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Trends and New Problems in Finite Element Methods

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1.1 Introduction

Trends and problems in finite element analysis technology are discussed from the points of view of engineering practice and academic research. Ideally, these two areas are closely related. Experience has shown, however, that with few exceptions it takes ten or more years before the results of research become available for use in professional practice.

Research is continuing at a substantial intensity. One measure of the level of research activity is the number of papers published on various aspects of the finite element method and its applications. The number of these papers is about 4,000 annually [Mac86].

One measure of the level of industrial activity is the license fees collected by vendors of finite element software products. This is estimated to be 400 million dollars annually, with a growth rate of about 18 percent. This figure does not reflect levels of expenditure related to the industrial use of finite element analysis software products, which is many times larger but very difficult to measure. At present the fastest growing major finite element analysis software product is Pro/Mechanica Structure[1], which is based entirely on the p-version of the finite element method.

There is a declining trend in the development and use of in-house finite element analysis codes in favor of commercial codes and there is a strong demand for close integration among commercial finite element analysis codes and computer-aided

[1] Pro/Mechanica Structure is a trademark of Parametric Technology Corporation.

design (CAD) software products. The proliferation of powerful computers makes the computation of larger and larger problems by an increasing number of users possible. The results of computation are typically presented in graphically very sophisticated and attractive ways, creating the usually false impression that the quality of the computed data is comparable to the quality of graphical presentation.

The prevailing trends and expectations were articulated recently by Mr. T. Curry, president of the MacNeal-Schwendler Corporation, as follows [Cur96]:

> *"Today, with immense business pressure to shorten product development cycles we find our customers' needs changing. Tools that require a significant investment in time and resources to master, let alone raw finite element technologies, are inconsistent with today's accelerated schedules."*

On one hand users demand software tools that do not require a significant investment in time and resources to master, on the other hand the consequences of wrong decisions can be very substantial. An example is the failure of the *Sleipner A-1* offshore platform. The finite element analysis, which was the basis of the design of the critical section, was inaccurate due to an inadequate mesh, elements of degree $p = 1$, and the absence of any accuracy control [Hol94], [RGA93], [JR94].

Clearly, engineers are responsible for their decisions based on computed information. They can exercise this responsibility well only with software tools that provide reliable a posteriori estimates of error in terms of all data of engineering interest. Given the competitive pressures noted by Mr. Curry, the demand for reliable a posteriori error estimation in terms of any data of interest is expected to increase.

The apparent economic value of finite element analysis technology based on the estimated levels of usage is very substantial. In this paper we address problems relating to the continued development and industrial use of finite element analysis technology. The discussion is restricted to solid mechanics. This restriction is made because the traditional applications are in solid mechanics where, relatively speaking, the finite element method (FEM) is most mature and has the widest range of applications. There is rapid progress in the application of FEM in computational fluid mechanics (CFD) as well.

1.2 FEM in Engineering Practice

The type of usage of FEM in engineering practice falls into two broad categories: (a) *design* computations and (b) *certification* computations related to various safety considerations and/or regulations. Design and certification computations have different objectives and requirements.

1.2.1 Design computations.

The most important potential use of FEM is in preliminary and conceptual design. It has been estimated that in typical aerospace projects at the end of the conceptual design phase 66% of life cycle costs are committed and 60% of design change opportunities are lost. Figure 1 shows a typical relationship between time and growth of decisions that affect life cycle costs and the consequential decline in design opportunities.

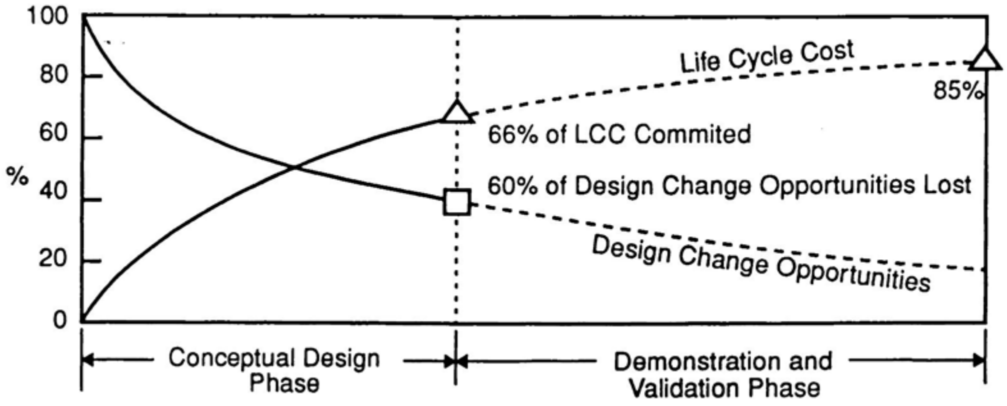

Figure 1 Typical relationship between decisions that affect life cycle costs and maturity of an aerospace project.

The conceptual phase of design is the most creative part of the design process. The objective is to investigate alternative design concepts from the points of view of feasibility, cost, expected performance, ease of manufacturing, and maintenance. Design considerations can change rapidly and widely as concepts are clarified and refined. The considerations are invariably multi-disciplinary in nature. This requires an effective framework for visual and quantitative communication among the various design groups. This role is performed by sophisticated CAD systems. An often quoted example is the design of the Boeing 777 aircraft. During the peak of the project, designers working on 7,000 workstations in 17 time zones managed to communicate effectively through the CAD system CATIA[2] [Pet95].

In general, finite element methods are either not employed in the conceptual design phase, or they are employed only for obtaining an estimate of the force distribution throughout a structure. Very coarse models, called 'stick' models, are created for this purpose. For example, an aircraft structure is typically modeled by very simple plate, beam and bar elements only. The expected output is a reliable distribution of forces acting on the major structural components. It is generally believed that the relative stiffnesses of the major components, and hence the load distribution, will be accurately captured by such models.

There are at least two reasons for the general under-utilization of FEA in the early part of the design cycle where it could provide the greatest benefit: First, a substantial amount of training is required for users of current FEA software products. Most designers lack the required expertise. Second, the time required for conducting finite element analyses is far too long in relation to the time available to designers for completing their work. Typically, design computations are based

[2] CATIA is a trademark of Dassault Systèmes.

on traditional procedures. Many organizations have incorporated these procedures in computer programs to make them more convenient to use but have not changed the underlying design methodology. An additional obstacle for many designers, especially for designers of aerospace structures, components and parts, is that design usually involves investigation of structural stability in the inelastic range which conventional finite element programs do not handle well.

The computational requirements for design are greatly varied from industry to industry, and even from department to department. For example, in automotive firms the requirements of car body designers are very different from those of power train designers. Present finite element software products attempt to meet all requirements across departments and industries. These software products are highly complex; difficult to use properly; slow in responding to new requirements, and often fail to meet specific engineering requirements.

Design computations generally involve parametric analysis. The objective is to select dimensions so that design criteria are met in a nearly optimal fashion. Designers usually focus on a few parameters they consider important and work with simple models in order to make parametric calculations possible in a short amount of time. Implied in this is the notion of hierarchic models: The simple model is understood as the lowest member of a family of models, capable of representing the real physical object with increasing reliability and detail.

Consider, for example, the problem of designing a reinforcement (boss) around a hole in a plate. The plate, made of ASTM-A36 steel (modulus of elasticity $E = 2.9 \times 10^7\ lbf/in^2$; Poisson's ratio $\nu = 0.295$), is subjected to tension (Fig. 2).

Assume that the parameters to which numbers are assigned are fixed. The parameters representing the diameter of the boss (B), the thickness of the boss (h) and the radius of the fillet r ($r < t$) are to be selected by the designer such that the stress concentration factor denoted by Ktg and defined by $Ktg := \sigma_{\max}/S$ is not grater than 2.5.

