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A General Technique for the Finite Element Shakedown and Limit Analysis of Axisymmetrical Shells Part I - Theory and Fundamental Relations

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Part 1 - Theory and Fundamental Relations

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Abstract

This paper describes the theory and the fundamental relations for the development of a displacement formulation for the finite element shakedown and limit analysis of axisymmetrical pressure vessels. The material is assumed to be elastic-perfectly plastic. The technique is developed based upon an upper bound approach using a reformulated kinematic shakedown theorem for a shell piecewiselinear yield conditions. The solution of the problem is obtained by discretizing the shell into finite elements. A consistent relationship between the kinematically admissible velocity fields and the pure plastic strain rate fields during collapse had to be enforced. Such requirement is satisfied by using the theory of conjugate approximations to minimize the residual of the two independent descriptions of the plastic strain increments; defined by the compatibility conditions in terms of nodal displacements (velocities) and by the flow law in terms of nodal plastic multipliers. In a subsequent paper the discretized problem is then reduced to a minimization problem and solved by linear programming. The class of displacement fields chosen assume plastic hinge lines forming at nodal points and only meridional and circumferential plastic strains occurring within the elements with no change in curvature. The applications are presented in Part 2 for different geometries, boundary and load conditions.

1 Introduction

The development of general numerical methods for structural analysis on the basis of shakedown and ratchetting concepts allows a more sophisticated treatment of structural components exposed to cyclic loading. Therefore, the application of shakedown theory to cyclic

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thermal loading problems has become a major task in the design of components operating at high temperatures and in the presence of steady mechanical loads. These problems have gained considerable significance with the advent of new technologies such as the Liquid Metal Fast Breeder Reactor, modern chemical plants, offshore structures under severe weather conditions and advanced aeronautical structures such as the solid propellant rocket booster of the NASA space shuttle. The thermal loading problem is now significantly more complicated by the occurrence of much higher temperatures and much more severe thermal loading conditions.

Emerging from the fundamental papers of Melan [1], Drucker [2], Koiter [3], Hill [4] and others, the shakedown theory and respective conceptual theorems have had a significant impact in engineering design of framed structures and pressure vessels. From an exhaustive list of major contributions in the application of cyclic mechanical and thermal loading to thin shells and framed structures it is worth mentioning the works of Leckie [6], for pioneering the use of mathematical programming, of Bree [7, 8] for the innovative and appealing diagram of results and Maier [9] for pioneering the use of the Finite Element Method. With the exception of some analytical works, only recently thermal loading problems have been included in the numerical approaches. The static shakedown theorem proposed by Melan [1] seems to have some preference. However, the works by Karadeniz and Ponter [10], Franco [11] and Franco and Ponter [12] show a formulation for pressure vessels subjected to cyclic thermal loading and steady mechanical loads based upon the kinematical principles of the classical shakedown theorem proposed by Koiter [3]. The complexity of the residual stress field for thermal loading problems in contrast with the simplicity of the classes of displacement field associated with plastic mechanism of shells justified the adoption of the upper bound approach.

Only quite recently attempts has been made to produce numerical techniques for shakedown analysis of 2D problems such as in [13]. The works by Stein et al. [14, 15] propose an extension of Besseling [16] overlay material model to simulate an elastic-plastic transient strain hardening behaviour. The Melan's static theorem is again used for the formulations.

In the present work a displacement formulation for the finite element limit and shakedown analysis of axisymmetric pressure vessels is developed in the form of a general technique for a shell piecewiselinear yield surface. The technique is developed based upon an upper bound approach making use of Koiter's kinematic shakedown theorem recounted by Franco [11] in its extended form to include cyclic thermal loading. By imposing a consistent relationship between the displacement (velocity) fields, and the kinematically admissible strain rate fields, the shakedown or limit analysis problems can be reduced to a minimization problem and solved by linear programming. Such requirement is satisfied by using the theory of conjugate approximations to minimize the residual between the two separate descriptions of the plastic strain increments, defined in terms of nodal displacements by the compatibility conditions and in terms of nodal plastic multipliers from the flow rule. The class of displacements fields chosen assume that all the meridional curvature occurs at nodal points, i.e., plastic hinge lines. The meridional and circumferential strains occurs within the element with no change in curvature, an assumption adopted by many authors [10, 18, 25, 27, 28, 31] for thin shell problems, which is consistent with the piecewiselinear form of the yield surface adopted here.

The numerical results obtained consist of the limit or shakedown load and the corresponding collapse mechanism, which are compared with other numerical or analytical results.

2 The Model Problem

Basic Assumptions : The problem can be described as the behaviour of an elastic-perfectly plastic body under a quasi-static process of cyclic deformation during collapse. The undeformed state of the body corresponds to an open bounded domain $V \subset R^3$ (or volume V) with smooth boundary or surface S disjointed in two parts S_p and S_u , i.e. $S = S_u \cup S_p$. The body is subjected to surface forces kp_i prescribed on Sp, where k is a proportional loading parameter, and to body forces per unit volume b_i . Over the remainder of the total surface, S_u , applied surface displacement u_i are also prescribed. In addition, a cyclic history of temperature $\theta(x, t)$ with a cyclic time Δt may occur at a point x within the domain V, given by:

$$\theta(x,t) = \theta_o + \theta(x,t) \tag{2.1}$$

where θ_o denotes some reference temperature and $\theta(x, t)$ is a quasi static cycling temperature given by

$$\begin{aligned} \theta(x,t) &= \Omega(x)\mu(t)\Delta\theta\\ 0 &< \mu < 1; 0 < \Omega < 1 \end{aligned}$$

$$(2.2)$$

where $\Delta \theta$ is the maximum temperature difference and $\Omega(x)$ is a non-dimensional shape function. If the variations of the elastic moduli with temperature are ignored, this implies that the thermo-elastic stress history follows a straight line path in the stress space. All the deformations are assumed to be sufficiently small so that changes in geometry can be disregarded

State of Stress: Let $\sigma_{ij}(x), 1 < i, j < 3$ denote stress tensor at a point $x = (x_1, x_2, x_3) \in V$ which must satisfy the equilibrium condition

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \qquad in \ V \ (i = 1, 2, 3) \tag{2.3}$$

and the following conditions are also prescribed on the boundary

$$u_i = 0 \qquad on \ S_u \sigma_{ij} n_j = k p_i \qquad on \ S_p$$

$$(2.4)$$

where n_j indicates the direction cosines of the outward normal to S_p .

State of Strain: The state of strain at a point is defined by the symmetric strain tensor ϵ_{ij} representing an elastic, a plastic and a possible thermal expansion component ϵ_{ij}^{θ} .

