(s),\tilde{G}(x,s) - v_h(s))$$ (2.2)

for all $v_h \in V_h$, where $V_h \subset V$ is the subspace of piecewise linear functions. Choosing $v_h = T_h G$ (piecewise linear interpolant of $G$), and denoting $\tilde{g} = G - T_h G$, we have

$$e_h = B(e_h,\tilde{g}) = \sum_{j=1}^N B_j(e_h,\tilde{g}_j)$$

$$= \sum_{j=1}^N r_{I_j}(\tilde{g}_j) = \sum_{j=1}^N B_j(\tilde{e}_j,\tilde{g}_j),$$

where

$$r_{I_j} := \left(-f - \frac{d^2u_h}{dx^2} - k^2 u_h\right)|_{I_j}$$ (2.3)

is the interior local residuum, $\tilde{g}_j$ is the restriction of $\tilde{g}$ to the element $I_j$ and $\tilde{e}_j$ is the residual indicator function defined as the solution of: Find $\tilde{e}_j \in H^1_0(I_j)$ such that

$$B_{I_j}(\tilde{e}_j,\tilde{v}) := \left(\frac{d\tilde{e}_j}{dx},\frac{d\tilde{v}}{dx}\right)_{I_j} - k^2(\tilde{e}_j,\tilde{v})_{I_j} = (r_{I_j},\tilde{v})_{I_j}$$ (2.4)
holds for all \( \hat{v} \in H^1_e(I_j) \). Here, \( H^1_e(I_j) \) denotes the subspace of \( H^1 \)-functions that vanish on the element boundaries and \( (u, v)_I = \int_I u \, \overline{v} \) is the \( L^2 \) inner product on interval \( I \).

For practical purpose, it still remains to replace the interpolation error \( \hat{g}_j \) by a function \( \hat{v} \in H^1_e(I_j) \) that is computable \textit{a posteriori}. The obvious choice is the local residual indicator \( \hat{e}_j^G \) of the FE error \( G - G_h \), where \( G_h \) is the finite element approximation of the Green’s function \( G \). More precisely, we define \( \hat{e}_j^G \) as the solution of: Find \( \hat{u} \in H^1_e(I_j) \) such that

\[
B_{I_j}(\hat{u}, \hat{v}) = r_j^G(\hat{v})
\]

for all \( \hat{v} \in H^1_e(I_j) \), where

\[
r_j^G(x, \cdot) = \delta(x) - k^2 G(x, \cdot).
\]

Since we have already established posteriori estimates for the element interiors, it is sufficient to compute the pollution estimate in \textit{just one point} of each element. We choose the right boundary by fixing \( x = x_i \) for computation of the pollution error in the element \( I_i \). Resuming, we obtain the entity

\[
E(x_i) := \sum_{j=1}^{N} B_{I_j}(\hat{e}_j, \hat{e}_j^G(x_i, \cdot))
\]

where \( \hat{e}_j (x_i, \cdot) \in H^1_e(I_j) \) satisfies

\[
B_{I_j}(\hat{e}_j^G(x_i, \cdot), \hat{v}) = (r_j^G(x_i, \cdot), \hat{v})_{I_j}, \quad \hat{v} \in H^1_0(I_j), \quad 1 \leq j \leq N
\]

with \( r_j^G(x_i, \cdot)(\cdot) = -k^2 G_h(x_i, \cdot) \). Note that the Dirac function in the residual is skipped here since the evaluation is done in nodal points and the functional \( \tau \) acts only on functions that vanish in the nodal points. The element of interest is left out in the summation since the pollution error is evaluated on the outside of the element. Similarly to \( E(x) \), we define an estimator for \( \frac{d \hat{e}_h}{dx}(\bar{x}_i) \), where \( \bar{x}_i = \frac{x_i + x_{i-1}}{2} \) is the midpoint of element \( I_i \), as follows:

\[
\frac{d \hat{e}_h}{dx}(\bar{x}_i) \approx \frac{e_h(x_i) - e_h(x_{i-1})}{h} = B(e_h, DG(\bar{x}_i, \cdot))
\]

where \( DG(\bar{x}_i, \cdot) := \frac{1}{h}(G(x_i, \cdot) - G(x_{i-1}, \cdot)) \). As before, we have the equality

\[
\frac{d \hat{e}_h}{dx}(\bar{x}_i) \approx \sum_{j=1}^{N} B_{I_j}(\hat{e}_j, DG(\bar{x}_i, \cdot) - I_h DG(\bar{x}_i, \cdot)).
\]

