On High Order Finite Element Method for Plasticity Problems

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Abstract

The quasistatic elastoplasticity problems with the constitutive law based on the so-called gauge function method are formulated as initial-boundary-value problems for both one dimensional and two dimensional cases. Some semi-discretization schemes using high order finite element method are presented. Numerical results with uniform mesh for one dimensional problem indicate that higher order finite element method proposed here produce higher convergence rate. However, our analysis also shows that to fully take the advantage of the high order finite element method, an adaptive scheme must be adopted because of the lack of regularities of the exact solution.

1 Introduction

Numerical treatment of cyclic plasticity problem has many intrinsic difficulties. The constitutive law plays an essential role in the computation of cyclic plasticity problem. It should be closely related to experimental data, and simultaneously it should guarantee that the mathematical model based on it is well posed, i.e., the existence, uniqueness of the solution is guaranteed as well as the continuous dependence on on the input data. Further, the numerical treatment should lead to accurate results and the method should be efficient.

In this paper we define a large class of constitutive laws which has desired properties. The commonly used constitutive laws such as linear kinematic and linear isotropic laws belong to this class. The paper elaborates on h-version of the finite element method with higher order elements, and on the p-version method also. The convergence of the finite element solutions has been proven theoretically in [16] [17].

In this paper we are focusing on the problem of the efficiency of higher order elements. For simplicity we restrict ourself to one dimensional problems. We show that the method with degree $p = 2$ is more efficient than degree $p = 1$, but increasing further the degree is not efficient (with respect to the computational work) if the procedure is not adaptive. We comment briefly on the problem of adaptive procedure which we will elaborate in the forthcoming paper [18].

The outline of the remainder of this paper is as follows. In Section 2 the mathematical model of the quasistatic elastoplasticity problem in both one and two dimensional is described. The class of constitutive laws based on the so called gauge function method is introduced in Section 3, where conditions for the well-posedness are also discussed. Section 4 is concerned with the semi-discrete finite element approximations with higher order elements, while Section 5 is devoted to the numerical examples.

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to show the efficiency of higher order elements. Finally in Section 6 we briefly comment on the problem of adaptive procedure and some issues related to the time discretization.

2 Mathematical formulation of the problem

Let $\tilde{\Omega} = [0, l]$ be the reference configuration of an elastoplastic bar with fixed two ends, and $f(x, t)$ be the body loading force acting on the bar. Denote by $\sigma(x, t)$ and $\epsilon(x, t)$ the stress and strain distribution functions of the bar, where $\epsilon(x, t) = \frac{\partial U}{\partial x}(x, t)$, and $U(x, t)$ here is the displacement function of the bar. Then the mathematical formulation of this elastoplastic bar problem can be written as follows.

To find the stress and displacement functions $\sigma(x, t)$ and $U(x, t)$, such that the stress distribution function satisfies the following equilibrium equation,

$$-\frac{\partial \sigma}{\partial x}(x, t) = f(x, t) \quad \text{for all } x \in \Omega \text{ and } t \in [0, T],$$

where $[0, T]$ is the time interval of interest. In the equilibrium equation (2.1), the relation between the stress and strain functions is defined by the following equation

$$\sigma(x, t) = \mathcal{A}(\epsilon)(x, t),$$

where $\mathcal{A}$ is an operator mapping the strain distribution function $\epsilon(x, t)$ into the stress distribution function $\sigma(x, t)$. The plastic deformation, especially cyclic plastic deformation is history dependent. The constitutive operator $\mathcal{A}$ in (2.2) depends not only on the current value of the strain function but also on the previous history of the strain, i.e. the current value of the stress distribution function $\sigma(x, t)$ depends on all the values of the strain distribution function $\epsilon(x, t)$ for all $t \leq t$. Here we assume that the operator $\mathcal{A}$ is rate independent, or the so-called hysteresis operator (see [4], for details).

From the fact that two ends of the bar are fixed, we have the following boundary conditions for the displacement function,

$$U(0, t) = U(l, t) = 0 \quad \text{for all } t \in [0, T].$$

For the sake of simplicity, we assume that both the stress and displacement functions satisfy the homogeneous initial conditions,

$$\sigma(x, 0) = 0, \quad U(x, 0) = 0 \quad \text{for all } x \in \Omega.$$
Of course we require here that the loading force satisfies the compatibility condition \( f(x, 0) = 0 \) for all \( x \in \Omega \).

For two dimensional problems, let's consider an elastoplastic body which occupies a bounded domain \( \Omega \) in \( \mathbb{R}^2 \) with Lipschitz boundary \( \Gamma \). We denote by \( U = (U_1, U_2) \) the displacement field of the body where \( U_1(x_1, x_2), U_1(x_1, x_2) \) are the displacement at the point \((x_1, x_2)\) in the \( x_1 \) and \( x_2 \) directions respectively. Let \( \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T \) be the stress vector, where \( \sigma_{11} \) and \( \sigma_{22} \) are the normal stress in the \( x_1 \) and \( x_2 \) directions, and \( \sigma_{12} \) is the shear stress. We use \( \epsilon = (\epsilon_{11}, \epsilon_{22}, \epsilon_{12})^T \) to denote the strain vector, where the normal strains \( \epsilon_{11}, \epsilon_{22} \) and the shear strain \( \epsilon_{12} \) are defined in the engineering notation:

\[
\begin{align*}
\epsilon_{11} &= \frac{\partial U_1}{\partial x_1}, & \epsilon_{22} &= \frac{\partial U_2}{\partial x_2}, & \epsilon_{12} &= \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}.
\end{align*}
\]

Again let \( I = [0, T] \) be the time interval of interest, and \( f = (f_1, f_2)^T \) be the body force acting on the elastoplastic body \( \Omega \). Then two dimensional elastoplastic problems can be formulated as follows.