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij} + \epsilon^\theta_{ij} \tag{2.5}$$

where the elastic part of the constitutive relation for a isotropic material take the form of Hooke's law

$$\sigma_{ij} = D_{ijkl} \epsilon^e_{ij} \tag{2.6}$$

Dijkl denotes the usual symmetric, positive definite elasticity tensor, which is also independent of temperature.

Assuming small strains, the strain tensor can be expressed in terms of displacements as

$$\epsilon_{ij}(u) = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$$
(2.7)

and likewise the strain rate tensor $\dot{\epsilon}_{ij}$ can be expressed in terms of the velocities \dot{u}_i by an identical expression.

3 Fundamental Relations for the Shakedown Theory

3.1 Koiter's Extended Shakedown Theorem to Cyclic Thermal Loading

The second general shakedown theorem referred to in the literature as Koiter's theorem, is based in kinematical principles and was proposed by Koiter [3]. Before the theorem is stated, it is convenient to give some preliminaries definitions of the terms involved :

Shakedown: A structure subjected to cyclic load is considered to be in a state of shakedown, if the response of the structure becomes elastic after the appearance of plastic deformation during the first cycles.

Reversed Plasticity: When a small volume of the structure is subjected to an elastic stress range larger than $2\sigma_y$, developing plastic strain, alternatively in tension and compression for each part of the cycle, that volume of the structure is said to be operating in a reversed plasticity condition.

Ratchetting: A structure is said to be ratchetting when its deformation increases over each loading cycle. The fundamental relations can now be stated as:

- "the concept of an arbitrary field of admissible plastic strain rate cycle $\dot{\epsilon}_{ij}^c$ (superscript c stands for collapse) is defined by its property that the plastic strain increments during a cycle $\Delta \epsilon_{ij}^c = \int_0^{\Delta t} \dot{\epsilon}_{ij}^c(t) dt$, constitute a kinematically admissible strain distribution, i.e., satisfies the strain-displacement relation 2.7 and the corresponding displacement field is assumed to be zero on S_u ".

- "the plastic strain rate field $\dot{\epsilon}_{ij}^c$ induces a unique residual stress rate distribution $\dot{\rho}_{ij}$ and its corresponding elastic strain rates. The kinematically admissible total strain rate field $\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p$ is obtained from the velocity field \dot{u}_i , by a relation identical to that of eq. 2.7".

- "in addition, the elastic part of the total elastic strain rate tensor is related to the stress tensor rate $\dot{\sigma}_{ij}$ by the Hooke's Law"

$$\dot{\sigma}_{ij} = D_{ijkl} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^c) \tag{3.8}$$

and also

$$\dot{\sigma}_{ij} = \hat{\sigma}^{\theta}_{ij} + \dot{\rho}_{ij} \tag{3.9}$$

where $\hat{\sigma}^{\theta}_{ij}$ is the rate of change of cyclic elastic stresses for the same loading programme".

- "the residual stresses at the end of a cycle return to their initial values at time t = 0since the increments of plastic strains are kinematically admissible; hence, $\int_0^{\Delta t} \dot{\epsilon}_{ij}^e(t) dt = 0$ while the displacement increments over a cycle time are $\Delta u_i^c = \int_0^{\Delta t} \dot{u}_i(t) dt$ ".

Koiter's theorem may now be formulated as follows: the structure will not reach a state of shakedown, in other words, it will either suffer ratchetting or reversed plasticity, if any system of external cyclic loads and any admissible plastic strain rate cycle $\dot{\epsilon}_{ij}^c(t)$ within prescribed limits satisfy

$$\int_{0}^{T} dt \int_{V} b_{i} \dot{u}_{i}^{c}(t) dV + k \int_{0}^{T} dt \int_{S_{p}} p_{i} \dot{u}_{i}^{c}(t) dS + \int_{0}^{T} dt \int_{V} \hat{\sigma}_{ij}^{\theta}(x,t) \dot{\epsilon}_{ij}^{c}(t) dV > \int_{0}^{T} dt \int_{V} \dot{\sigma}_{ij}^{c}(t) \dot{\epsilon}_{ij}^{c}(t) dV$$

$$(3.10)$$

where b_i and p_i are respectively body forces and surface forces assumed to be time independent, $\hat{\sigma}_{ij}^{\theta}(x,t)$ denotes the elastic stresses due to time dependent loads which could include some variable mechanical loads and $\dot{\sigma}_{ij}^{c}$ is a state of stress on the yield surface associated with the pure plastic strain rates $\dot{\epsilon}_{ij}^{c}(t)$.

This inequality depends only on the plastic strain rates $\dot{\epsilon}_{ij}^c(t)$ and the corresponding displacement increments \dot{u}_i^c with no involvement of the residual stress field ρ_{ij} induced by $\dot{\epsilon}_{ij}^c(t)$ during the transient part of the cycle. Once a consistent relationship between $\dot{\epsilon}_{ij}^c$ and \dot{u}_i^c is defined, corresponding to each instant of the cyclic elastic stress history, Eq. 3.10 may be used to calculate upper bounds on the load parameter. A systematized technique using the finite element method, to define such a consistent relationship for the shakedown analysis of pressure vessels is described in Section 6. Optimization of Eq. 3.10 on the basis of the non-variation of stress rates $\dot{\sigma}_{ij}$, Eq. 3.8, during collapse can be achieved by reducing the problem to a mathematical problem for certain classes of linear yield surfaces, which can be formulated as follows:

Let the Koiter's shakedown theorem (Eq. 3.10) be rearranged, assuming the body forces bi = 0, to give

$$k\int_{0}^{T} dt \int_{S_{p}} p_{i} \dot{u}_{i}^{c}(t) dS > \int_{0}^{T} dt \int_{V} \left[\sigma_{ij}^{c}(t) - \hat{\sigma}_{ij}^{\theta}(x,t) \right] \dot{\epsilon}_{ij}^{c}(t) dV$$
(3.11)

The minimization problem can now be stated as For $k^s \ge k$ Minimize

$$k^{s} = \frac{\int_{0}^{T} dt \int_{V} \left[\sigma_{ij}^{c}(t) - \hat{\sigma}_{ij}^{\theta}(x, t)\right] \dot{\epsilon}_{ij}^{c}(t) dV}{\int_{0}^{T} dt \int_{S_{p}} p_{i} \dot{u}_{i}^{c} dS}$$
(3.12)

subjected to

$$\mathbf{u} = constant$$
 on S_u (BoundaryConditions) (3.13)

$$\dot{\epsilon}_{ij} = \mathbf{B}\dot{\mathbf{U}}$$
 (Compatibility Conditions) (3.14)

$$\dot{\epsilon}_{ij}^c = \mathbf{N}\dot{\lambda} \quad (Flow \ Law)$$
(3.15)

$$\dot{\lambda} > \mathbf{0} \tag{3.16}$$

$$\dot{\sigma}_{ij} = \mathbf{D}_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^c) = 0 \text{ or } \dot{\epsilon}_{kl} = \dot{\epsilon}_{kl}^c (During \ Collapse)$$
(3.17)

In order to solve this minimization problem the yield conditions governing the material plastic behaviour for thin shells are now described.