Similarly to (2.6), we define the residual

\[
r_{I_j}^{DG}(\bar{x}_i, \cdot) = -k^2 DG_h(\bar{x}_i, \cdot)
\]

and compute the local estimators \( \hat{e}_j^{DG(x_i, \cdot)} \in H^1_0(I_j) \) satisfying

\[
B_{I_j}(\hat{e}_j^{DG(x_i, \cdot)}, \hat{v}) = (r_{I_j}^{DG(x_i, \cdot)}, \hat{v})_{I_j}
\]
for all $\hat{v} \in H^1_0(I_j)$. Finally, we define now an estimator for the derivative of the error at $\bar{x}_i$ as
\[
E'(\bar{x}_i) = \sum_{j=1, j \neq i}^N B_{ij} (\hat{e}_j, \hat{e}_j^{DG}(\bar{x}_i, \cdot))
\]

(2.9)

**Remark 1:** The computation of the discrete Green's function $G_h(x, \cdot)$ is inexpensive when a direct solver is employed for the solution of the system of linear equations for the finite element solution. It involves only the forward elimination of an additional right-hand side and a back-substitution.

### 2.2 Analytical and numerical evaluation

To analyze the quality of $E'(x_i)$, consider the difference $|E'(x_i) - e'(x_i)|$. We have
\[
|E'(x_i) - e'(x_i)| = \left| \sum_{j=1, j \neq i}^N B_j (\hat{e}_j, \hat{e}_j^{DG}(\bar{x}_i, \cdot)) \right|
\]

by definition of $\hat{e}^{DG}$ and $\hat{g}'_j := G_{x} - I_hG_{x}$. By excluding the element $I_i$ from the summation we make sure that both the trial and test functions are $H^1$-functions. Denoting by $\hat{e}$ the function that is identical with $\hat{e}_i$ on each element $I_i$ and letting $z(\cdot) := DG_h(\bar{x}_i, \cdot) - I_hG_{x}(\bar{x}_i, \cdot)$, we have equivalently
\[
|E'(x_i) - e'(x_i)| = |\hat{B}(\hat{e}, z)|
\]

where the form $\hat{B}$ is defined on $\tilde{I} := I - I_i$. Making use of the fact that $\hat{e}$ vanishes in all nodal points of the mesh $\Delta$, we can integrate by parts as follows:
\[
|\hat{B}(\hat{e}, z)| = |\int_{\tilde{\Omega}} \hat{e}' z' - k^2 \int_{\tilde{\Omega}} \hat{e} z| \\
\leq |\int_{\tilde{\Omega}} \hat{e} z''| + k^2 |\int_{\tilde{\Omega}} \hat{e} z| \\
\leq ||\hat{e}|| (||z'|| + k^2 ||z||)
\]

where we applied the Schwartz inequality for the $L^2$ inner product. It can be shown (cf. [11], proof of Theorem 3.1) that $z$ solves the variational problem: find $z$ such that
\[
\hat{B}(z, v) = k^2 (G_{x} - I_hG_{x}, v)_{\tilde{\Omega}}
\]

for all $v \in V_h$. Then the norms of $z$ and its derivatives can be estimated by the $L^2$-norm of the data $k^2 \hat{g}' = k^2 (G_{x} - I_hG_{x})$ using Lemma 1 from [11]. We assume that this Lemma holds also on the subspace (cf. Lemma 3 of [11]). Then we have (all $L^2$-norms are computed over the reduced domain $\tilde{\Omega}$ and $C$ is a generic constant which may be different in different estimates)
\[
||z|| \leq C k ||\hat{g}'|| \\
||z''|| \leq C k^3 ||\hat{g}'||
\]

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and, by the standard interpolation result

\[ \| \hat{g}' \| = \| G_{\infty} - I_h G_{\infty} \| \leq C h^2 \| G_{\infty,\text{ess}} \| \]

where \( \| G_{\infty,\text{ess}} \| \leq C k^2 \). Further, we know from our local analysis that \( \| \hat{e} \| \leq C h^2 \| u'' \| \). Assembling the results we finally obtain

\[ | E'(x) - e'(x) | \sim k^5 h^4 \| u'' \| . \]