To find the displacement field function \( U(x_1, x_2, t) \) and the stress vector function \( \sigma(x_1, x_2, t) \), such that \( \sigma(x_1, x_2, t) \) satisfies the following equilibrium equations:

\[
\begin{cases}
-\frac{\partial \sigma_{11}}{\partial x_1}(x_1, x_2, t) - \frac{\partial \sigma_{12}}{\partial x_2}(x_1, x_2, t) = f_1(x_1, x_2, t) \\
-\frac{\partial \sigma_{21}}{\partial x_1}(x_1, x_2, t) - \frac{\partial \sigma_{22}}{\partial x_2}(x_1, x_2, t) = f_2(x_1, x_2, t)
\end{cases}
\quad \text{for all } (x_1, x_2) \in \Omega \text{ and } t \in [0, T],
\]

subject to the homogeneous boundary conditions:

\[
U_1(x_1, x_2, t) = U_2(x_1, x_2, t) = 0 \quad \text{for all } (x_1, x_2) \in \Gamma \text{ and } t \in [0, T],
\]

and the homogeneous initial condition:

\[
\sigma(x_1, x_2, 0) = 0, \quad \text{and} \quad U(x_1, x_2, 0) = 0 \quad \text{for all } (x_1, x_2) \in \Omega.
\]

Again in equation (2.6), the stress vector function \( \sigma(x_1, x_2, t) \) satisfies the constitutive equation:

\[
\sigma(x_1, x_2, t) = A(\epsilon)(x_1, x_2, t),
\]

with the constitutive operator \( A \) depending on the strain vector function \( \epsilon(x_1, x_2, s) \) for all \( s \leq t \).
As we have seen so far, here we are studying the quasi-static behavior of the elastoplastic bar. The quasi-static nature of the problem is due to the fact that because of the plastic deformation, the constitutive relation can only be correctly described in terms of the rates of change of certain variable (such as the strain, as we will see later); thus these contribute to the presence of rate quantities, and the problem we get is an initial-boundary-value problem instead of merely a boundary-value problem. On the other hand, the processes of change are assumed to occur sufficiently slowly so that the inertia effects may be ignored. Therefore acceleration term does not appear in the equilibrium equation. As an approximation, the quasi-static elastoplastic problem has been confirmed as an important special case by the large number of papers both from the mathematical point of view and the practical point of view.

3 Existence and uniqueness of the solution of the problem

Obviously mathematical properties of the initial-boundary-value problem (2.1) ~ (2.4) (or (2.6) ~ (2.9) in 2-D cases), such as the existence and uniqueness of the solution, depend on the mathematical formulation of the constitutive operator $A$. Under various assumptions on the the constitutive operator $A$, a large amount of literature addressing this problem exists (cf. e.g. [5] [9] [10] [13] [14] [15] [16] [17] [19]).

In this section we will present the formulation of the constitutive operator based on the so-called gauge function method which is first proposed by Bonnetier [5], and later used in our paper [16] & [17]. A theoretical result about the existence and uniqueness of the solution to the continuous problem will also be presented.

Basically we assume that the elastic set of the material can be described as a convex set in the product space of the stress variables and the so-called internal parameter variables. More precisely, for 1-D problems, we assume that there exists a yield surface function or the so-called gauge function $F(\sigma, \alpha)$ of the convex set, where $\sigma \in \mathbb{R}^1$ and $\alpha \in \mathcal{U} \subset \mathbb{R}^m$, and $\mathcal{U}$ is a convex set in $\mathbb{R}^m$, such that $F(\sigma, \alpha)$ satisfies the following three assumptions

A1 ) $F(\sigma, \alpha)$ is a convex, continuous and piecewise analytic function of the variables $\sigma$ and $\alpha$.

A2 ) $F(0, 0) = 0$.

A3 ) There exists constants $\Gamma, \gamma > 0$ such that

$$\gamma \leq \left| \frac{\partial F}{\partial \sigma} \right|, \quad \left| \frac{\partial F}{\partial \alpha} \right| \leq \Gamma$$

uniformly on the set,

$$\left\{ (\sigma, \alpha) \in \mathbb{R}^1 \times \mathcal{U} \mid F(\sigma, \alpha) = Z_0 \right\}$$

for some $Z_0 > 0$.

Remark 3.1 The definition of the gauge function here is not the conventional definition of a gauge function (see [20]), where the function is assumed to be non-negative and positively homogeneous.

Using the gauge function, we can easily define elastic set and plastic set. A point $(\sigma, \alpha)$ is said to be in an elastic set if it belongs to
\[ \mathcal{E} = \left\{ (\sigma, \alpha) \in \mathbb{R}^1 \times \mathcal{U} \left| \begin{array}{l} F(\sigma, \alpha) < Z_0 \quad \text{or} \\ F(\sigma, \alpha) = Z_0 \quad \text{and} \quad \frac{\partial F}{\partial \sigma}(\sigma, \alpha) \dot{\sigma} \leq 0 \end{array} \right. \right\}. \]

In this case the constitutive equation is defined by
\[
\begin{cases}
\dot{\sigma} &= E \dot{\varepsilon}, \\
\dot{\alpha} &= 0,
\end{cases}
\]
the overdots here are the derivatives with respect to the time \( t \), and \( E \) is the Young's modulus of the material.