In a case like this the designer would first investigate the parameters B and h because these parameters can be selected using a simple two-dimensional model, assuming plane stress conditions. The domain is defined parametrically and the mesh is associated with the domain so that when a parameter is changed, the mesh changes with it. In this case p-extension is the logical choice for controlling the errors of discretization. The reasons for this are discussed in Section 1.2.5. Of course, the two-dimensional model cannot account for the effects of the fillet, but is capable of approximating the maximum stress and its dependence on the dimensions B and h well. The computations were performed with Stress Check[3]. Varying one parameter at a time, the design curves shown in Fig. 3 were obtained. From the design curves the dimensions B and h were selected, so that the design conservatively satisfies the constraint $Ktg \leq 2.5$.

The next problem is to select the fillet radius such that the design criterion is met. This requires the use of a three-dimensional model which either represents the fillet or, alternatively, the program has to have a capability to compute the stress intensity factor along the edge to be filleted. It is possible to derive an asymptotic formula for the stress at the surface of the fillet as a function of the stress intensity factor and

[3] Stress Check is a trademark of Engineering Software Research and Development, Inc.

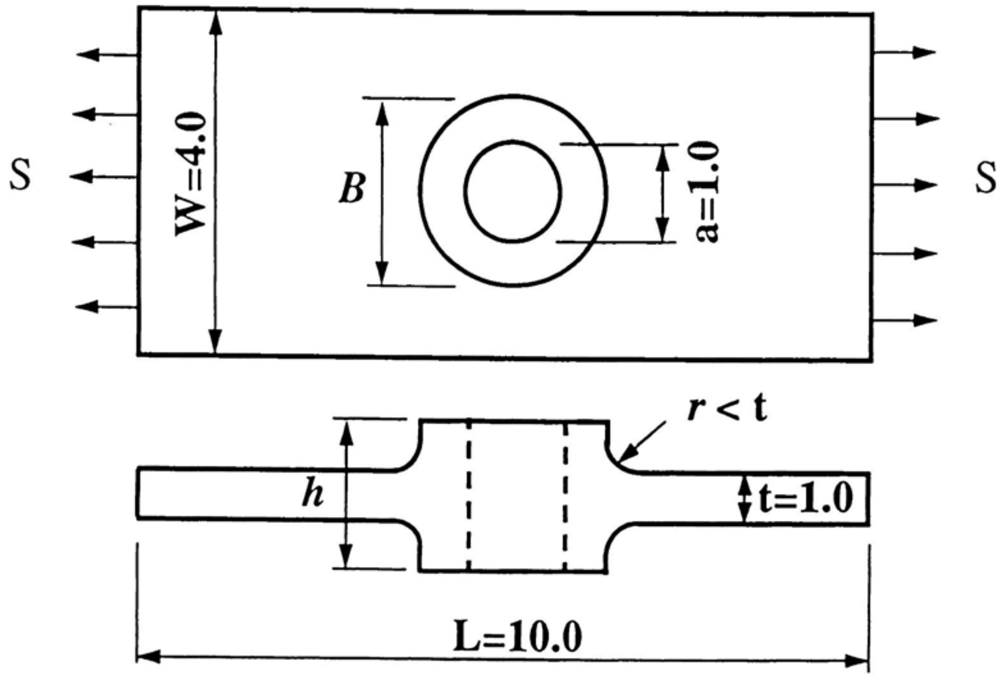

Figure 2 Plate with a reinforced circular hole.

the radius of the fillet (see [AFBP95]). The stress concentration factor (or the stress intensity edge function) can be quickly estimated and the radius of the fillet selected such that the maximal stress in the fillet does not exceed the maximal stress on the surface of the hole.

Error estimation and control are an important part of analyses performed in support of the design decisions. This point is discussed next.

1.2.2 Certification computations

Computations performed to verify that a particular set of design criteria are met involve the use of finite element analysis programs together with the data that characterize the problem (e.g., domain, material, load, etc.). The goals of the computation are to determine maximal stresses, stress intensity factors, buckling stresses, etc.; compare them against allowable values, and report the results in terms of factors or margins of safety.

In general, a discretization can be considered adequate if the following four criteria are met:

1. the data of interest (in this case the maximum stress) are known to be finite;
2. the error in energy norm is small;

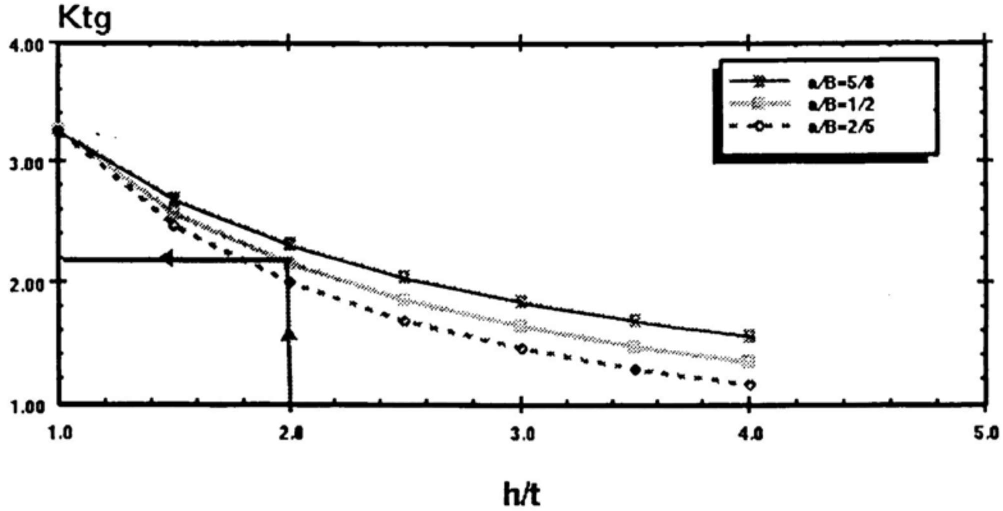

Figure 3 Variation of Ktg with respect to the design parameters h/t and a/B.

3. there are no significant jumps in those stress contours which correspond to high stress levels at the interelement boundaries;
4. the data of interest (in this case the maximum normal stress) appear to have converged to a limit with respect to h-, p- or hp-extension, or a reliable a posteriori error estimator indicates that the error is small.

These rules work well in practice, nevertheless it is possible to conceive counter-examples because the validity of the estimators is established for the asymptotic range but many engineering computations are performed in the pre-asymptotic range. Also, some estimators are not sufficiently robust.

These rules are satisfied in the case of the reinforced hole problem which was analysed using the p-version: Item 1 is satisfied, since there are no sharp re-entrant corners or other causes for stress singularities to occur. That item 2 is satisfied is seen from see Table 1 which shows the convergence of the potential energy with respect to increasing p; the estimated rate of convergence β, and the estimated relative error in energy norm (e_r), which is related to the root-mean-square error in stresses. For details we refer to [SB91]. Item 3 is satisfied, see Fig. 4, and item 4 is satisfied, see Fig. 5.

Recognizing the trends in industry toward abandoning the use of in-house programs, vendors of finite element analysis software products promote codes that satisfy needs previously met by in-house codes. The demand for a posteriori error analyses and adaptive approaches has increased substantially in the past few years. In Fig. 6 we present some results of a recently conducted survey of industrial users [Hal96] in comparison with a survey conducted in 1991 [HBD91].

The results of this survey show that approximately fifty percent of FEA users now employ some form of error estimation and adaptive methods.