On the other hand, it is well-known that \( C h \| u'' \| \geq |e|_1 \geq C h \| u'' \| \). Hence, writing \( |E'/e'| = |E' - e' + e'| \) and putting \( |e'| \sim |e|_1 \) we obtain the bounds

\[ 1 - c(k^5 h^4) \frac{\| \hat{u}'' \|}{\| u'' \|} \leq \frac{|E'(x)|}{|e'(x)|} \leq 1 + C(k^5 h^4) \frac{\| \hat{u}'' \|}{\| u'' \|} \]

By similar argument, we may show that

\[ 1 - c(k^4 h^2) \frac{\| \hat{u}'' \|}{\| u'' \|} \leq \frac{|E(x)|}{|e(x)|} \leq 1 + C(k^4 h^2) \frac{\| \hat{u}'' \|}{\| u'' \|} \]

Assuming that the shifted and the exact solution have similar curvatures in average we have shown

**Proposition 2.1** Let \( E_i := E(x_i) \) and \( E'_i := E'(\hat{x}_i) \) be the pollution estimators for the element \( I_i \) as defined in eqs (2.7), (2.9), respectively, and let \( e(x) \) be the exact error of the FEM. Then the local pollution effectivity indices can be estimated as

\[ 1 - c(k^4 h^2) \frac{\| E_i \|}{\| e \|_1} \leq 1 + C(k^4 h^2) \]

\[ 1 - c(k^5 h^3) \frac{\| E'_i \|}{\| e \|_1} \leq 1 + C(k^5 h^3) \]

where \( c, C \) are generic constants not depending on \( k, h \).

The estimates show that one can expect superconvergence in \( h \), as in the non-oscillatory case. Eq (2.11) indicates a slight improvement in terms of \( h, k \) compared to the local estimate in Theorem 3.1, part I, but it is still unsatisfactory since the constants \( c, C \) are not known and the bounds may be extensively large if \( k \) is large and only \( hk \) is restricted. On the other hand, the broad band for the effectivity index given by inequality (2.11) indeed reflects the oscillatory behavior of the pointwise pollution estimator \( E'(x) \). In Fig. 1, we show the true pollution error and the pollution estimator \( E''(x) \) for \( k = 100 \) as a function of \( x \). Though the estimator is close to the solution everywhere, still the effectivity index \( |E'(x)/e'(x)| \) oscillates between peak values of \( \approx 2.2 \) and \( \approx 0.8 \). Observe that the upper peak values are achieved at coordinates where both the error and the estimator are close to zero. Note that the computation is carried out on a refined (w.r. to the recommended relation for \( hk \)) mesh. Results from a computation with the “rule of thumb” \( hk = 0.6 \) – but for smaller wavenumber \( k = 30 \) – are shown in Fig. 2. Observe that the maximal value of the pollution effectivity index is larger than 7.
Figure 1: Pollution estimator to exact error and pollution effectivity index as functions of $x$ for $k = 100, \frac{hk}{x} = 0.3$. 
Figure 2: Pollution estimator to exact error and pollution effectivity index as functions of $x$ for $k = 30, \ hk = 0.6$. 
2.3 A posteriori estimation of locally smoothened pollution errors

From our analysis and numerical observations, we draw the following conclusions with respect to the reliable estimation of the pollution error.

1. The error estimator $E(x)$ ($L^2$-norm) is not suited for reliable estimation of the pollution error.

2. The pollution error is oscillating with the same frequency as the solution itself. The effectiveness index becomes infinite or very large whenever the exact pollution error is zero or close to zero.

Hence, due to the oscillatory behaviour of the error and the estimators we cannot, in general, obtain reliable estimates of the pollution error at a point or in an element. As a remedy, we propose to evaluate an *smoothened value of the pollution error* instead. The method of smoothening must of course be local. Further, it should lead to a new function that is closely related to the function of interest but is less oscillatory and bounded away from zero. With these goals in mind, we consider (a) taking sliding averages of the estimator $E'(x)$ and (b) approximating the upper envelope of the oscillatory curve by taking local maxima over a patch of elements. Both methods lead to new functions that have lower frequency and amplitude of oscillation than the original function. By taking the upper envelope we make sure that the smoothened function is sufficiently bounded from below.