4 Flow Law Associated with the Yield Surfaces for Thin Shells of Revolution

The class of displacement fields proposed here was originally defined from the simplified versions of yield surfaces for shells of revolution by Drucker [17] and Drucker and Shield [18], where the meridional curvature is assumed to occur only at nodal points and the meridional and circumferential membrane strains occur within the element with no change in curvature. No change in curvature implies the material behaviour following the Tresca yield condition within the element as shown by Drucker and Shield [18]. Such a assumption produces upper bounds on the shakedown or limit load [17], and some error is induced, associated with the energy dissipation due to the change in curvature of the element during the plastic deformation, which is not considered.

The derivation of appropriate yield surfaces for axially symmetric loaded shells of revolution in terms of force and moments resultants is rather more complex than it is in the case of a biaxial state of plane stress. The problem of a symmetrically loaded cylindrical shell was extensively studied by several authors with the inclusion of axial and circumferential forces acting individually or simultaneously with bending moment [17, 20, 21, 22, 23]. Drucker [17] described a technique for deriving appropriate yield conditions for symmetrically loaded cylindrical shells without axial load (Fig. 1). The relevant variables in this case are the circumferential force per unit length N_{θ} and the meridional bending moment per unit length, which will be referred to as generalized forces N_{ij} . The yield surface is then described by

$$\phi(N_{ij}) = \phi(N_{\theta}, M_{\phi}) \le 0 \tag{4.18}$$



Figure 1: Cylindrical Element:a)Without Axial Load, b)With a Complete Set of Forces

The extension of the work to describe the yield surface for the general case of a cylindrical element subjected to a complete set of forces, Fig. 1, was proposed by Onat [20] and by Hodge [22] as a 3D yield surface as shown in Fig. 2.

$$\phi(N_{ij}) = \phi(N_{\theta}, N_{\phi}, M_{\phi}) \le 0 \tag{4.19}$$



Figure 2: 3D Yield Surface for Cylindrical Vessels

Onat and Prager [25] and Hodge [24] extended the investigation to general shells of revolution, Fig. 3, producing a yield surface in a four-dimensional space.

$$\phi(N_{ij}) = \phi(N_{\phi}, N_{\theta}, M\phi, M_{\theta}) \le 0 \tag{4.20}$$

In general, due to its complexity, the applicability of such yield criterions is restricted to simple problems [25]. For this reason it was essential to develop simplified versions of these yield surfaces which would allow its direct application to practical problems without compromising the reliability of the results. The search for a more practical yield surface started from the observation that for the cylindrical case, within the framework of small displacement theory, the circumferential bending moment M_{θ} plays no role in the load bearing capacity of the shell. It has been described as an induced or passive moment although equal to $M_{\phi}/2$ if a Poissons ratio of 0.5 (i.e., incompressibility) is assumed for the plastic range. The yield surface may then be described in three dimensions as in Eq. 4.19 and M_{θ} does not appear in the equations of equilibrium. On the other hand, for the general axisymmetric case as in Fig. 3, within the framework of small displacement theory, the circumferential bending moment M_{θ} plays no role in the load bearing capacity of the shell. It has been described as an induced or passive moment although equal to $M_{\phi}/2$ if a Poisson's ratio of 0.5 (i.e., incompressibility) is assumed for the plastic range. The yield surface may then be described in three dimensions as in Eq. 4.19 and M_{θ} does not appear in the equations of equilibrium. On the other hand, for the general axisymmetric case as in Fig. 3, the moment M_{θ} does enter into the equilibrium equations (4.21-4.23) and also does work during deformation due to changes in the circumferential curvature.

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Figure 3: General Thin Shell Element of Revolution

$$r\frac{dN_{\phi}}{d_{\phi}} + (N_{\phi} - N_{\theta})R_1 \cos\phi - rQ + rR_1T = 0$$
(4.21)

$$rN_{\phi} + R_1 N_{\theta} \sin \phi + \frac{d}{d_{\phi}} (rQ) + rR_1 p = 0$$
(4.22)

$$r\frac{dM_{\phi}}{d_{\phi}} + R_1(M_{\phi} - M_{\theta})\cos\phi - rR_1Q = 0$$
(4.23)

However, for thin shells, i.e., $h/r \ll 1$ where h is the thickness and r is shown is Fig. 3, the contribution of M_{θ} to the yield criterion becomes negligible and the term containing $(M_{\phi} - M_{\theta})$ in the equilibrium Equation 4.23 can be ignored. Two interpretations of that assumption are given in [18] and it enables the use of the yield condition 4.19 or Fig. 5 in the analysis of general shells of revolution. The way M_{θ} should be treated in 4.23 as proposed in [18] is as follows: In the equilibrium Equation 4.23, the order of magnitude of the term involving $(M_{\phi}-M_{\theta})$ can be neglected when compared with the other two terms, except for portions of the shell near the axis of revolution where h/r may not be small [18]. The second interpretation, which was later reinforced by Dinno and Gill [28], corresponds

to putting $M_{\phi} = M_{\theta}$ in Equation 4.23. The theorems of limit or shakedown analysis in [26, 3] respectively, could then be applied using the resulting equations of equilibrium in conjunction with any particular four-dimensional interaction yield surface to calculate the true lower bound. In addition, as pointed out in [28] since $M_{\phi} = M_{\theta}$ the one moment limited interaction surface will be contained within, or on such yield surface and therefore may be used. Drucker and Shield [18] have also shown that the yield surface for a thin cylindrical shell can be used for any axisymmetric thin shell of revolution as a very good approximation.

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Figure 4: Hexagonal Prism Yield Surface for Thin Shells

The Hexagonal Prism: Despite all the simplification, the use of the one moment limited interaction yield surface (Fig. 2) may bring to the analysis of general shells of revolution, its original form presents a high level of complexity which is still incompatible with the necessary practicability of analysis as an aid to design. Therefore, for the sake of simplicity and convenience of practical design, inscribed and circumscribed linearized surfaces (Hexagonal Prism, Fig. 4) which give rise to lower and upper bounds respectively, have been proposed.