Table 1 The reinforced hole problem. Three-dimensional model. Convergence of the potential energy.

p	N	$\Pi(u_{FE})$ (lbf in)	β	e_r %
1	69	$-3.424467054 \times 10^{-8}$	-	12.59
2	226	$-3.466245235 \times 10^{-8}$	0.60	6.21
3	383	$-3.473543257 \times 10^{-8}$	0.75	4.19
4	655	$-3.477141146 \times 10^{-8}$	0.83	2.68
5	1042	$-3.478937791 \times 10^{-8}$	1.36	1.43
6	1571	$-3.479397501 \times 10^{-8}$	1.27	0.85
7	2269	$-3.479574810 \times 10^{-8}$	1.69	0.46
8	3163	$-3.479623475 \times 10^{-8}$	1.69	0.26
∞	∞	$-3.479647034 \times 10^{-8}$	-	0.0

1.2.3 The danger of the uncritical use of FEM.

Vendors of commercial finite element codes claim to provide for "painless" use of their code by nonspecialists as much as possible. Many users treat the code as a "black box". The results, made available in graphical form, often appear plausible for nonspecialists, even when the results are grossly inaccurate. An interesting case study was presented in [Kur95].

Some commercial codes provide a posteriori error estimation in terms of a percentage the data of interest, but the method by which the estimation is obtained is not specified exactly. As seen from Fig. 6, there is an understanding of need for accuracy and hence information about the error (often in the energy norm or the similar L_2 norm) is interpreted as accuracy of *any* data of interest. Many users are not familiar with the notions of error in the energy norm and error in strain energy, or the difference between them. In addition, the notion of the error as the difference between the exact solution of the mathematical problem and the FE solution as its approximation is often not appreciated, because many users still think of the FEM as finite element modeling and not as a method for obtaining an approximate solution. This is illustrated by an example in the following.

The bracket problem shown in Fig. 7 was computed with various FEA codes. (Fig. 7 was generated by Stress Check, $p = 8$.) The analysts were asked to report the location and magnitude of the *maximal* equivalent (von Mises) stress. Inspection of the domain reveals that there is an edge (with an internal angle of about 270°) and hence the maximal stress corresponding to the linear theory of elasticity is infinity. Often it is argued that infinitely large stresses cannot exist and therefore the results are "correct". The authors were shown reports that gave some specific numerical value at the (reentrant) edge, where the exact stress is known to be infinity, yet it was stated that the error was 5% in maximal stresses.

The graphical representation looked very plausible to persons who do not view the

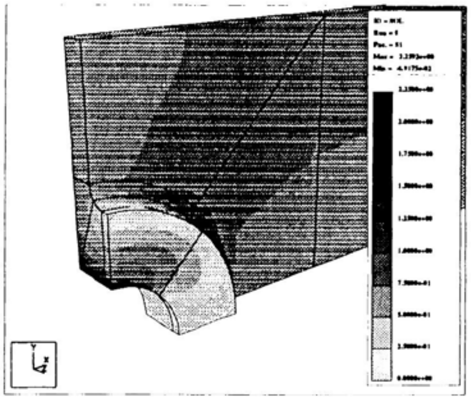

Figure 4 Three-dimensional model of the reinforced hole problem. Contours of the first principal stress.

results of finite element analysis as an approximation to a problem of elasticity but rather as a numerical modeling exercise.

Using the mesh shown in Fig. 7 and p-extension in the range of $p = 1$ to $p = 8$ (trunk space), the maximal stress plotted against the number of degrees of freedom will appear to have converged to a finite value and the indicated location of the maximal stress will be at the largest circular cut-out in the web. The reason for this is that the stress intensity factor along the re-entrant edge is a small number and hence the effects of the singularity are not visible on this mesh within the range of p-values employed. This example demonstrates that it is not sufficient to look at the numerical results only. It is also necessary take into consideration a priori information concerning the nature of the solution, as noted in Section 1.2.2. In this case the stress report would be correctly phrased as follows: "Excluding the neighborhood of the re-entrant edge, the location and magnitude of the maximum principal stress are ..." Without such qualification, the report would be false.

Stresses are computed in order to determine a factor of safety or margin of safety with respect to failure. Therefore an infinite maximal stress predicted by a mathematical model based on the linear theory of elasticity has to be interpreted from the point of view of failure, such as crack initiation. In the case of the bracket shown in Fig. 7 the re-entrant edge may not be critical from the point of view of failure, nevertheless this cannot be decided without a properly formulated failure criterion.

1.2.4 The problems of mesh design and pollution.

Mesh generation is typically the most laborious part of finite element analysis in three dimensions. Various sophisticated mesh generators are available. These mesh generators are connected with computer-aided design (CAD) software because usually the geometric representation is first constructed in a CAD environment. The quality of

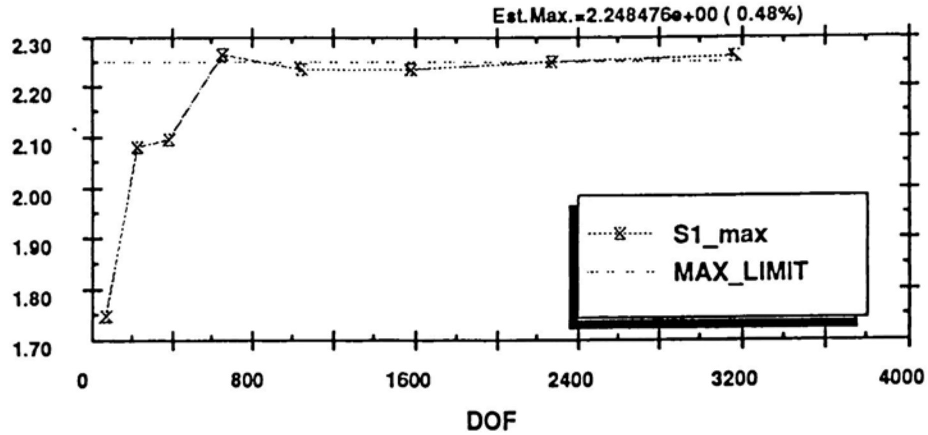

Figure 5 Three-dimensional model of the reinforced hole problem. Convergence of the maximum normal stress.

the mesh from the point of view of computational accuracy is not considered in general. In addition some details, such as as fillets (the rounding of corners and edges), are usually omitted.

A mesh, typical of meshes currently used in industrial applications for simple three-dimensional domains, is shown in Fig. 8. Note that some edges were filleted while others were not. Typically, finite element meshes are constructed for representation of the main topological features, rather than for ensuring the accuracy of the finite element solution. In this case the solution is singular along the edges which leads to a potentially large error in the finite element solution. The size of the error depends on the loading. In this regard we note that the errors in design that led to the sinking of the Sleipner A-1 platform were caused by improper meshing which described the geometry well but was grossly inadequate from the point of view of accuracy.

It is generally thought that one does not need to be concerned about the mesh in those areas which are far from the area of interest when a posteriori local extraction of the stresses with error estimation (recoveries principle) is used. This can lead to very unreliable results due to pollution caused by coarse meshes in the neighborhoods of the edges and corners. Capabilities for a posteriori estimation of pollution errors are needed. Unfortunately, there is a general lack of awareness of pollution errors in the engineering community.

Problems of a similar character but different nature occur in the modeling of structural connections. Typically, fasteners (rivets) are modeled by point constraints or point attachments. In such cases the results strongly depend on the mesh not only in the neighborhood of the fasteners but also far from the fasteners. For details we refer to [Sza90], [SB91], [BS92a], [BS92b].