Similarly a point \((\sigma, \alpha)\) is said to be in a plastic set if it belongs to
\[
\mathcal{P} = \left\{ (\sigma, \alpha) \in \mathbb{R}^1 \times \mathcal{U} \left| \begin{array}{l} F(\sigma, \alpha) = Z_0 \quad \text{and} \quad \frac{\partial F}{\partial \sigma}(\sigma, \alpha) \dot{\sigma} > 0 \end{array} \right. \right\}.
\]

From the normality condition, we can derive the following constitutive equation,
\[
\begin{cases}
\dot{\sigma} &= \frac{E}{\frac{\partial F}{\partial \alpha} \cdot \frac{\partial F}{\partial \alpha}} \frac{\partial F}{\partial \dot{\varepsilon}} + \frac{E}{\left( \frac{\partial F}{\partial \sigma} \right)^2} \frac{\partial F}{\partial \alpha} \\
\dot{\alpha} &= -\frac{E}{\frac{\partial F}{\partial \dot{\varepsilon}} \cdot \frac{\partial F}{\partial \alpha} + \frac{E}{\left( \frac{\partial F}{\partial \sigma} \right)^2}} \frac{\partial F}{\partial \alpha} 
\end{cases}
\]

(3.2)

For two dimensional problems, the gauge function is given by a function \( F(\sigma, \alpha) \) which satisfies all three assumptions A1)~A3). The elastic and plastic sets are defined in the similar way, and the constitutive equations are defined by
\[
\begin{cases}
\dot{\sigma} &= D \dot{\varepsilon}, \\
\dot{\alpha} &= 0,
\end{cases}
\]
for elastic deformation, and
\[
\begin{cases}
\dot{\sigma} &= (D - \frac{D \frac{\partial F}{\partial \alpha} \frac{\partial F^T}{\partial \alpha} D}{\frac{\partial F}{\partial \alpha} \cdot \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \sigma} \cdot D \frac{\partial F}{\partial \sigma}}) \dot{\varepsilon}, \\
\dot{\alpha} &= -\frac{\partial F}{\partial \dot{\alpha}} \cdot \dot{\varepsilon} + \frac{\partial F}{\partial \sigma} \cdot D \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \alpha} 
\end{cases}
\]

(3.4)
for plastic deformation, where $D$ is a symmetric, positive definite matrix. In plane stress problems, for instance, $D$ has the form

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix},$$

where $\nu$ is the Poisson's ratio of the material.

If the constitutive operator $A$ is given by (3.1) and (3.2) (or (3.3) and (3.4), for two dimensional problems), and if the gauge function $F(\sigma, \alpha)$ satisfies all the three assumptions $A1 \sim A3$, we will then say that the constitutive operator $A$ is admissible. For the admissible constitutive operator, we have the following existence and uniqueness result (for details of the proof, see [5], [16] or [17]).

**Theorem 3.1** For the initial-boundary-value problem (2.1)~(2.4), if the constitutive operator $A$ is admissible, and if the loading force function $f(x,t)$ satisfies the following regularity conditions: for every fixed $t \in [0,T]$, $f(\cdot,t) \in L^2(\Omega)$, and for every fixed $x \in \Omega$ $f(x,\cdot)$ is continuous and has uniformly bounded right derivatives of any order with respect to time $t$. Then the problem must have a unique solution $(\sigma, U)$ with $U(x,t), \dot{U}(x,t) \in L^\infty([0,T],H^1(\Omega))$ and $\sigma(x,t), \dot{\sigma}(x,t) \in L^\infty([0,T],L^2(\Omega))$.

Admissible constitutive operators are actually generalizations of some most commonly used engineering constitutive models. For example, the commonly used linear kinematic or isotropic hardening laws are the special cases of the admissible constitutive operators, where the gauge function is of the form

$$F(\sigma, \alpha) = |\sigma| - \sqrt{\frac{E E_p}{E - E_p}} \alpha$$

or

$$F(\sigma, \alpha) = |\sigma| - \sqrt{\frac{E E_p}{E - E_p}} \alpha$$

respectively, here $E_p$ is the plasticity modulus of the material. Thus for the elastoplastic bar problem (2.1)~(2.4), if the constitutive operator takes the form of linear kinematic law, or the linear isotropic law, from theorem 3.1, we know that this problem has a unique solution $(\sigma, U)$ with $U \in L^\infty([0,T],H^1(\Omega))$ and $\sigma \in L^\infty([0,T],L^2(\Omega))$, under the assumption that the body loading function satisfies the required regularity conditions.

**Remark 3.2** As we have indicated here that the mathematical properties such as the existence and uniqueness of the solution to the continuous problem is closely related to the mathematical formulation of the constitutive law. On the other hand, the reliability of the computed results depends not only on the mathematical formulation of the constitutive law, but also on the numerical treatment of parameter values involved in a constitutive law. For more details about the reliability analysis of various constitutive laws, we refer to [1] and [2].
4 Semi-discretization scheme using high order finite element spaces

Various finite element methods for elastoplastic problems have been proposed in much of the engineering and mathematical literature (see, for example, [3] [11]-[12]). Based on the different assumptions, some theoretical results on the convergence of the finite element solutions were presented in the work of Han [10], Johnson [13]-[15], and Miyoshi [19]. Some convergence rate of fully discretized scheme were also presented in some of the work mentioned above under the extra assumption that the exact solution satisfies some regularity conditions. Unfortunately, only few result about the regularity of the exact solution for one dimensional problems is available, and there is no any theoretical results available about the regularity of the exact solution for two dimensional problems. With the gauge function method in particular, a convergence analysis of the finite element solutions was first presented by Bonnetier [5] for low order finite element spaces, and then in our papers [16] [17] for some high order finite element spaces. In this section, we will present high order discretization schemes we used for both one and two dimensional problems.