More precisely, we define – for a given interior element number $i$ – the local index-set

$$J_{loc} = \{i - n_{loc}, i - n_{loc} + 1, \ldots, i + n_{loc}\}$$

with $n_{loc} = \left\lfloor \frac{\lambda}{4h} \right\rfloor$ where $\lambda$ is the wavelength and the square brackets mean the biggest integer not exceeding the argument. With this, we define smoothened errors in the element $i$ as

$$E^\text{av}_{i} = \frac{1}{|J_{loc}|} \sum_{j \in J_{loc}} |E'(x_j)|$$

$$E^\text{env}_{i} = \max_{j \in J_{loc}} |E'(x_j)|$$

Both methods of smoothening are illustrated in Fig. 3 and Fig. 4. We note that for the sliding averages the best smoothening effect is achieved if the averaging interval is chosen to be one-half the wavelength. In Fig. 4 we consider averaging by means of the upper envelope. As another illustration, we show in Fig. 5 the absolute value of the exact error for $k = 100$ and its mean value computed by sliding averages. We see that the oscillations are significantly reduced in amplitude and that the mean value curve is bounded away from zero. We will hence measure the averaged error over a patch by an averaged estimator which is designed such that it is overestimating. In the next section we will give the asymptotic study of the estimator and its averaged value.

3 Asymptotic behavior of the pollution index

We seek asymptotic expressions for the pollution estimator and the true error which are suitable as an analytical tool to investigate the behavior of the pollution effectivity index as a function of
Figure 3: Effect of averaging on estimator and effectivity index as functions of $x$ for $k = 100$, $hk = 0.1$. 

(a) Absolute value of pollution estimator using the sliding averages

(b) Effectivity index using the sliding averages
Figure 4: Effect of averaging on estimator and effectivity index as functions of $x$ for $k = 100$, $hk = 0.1$. 

(a) Absolute value of pollution estimator using the upper envelope

(b) Effectivity index using the upper envelope
Figure 5: Smoothed exact error obtained by sliding averages over 1/2 wavelength

$k$. We begin by defining from eq (2.9) the normed expression

$$E_1 := \left( \frac{1}{h} \sum_{j=1 \atop j \neq i}^{N} |B_{ij}^* (\hat{e}_j, \overline{DG(x_{i+\frac{h}{2}})})|^2 \right)^{1/2} \tag{3.1}$$

as the pollution estimator of the finite element solution $u_h(x, k)$ in $H^1$-norm. Comparing to the exact pollution error, we define the pollution effectivity index

$$\theta_p = \frac{E_1}{e_1} \tag{3.2}$$

with

$$e_1 := \left| \sum_{j=1 \atop j \neq i}^{N} B_{ij} (\hat{e}_j, DG - T_h DG) \right|. \tag{3.3}$$

Let us analyse the effectivity index for the model problem, considering first the asymptotics $h \to 0$ and then the asymptotics $k \to \infty$. Using difference calculus, one finds that, for data $f = 1$, the finite element solution in the nodal points $x_h$ is [11]

$$u_h = u_{fe}(x_h) = \frac{1}{k^2} \left( -1 + A_1 \sin \tilde{k} x_h + \cos \tilde{k} x_h \right)$$

with a “discrete wavenumber” $\tilde{k}(k, h) \neq k$. Similarly, the discrete Green’s function is

$$G_h = \frac{h}{\sin k h} \left\{ \begin{array}{ll}
\sin \tilde{k} x_h (A_2 \sin \tilde{k} s_h + \cos \tilde{k} s_h) & x_h \leq s_h \\
\sin \tilde{s}_h (A_2 \sin \tilde{k} x_h + \cos \tilde{k} x_h) & s_h \leq x_h
\end{array} \right. \tag{3.4}$$

where the coefficients $A_1(k, h), A_2(k, h)$ are computed from the boundary conditions – see [11].
The shifted solution \( \hat{u} \) is simply obtained by extending the nodal function \( u_h \) to all \( x \in (0,1) \),

\[
\hat{u}(x, k, h) = \frac{1}{k^2} \left( -1 + A_1 \sin kx + \cos kx \right).
\]