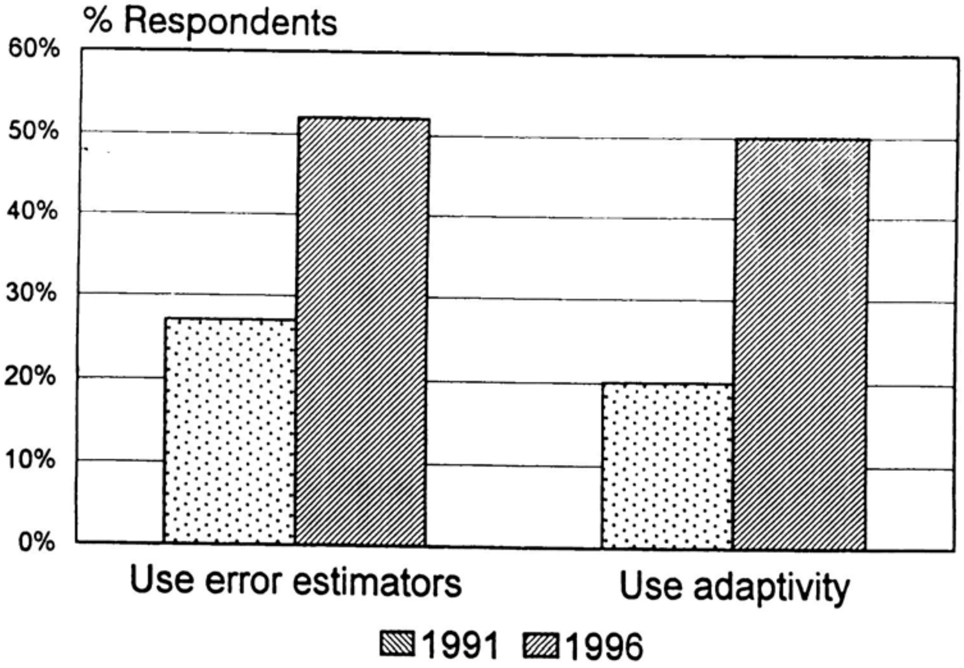

Figure 6 Increased use of error estimation and adaptivity in industry within the past five years.

1.2.5 The p- and hp-versions of FEM.

There are three versions of FEM: In the classical h-version the errors of discretization are controlled by making the mesh sufficiently fine while keeping the degree p of the elements fixed, usually at $p = 1$ or $p = 2$. In the p-version the errors of discretization are reduced by keeping the mesh fixed, and increasing the polynomial degrees of elements (p). The hp-version combines both approaches. The p- and hp-versions are of substantial interest to software houses today because they provide effective means for controlling errors of discretization for a large and important class of problems.

The leading vendors of conventional finite element analysis software have implemented the p-version so that today the largest commercial finite element codes MSC/NASTRAN[4], ANSYS[5], I-DEAS[6] offer at least some p-version capabilities. There are several other codes which are based entirely on the p-version. These include Pro/MECHANICA; PolyFEM[7]; STRESS CHECK; PHLEX[8], and STRIPE[9]. The p-version has some important advantages:

[4] MSC/NASTRAN is a trademark of the MacNeal-Schwendler Corporation.
[5] ANSYS is a trademark of Ansys, Inc.
[6] I-DEAS is a trademark of Structural Dynamics Research Corporation.
[7] PolyFEM is a trademark of IBM Corporation.
[8] PHLEX is a trademark of Computational Mechanics Company.
[9] Flygtekniska Försöksanstalten, Bromma, Sweden.

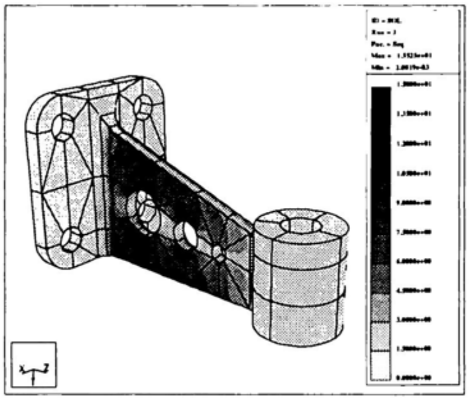

Figure 7 Model problem. Contours of the equivalent (von Mises) stress.

1. Although the mesh design has to follow some rules, the meshing is much simpler than in the case of the h-version and there is much less sensitivity to the mesh than in the h-version.
2. The method is robust with respect to locking.
3. In practical cases a much faster rate of convergence can be achieved.
4. It is possible to obtain in practice a guarantee of the accuracy of *any* value of interest by comparing the values for increasing p.
5. The method is useful and effective for both design and certification computations.

Mesh generators appropriate to the p-version have not yet been developed. An adaptive hp-version meshing is available in the program PHLEX.

1.2.6 The problem of modeling

Proper selection and definition of the mathematical model is the most important prerequisite of reliability in numerical simulation. This involves (a) the formulation of the mathematical problem to be solved and (b) the selection of specific numerical data.

By the term "formulation" we mean various simplifications, such as selection of plate or shell theories; the formulation of boundary conditions; modeling of fasteners; neglecting nonlinear effects; selecting constitutive laws; describing composite materials, etc. Identification of the goals of computation, such as, for example, determination of maximum equivalent stress; displacement; stress intensity factors; limit loads etc. is part of the formulation.

In many cases the data of interest are very sensitive to modeling decisions. An interesting example is the Girkmann-Timoshenko problem. An axially symmetric spherical shell with a stiffening ring shown in Fig. 9, was analysed by eminent engineers using the classical methods of structural analysis ([Gir56], [TWK59]) before computer-based methods were available to solve such problems. Fig. 9 is not to scale,

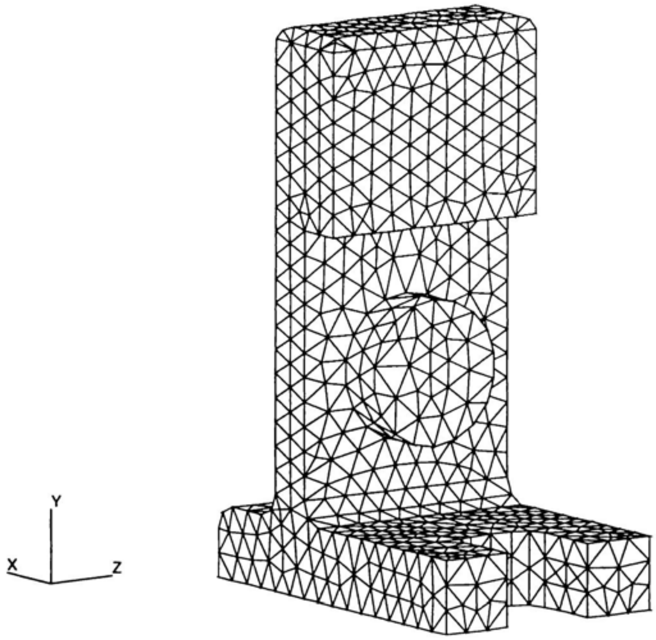

Figure 8 An example of meshes produced by automatic mesh generators.

the shell is quite thin, the radius to thickness ratio is 389.4. The material is isotropic, elastic, the modulus of elasticity is 3×10^6 lbf/in^2 and Poisson's ratio is zero. The shell is loaded by a uniformly distributed load of 0.28472 lbf/in^2 acting in the middle surface of the shell parallel to the axis of symmetry. This load is equilibrated by uniformly distributed normal traction acting on the base of the ring. The goals of the computation are to determine the radial force (F_r) and the bending moment (M) acting between the shell and the stiffening ring. These data are very sensitive to the stiffness of the ring and the boundary conditions used.

Analyses were performed using a fully axisymmetric model, i.e., the shell and stiffening ring were modeled as an axisymmetric elastic body, using three types of boundary conditions: (a) equilibrium loading (the stiffening ring was not constrained against rotation or radial displacement); (b) simple support (the stiffening ring was constrained against rotation but not against radial displacement); and (c) rigid support (the stiffening ring was constrained against both radial displacement and rotation). Using various meshes (one of which in shown in Fig. 9) and p-extension, the possibility of a significant discretization error was eliminated [Sza88] [SB91]. The results of the analysis are shown in Table 2.