An important feature of thin shells is their high bending flexibility which leads to a primary membrane behaviour, except for some localized curvature. This characteristic has been widely used [10, 18, 19, 27, 28, 29, 30, 31] to separate the distinct volumes of the shell where either bending or membrane behaviour are independently relevant. These volumes are the plastic hinges where curvatures ($\kappa_{\theta}, \kappa_{\phi}$) are unrestricted and the regions between two adjacent hinges where only circumferential (ϵ_{θ}) and meridional (ϵ_{ϕ})strains occur. This important conclusion was drawn from the normality condition and from the convexity of hexagonal prism yield surface [18]. The several analytical solutions for plastic collapse problems referenced above had a deformation pattern consisting of localized three hinges mechanism where the positions of the hinges are obtained generally from the lower bound approach by finding the positions where the bending moment is maximum, i.e. where $dM_{\phi}/d\phi = 0$. Correspond-

ingly, for the application of the present upper bound approach, the use of a displacement field consisting of discrete hinges at the nodes of a finite element subdivision and zero curvature within the element (hinge-cone displacement field), results in identical upper bounds. Such a discretized displacement field separates membrane action from bending action as in the case of the analytical approaches using the one moment limited interaction yield surface (Fig. 4).

Since the hexagonal prism (Fig. 4) is singular due its edges and corners, it has to be described by a finite number of yield functions

$$\phi_k(N_{ij}) = \phi_k(N_{\phi}, N_{\theta}, M_{\phi}) \le 0 \qquad for \ k = 1,8 \tag{4.24}$$

The elastic domain is defined by the negative values of all yield functions and a zero value of one or more yield functions represents a state of stress on the yield surface.

The plastic strains and curvature are related to the rate of change of stress by the flow rule, associated with the hexagonal prism yield surface, in the form:

$$\dot{\epsilon}_{ij}^{c} = \sum_{k=1}^{6} \dot{\lambda}_{k} \frac{\partial \phi_{k}}{\partial N_{ij}} \qquad between \ hinges \tag{4.25}$$

where $\dot{\lambda}_k$ are positive plastic multipliers and

$$\dot{\kappa}_{ij}^c = \dot{\theta}_k \frac{\partial \phi_k}{\partial N_{ij}} \qquad at \ hinges \ circles \tag{4.26}$$

where $\dot{\theta}_k$ are positive rotations rates induced by the plastic unrestricted curvatures rates at the hinge circles.

The distinct behaviour of material points within the elastic-perfectly plastic body can be described as follows:

$$Elastic Behaviour \begin{cases} \phi < 0 \quad loading \text{ or unloading} \\ \phi_k = 0; \ \dot{\phi}_k < 0 \quad unloading \end{cases}$$

$$Plastic Behaviour \{\phi_k = \dot{\phi}_k = 0 \quad loading \quad (4.27)$$

where $\dot{\phi}_k = \frac{\partial \phi_k}{\partial N_{ij}} \dot{N}_{ij} < 0$, defines the unloading condition.

The minimization problem stated in 3.12 can only be solved for practical problems by the resort of finite element techniques. The discretization of axisymmetric shells is discussed next.

5 Discretization of Axisymmetric Shells via Finite Elements

The elastic-plastic deformation pattern for thin shells may be defined in terms of the displacement field of its middle surface $U_e(s)$ and the plastic strains in terms of plastic multipliers $\lambda_k(s)$ which characterize the plastic behaviour of the material. It is convenient to describe the displacement field using global displacements components in the outward horizontal direction W(s) and in the downward vertical direction U(s), respectively perpendicular and parallel to the axis of symmetry as shown in Fig. 5. The local displacement components normal w(s)and tangential u(s) to the meridional direction can be obtained by simple transformation. The finite element formulation for the elastic-plastic problem of thin shells may be derived from suitable interpolation of the displacement field and of the plastic multipliers as function of nodal values. For this purpose, let the shell be discretized into NE finite elements. The displacement field within the $i^t h$ element may be expressed in terms of nodal global displacements $\{U_N\}^i$. The element nodal displacements $\{U_N\}^i$ can be divided into two independent parts; constant rigid body motions $\{U_0^i\}$ which gives no contribution to the strain field and the so called natural displacements which generates the straining modes $\{U_n^i\}$.



Figure 5: Global and Local Displacements for Thin Shells of Revolution

Thus,

$$\left\{U_e^i(s)\right\} = \left\{U_0^i\right\} + \left[\Omega^i(s)\right] \left\{U_n^i\right\}$$
(5.28)

where $[\Omega^{i}(s)]$ is a matrix of suitably chosen interpolation functions which ensures the continuity between adjacent elements when the assemblage is performed.

Multiplying the global displacement by an appropriate transformation matrix [T] give rise to the local displacement field (Equation 5.29) with components normal and tangential to the meridional surface of the shell (Fig. 5).

$$\left\{u_e^i(s,\phi)\right\} = [T]^i \left\{U_e^i(s)\right\}$$
(5.29)

where

$$[T]^{i} = \begin{bmatrix} \sin \phi_{i} & \cos \phi_{i} \\ -\cos \phi_{i} & \sin \phi_{i} \end{bmatrix}$$
(5.30)

Substituting 5.28 into 5.29 gives

$$\left\{u_e^i(s,\phi)\right\} = [T]^i \left\{U_0^i\right\} + [T]^i \left[\Omega^i(s)\right] \left\{U_n^i\right\}$$

$$(5.31)$$

Considering the Kirchhoff-Love assumptions and neglecting the effects of M_{θ} on the circumferential curvature as discussed in Section 4, the compatibility conditions give rise to three strain components :

$$\left\{\epsilon_{ij}^{1}(s,\phi)\right\}^{i} = \left\{\begin{array}{c}\epsilon_{\phi}^{1}\\ \epsilon_{\theta}^{1}\\ \kappa_{\phi}^{1}\end{array}\right\} = \left[\begin{array}{c}\frac{d}{ds} & \frac{1}{R_{1}}\\ \frac{\sin\phi}{r} & \frac{\cos\phi}{r}\\ \frac{1}{R_{1}}\frac{d}{ds} & -\frac{d^{2}}{ds^{2}}\end{array}\right] \left\{u_{e}^{i}(s,\phi)\right\}$$
(5.32)

where r, R_1 and ϕ are geometric parameters shown in Fig. 5.

The classical relation independent of the rigid body motions, for points within the element only, is now

$$\left\{\epsilon_{ij}^{1}(s,\phi)\right\}^{i} = \left\{\begin{array}{c}\epsilon_{\phi}^{1}\\\epsilon_{\theta}^{1}\end{array}\right\}^{i} = \left[B\right]^{i}\left\{U_{n}^{i}\right\}$$
(5.33)

Change in curvature is considered to be localized at the plastic hinge circles. Details of such displacement and strain fields assumed for the case of axisymmetric shells, is given in Appendix A.