For \( h \to 0 \), the function \( \hat{u} \) converges to the exact solution

\[
u(x) = \frac{1}{k^2} \left( -1 + \cos kx + \sin k \sin kx + i(1 - \cos k) \sin kx \right).
\]

Similarly, we can write the estimator \( \hat{e}^G \) as

\[
\hat{e}^G = \hat{G}(x, s) - G_h(x, s)
\]

where \( \hat{G} \) is obtained from \( G_h \) in eq (3.4) by replacing the discrete \( x_h \) with continuous \( x \). (For simplicity of notation, we use here \( G_h \) both for the nodal function defined above and the piecewise linear function connecting the nodal values of \( G_h \).) Now, the interpolation error of a twice differentiable function on the interval \( \Delta_i = [x_{i-1}, x_i] \) is

\[
\hat{e}^f(s) = (s - s_{i-1})(s - s_i) f''(\xi(x)) / 2
\]

for some \( \xi(x) \in \Delta_i \). Assuming that \( h \) is small such that \( f'' \) does not change much over \( \Delta_i \) we put

\[
\hat{f}''(\xi) \approx f''(\bar{s})
\]

where \( \bar{s} = (s_i + s_{i-1})/2 \) is the mid-point of the interval \( \Delta_i \). With this assumption, we compute the local forms in the estimator \( E_1 \) as

\[
B_{I_j}(\hat{e}(s), \hat{e}^D G(x, \cdot)(s)) =
\]

\[
\frac{1}{4} \hat{u}''(\bar{s}) \hat{G}_{xss}(\bar{s}) B_{I_j}((s - s_j)(s - s_{j-1}), (s - s_j)(s - s_{j-1})).
\]

On uniform mesh, the local forms are constant, \( B_{I_j} \equiv h^3/12 (1 - k^2 h^2/10) \), and we have

\[
\left( \sum_{j=1, j \neq i}^{N} |B_{I_j}(\hat{e}(s), \hat{e}^D G(x, \cdot)(s))|^2 \right)^{1/2} =
\]

\[
\frac{h^3/2}{12} \left( 1 - \frac{k^2 h^2}{10} \right) \left( \sum_{j=1, j \neq i}^{N} |\hat{u}''(s_j)|^2 |\hat{G}_{xss}(s_j)|^2 \right)^{1/2}.
\]

Considering this sum as a Riemann sum we see that the corresponding integral is

\[
E_1(x) \approx \frac{h^2}{12} \left( 1 - \frac{k^2 h^2}{10} \right) \left( \int_{I \setminus \{x\}} |\hat{u}''(s)|^2 |\hat{G}_{xss}(x, s)|^2 ds \right)^{1/2}.
\]

Similarly, we write the absolute value of the exact pollution error in \( \bar{x} \) as

\[
e(\bar{x}) = \sum_{j=1, j \neq i}^{N} B_{I_j}(\hat{e}(s), G_{x}(\bar{x}, s) - T G_{x}(\bar{x}, \cdot)(s))\].
which leads to

\[
e_1(x) \approx \frac{h^2}{12} \left(1 - \frac{k^2 h^2}{10}\right) \left|\int_{I-\{x\}} \hat{u}''(s) G_{xsx}(\bar{x}, s) \, ds\right| =
\]

\[
\frac{h^2}{12} \left(1 - \frac{k^2 h^2}{10}\right) \left|\int_{I-\{x\}} (f(s) + k^2 u_h(s)) k^2 G_x(\bar{x}, s) \, ds\right|.
\]

Here, the second derivative of the shifted solution \(\hat{u}''\) has been replaced using its definition in part I. \textbf{Remark 2}: This expression can be used as an explicit estimator of the error in the derivative of the error, i.e., one can compute an estimate of the error directly without solving any local problems.