For one dimensional problems, we assume that the interval $\Omega = (0, l)$ is partitioned into several subintervals and denote by $\Delta$, the corresponding mesh:

$$\Delta : \quad 0 = x_1 < x_2 < \cdots < x_{M(\Delta)} < x_{M(\Delta)+1} = l,$$

(4.1)

where $M(\Delta)$ is the number of subintervals associated with the partition. The $k$th element is denoted by $\Omega_k$:

$$\Omega_k = \{ x \mid x_k < x < x_{k+1} \} \quad k = 1, 2, \ldots, M(\Delta).$$

Each subinterval is mapped by a function $Q_k^{-1}(x)$ into the standard element:

$$\Omega_{st} = \{ \xi \mid -1 < \xi < 1 \}.$$

Denote by $S^p$ the space of polynomials of degree $p$ defined on $\Omega_{st}$. Then the finite element space $S^p_0(\Omega, \Delta, Q)$ is the set of all functions $u(x)$ which lie in the space $H^1_0(\Omega)$ and on the $k$th element $u(Q_k(\xi)) \in S^{p_k}$, i.e.,

$$S^p_0(\Omega, \Delta, Q) = \left\{ u \mid u \in H^1_0(\Omega), \quad u(Q_k(\xi)) \in S^{p_k}, \quad k = 1, 2, \ldots, M(\Delta) \right\},$$

where $p$ is the vector of polynomial degrees assigned to the elements:

$$p = \left\{ p_1, p_2, \ldots, p_{M(\Delta)} \right\},$$

(4.2)

and $Q$ is the vector of mapping functions assigned to the elements:

$$Q = \left\{ Q_1, Q_2, \ldots, Q_{M(\Delta)} \right\}.$$

(4.3)

On the standard element $\Omega_{st}$, the following basis functions will be used for $S^p$:

$$N_1(\xi) = \frac{1-\xi}{2}; \quad N_2(\xi) = \frac{1+\xi}{2}; \quad N_i(\xi) = \psi_{i-1}(\xi), \quad i = 3, 4, \ldots, p+1.$$
where $\psi_j$ is defined in terms of the Legendre polynomials $P_{j-1}$:

$$\psi_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(t) \, dt, \quad j = 1, 2, \ldots$$  \hspace{1cm} (4.4)

Corresponding to the space $S^0_n(\Omega, \Delta, Q)$, we can define another finite element space $\tilde{S}^{p-1}(\Omega, \Delta, Q)$ such that

$$\tilde{S}^{p-1}(\Omega, \Delta, Q) = \left\{ u \mid u \in L^2(\Omega), \quad u(Q_k(\xi)) \in \tilde{S}^{p_k-1}, \quad k = 1, 2, \ldots, M(\Delta) \right\},$$

where $\Delta, p, Q$ are defined as in (4.1), (4.2) and (4.3). However on the standard element $\Omega_{st}$, the following basis functions will be used for $\tilde{S}^p$:

$$\tilde{N}_1(\xi) = 1; \quad \tilde{N}_i(\xi) = \psi_{i-1}(\xi), \quad i = 2, 3, \ldots, p + 1.$$  

From the definition of the spaces $S^0_n(\Omega, \Delta, Q)$ and $\tilde{S}^{p-1}(\Omega, \Delta, Q)$ we can see that the space $S^0_n(\Omega, \Delta, Q)$ is a set of continuous piecewise polynomial functions with the total degree of freedom $N = \sum_{i=1}^{M(\Delta)} p_i - 1$, but the space $\tilde{S}^{p-1}(\Omega, \Delta, Q)$ is a set of piecewise polynomial functions not necessarily continuous over $\Omega$ with the total degree of freedom $\tilde{N} = \sum_{i=1}^{M(\Delta)} p_i$.

Now let $\{ \hat{\phi}_j(x) \}_{j=1}^N$ (resp. $\{ \tilde{\phi}_j(x) \}_{j=1}^N$) be the set of the basis function of the space $S^0_n(\Omega, \Delta, Q)$ (resp. $\tilde{S}^{p-1}(\Omega, \Delta, Q)$). Then the semi-discrete approximation of the continuous problem can be formulated as follows:

To seek the solution $U(x, t), \sigma(x, t)$ and $\alpha(x, t)$ in the form

$$U(x, t) = \sum_{n=1}^{N} U_n(t) \phi_n(x) \quad \text{with} \quad U_n(t) \in C^0_{\infty}(0, T),$$  

$$\sigma(x, t) = \sum_{n=1}^{N} \sigma_n(t) \phi_n(x) \quad \text{with} \quad \sigma_n(t) \in C^0_{\infty}(0, T),$$  \hspace{1cm} (4.5)  

$$\alpha(x, t) = \sum_{n=1}^{N} \alpha_n(t) \tilde{\phi}_n(x) \quad \text{with} \quad \alpha_n(t) \in C^0_{\infty}(0, T),$$

such that the stress function function satisfies the discretized weak form of the equilibrium equation:

$$\int_0^l \sigma \frac{d\phi_n}{dx} \, dx = \int_0^l f \phi_n \, dx \quad \forall t \in [0, T] \quad \text{and} \quad 1 \leq n \leq N,$$  \hspace{1cm} (4.6)

subject to the homogeneous initial conditions

$$U(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \alpha(x, 0) = 0.$$  \hspace{1cm} (4.7)

Here we assume that the body loading function $f(x, t)$ is piecewise analytic in time $t$, and for each $t \in (0, T), f(\cdot, t) \in L^2(\Omega)$.  

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Notice that on each element $\Omega_j$, the stress and internal parameter functions $\sigma(x, t)$, $\alpha(x, t)$ are polynomials of degree $p_j - 1$. Therefore, these functions are uniquely determined by their values at the $p_j$ Gaussian points of the element. Using this fact, we can then define the constitutive equations as

$$
\begin{cases}
\sigma(\hat{x}_k) = E \dot{\epsilon}(\hat{x}_k) & \text{if } F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) < Z_0 \\
\alpha(\hat{x}_k) = 0 & \text{if } \partial_\sigma F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \leq 0
\end{cases}
$$

(4.8)

$$
\begin{cases}
\sigma(\hat{x}_k) = (E - E')(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \dot{\epsilon}(\hat{x}_k) & \text{if } F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) = Z_0 \\
\alpha(\hat{x}_k) = -\frac{\partial_\sigma F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \dot{\sigma}(\hat{x}_k)}{(\partial_\alpha F \cdot \partial_\alpha F + E (\partial_\sigma F)^2)(\sigma(\hat{x}_k), \alpha(\hat{x}_k))} \partial_\sigma F(\hat{x}_k) & \text{if } \partial_\sigma F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \dot{\sigma}(\hat{x}_k) > 0
\end{cases}
$$