The flow law associated with the hexagonal prism (Fig. 4) relates the plastic strains to the plastic multipliers via a suitable matrix [N]. Within the element such yield conditions are identical to the Tresca yield conditions. Assuming no variation of strain through the shell thickness the relations between plastic strains and plastic multipliers can be defined as follows:

$$\left\{\epsilon_{ij}^{2}(s)\right\}^{i} = \left\{\begin{array}{c}\epsilon_{\phi}^{2}\\\epsilon_{\theta}^{2}\end{array}\right\}^{i} = [N]\left\{\lambda_{k}^{i}(s)\right\} \quad for \ k = 1, 6$$

$$(5.34)$$

where

$$[N] = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}$$
(5.35)

Similarly to the displacement field, the plastic multipliers field for the element can be interpolated in terms of nodal parameters in a finite element approximation

$$\left\{\lambda_k^i(s)\right\} = \left[\Lambda^i(s)\right]\left\{\lambda_k^n\right\}$$
(5.36)

where the only restrictions on $\lambda_k^i(s)$ is that it has to be non-negative. This condition can be enforced by imposing the restriction on the interpolation matrices within the element to be $[\Lambda^i(s)] \geq 0$ for any $\{\lambda_k^n\} > 0$. The localized nature of the nodal plastic hinges requires no interpolation. The flow rule, Eq. 4.26, describe the unrestricted curvature κ_{ϕ} in terms of the rotation θ , which are defined in terms of the displacement field.

The reduction of the upper bound theorem (Equation 3.10) to a Linear Programming problem requires that the two separate description of the plastic increment given by Equations 5.33 and 5.34, must be consistent with each other at both nodes and within the element. Karadeniz and Ponter [10] showed that, for cylindrical shells, suitable shape functions Ω^i and Λ^i could be found so that forcing equality of strains for 5.33 and 5.34 at the nodes would imply that equality is also satisfied within the element. Generally however, this cannot be achieved and there does not exist shape functions which give such consistency for other shape of shell element. Therefore, a more systematic mean by which 5.33 and 5.34 may be made consistent has to be found. This problem is similar to the problem discussed by Corradi [32] for incremental finite element solutions, where evaluation of the stress within an element is given by the relationship

$$\{\sigma_{ij}\}^{i} = \{D\}^{i} \{[B] \{U\} - [N] \{\lambda_{k}\}\}^{i}$$
(5.37)

where $\{D\}^i$ is the elastic stiffness coefficient matrix. During collapse there is no increment of stress due to the pure plastic nature of the deformation and consequently the stress given by 5.37 has to be zero. Thus, the condition $[B]\{U\} = [N]\{\lambda_k\}$ must be satisfied at the element level. Such an approach can give poor results, if separate descriptions of the total plastic strain rates are adopted. Corradi used a procedure suggested by Oden, et al. [33, 34] to obtain improved results. Oden and Brauchli [33] described a procedure, which produces a continuous stress field $\{\sigma_{ij}\}$ that is consistent with the discontinuous stress field $\{\sigma_{ij}\}$, obtained form the elastic strains using the traditional finite element procedure, by imposing the condition

$$\int_{V} \left\{ \epsilon_{ij} \right\}^{T} \left\{ \sigma_{ij} - \sigma_{ij}^{e} \right\} dV = 0$$
(5.38)

Such a condition has to be satisfied for all possible strain fields $\{\epsilon_{ij}\}$ belonging to the class assumed in the finite element method. The solution to this problem is claimed by Oden, et al. [33, 34] to be unique, and the procedure is discussed in the following section.

6 On a Consistent Relationship Between Nodal Displacement Variables and Nodal Plastic Multipliers

6.1 Stress and Strain Fields as Function of Generalized Quantities

Assuming that the behaviour of individual elements may be described in terms of generalized variables, the stress and strain fields within the element can be expressed as follows:

Let [b(s)] be a suitable interpolation matrix which give rise to an internal distribution of strain in terms of generalized strain quantities $\{E\}$.

$$\left\{\varepsilon_{ij}^{i}\right\} = \left[b^{i}\right]\left\{E_{i}\right\} \tag{6.39}$$

Similarly, let the stress field be written in terms of generalized stress quantities $\{\Sigma\}$ by means of matrix $[\psi(s)]$

$$\left\{\sigma_{ij}^{i}\right\} = \left[\psi_{ij}^{i}\right]\left\{\Sigma_{i}\right\}$$
(6.40)

Note that the stress distribution defined by Equation 6.40, although apparently independent of the constitutive law, when applied to an elastic material, the constitutive Equation imposes constraints on the form of [b] and $[\psi]$, which then define the stiffness matrix of the element. By the following argument Oden suggests a relationship between [b] and $[\psi]$ independent of the constitutive law : The generalized stresses and strains within the element must satisfy the principle of virtual displacement [35] which states that

$$\{\Sigma_i\}^T \{E_i\} = \int_V \left\{\sigma_{ij}^i\right\}^T \left\{\varepsilon_{ij}^i\right\} dV$$
(6.41)

Equations 6.39 to 6.41 imply the biorthogonality condition

$$\int_{V} \left[\psi_{ij}^{i} \right]^{T} \left[b^{i} \right] dV = [I]$$
(6.42)

A solution of Equation 6.42 can be obtained by imposing

$$\left[\psi_{ij}\right] = \left[b^i\right] \left[C\right]^i \tag{6.43}$$

as proposed in [33, 34], where $[C]^i$ is a symmetric and non-singular matrix defined as

$$[C]^{i} = \left\{ \int_{V} \left[b^{i} \right]^{T} \left[b^{i} \right] dV \right\}^{-1}$$
(6.44)

Note that this result is independent of the constitutive relationship. Corradi [32] uses the functions $[\psi]$ of 6.40 to calculate internal stresses from nodal values $\{\Sigma\}$ which have been generated by the solution of the finite element formulation. He obtained improved results compared to the conventional approach. For the formulation that follows, the subscript "*i*" referring to the *i*th element will be dropped for simplicity.