It is seen that very substantial differences exist between the solutions obtained with

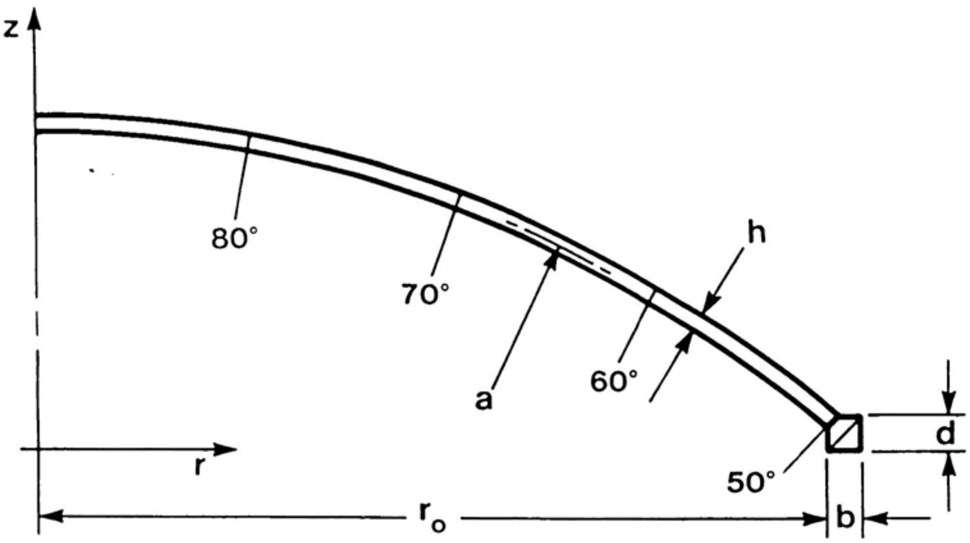

Figure 9 The Girkmann-Timoshenko problem: Generating section of a spherical shell with stiffening ring (not to scale). The dimensions are: $a = 919.20$ in, $h = 2.36$ in, $b = 23.64$ in, $d = 19.68$ in.

the Girkmann-Timoshenko model and the fully axisymmetric model. The reason for this is that the reactions F_r, M are very sensitive to the stiffness of the ring and the boundary conditions imposed on the ring. In the classical solution ([Gir56], [TWK59]) equilibrium loading was assumed. By the methods available at that time, the stiffness of the ring could not be estimated with sufficient accuracy.

The Girkmann-Timoshenko problem is an example that demonstrates the great importance of proper definition of the model and taking the sensitivity of the computed data to modeling decisions into consideration. Other examples are available in [SB91], [BS92a], [BS92b], [SO96].

Another source of uncertainties is the constitutive law, especially for nonlinear material behavior. There is a large literature about constitutive laws but there is insufficient information about the *values* of the parameters in the laws. In addition, there is substantial sensitivity to various factors. For example, it has been reported that the yield stress for the 5454-H32 aluminum alloy differs by 22% for the sheet (0.2 in. normal thickness) and plate (0.4 in. normal thickness) [WBSC87].

An extensive analysis of the properties of aluminum alloy 5454-H32 in the one dimensional setting and cyclic strain was performed [BJLS93]. Denoting $\sigma^{MEAS}(t)$ (resp. $\sigma^{PRED}(t)$) the measured stress (resp. the predicted stress) from a particular law, the accuracy measure Θ (in percent) was defined as

$$\Theta = 100 \frac{\max_{i=1\to 20000} |\sigma^{MEAS}(t_i) - \sigma^{PRED}(t_i)|}{\max_{i=1\to 20000} \frac{1}{2}|\sigma^{MEAS}(t_i) + \sigma^{PRED}(t_i)|}. \tag{1.1}$$

It was found that for the kinematic law and data taken from the material handbook

Table 2 The Girkmann-Timoshenko problem.

Description of the model	Force F_r (lbf/in)	Moment M (lbf in/in)
Fully axisymmetric model		
(a) Equilibrium loading	-105.2	-7.84
(b) Simple support	-93.0	-231.1
(c) Rigid support	-109.9	-43.8
The Girkmann-Timoshenko model	-8.95	-24.84

leads to the mean value $\bar{\Theta} = 33\%$ and the standard deviation is 15% of the mean. Replacing σ^{MEAS} and σ^{PRED} in eq. 1.1 by the two measured stresses for the same strain, a measure of the reproducibility is obtained. The mean was about 5% and the standard deviation was 20% of the mean. See [BJLS93] for additional data and assessment of other laws.

Obviously, the reliability of the computed data depends strongly on the reliability of the constitutive law, which is very low in the case of inelastic deformation.

1.2.7 Summary of Trends

1. A major emphasis in the last few years has been on the development of FEM for design purposes. No consensus has emerged concerning an optimal approach. Various alternatives are under consideration.
2. There is a decline in the development and use of in-house software. They are being replaced by commercial programs. Commercial programs have many proprietary details which complicate the assessment of their quality and the results are prone to misinterpretation.
3. There is a widespread and growing demand for a posteriori error estimation and adaptive approaches. The presently available error measures are often inadequate with respect to the goals of computation and are prone to severe misinterpretation. The programs do not indicate points where the stresses are infinite. Very likely, various expert systems will appear in the future.
4. The simplified formulation used in the design phase needs some approaches for the assessment of the reliability of the conclusions from these models. Also, there is a need for estimating the effects of uncertainties on the computed data.

1.3 Research topics.

Some current research topics are discussed in the following which are likely to impact professional practice in the not-too-distant future.

1.3.1 The p- and hp-versions of FEM

During the last few years significant progress in the theory of the p- and hp-versions of the finite element method has been made in the fields of elliptic equations (solids) and hyperbolic equations (fluids). The progress in industrial applications was discussed in Section 1.2.5. In this section some recent results and research directions for elliptic equations are discussed.

The essential feature of a very large class of problems in solid mechanics is that the input data (i.e., the boundary, the coefficients, the boundary conditions, etc.) are piecewise analytic. For this type of problems it is possible describe the regularity of the solution in the framework of countably normed spaces. See [BG88b], [GB93], [Guo94], [GB96]. Exact and accurate characterization of the regularity of the solution, which is as narrow as possible, but encompasses the majority of engineering problems, is essential for the analysis of the performance of the p- and hp-versions. Using these results, it was possible to prove that the hp-version converges exponentially for problems characterized by piecewise analytic data. More precisely, denoting the energy norm by $\|\cdot\|_E$, the exact (resp. finite element) solution by u (resp. u_{FE}), the number of degrees of freedom by N, then for the hp-version we have the estimate:

$$\|u - u_{FE}\|_E < Ce^{-\gamma N^{1/\alpha}} \tag{1.2}$$

where the coefficient α depends on the dimension of the problem. In two dimensions $\alpha = 3$ and in three dimensions $\alpha = 5$.

The coefficient $\gamma > 0$ depends on the strength of the singularity of the solution in the neighborhood of the boundary (edges, corners) and interfaces. There exist cases where γ is very small. The meshes must be refined in the neighborhood of the corners of the domain in two dimensions and vertices and edges in three dimensions. The elements in the neighborhood of the edges have very large aspect ratios ("needle elements") reflecting the fact that the solutions are smooth along the edges but singular in the direction perpendicular to the edge. For additional information see [BG86a], [BG86b], [BG88a], [Guo94], [BG92], [BG96].

In [GB86] a detailed analysis of a one-dimensional model problem was performed. It was shown that $\alpha = 2$ and it cannot be improved, more precisely, no sequence of meshes and elements of arbitrary degrees can lead to smaller α. We conjecture that $\alpha = 3$ (resp. $\alpha = 5$) in two (resp. three) dimensions are the optimal values, but this has not been proven. Further, in [GB86] we showed that the optimal value of γ depends on the strongest singularity of the solution. The first results about the hp-version were obtained in [BD81].