(4.9)

where $\hat{x}_k$ $(k = 1, 2, \ldots, p_j)$ are the Gaussian points of each subinterval $\Omega_j$ and

$$
E'(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) = \frac{E^2 (\partial_\alpha F(\sigma(\hat{x}_k), \alpha(\hat{x}_k))^2}{\partial_\beta F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \cdot \partial_\beta F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) + E (\partial_\sigma F(\sigma(\hat{x}_k), \alpha(\hat{x}_k))^2}
$$

(4.10)

For the discretized problem (4.6)~(4.10), we have the following theorem (for the details of the proof, cf. [16])

**Theorem 4.1** If the loading force $f(x, t)$ is piecewise analytic in time $t$, and for each $t \in (0, T)$, $f(\cdot, t) \in L^2(\Omega)$, and if the gauge function $F(\sigma, \alpha)$ satisfies all the three assumptions A1)~A3), the semi-discretized problem (4.6)~(4.10) has a unique solution $(\sigma_{FE}, U_{BF})$. Moreover, as the mesh size $h \to 0$, the sequence of the finite element solution, $\{ \sigma_{FE} \}$ and $\{ U_{FE} \}$, will converge to the solution of the continuous problem weakly in spaces $L^\infty([0, T], L^2(\Omega))$ and $L^\infty([0, T], H^1(\Omega))$.

In particular, if the constitutive law is linear kinematic or linear isotropic or the mixture of linear kinematic and isotropic, we have the following $p$-version convergence result.

**Theorem 4.2** If the loading force $f(x, t)$ has the same regularity as stated in Theorem 4.1, and if the constitutive equation is linear kinematic or linear isotropic or the mixture of linear kinematic and isotropic. For properly refined fixed mesh, the finite element solutions, $\{ \sigma_{FE} \}$ and $\{ U_{FE} \}$, will converge to the solution of the continuous problem weakly in spaces $L^\infty([0, T], L^2(\Omega))$ and $L^\infty([0, T], H^1(\Omega))$, as the polynomial degree $p_j$ of each element goes to infinity.

**Remark 4.1** Using the compact embedding theorem and uniqueness of the solution of the continuous problem, we can easily see that the finite element solution $U_{FE}$ will converge strongly to the solution of the continuous problem in the space $L^\infty([0, T], L^2(\Omega))$.

For two dimensional problems, we first assume that the domain $\Omega \subset \mathbb{R}^2$ can be partitioned into rectangles $\{ T_n \}$ of size $h$ and the aspect ratios satisfy $\alpha_0 < \alpha < \alpha_1$ where $\alpha_0$ and $\alpha_1$ are independent of the element size $h$. Let $T_h$ be such a partition. Denote by $Q_p(x_1, x_2)$ the set of polynomials in $x_1$ and $x_2$ of the form
We can then define a finite element space, \( Q_{p,0}(\Omega_h) \), the set of continuous piecewise \( Q_p \) polynomials defined over \( \Omega \), and vanishing on the boundary \( \partial \Omega \). i.e.

\[
Q_{p,0}(\Omega_h) = \left\{ h(x_1, x_2) \in H^1_0(\Omega) \mid h(x_1, x_2) \big|_{T_n} \in Q_p(x_1, x_2), \quad \forall T_n \in T_h \right\}.
\]

Meanwhile denote by \( \tilde{Q}_p(\Omega_h) \) the set of all piecewise \( Q_p \) polynomials defined over \( \Omega \), i.e.,

\[
\tilde{Q}_p(\Omega_h) = \left\{ \tilde{h}(x_1, x_2) \in L^2(\Omega) \mid \tilde{h}(x_1, x_2) \big|_{T_n} \in Q_p(x_1, x_2), \quad \forall T_n \in T_h \right\}.
\]

Notice that the functions in \( \tilde{Q}_p(\Omega_h) \) are not necessarily continuous over the domain \( \Omega \). Similar to the one dimensional case, any function in \( \tilde{Q}_p(\Omega_h) \) are uniquely determined by its values at the \( p+1 \) Gaussian points of each element. Now let \( \mathcal{N} \) (resp. \( \tilde{\mathcal{N}} \)) be the total degrees of freedom of the finite element space \( Q_{p,0}(\Omega_h) \) (resp. \( \tilde{Q}_p(\Omega_h) \)), and \( \{ \phi_n \}_{n \in \mathcal{N}} \) (resp. \( \{ \tilde{\phi}_n \}_{n \in \tilde{\mathcal{N}}} \)), be the piecewise \( Q_p \) polynomial basis functions in the space \( Q_{p,0}(\Omega_h) \) (resp. \( \tilde{Q}_p(\Omega_h) \)).

Now we can formulate two dimensional semi-discrete problems as follows. To simplify our notation, from now on, we also use \( x \) to stand for \( (x_1, x_2) \). To seek the solution \((U, \sigma, \alpha)\), such that for any fixed \( t, U = (U_1, U_2)^T \in (Q_{p,0}(\Omega_h))^2, \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T \in (\tilde{Q}_p(\Omega_h))^3 \) and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T \in (\tilde{Q}_p(\Omega_h))^m \), i.e., \((U, \sigma, \alpha)\) of the form

\[
U_i = \sum_{n \in \mathcal{N}} U_{i,n}(t) \phi_n(x) \quad \text{with} \quad U_{i,n} \in C^{0,\infty}_+(0, T) \quad i = 1, 2
\]

\[
\sigma_{ij} = \sum_{n \in \tilde{\mathcal{N}}} \sigma_{ij,n}(t) \tilde{\phi}_n(x) \quad \text{with} \quad \sigma_{ij,n} \in C^{0,\infty}_+(0, T) \quad i, j = 1, 2
\]