In the present formulation, the generalized strains in 6.41 are replaced by the displacement with the rigid body translation removed and the generalized stresses are replaced by forces, so that

$$\{F\}^T \{U_n\} = \{\Sigma\}^T \{E\} = \int_V \left\{\sigma_{ij}^i\right\}^T \left\{\varepsilon_{ij}^i\right\} dV$$
(6.45)

where the strain distribution field $\{\epsilon_{ij}\}$ within the element is defined by Equation 2.7 and the stress distribution field is assumed to be described in terms of the generalized forces by means of a matrix $[\psi]$ of suitably interpolation functions

$$\left\{\sigma_{ij}^{i}\right\} = \left[\psi_{ij}^{i}\right]\left\{F\right\}$$
(6.46)

Equality 6.45 will now always be satisfied for any $\{U_n\}$ and $\{F\}$ if

$$[\psi_{ij}] = [B] \left[\overline{C}\right] \tag{6.47}$$

where $\left[\overline{C}\right]$ is given by

$$\left[\overline{C}\right] = \left\{\int_{V} \left[B\right]^{T} \left[B\right] dV\right\}^{-1}$$
(6.48)

A consistent relationship between the two independent definitons of the strain fields $\{\epsilon_{ij}^1\}$ and $\{\epsilon_{ij}^2\}$ can now be defined by requiring that

$$\{F\}^{T}\{U_{n}\} = \int_{V} \{\sigma_{ij}\}^{T} \{\varepsilon_{ij}^{1}\} dV = \int_{V} \{\sigma_{ij}\}^{T} \{\varepsilon_{ij}^{2}\} dV$$
(6.49)

where $\{\sigma_{ij}\}$ is defined in 6.46 for arbitrary values of $\{F\}$. Substituting 2.7, 5.34, 5.36, 6.46, 6.47 and 6.48 into 6.49 gives the result

$$\{U_n\} = [L]\{\lambda_k^n\} \tag{6.50}$$

where

$$[L] = \left[\overline{C}\right]^T \int_V \left[B\right]^T \left[K(s)\right] dV$$
(6.51)

and

$$[K(s)] = [N][\Lambda(s)] \tag{6.52}$$

This gives the required relationship between the nodal values $\{\lambda_k^n\}$ of the plastic multipliers and the nodal displacements $\{U_n\}$ so that the consistent relationship 6.49 is always satisfied.

7 A General Solution for the Biorthogonality Condition

In the case of axisymmetrical shells, the elements which show constant and conical deformation pattern are those whose meridian angles are constant and in this case it might be expected that Equations 6.46, 6.47 and 6.48 satisfy the required biorthogonality condition 6.45. For curved elements, however, the deformation pattern $\{\epsilon_{ij}^1\}$ varies as function of the meridian angles along the element and it is no longer linear. Although the approximate solution given by 6.47 will also satisfy the required condition 6.45 for a non-linear deformation pattern, it has been found that, using 6.47 the difference between the two strain fields $\{\epsilon_{ij}^1\}$ and $\{\epsilon_{ij}^2\}$ is not negligible. However, a general solution for 6.45 different from 6.50 can be obtained as follows: let [R] be a matrix of the same size as [B] so that

$$[H] = \left\{ \int [B]^T [R] dV \right\}^{-1}$$
(7.53)

is a non-singular matrix. Hence, the following statement can be proved: Equation 6.45 can be satisfied if a matrix [R] can be found such that

$$[\psi] = [R][H] \tag{7.54}$$

Then

$$\int_{V} [\psi]^{T} [B] dV = \int_{V} [H]^{T} [R]^{T} [B] dV$$
(7.55)

where

$$[H]^{T} = \left\{ \int_{V} [R]^{T} [B] dV \right\}^{-1}$$
(7.56)

which leads to

$$\int [\psi]^T [B] dV = [I] \tag{7.57}$$

Thus, the solution given by 6.47 is a special case for [R] = [B].

Let the consistent relationship between the two strain fields, Equation 6.49, be written in the form of the orthogonality condition

$$\int_{V} \{\sigma\}^{T} \left(\{\varepsilon_{1}\} - \{\varepsilon_{2}\}\right) dV = 0$$
(7.58)

where $\{\sigma\}$ is defined by 6.46 and 6.47 so that 7.58 holds for all $\{U_n\}$, $\{\lambda_k^n\}$ and $\{F\}$. It seems that this condition will only provide reliable displacement-plastic multipliers relation for [R] = [B] if $\{\epsilon_{ij}^1\}$ is linear, as $\{\epsilon_{ij}^2\}$ is assumed to be for the present case. Hence, Equations 6.50 to 6.52 provide a consistent relation between displacement field and the plastic multipliers field only if the entries of matrix [B] are at most linear.

However, when the entries of [B] are polynomials of higher order or not polynomials, a more general solution for the biorthogonality condition 7.57, such as that given by 7.54, is required so that

$$\{U_n\} = [H]^T \left\{ \int_V [R]^T [K] dV \right\} \{\lambda_k^n\}$$
(7.59)

when 6.46 and 7.54 are substituted into 6.49. Let matrix now be

$$[L] = [H]^T \left\{ \int_V [R]^T [K] dV \right\}$$
(7.60)

Hence,

$$\{U_n\} = [L]\{\lambda_k^n\} \tag{7.61}$$

Equation 7.61 enforces a general consistency relation between displacement field and plastic multipliers field for each individual element and therefore gives rise to two such relation for each nodal displacement. Once assemblage is performed, interelement continuity may be ensured by taking the average of the nodal values, relative to the sharing node of adjacent elements. Details of this averaging process are given in Appendix B.

The difficulty of this approach is that the matrix [R] is arbitrary and there is no obvious criteria for its choice. In order to discuss a procedure to choose a consistent matrix [R], let the relationship 7.59 be rearranged in simpler form by multiplying both of its sides by $\{\int_{V} [R]^{T} [B] dV\}$, which gives the condition

$$\int [R]^T \left([B] \{ U_n \} - [K] \{ \lambda_k^n \} \right) dV = 0$$
(7.62)

i.e.,

$$\int_{V} [R]^{T} \left(\{ \varepsilon_1 \} - \{ \varepsilon_2 \} \right) dV = 0$$
(7.63)

Such a condition requires that the difference between $\{\epsilon_{ij}^1\}$ and $\{\epsilon_{ij}^2\}$ shall be orthogonal to the arbitrary matrix [R]. The elements of [R] need, therefore, to be chosen so that the strain difference is as small as possible. Expanding 7.63 into its individual components results in

$$\int_{V} R_{11} \left(\varepsilon_{\phi}^{1} - \varepsilon_{\phi}^{2} \right) dV + \int_{V} R_{21} \left(\varepsilon_{\theta}^{1} - \varepsilon_{\theta}^{2} \right) dV = 0$$
(7.64)

$$\int_{V} R_{12} \left(\varepsilon_{\phi}^{1} - \varepsilon_{\phi}^{2} \right) dV + \int_{V} R_{22} \left(\varepsilon_{\theta}^{1} - \varepsilon_{\theta}^{2} \right) dV = 0$$
(7.65)

$$\int_{V} R_{13} \left(\varepsilon_{\phi}^{1} - \varepsilon_{\phi}^{2} \right) dV + \int_{V} R_{23} \left(\varepsilon_{\theta}^{1} - \varepsilon_{\theta}^{2} \right) dV = 0$$
(7.66)

where $dV = 2\pi R(s)h(s)ds$. The components of [R] can now be seen to be a set of functions in a Galerkin type procedure for the minimization of the strain differences. The final choice was as follows