In [BSK81] the p-version was analysed and it was shown that the rate of convergence of the p-version is at least twice that of the h-version with uniform or quasi-uniform meshes for problems characterized by piecewise analytic data.

Due to difficulties in implementation in its asymptotically correct form, the hp-version is very often the p-version on judiciously constructed finite element meshes in practice. The principles of mesh construction are simple. When the correct mesh is used then, in the range of accuracy normally expected in engineering computations, the performance of the p-version is similar to that of the hp-version.

In [BS87] the results of [BSK81] were generalized and in [BS82] some aspects of p-version with emphasis on applications were discussed.

As noted above, the coefficient γ in equation 1.2 can be very small in some cases. In such cases neither the hp-version nor p- or h-versions can be effective. Nevertheless it was shown in [OB95] that the problem can be circumvented by using non-polynomial basis functions.

Although the basic properties of the hp-version are well understood with respect to the error in the energy norm, some important problems remain. One such problem is error estimation in L_∞ norm. For example, it is known that the derivatives (stresses) computed directly from a finite element solution obtained with the p-version have very low accuracy in those elements where the exact solution is singular, such as at re-entrant corners, vertices or edges. A detailed theoretical investigation of this subject is needed.

Typically, a-posteriori estimation of error is based on extrapolation which involves comparison of the solutions with different polynomial degrees. This approach usually gives good results not only for the energy norm but also for any other value of interest. Nevertheless, theoretical analysis is needed. See, for example, [BG86a], [BG86b], [Sza86], [YS94]. Also, various approaches described in the next section are applicable for the estimates in the energy norm.

In p-adaptive procedures polynomial degrees are adaptively assigned to finite elements. There can be large differences in polynomial degrees when elements of high degree are used. Experience has shown that the a posteriori error estimation of stresses based on comparing solutions with various degrees of freedom are less reliable when non-uniform p-distribution is used than in the case of uniform p-distributions.

In hp-adaptive procedures either the polynomial degree is increased or the mesh is refined. The only finite element code at present which has such a capability is PHLEX.

There are important aspects of implementation of the p- and hp-versions. One is to compute the stiffness matrix efficiently. The method used in the h-version for low p is very ineffective for the p-version but can be improved significantly [AFBP95].

The problem of solving large systems of linear equations generated by hp-extensions is complicated by the fact that the condition number is typically large. For various aspects of this problem we refer to [Man92], [BEM92], [OPF94], [GC96], [GM96].

The p-version has excellent robustness with respect to locking in shells and plates, see [HLP96]. For surveys of the hp-method we refer to [BG92], [BS94].

1.3.2 A-posteriori Error Estimation and Adaptive Procedures for the h-version

In the last few years large emphasis has been placed on a-posteriori error estimation and adaptive procedures in both academic research and industrial applications. Error estimates in the energy norm measure are most typical although estimation of other data of interest have been introduced. A large number of papers have been written on this subject. Essentially there are two types of estimators in the h-version:

1. residual estimations;
2. estimations based on the smoothing (recovery).

Various estimators have been proposed and analysed either theoretically or computationally. A major problem in the selection of an estimator for practical use is the assessment of its quality and robustness.

Error indicators are defined locally and hence assessment of an indicator cannot depend on the pollution i.e., on the effects of the finite element error outside of the elements, because this effect is a global one while the estimation itself is local and hence it cannot "see" the pollution.

In [BSU94a], [BSU+94b], [BSU96] a theory was developed which allows the assessment of a particular error indicator. Given a translation-invariant mesh (e.g., Fig. 10), the asymptotic lower and upper bound C_L and C_U of the (elemental) effectivity index for particular estimators can be computed. The bound is over a set of solutions, (smooth ones or ones that satisfy, in addition, a differential equation or a set of equations e.g., characterized by the coefficients) and over a class of meshes. It was shown that in the case of smooth solutions it is possible to consider only the solutions which are locally polynomials of degree $p+1$, when p is the degree of the element used.

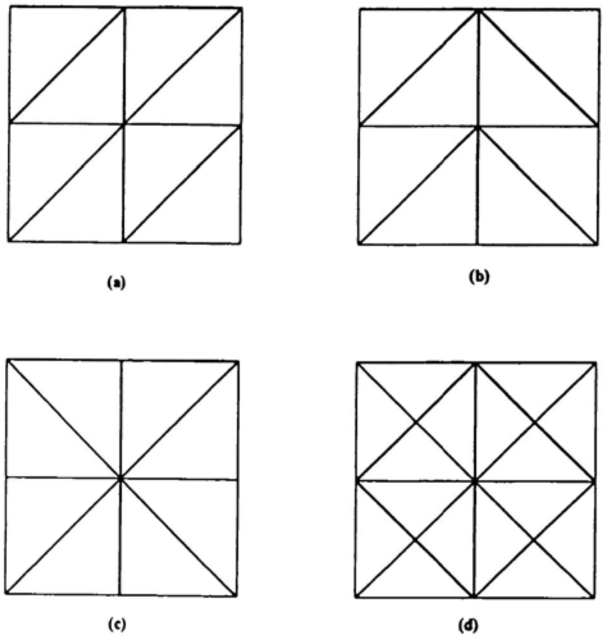

Figure 10 Types of mesh patterns considered: a) regular pattern, b) chevron pattern, c) union jack pattern, d) criss-cross pattern.

The robustness index R is introduced

$$R := \min\left(|1-C_U|+|1-C_L|,\ |1-\frac{1}{C_U}|+|1-\frac{1}{C_L}|\right). \tag{1.3}$$

Obviously, $R = 0$ is the optimal value of the robustness index. In [BSU94a] the indicators inside the domain were considered and in [BSU96] the elements at the boundary were investigated. In [BSU$^+$94b] it was shown that the robustness index depends on the pattern but only weakly depends on the mesh in the neighborhood of the elements under consideration. The robustness index characterizes the quality of the estimators well. Various estimators have large robustness index, especially for elements with large aspect ratios, anisotropic materials, etc. and therefore should not be used in practice. Based on the results of these investigations, it is possible to conclude the following:

1. Comparing the existing estimators for the error measured in energy norm based on the residual and smoothening (recovery) approach, for large classes of solutions, meshes and materials, the Zienkiewicz-Zhu (ZZ) estimator proposed in [ZZ92a], [ZZ92b] is the most robust one, although sufficiently rigorous mathematical theory that would explain this conclusion is not available at present.
2. Among the residual estimators the equilibrated one is in general the most robust one. The unequilibrated ones are not robust, in general. The theoretical understanding of this type of estimators is well advanced. The estimate (in energy norm) could be a guaranteed upper bound on the entire domain although the robustness index is not necessarily small.

The above theory is asymptotic for the h-version (with respect to $h \to 0$), assuming that the exact solution is smooth. For elements of high degree the asymptotic theory can be misleading because the prescribed accuracy is achieved in practice in the preasymptotic range. Also, conclusions from the asymptotic theory cannot be used directly when the solution is not smooth, as in the neighborhoods of corners. The theory has to be adjusted and we expect that this will be done in the future.

Although a-posteriori error estimation is applied mostly in the energy norm, it is possible to derive the estimators for the other norms also because in the absence of pollution the recovered solution is closer to the exact solution than the finite element solution. We emphasize that this pollution control is an essential attribute of the estimator. The problem of pollution is discussed in the Section 1.3.4. The indicators (estimators) derived for linear equations can be used also for nonlinear ones because essentially we estimate the error for the linearized problem. For analyses of nonlinear equations we refer to [Ver94], [Ver95].