In order to further simplify the asymptotic expression consider the asymptotics \(\tilde{k} \to k\). From our analysis of the phase lag \([11, 12]\) we know that \(\hat{k} = k + O(k^3 h^2)\). We assume that \(h\) and \(k\) are such that the phase lag is small. Then \(\hat{u} \to u, \; \hat{G} \to G\) and the numerator of the effectivity index becomes

\[
\left(\int_{I-\{x\}} |u''(s)|^2 |G_{xsx}(x, s)|^2 \, ds\right)^{1/2} = k^2 \left(\int_0^x |u''(s)|^2 \sin^2 ks + \cos^2 kx \int_x^1 |u''(s)|^2 \right)^{1/2}
\]

with

\[
|u''(s)|^2 = \sin^2 ks(1 - 2 \cos k) + \sin 2ks \sin k + 1.
\]

We now compute the integrals in eq (3.5) under the assumption that \(k\) is large, neglecting terms of the form \(f(k, x)/k\) with bounded \(|f|\); e.g., we put \(f \sin^2 ks = 1/2x - 1/2k \sin kx \cos kx \approx 1/2x\). Carrying out a similar computation for the denominator of the effectivity index \(\theta_p\), we show

\textbf{Proposition 3.1} Consider the model problem (2.1-3) and let \(E_1, e_1\) be the pollution estimator and exact pollution error as defined in eqs (3.1), (3.3), respectively. Then, for \(0 < x < 1\), the equalities

\[
E_1 = A \left(x \left(\frac{7}{8} - \frac{3}{4} \cos k\right) + (1 - x) \cos^2 kx \left(\frac{3}{2} - \cos k\right)\right)^{1/2}
\]

\[
e_1 = A \left(\frac{1}{4} \cos^2 kx \left((\cos k - x)^2 - x^2(\cos k - 1)^2\right)
+ \frac{x(x - 1)}{4} \cos kx \sin kx \sin 2k + \frac{x^2}{2}(1 - \cos k)\right)
\]

hold asymptotically for small \(h\) and large \(k\), where

\[
A = k^2 h^2 \left(1 - \frac{k^2 h^2}{10}\right).
\]

We see that, asymptotically, the error norm \(e_1(x)\) oscillates with frequency \(k\). Furthermore, \(e_1(x)\) vanishes at all zeros of \(\cos kx\) if \(\cos k = 1\). On the other hand, the error norm is bounded away from zero for all \(x > 0\) if \(\cos k - 1 > a > 0\). This explains observations from the finite element computations that the (unaveraged) effectivity index is highly sensitive not only to the magnitude of \(k\) and the relation of \(k\) and \(h\) but also to slight changes in the value of \(k\) on fixed mesh. This sensitivity is clearly removed by the proposed ideas of smoothening.
4 Numerical evaluation

The goal is to show the reliability of the pollution error estimator compared to the local estimators analyzed in part I. In particular, we want to see the effect of smoothening on realistic (preasymptotic) meshes. By reliability in this context we mean if the estimator leads to good (i.e., sufficiently close to one) effectivity indices on meshes with \( kh = \text{const} \). It is also favorable that the estimator overestimates the true error.

Let us first evaluate the asymptotic behavior of the estimator. In Fig. 6, we show the asymptotic expressions of \( E_1(x), e_1(x) \) for \( k = 100 \), together with the mean values and envelopes. The envelope of the numerator \( E_1 \) is obtained if we set \( |\cos^2 k x| = 1 \) or \( = 0 \), respectively.

\[
Env(E_1) = \begin{cases} 
(-5/8 + \frac{\cos(k)}{4}) x + 3/2 - \cos(k) & \text{upper} \\
\frac{\pi}{2} \left( \frac{7}{4} - \frac{3 \cos(k)}{2} \right) & \text{lower}
\end{cases}
\]

Approximate envelopes of the denominator are obtained by assuming that the phases \( \cos^2 k x \) and \( \sin 2k x \) oscillate with equal amplitude. The mean values of the numerator and denominator are then obtained as the arithmetic means of the envelopes. The effectivity indices, as functions of \( x \), with and without averaging, are shown in Fig. 7. We see the positive effect of smoothening on the effectivity index. The asymptotic mean value effectivity index, as a function of \( k \), is shown in Fig. 8. The index oscillates but does not grow with \( k \). Unlike the local estimators which are asymptotically exact, the pollution estimator is asymptotically overestimating, as intended by its design.