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1 - s/\ell_i)/R & (s/\ell_i)/R \end{bmatrix}$$
(7.67)

This choice implies that Equation 7.64 becomes

$$\int_{V} \left(\varepsilon_{\phi}^{1} - \varepsilon_{\phi}^{2} \right) dV = 0 \tag{7.68}$$

i.e., $(\epsilon_{\phi}^1 - \epsilon_{\phi}^2)$ is zero in the mean within each element and the combination of Equations 7.65 and 7.66 is equivalent to

$$\int_{V} \left[A \left(1 - s/\ell_i \right) + Bs/\ell_i \right] \left(\varepsilon_{\theta}^1 - \varepsilon_{\theta}^2 \right) / R_2 dV = 0$$
(7.69)

which represents the orthogonality of $(\epsilon_{\theta}^1 - \epsilon_{\theta}^2)$ to an arbitrary linear function. It has been found that this gives the identical result to [R] = [B] for cylindrical and conical elements, but a markedly different result for any curved element. The numerical results are discussed in Part 2.

8 Approximate Displacement and Strain Fields for Axisymmetric Shells

For the current stage of the present technique, the global displacement field within the element will be assumed to be interpolated by linear functions. The Equations defining the displacements and strain fields will be given here directly in a matrix form and all the algebraic development is given in Appendix A. Equation 5.28 represents the global displacement field (defined in Section 5) which for an axisymmetric shell element (Fig. 5) becomes

$$\{U_e(s)\} = \left\{ \begin{array}{c} U(s) \\ \\ \\ W(s) \end{array} \right\} = \left\{ \begin{array}{c} U_i \\ \\ 0 \end{array} \right\} + \left[\begin{array}{c} s/\ell_i & 0 & 0 \\ \\ \\ 0 & (1-s/\ell_i) & s/\ell_i \end{array} \right] \left\{ \begin{array}{c} U_{i+1} - U_i \\ \\ W_i \\ \\ W_{i+1} \end{array} \right\}$$
(8.70)

where l_i is the length of a generic element "i" and the rigid body translation and the nodal displacements are defined by the vectors

$$\{U_0\} = \left\{\begin{array}{c} U_i \\ 0 \\ 0 \end{array}\right\}; \{U_n\} = \left\{\begin{array}{c} U_{i+1} - U_i \\ W_i \\ W_{i+1} \end{array}\right\}$$
(8.71)

A simple transformation gives the local displacement field as

$$\begin{cases} u(s,\phi) \\ w(s,\phi) \end{cases} = \begin{cases} U_i \sin\phi \\ -U_i \cos\phi \end{cases} + \begin{bmatrix} \frac{s}{\ell_i} \sin\phi & \left(1 - \frac{s}{\ell_i}\right) \cos\phi & \frac{s}{\ell_i} \cos\phi \\ -\frac{s}{\ell_i} \cos\phi & \left(1 - \frac{s}{\ell_i}\right) \sin\phi & \frac{s}{\ell_i} \sin\phi \end{bmatrix} \begin{cases} U_{i+1} - U_i \\ W_i \\ W_{i+1} \end{cases}$$

where $\phi = \phi(s) = \phi_i + s/R_1$ (Fig. 5).

The strain field, independent of the rigid body motion (see Appendix A), may now be written as

$$\begin{cases} \varepsilon_{\phi}(s,\phi) \\ \varepsilon_{\theta}(s,\phi) \end{cases} = \begin{cases} \frac{\sin\phi}{\ell_{i}} & -\frac{\cos\phi}{\ell_{i}} & \frac{\cos\phi}{\ell_{i}} \\ 0 & \left(1-\frac{s}{\ell_{i}}\right) / \left(R_{2}\sin\phi\right) & \left(\frac{s}{\ell_{i}}\right) / \left(R_{2}\sin\phi\right) \end{cases} \begin{bmatrix} U_{i+1} - U_{i} \\ W_{i} \\ W_{i+1} \end{bmatrix} (8.73)$$

or simply

$$\{\varepsilon(s,\phi)\} = [B] \{U_n\}$$
5.33

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A Displacement and Strain Fields for Axi-Symmetric Shells Discretized into Finite Elements

The displacement field for an element "i" of such type of structures will be described in terms of global displacements in the outward horizontal direction W(s) and in the downward vertical direction U(s) as shown in Fig. 5. Such global displacements is assumed to be interpolated by linear functions in terms of nodal values as

$$U(s) = U_i + \frac{s}{\ell_i} \left(U_{i+1} - U_i \right)$$
(A.74)

$$W(s) = \left(1 - \frac{s}{\ell_1}\right) W_i + \frac{s}{\ell} W_{i+1} \tag{A.75}$$

In a matrix form, the vectors representing the total displacement field and the rigid body translation may be respectively written as

$$\{U_e(s)\} = \left\{\begin{array}{c} U(s)\\ W(s) \end{array}\right\}; \{U_0\} - \left\{\begin{array}{c} U_i\\ 0 \end{array}\right\}$$
(A.76)

The global displacement field can be written in a matrix form, in terms of a nodal values vector

$$\{U_n\} = \left\{ \begin{array}{c} U_{i+1} - U_i \\ W_i \\ W_{i+1} \end{array} \right\}$$
(A.77)

and a matrix of shape functions

$$[\Omega(s)] = \begin{bmatrix} \frac{s}{\ell_i} & 0 & 0\\ 0 & \left(1 - \frac{s}{\ell_i}\right) & \frac{s}{\ell_i} \end{bmatrix}$$
(A.78)

as follows

$$\{U_e(s)\} = \left\{\begin{array}{c} U(s)\\ W(s)\end{array}\right\} = \left\{\begin{array}{c} U_i\\ 0\end{array}\right\} + \left[\begin{array}{c} s/\ell_i & 0 & 0\\ 0 & (1-s/\ell_i) & s/\ell_i\end{array}\right] \left\{\begin{array}{c} U_{i+1} - U_i\\ W_i\\ W_{i+1}\end{array}\right\}$$
(A.79)

or simply

$$\left\{U_e^i(s)\right\} = \left\{U_0^i\right\} + \left[\Omega^i(s)\right] \left\{U_n^i\right\}$$
(A.80)

The local displacement field is obtained by simple transformation as

$$\left\{u_e^i(s,\phi)\right\} = [T]^i \left\{U_e^i(s)\right\}$$
(A.81)

.