\[
\alpha_i = \sum_{n \in \tilde{\mathcal{N}}} \alpha_{i,n}(t) \tilde{\phi}_n(x) \quad \text{with} \quad \alpha_{i,n} \in C^{0,\infty}_+(0, T) \quad i = 1, \ldots, m,
\]

such that the stress function satisfies the discretized weak form of the equilibrium equation:

\[
\begin{align*}
\int_{\Omega} \left[ \sigma_{11} \frac{\partial \phi_n}{\partial x_1} + \sigma_{12} \frac{\partial \phi_n}{\partial x_2} \right] dx &= \int_{\Omega} f_1 \phi_n dx & \forall \ t \in [0, T] \text{ and } n \in \mathcal{N}, \\
\int_{\Omega} \left[ \sigma_{21} \frac{\partial \phi_n}{\partial x_1} + \sigma_{22} \frac{\partial \phi_n}{\partial x_2} \right] dx &= \int_{\Omega} f_2 \phi_n dx
\end{align*}
\]

with the homogeneous initial conditions:

\[
U(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \alpha(x, 0) = 0.
\]

Of course here we assume that the body force functions satisfy the compatibility conditions:

\[
f_1(x, 0) = f_2(x, 0) = 0.
\]

In (4.11), the stress function \( \sigma \) is determined by the following constitutive equations:
\[
\begin{align*}
\dot{\sigma}(\hat{x}_k) &= D \dot{\epsilon}(\hat{x}_k) \\
\alpha(\hat{x}_k) &= 0
\end{align*}
\]

and

\[
\begin{align*}
\dot{\sigma}(\hat{x}_k) &= (D - D') (\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \dot{\epsilon}(\hat{x}_k) \\
\alpha(\hat{x}_k) &= \tilde{D} (\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \partial_\alpha F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \\
& \quad \text{if } \begin{cases} 
F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) < Z_0 \\
\partial_\sigma F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \cdot \dot{\sigma}(\hat{x}_k) \leq 0
\end{cases}
\end{align*}
\]

where

\[
D'(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) = \frac{D \partial_\sigma F \partial_\sigma F^T D}{\partial_\alpha F \cdot \partial_\alpha F + \partial_\sigma F \cdot D \partial_\sigma F}(\sigma(\hat{x}_k), \alpha(\hat{x}_k))
\]

and

\[
\tilde{D} (\sigma(\hat{x}_k), \alpha(\hat{x}_k)) = -\frac{\partial_\sigma F(\sigma(\hat{x}_k), \alpha(\hat{x}_k)) \cdot \dot{\sigma}(\hat{x}_k)}{[\partial_\alpha F \cdot \partial_\alpha F + \partial_\sigma F \cdot D \partial_\sigma F](\sigma(\hat{x}_k), \alpha(\hat{x}_k))}.
\]

Again \( \hat{x}_k \in \Omega_j \), \( k = 1, 2, \ldots, 2p_j + 2 \), are all the Gaussian points of the element \( \Omega_j \).

Similar to the one dimensional case, for the discretized problem (4.11)~(4.14), we have the following theorem:

**Theorem 4.3** If the loading force \( f(x, t) \) is piecewise analytic in time \( t \), and for each \( t \in [0, T] \), \( f(x, t) \in (L^2(\Omega))^2 \), and if the gauge function \( F(\sigma, \alpha) \) satisfies all the three assumptions \( A1 \)~\( A3 \), the semi-discretized problem (4.11)~(4.14) has a unique solution \( (\sigma_{FE}, U_{BF}) \). Moreover, as the mesh size \( h \to 0 \), the sequence of the finite element solution, \( \{\sigma_{FE}\} \) and \( \{U_{FE}\} \), will converge to the solution of the continuous problem weakly in spaces \( (L^\infty([0, T], L^2(\Omega)))^3 \) and \( (L^\infty([0, T], H^1(\Omega)))^2 \).

Finally, for polygonal domains \( \Omega \), we can partition the domain into triangles. In this case, some high order finite element methods based on some symmetrical Gaussian quadrature rules are also available. Some symmetrical Gaussian quadrature rules for triangular element were presented in [6], [7] and [8], where an integration is performed by a Gaussian quadrature rule of the form:

\[
\int_A f(\alpha, \beta, \gamma) dA = A \sum_{i=1}^{ng} \omega_i f(\alpha_i, \beta_i, \gamma_i).
\]

In (4.15), \((\alpha_i, \beta_i, \gamma_i)\) are the natural coordinates of the \( i \)-th Gaussian point, \( \omega_i \) the corresponding Gaussian weight and \( ng \) is the number of Gaussian points used in the rule. Using Gaussian quadrature rule of the form (4.15) over triangular elements, we come up with the following high order finite element spaces for \( U, \sigma \) and \( \alpha \).

Denote by \( P_m(x_1, x_2) \) the set of polynomials of the form

\[
P_m(x_1, x_2) = \left\{ h(x_1, x_2) \mid h(x_1, x_2) = \sum_{i+j \leq m} a_{ij} x_1^i x_2^j \right\},
\]

and

\[
P_m,0(\Omega_h) = \left\{ h(x_1, x_2) \in H_0^1(\Omega) \mid \left. h(x_1, x_2) \right|_{T_n} \in P_m(x_1, x_2) \quad \forall T_n \in T_n \right\}.
\]
and
\[
\tilde{P}_m(\Omega_h) = \left\{ h(x_1, x_2) \in L^2(\Omega) \mid h(x_1, x_2)\big|_{T_n} \in P_m(x_1, x_2) \quad \forall T_n \in T_n \right\}.
\]

Then any functions in \( \tilde{P}_m(\Omega_h) \) will be uniquely determined by their values at \((m+1)(m+2)/2\) Gaussian points. Again we can define the constitutive equations at those Gaussian points, and get the following possible combinations of finite element spaces for \( U, \sigma \) and \( \alpha \).