Error estimators used in conjunction with adaptive procedures should be related to the goals of analysis, not to the accuracy in energy norm only. There are various adaptive procedures, but essentially all are aiming to create meshes in which all elements have approximately the same error, i.e., the mesh is equilibrated. In [BV84] the optimality of various adaptive procedures was analysed on the basis of a simple one-dimensional model problem. It was shown that the equilibration (equidistribution) principle does not lead to asymptotically optimal meshes when the solution is not "reasonable". The a-posteriori error estimators were first addressed in conjunction with elliptic equations (see [BR78], [LL83], [BW85], [BM87], [Bab86], [OD87], [EJ88], [Ewi90], [AC91], [DMR92], [AO92], [BDR92], [JPH92], [AO93], [Rod94], [Ain94]. For survey papers on the state of the art we refer to [OD87] and [Ver95].

The a-posteriori error estimators and adaptive techniques were developed also for other types of equations but we will not discuss them here. We refer, for example, to [OWA94].

Although significant progress has been made in the last 10 years, especially with respect to the error measured in the energy norm, many important questions still remain, because the aim is to have reliable and accurate estimates of *any data of interest* computed from a finite element solution. The estimators should be of high quality in the preasymptotic range and should not contain any constants which are not computable.

1.3.3 Superconvergence and Recovery Techniques

Superconvergence means that some values can be locally extracted so that higher accuracy is obtained from a finite element solution than with direct computation. This possibility exists because the finite element solution oscillates about the exact solution. Typically, one is interested in the points where the accuracy in (for example) the first derivative of the finite element solution is better by an order than in a general point.

For a survey of the state of the art in the theory we refer to [Wah95]. In [BSUG96] the theory of the computer-based proof of superconvergence was developed. It was shown that the superconvergence point in an element (if such a point exists) can be determined by a finite number of operations. As in the case of a posteriori error estimation, it is essential that the pollution error be negligible and the exact solution must be smooth.

We considered the pattern of the meshes shown in Figure 10 and analysed the superconvergence points for the derivatives of the Laplace and elasticity equations. We analysed the classical superconvergence when the set of admissible solutions is the entire set of smooth solutions. We also analysed superconvergence for the set of the solutions which satisfy the homogenous differential equation under consideration. This is obviously a very practical case because in most applications the "volume forces" are zero. Such sets are said to be "harmonic".

Let us consider for example the Poisson problem $-\Delta u = f$ for elements of degree 3 and let us be interested in the superconvergence point for the value $\partial u/\partial x$ and the regular pattern shown in Fig. 10. For general f there is only one superconvergence point while for the set of "harmonic" functions we have exactly six superconvergence points, see [BSUG96].

In the general case, especially for the elasticity equations, no superconvergence point has to exist even for "harmonic" solutions. Consider, for example, the problem of the superconvergence for the stress component σ_{12} for Poisson's ratio $\nu = 0.3$ and the chevron pattern shown in Fig. 10. In this case no superconvergence point exists. We define η % superconvergence set of points in the element, see [BSUG96], [BSGU96], [BSU95]. The points $x \in \tau$ (τ is the element) belong to the η % superconvergence set of σ_{12} if

$$100 \frac{|\sigma_{12}(x)|}{\max |\sigma_{12}(x)|_{x \in \tau}} \leq \eta(x)\% \tag{1.4}$$

for all admissible solutions whether general or the "harmonic". Further, we define

$$\eta_{\min}\% = \min \eta(x)\%, \qquad x \in \tau. \tag{1.5}$$

Obviously a superconvergence point exists only if $\eta_{\min} = 0$. In Fig. 11 we show the η%

superconvergence set for "harmonic" solutions (for σ_{12}, $\nu = 0.3$) and element degree $p = 2$. In this case $\eta_{\min}\% = 11.79$. Because $\eta_{min}\% > 0$, there is no superconvergence point.

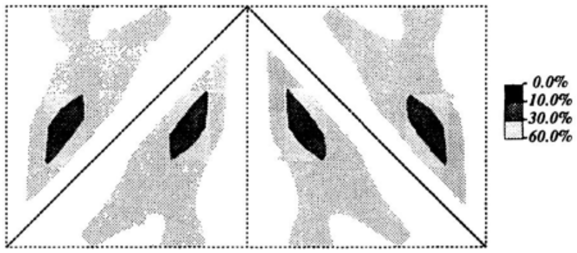

Figure 11 The $\eta\%$ superconvergence regions for the elasticity problem ($\nu = 0.3$) for σ_{12}

We have to distinguish between superconvergence in the interior elements or the elements at the boundary or on the mesh with refinement pattern, see [BSGU94]. It is also possible to recover the derivatives of the solutions or stresses by various approaches suggested in the literature. Some of those were analysed in [BSGU95].

This discussion was restricted to pointwise superconvergence. Note that L_2-type superconvergence has been analysed also, see e.g., [Zlá77] [LZ79]. It is also possible to obtain the data of interest by special techniques, called *extraction methods.* These techniques yield more accurate results than direct computations do and do not require special meshes. Nevertheless, this extraction technique is more complicated than the simple pointwise superconvergence or the recovery technique. For more see [BM84], [Sza90], [SY96a], [YS95], [SY96b].

For a survey of the results up to 1987 we refer to [KN87], [ZL89]. For some additional results we refer to [GW89], [SSW96].

1.3.4 The Pollution Problem

A posteriori error estimators and superconvergence points in the elements are defined locally utilizing only the element under consideration and possibly some of its neighbors. The quality of the estimator or $\eta\%$ superconvergence was defined elementwise. This definition has meaning only if the pollution in the given element is negligible. Pollution describes the effects of errors originating outside of the element under consideration on the error within the element. More precisely, denoting the error in the element τ by V, we can write

$$V = V_1 + V_2 \tag{1.6}$$

where V_1 is the error governed by the finite element solution in the element under consideration and its small neighborhood and V_2 is the effect of the errors outside of the element.

For example, consider the problem of solving the Laplace or the elasticity equations on $\Omega \in R^2$, where Ω has a boundary with reentrant corners. Assuming that the mesh is quasiuniform, the pollution error is dominant [BSMU94], [BSUG95]. Even in the presence of reentrant corners elements of high polynomial degree will be subject to pollution because Green's function is not smooth in the reentrant corners of Ω.

The pollution is said to be negligible if $||V_2||/||V_1|| << 1$. If the pollution error is dominant then the a-posteriori error estimation, recovered values, superconvergence, etc., are not reliable because $||V_1|| << ||V_2||$. For example, the (elemental) effectivity index of the error indicator (estimator) could be of order 0.1 or even 0.01 in practical cases.

It is virtually impossible to estimate the size of the pollution error a priori. Therefore an a posteriori estimator of pollution error has to be used. The estimator is necessarily global, nevertheless it has to be computable efficiently. Such an estimator was proposed and analysed in [BSUG95], [BSGU96], [BSG96]. Such estimate can then be used in an adaptive procedure which leads to prescribed accuracy in the data of interest (for example, the stress in the single element) even if the error in the energy norm is large.

Consider, for example, an L-shaped domain with a crack. In Fig. 12 we show a sequence of meshes adaptively constructed to increase the accuracy only in the single (shaded) element in a given patch of elements. Observe that the refinement changes substantially in the neighborhoods of the corner and tip of the crack as the accuracy is increased. For details we refer to [BSG96].

Understanding pollution errors, and having a capability for assessing their magnitude, are of major practical importance because in practice refinements far from the area of interest are generally neglected. The so-called "global-local approach" is of this type when the refined mesh is used only in the area of interest. See also [Ode95].

Although the basic principles of pollution estimates and control are available, further theoretical and practical investigations are needed.

1.3.5 The Partition of Unity Method

A frequently asked question is: "why are polynomial basis functions used in the h-, p- and hp-versions of the finite element method?". There are a few reasons for this:

a) polynomials approximate *smooth* functions well;

b) monomials are homogenous with respect to scaling;

c) polynomials have certain advantages from the point of view of implementation.