where

$$\left\{u_e^i(s,\phi)\right\} = \left\{\begin{array}{c}u(s,\phi)\\w(s,\phi)\end{array}\right\}$$
(A.82)

which gives

$$\begin{cases} u(s,\phi) \\ w(s,\phi) \end{cases} = \begin{cases} U_i \sin\phi \\ -U_1 \cos\phi \end{cases} + \begin{bmatrix} \frac{s}{\ell_i} \sin\phi & \left(1 - \frac{s}{\ell_i}\right) \cos\phi & \frac{s}{\ell_i} \cos\phi \\ \frac{s}{\ell_i} \cos\phi & \left(1 - \frac{s}{\ell_i}\right) \sin\phi & \frac{s}{\ell_i} \sin\phi \end{bmatrix} \begin{cases} U_{i+1} - U_i \\ W_i \\ W_{i+1} \end{cases}$$
A.83)

In a matrix form it becomes

.

$$\{u_e(s)\} = \{u_0\} + [F(s)]\{U_n\}$$
(A.84)

The strain field is obtained from such a local displacement field, without the rigid body motion vector $\{u_0\}$, by means of the strain displacement relation for the linear theory (Equation 2.7). In the case of thin shells of revolution, it will be defined within the element by the meridional and circumferential strains only, with no curvature involved. Equation 2.7 is given for thin shells by, for example, [37] in terms of nodal values as

$$\{\varepsilon(s,\phi)\} = \left\{ \begin{array}{c} \varepsilon_{\phi}(s,\phi) \\ \varepsilon_{\theta}(s,\phi) \end{array} \right\} = \left[\begin{array}{c} \frac{d}{ds} & \frac{1}{R_1} \\ \frac{\cot\phi}{R_2} & \frac{1}{R_2} \end{array} \right] [F] \{U_n\}$$
(A.85)

Matrix [B] in equation (2.5) is defined as

$$[B] = \begin{bmatrix} \frac{d}{ds} & \frac{1}{R_1} \\ \\ \frac{\cot \phi}{R_2} & \frac{1}{R_2} \end{bmatrix} [F]$$
(A.86)

which gives rise to

$$[B] = \begin{bmatrix} \frac{\sin\phi}{\ell_i} & \frac{-\cos\phi}{\ell_i} & \frac{\cos\phi}{\ell_i} \\ 0 & \frac{1}{R_2} \left(1 - \frac{s}{\ell_i}\right) \frac{1}{\sin\phi} & \frac{1}{R_2} \frac{s}{\ell_i} \frac{1}{\sin\phi} \end{bmatrix}$$
(A.87)

when [F] is given by A.83.

B Averaging the Nodal Consistent Relationship between the Global Displacements 's and the Plastic Multipliers

The consistent displacement formulation proposed in Section 7 gives rise to a relationship between nodal values of W's and λ 's for each element. Consequently, two of such relationship for each node is produced since adjacent elements share one node as illustrated in Fig. B1. In order to obtain a single nodal relation and therefore a continuous displacement field, the average the two equations corresponding to the adjacent elements is adopted. From Equation 7.61 and Fig. B1 the consistent relationships for such adjacent elements can be written as

Element i-1

$$\begin{cases} U_{i} - U_{i-1} \\ W_{i-1} \\ W_{i} \end{cases} \} = \begin{bmatrix} L_{11}^{i-1} & L_{12}^{i-1} \dots & L_{16}^{i-1} & L_{17}^{i-1} & L_{18}^{i-1} \dots & L_{112}^{i-1} \\ L_{21}^{i-1} & L_{22}^{i-1} \dots & L_{26}^{i-1} & L_{27}^{i-1} & L_{28}^{i-1} \dots & L_{212}^{i-1} \\ L_{31}^{i-1} & L_{32}^{i-1} \dots & L_{36}^{i-1} & L_{37}^{i-1} & L_{38}^{i-1} \dots & L_{312}^{i-1} \end{bmatrix} \begin{cases} \lambda_{1}^{i-1} \\ \vdots \\ \lambda_{6}^{i-1} \\ \lambda_{1}^{i} \\ \vdots \\ \lambda_{6}^{i} \end{cases}$$
(B.88)

. .

Element i

$$\begin{pmatrix} U_{i+1} - U_i \\ W_i \\ W_{i+1} \end{pmatrix} = \begin{bmatrix} L_{11}^i & L_{12}^i \dots L_{16}^i & L_{17}^i & L_{18}^i \dots L_{112}^i \\ L_{21}^i & L_{22}^i \dots L_{26}^i & L_{27}^i & L_{28}^i \dots L_{212}^i \\ L_{31}^i & L_{32}^i \dots L_{36}^i & L_{37}^i & L_{38}^i \dots L_{312}^i \end{bmatrix} \begin{cases} \lambda_1^i \\ \vdots \\ \lambda_6^i \\ \lambda_i^{i+1} \\ \vdots \\ \lambda_6^{i+1} \end{cases}$$
(B.89)

A single general equation representing a single relationship between W's and λ 's for a node "i" can thus be written from B.88 and B.89 as

$$W_{i} = \frac{1}{2} \left\{ \sum_{k=1}^{6} \left[L_{3k}^{i-1} \lambda_{k}^{i-1} + \left(L_{3k+6}^{i-1} + L_{2k}^{i} \right) \lambda_{k}^{i} + L_{2k+6}^{i} \lambda_{k}^{i+1} \right] \right\}$$
(B.90)

which involves plastic multipliers from three nodes i - 1, iandi + 1. In a matrix form such single relationships for all the nodes can be written as

$$\begin{cases} W_{1} \\ \vdots \\ W_{i} \\ \vdots \\ W_{N} \end{cases} = \frac{1}{2} \begin{bmatrix} 0 \dots 0 & 2 \times L_{21}^{1} \dots 2 \times L_{26}^{1} & 2 \times L_{27}^{1} \dots 2 \times L_{212}^{1} \\ L_{31}^{i-1} \dots L_{36}^{i-1} & \left(L_{37}^{i-1} + L_{21}^{i}\right) \dots \left(L_{312}^{i-1} + L_{26}^{i}\right) & L_{27}^{i} \dots L_{212}^{i} \\ 2 \times L_{31}^{N-1} \dots 2 \times L_{36}^{N-1} & 2 \times L_{37}^{N-1} \dots 2 \times L_{312}^{N-1} & 0 \dots 0 \end{bmatrix} \begin{bmatrix} \lambda_{1}^{i} \\ \vdots \\ \lambda_{6}^{i} \\ \lambda_{1}^{i} \\ \vdots \\ \lambda_{6}^{i} \\ \lambda_{1}^{i+1} \\ \vdots \\ \lambda_{6}^{i+1} \end{bmatrix}$$