<table>
<thead>
<tr>
<th>Space for ( U )</th>
<th>Space for ( \sigma ) and ( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{1,0}(\Omega_h) )</td>
<td>( \tilde{P}_0(\Omega_h) )</td>
</tr>
<tr>
<td>( P_{2,0}(\Omega_h) )</td>
<td>( \tilde{P}_1(\Omega_h) )</td>
</tr>
<tr>
<td>( P_{3,0}(\Omega_h) )</td>
<td>( \tilde{P}_2(\Omega_h) )</td>
</tr>
</tbody>
</table>

For discretized problems associated with one of the above combination of the finite element spaces, we have the similar existence, uniqueness and convergence result for the finite element solutions as the rectangular case.

Remark 4.2 We can not get higher order finite element spaces with the complete polynomial subspaces and symmetrical Gaussian quadrature rules described above. However, same idea can be used to derive even higher order finite element spaces with incomplete polynomial subspaces, or nonsymmetrical Gaussian quadrature rules. What essential here is that we need a kind of \( np \) point Gaussian quadrature rule and some finite dimensional spaces \( S_\sigma, S_\alpha \) and \( S_U \) such that functions in the spaces \( S_\sigma \) and \( S_\alpha \) are uniquely determined by their values at the set of \( np \) Gaussian points, and the

5 Some computational results

It is well known that the convergence rate of the finite element solutions is closely related to the regularity of the solution of the continuous problem. However, due to the nature of the plastic deformation associated with cyclic loading, the solution of the continuous problem does not have a nice regularity. For example, in one dimensional case, if the constitutive equation is defined as the linear kinematic law, we can only expect that the displacement function \( U(x, t) \) is in the space \( L^\infty([0, T], L^2(\Omega)) \). On the other hand, from the best approximation theorem, we know that the best approximation error in \( H^1 \) norm of piecewise quadratic polynomial spaces is of the order \( h^2 \), if the solution of the continuous problem is in the space \( L^\infty([0, T], H^3(\Omega)) \). Therefore, for one dimensional problem with uniform mesh, we can not expect the convergence rate of the finite element solutions to be of the order 2, as in linear elastic case. However, as we will see in the following examples, our numerical results do indicate that we can get higher convergence rate if high order finite element spaces were used.

Based upon our approach with high order finite element spaces, we developed a package of Fortran programs capable of solving one dimensional problems with piecewise polynomials of degree up to 50 for displacement functions (49 for stress and internal parameters). The so-called subincremental method is
used to solve the corresponding ODE system. At every time level, shooting method is used to solve the equilibrium equation. We tested the following three cyclic loading functions:

\[ f_1(x, t) = 6 \sin(t) x, \]
\[ f_2(x, t) = \sin(t) [1 - 24 x^2 + 300 x^4], \]
\[ f_3(x, t) = 4 \sin(t) x + 3 [1 - \cos(e t)] (x - 3 x^2), \]

where \( f_1 \) is chosen in such a way that a "closed form solution" can be found by using free boundary method for \( t \) within a certain range. We choose \( f_2 \) of this form, because for every fixed time \( t \), it is a polynomial of degree 4 in \( x \). Finally, unlike \( f_1 \) and \( f_2 \) which are both proportional function of time \( t \), we select \( f_3 \) to be an unproportional function of time \( t \). In the following computational results, except for the case \( f_1 \) at the time level \( t = \pi / 4 \) in which a closed form solution is used to evaluate the error of the finite element solutions, a very fine mesh of 10000 elements with piecewise quadratic polynomials were used as the "exact solution".

![Fig5.1 Convergence rate of error in displacement (I).](image1)

![Fig5.2 Convergence rate of error in displacement (II).](image2)

![Fig5.3 Convergence rate of error in stress (I).](image3)

![Fig5.4 Convergence rate of error in stress (II).](image4)
Figure 5.1 and 5.2 show the convergence rate of the error in displacement measured in $H^1$ norm at the time levels $t = \pi/4$ and $t = 5\pi/4$ respectively, where the loading function is $f_1(x, t)$, and the solid (respectively dashed and dashdotted) lines is computed with the numerical solutions that are chosen as piecewise polynomial of degree one (two and three). The slope $K$ in the pictures is the convergence rate. Figure 5.3 and 5.4 show the convergence rate of the error in stress measured in $L^2$ norm at the same time levels $t = \pi/4$ and $t = 5\pi/4$. The loading function is also $f_1(x, t)$, and the solid (respectively dashed and dashdotted) lines is computed with the numerical solutions that are chosen as piecewise polynomial of degree zero (one and two).

Figure 5.5 and 5.6 show the convergence rate of the error in displacement measured in $H^1$ norm at the time levels $t = \pi/4$ and $t = 5\pi/4$, but the loading function is chosen as $f_2(x, t)$. Figure 5.7 and 5.8 show the convergence rate of the error in stress measured in $L^2$ norm corresponding to the loading function.
function $f_2(x, t)$. Again the time levels are $t = \pi/4$ and $t = 5\pi/4$ respectively.

Figure 5.9 and 5.10 show the convergence rate of the error in displacement measured in $H^1$ norm at the time levels $t = 4\pi$ and $t = 6\pi$ corresponding to the unproportional loading function $f_3(x, t)$. Figure 5.11 and 5.12 show the convergence rate of the error in stress measured in $L^2$ norm corresponding to the same loading function $f_3(x, t)$ at the time levels $t = 4\pi$ and $t = 6\pi$.