Nevertheless, if the exact solution is highly irregular then the polynomials *do not* approximate it well. For example, consider the problem

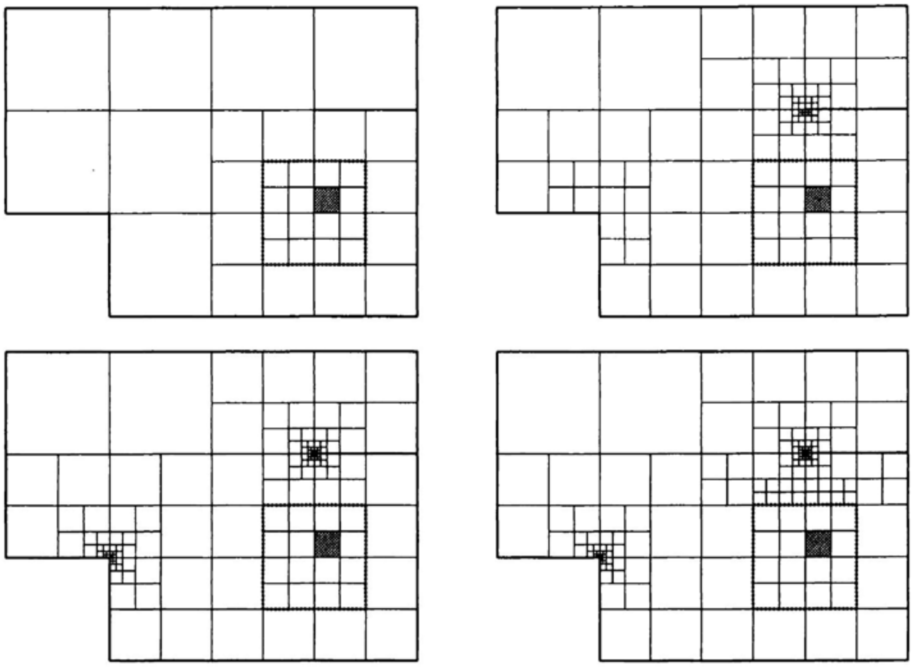

Figure 12 Meshes constructed adaptively with the objective to increase the accuracy in the single (shaded) element.

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = f(x) \qquad 0 < x < 1, \qquad u(0) = u(1) = 0 \tag{1.7}$$

with $0 < \alpha_1 \leq a(x) \leq \alpha_2$, but only measurable, then in [BO] we have shown that the finite element method using polynomial functions converges arbitrarily slowly. More precisely, given $\psi(N) \to 0$, $N = 1, 2, \ldots$, it is possible to find $a(x)$, $0 < \alpha_1 \leq a(x) \leq \alpha_2$ corresponding to ψ such that

$$\lim_{N\to\infty} \sup \|u - u_{FE}\|_{H^\ell}/\psi(N) \geq C > 0, \qquad \ell = 0, 1. \tag{1.8}$$

Here u_{FE} is the finite element solution and N is number of degrees of freedom. Furthermore, adaptive selection of the mesh affects only the constant C in eq. 1.8.

In higher dimensions the finite element method based on polynomial basis functions can perform poorly also. The reason is that coefficients can be selected so that the solution is very irregular, more precisely, the solution is not in any $H^\alpha(\Omega^\star)$, $\alpha > 1$ in any $\Omega^\star \subset \Omega$. Analogous difficulties occur when solving the Helmholtz equation with high wave number because the solution is highly oscillatory.

Consider solving the Laplace equation $\Delta w = 0$. In this case the harmonic polynomials have the same approximation properties as the set of all polynomials. Use of the set of all polynomials is necessary in the standard FEM because otherwise the requirement of continuity could not be satisfied. Therefore two separate questions have to be addressed in selecting the basis functions:

a) What set of basis functions approximate the solution well?

b) How to enforce continuity?

In [BCO94], [Mel95], [BM96], it was shown that by the method of partition of unity conforming shape functions can be constructed from local nonconforming ones. In addition, if the solution can be approximated well by the local (nonconforming) shape functions then the conforming shape functions constructed by the partition of unity method approximate the solution with the same accuracy. Therefore the method gives freedom in selecting the local shape functions. In [BCO94] it was shown that in two dimensions the rate of convergence $0(N^{-1/2})$ can be achieved even if the coefficients are arbitrarily rough, provided that the proper shape functions are used.

In [Mel95], [BM96] it was shown that when solving the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1.9}$$

with a high wave number, it is very advantageous to use as the (local) basis functions the generalized harmonic functions of Bergman and Vekua, or a set of plane wave solutions of eq. 1.9. Note that these shape functions satisfy the differential equation eq. 1.9. If the right hand side of eq. 1.9 is not zero, additional shape functions are necessary. In [Mel95], [BM96] the a priori estimates were given and numerical examples showing effectiveness of this method were presented.

Because of its flexibility, this method has a high potential. The partition of unity method has some commonality with *the mesh-free finite element method* [BKO+96] and the "cloud method" described in [OD95].

1.3.6 The Problem of Thin Domains

Models of homogeneous or laminated plates and shells are understood as approximate solutions of three-dimensional problems, which have to be developed adaptively based on an a-posteriori error estimation. Each model is, of course, solved numerically and hence there is an interplay between the error in the model and the error of the approximate solution of the model.

The adaptive selection of the model must account for singularities and boundary layer effects associated with the model.

This approach is different from both the classical modeling based on physical derivation or asymptotic approaches. See [GR91], [Amb91], [MNP91], [Cia92]. Adaptive selection of models is based on the dimensional reduction method (see, for example, [VB80], [SS88], [Sza89], [Sch89], [SA92], [BSA92], [BL92], [BLS94], [BS96]) which establishes a theoretical basis and practical means for the construction of hierarchic models.

The hierarchic dimensional reduction method for homogeneous plates and shells is closely related to the hp-version of FEM where the model is characterized by the use of the polynomial shape functions through the thickness. The p-version is well suited for the treatment of plates and shells because it is very robust with respect to locking. For more information we refer to [ZD94], [HLP96], [OC96], [CO96].

Hierarchic modeling, as defined above, is a methodology for obtaining approximate solutions for three-dimensional problems. It is assumed that this three-dimensional problem is exactly formulated and the aim is to find its solution accurately. A major difficulty is that large uncertainties may have to be incorporated in the formulation. Consider, for example, zero displacement prescribed on the boundary of an elastic body. This is a crude idealization, especially with respect to any data of interest in the neighborhood of the boundaries. Fixed (built-in) boundaries should be modeled more precisely, nevertheless some uncertainty is usually unavoidable. Therefore quantitative assessment of the effects of uncertainties is an important aspect of computation. Theoretically and practically justified approaches are needed. We also refer to the Girkmann-Timoshenko problem discussed in Section 1.2.6 which exemplifies the high degree of sensitivity of plate and shell-like bodies to various boundary conditions.

1.3.7 Summary

We have discussed a few directions of theoretical research on the finite element method, which, in our opinion, are important for the practical uses of FEM and therefore new results are desirable. The common focus of the research topics discussed herein is reliability of the computed data. An essential prerequisite of reliability in engineering computations is that the mathematical problem must be properly formulated.

1.4 Conclusions

The authors' objective in writing this paper was to identify some trends in engineering computation and their relation to research in FEM. It is obvious that successful use of FEM depends on the reliability of the formulation; the reliability of the parameters used; the reliability of the numerical solution, and proper interpretation of the results. Failure to ensure reliability in either one of these areas can have disastrous consequences. The Sleipner disaster was mentioned as an example. In many cases the consequences are less obvious: Lack of confidence in the reliability of computed information causes engineering managers to undertake expensive and time-consuming tests, or bypass finite element analysis entirely.

Only a few aspects of the finite element method could be discussed within the framework of this paper. Many relevant references were cited but certainly not all. Additional details can be found in these references.

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