Finally, we show FE-computations in the preasymptotic range on meshes designed by the refined “rule of thumb” \( hk = 0.3 \). We wish to compare the pollution estimator to the local estimators analyzed in part I. On the example of two large wavenumbers, \( k = 100 \) and \( k = 500 \), we show in Fig. 9 the local residual indicators and the estimator \( E_{pol} \) versus the smoothened true error \( |e_h'(x)| \). Note that the errors are plotted in log-scale. The pollution estimator overestimates the true error in average by a factor 2-3 whereas the local indicators underestimate the true error by a factor \( 100^{-1} \ldots 10^{-1} \). The dependence of the effectivity indices \( x \) is illustrated in Fig. 10 for different \( k \). Firstly, we observe that the averaged effectivity index behaves smoothly w.r. to variation of the coordinate \( x \). The variation of the index w.r. to \( k \) are moderate and in the range predicted by the asymptotic analysis (Fig. 8).
Figure 6: Pollution estimator and pollution error, asymptotic expressions from Proposition, smoothed by wave-enveloping and mean-value approach as functions of $x$ for $k = 100$. 
5 Conclusion

Resuming our analysis, we propose the following methodology of a posteriori error estimation. To estimate the pollution error for an element \( \Delta_i \) in the interior of \( \Omega \), do:

(1) Define \( n_{loc} := \left\lfloor \frac{\lambda}{4h} \right\rfloor \), where the square brackets mean the largest integer not exceeding the argument. Thus \( n_{\lambda} := 2n_{loc} + 1 \) is approximately the number of elements per half-wave. Compute, for the elements \( \Delta_m \) with \( i - n_{loc} \leq m \leq i + n_{loc} \), the pointwise estimator:

\[
E_1(\bar{x}_j) := \left( \frac{1}{h} \sum_{j=1}^{N} B_{kj}(\bar{\varepsilon}_j, \bar{\varepsilon}_j^{DG}(\bar{x}_m, \cdot)) \right)^{1/2},
\]

where \( \bar{x}_m = \frac{x_m + x_{m-1}}{2} \),

\[
\bar{\varepsilon}_j^{DG}(\bar{x}_m, \cdot) \in S_0(I_j) : B_{kj}(\bar{\varepsilon}_j^{DG}(\bar{x}_m, \cdot), v) = \left(-k^2DG(\bar{x}_m, \cdot), v(\cdot)\right)_{I_j} \quad \forall v \in S_0(I_j),
\]

and

\[
DG(\bar{x}_m, \cdot) \in V_h : B(v, DG(\bar{x}_m, \cdot)) = \frac{v(x_m) - v(x_{m-1})}{h} \quad \forall v \in V_h.
\]

Here, \( V_h \) is the usual (global) FE-space of piecewise linear approximation whereas \( S_0(I_j) \) are local approximation spaces of order \( q > 1 \).

(2) Compute the arithmetic mean

\[
\frac{1}{n_{\lambda}} \sum_{m=i-n_{loc}}^{i+n_{loc}} E_1(\bar{x}_m) := E_{pol}(\Delta_i)
\]

and take this average to be the averaged pollution estimator for the element \( \Delta_i \).

Accordingly, we define the averaged pollution effectivity index as

\[
\Theta_{pol}(\Delta_i) := \frac{E_{pol}(\Delta_i)}{\frac{1}{n_{\lambda}} \sum_{m=j-n_{loc}}^{j+n_{loc}} |d_{e_h}(\bar{x}_m)|}.
\]

We thus propose a methodology that does not access the pollution error in one element only but rather over a patch of elements covering approximately one wavelength. Our evaluation shows that this approach guarantees for high wavenumber:

- Overestimation of the true error instead of severe underestimation by the local estimators;
- Effectivity indexes in the range of 1 \ldots 3 compared to 100^{-1} \ldots 10^{-1};
- Smoothening of the effectivity index w.r. to the coordinate \( x \); i.e., the estimator is equally reliable throughout the domain.

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References


Figure 7: Effectivity index, asymptotic expression, original and averaged by mean-value approach as functions of $x$ for $k = 100$. 
Figure 8: Mean-value effectivity index in $x = 0.4$ as a function of $k$
Figure 9: Reliability of averaged estimator compared to local a-posteriori estimators, Estimator $E_{pol}$ vs. averaged true error vs. local residual indicator as functions of $x$ for $k = 100, k = 500$. 

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Figure 10: Averaged pollution effectivity index as function of $x$ for increasing $k$