In Figure 5.1~5.12, all the computation are based on linear kinematic law. As we have seen through those figures, with the uniform mesh, high order element spaces do improve the convergence rate by 1/2 in the error of displacement measured in $H^1$ norm, and 1 in the error of stress measured in $L^2$ norm. Since the exact solution $U$ is only in space $L^\infty([0, T], H^2(\Omega))$, $U''(x, t)$, the second derivative of the displacement function with respect to $x$ is a discontinuous function in $x$. The convergence rate 3/2 is
just the best approximation error of such a function with piecewise polynomials of degree greater than or equal to two measured in $H^1$ norm. The oscillation in the slope of the curves is related to the location of the discontinuous point of the function $U''(x, t)$. If the discontinuous point happens to be at the mesh point, we may get better accuracy. Otherwise, we get the error of order $h^{3/2}$. Figure 5.13 and 5.14 show the log-log pictures of error in displacement measured in $H^1$ norm versus $nelem$, the number of elements, with $U_h$ chosen as piecewise polynomials of degree one to four respectively. In Figure 5.13, the loading force function is $f_2(x, t)$ at the time level $t = 5\pi/4$, and in Figure 5.14, the loading force function is $f_3(x, t)$ at the time level $t = 4\pi$. As we can see, the oscillation in the slope has the same pattern for $U_h$ of different degrees, which indicates that the oscillation is caused by the location of the discontinuous points.

As we mentioned earlier, if the constitutive equations is based on linear kinematic or linear isotropic or mixture of linear kinematic and isotropic laws, we have $p$-version convergence for one dimensional
problems. Figure 5.15 and 5.16 show the convergence of the errors in displacement and stress of finite element solutions with fixed number of elements and increasing polynomial degree. The constitutive equations used in this computation are based on linear kinematic law. Eight elements are used, and the polynomial degrees for displacement functions are raised from 1 to 50. As we can see, the convergence rate is again 3/2 for displacement function, and 2 for stress function.

Finally, Figure 5.17 and 5.18 show the convergence rate of finite element solutions based on linear isotropic law. The loading force function used in this computation is $f_1(x, t)$, and the time level is $t = 3\pi/2$. Again, convergence rate 3/2 and 2 are observed for displacement and stress functions with high order finite element method. Unlike the case with linear kinematic law, in which the solution would be periodic if the loading force is a periodic function of time $t$, for linear isotropic law, the solution would not be periodic even if the loading force function is periodic. However, as the size of the yield surface getting bigger and bigger, finally all the points will be inside the yield surface, and the material will be in the elastic region everywhere.

6 Remarks and conclusions

As we have seen in the previous section, for one dimensional problems, higher order elements produce higher convergence rate with the uniform mesh. However, since the unsmoothness of the exact solution, the convergence rate does not change when we raise the polynomial degree from 2 to higher numbers for displacement functions. Actually it can be shown, for example, that for proportional loading functions, if for every fixed time $t$, $f(\cdot, t)$ is continuous on the interval $[0, t]$, the exact solution for the displacement function, $U(x, t)$, is in the Besov space $B_{\infty}^{1/2}(H^3(I), H^3(I))$ for every fixed time $t$. From the approximation theory, on every subinterval $\Omega_j$, there exits $\chi \in S_2^p(\Omega_j)$ such that

$$\| U - \chi \|_{H^1(\Omega_j)} \leq C h_j^\nu \| U \|_{B_{\infty}^{1/2}(H^3(\Omega_j), H^3(\Omega_j))}$$

with $\nu = \min(p, 3/2)$. The above approximation error analysis indicates that we can not expect convergence rate to be higher than 3/2 with the uniform mesh. So in order to fully take the advantage of high
order elements, an adaptive scheme has to be adopted.

The lack of regularity of the displacement function \( U \) is caused by the difference between the elastic and plastic modulus. The discontinuity points of the function \( U'' \) in \( x \) are those which divide the elastic and plastic sets on the bar. So if we can trace the fronts (or the free boundaries) which divide the elastic and plastic sets along the time, we can then simply apply the following h-version adaptive method. Assuming that on every element, we uniformly use polynomials of degree \( p \). Then on the elements where the solution is smooth, from the best approximation theory, we have the error of the order \( h^p \) if measured in \( H^1 \) norm. On the other hand, on the elements where the solution is not smooth, the error is of the order \( h^{3/2} \) (of course we assume here \( p \geq 2 \)). So we can have the error evenly distributed over the whole interval \([0, l]\), by refining the mesh over the elements where the solution is not smooth. For example, we can divide such an element into \( N \) subintervals, and the error on those refined elements would be of the order \((h/N)^{3/2}\). Let \((h/N)^{3/2} = h^p\), we get

\[
N = \left\lfloor \frac{1}{h} \frac{2p}{3-1} \right\rfloor + 1. \tag{6.1}
\]

here \( \lfloor d \rfloor \) is the integer part of the number \( d \). So using (6.1) to refine elements where solution is not smooth, we can achieve higher convergence rate. Since the fronts are moving along the time \( t \), some refined elements may not have discontinuity point of \( U'' \) at some other time level. Therefore, procedures of combining refined subintervals into one subinterval should also be used in the adaptive scheme in order to get better convergence rate. Of course, we can also use p-version adaptive methods which always keep the mesh size fixed, and only raise the polynomial degree on the elements where the solution is not smooth. For more details about the adaptive scheme for elastoplastic problems and numerical results, we refer to our forthcoming paper \cite{18}. The procedure has some similarity with the adaptive approaches for solving parabolic differential equations by method of lines with adaptation of meshes or some procedures for solving problems with moving fronts. Finally, we mention that quite analogues results are obtained for other constitutive laws.

Because subincremental method was used to solve the corresponding ODE system, error in the numerical solution caused by the discretization in time space is not a dominant fact. However, because of the nature of the cyclic plastic deformation, it is very important to always catch the “peak point” of the loading force function in the time discretization. For example, if the loading force has the form

\[
f(x, t) = 4 \sin(t) x + 3 (1 - \cos(\pi x))(x - 3 x^2).
\]

Then the points \( t = \pi/2 + k\pi \) and \( t = k\pi/e \) for \( k = 0, 1, \ldots \) should always be included as level points, and the solution of the fully discretized problem must be found at those time levels.

Finally all those computational results are only for one dimensional problems, but we expect that high order method will perform similarly for two dimensional problems.

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