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Computational Methods in Nonlinear Mechanics

by

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Plan of this Report

This report is divided into two major parts. In PART I, brief summaries of the principal research results are given together with lists of published papers, reports, dissertations, theses, and oral presentations that resulted from the work. Also, the personnel who worked on the project are listed. In PART II, summary accounts of three selected technical areas are given and suggestions for fruitful areas of future work are listed. Many results uncovered during the course of the project are not dealt with in PART II; rather, three subjects representative of what are felt to be of principal importance in the overall work are discussed. Further details of the technical results can be found in the papers and reports listed in PART I.

PART I

SUMMARY OF RESEARCH RESULTS

1. INTRODUCTION

This document represents the concluding report of research results obtained on the project, Computational Methods in Nonlinear Mechanics, during the period June 1, 1980 to September 30, 1982. The project was supported by the Air Force Office of Scientific Research under Contract F-49620-C-0083 and was monitored by Dr. Anthony Amos of AFOSR.

All objectives originally listed in the Statement of Work of the proposal for this work have been accomplished. Indeed, a number of research results which are regarded as very important in the study of the nonlinear mechanical behavior of complex structures by finite element methods have been established. These have represented advances in specific areas of numerical analysis, computational algorithms, computer codes for nonlinear structural problems, modelling and characterization of various nonlinear effects, and resolving theoretical questions connected with nonlinear theories of structural behavior. More details on these results are given later.

During the two year contract period, the research effort led to the publication of 33 journal articles and scientific papers, 11 technical reports, 5 Ph.D. dissertations, 2 M.S. theses, and 50 oral presentations at conferences, seminars, and invited lecture series. The principal investigator of the project was Professor J. Tinsley Oden and the co-principal investigator was Professor E. B. Becker, both of the University of Texas at Austin. Lists of other personnel who worked on the project during the contract period are given in Chapter 3.

2. PRINCIPAL ADVANCES

Significant research results in a number of distinct areas were obtained during the course of this project. However, it is felt that especially penetrating and important results in several specific areas were obtained. In the following, we list what are considered some of the principal contributions made in this research project.

1. Finite Element Methods for Contact Problems in Elastostatics

A large volume of work was done on the problem of contact between deformable elastic bodies. In this general area, a variety of results were obtained on the description of the physical phenomena, development of mathematical models using the theory of variational inequalities, the study of existence, uniqueness, and regularity of solutions of the variational inequalities, development of corresponding approximation theories, the development of numerical stability criteria, error estimates, convergence criteria, development of families of completely new algorithms for solving resulting systems of linear and nonlinear inequalities, development of programming strategies and finite element codes for the analysis of such problems, and the actual analysis of numerous example problems drawn from elasticity, plasticity, and elastoplasticity in which contact conditions are encountered. Included in this large collection of results are several completely new variational principles for contact problems in elasticity, new finite element methods which employ such devices as interior and exterior penalty methods, reduced integration, mixed finite element formulations, and techniques drawn from linear and nonlinear programming.

The theoretical results were not limited to simple unilateral contact. Indeed, new results were obtained for problems of elastic stability and buckling of thin elastic plates and shells supported by unilateral constraints and studies were made of a complex family of bifurcation problems characterized by nonlinear variational inequalities.

Many of the results obtained in this general area are summarized in a forthcoming treatise entitled, "Contact Problems in Elasticity," by the principal investigator and Professor N. Kikuchi, who worked on the project periodically during the contract period. This volume is to be published by SIAM Publications (Society of Industrial and Applied Mathematics, Philadelphia, PA) and should appear in early 1983. It is believed that there has never before been a more detailed and thorough analysis of this difficult class of nonlinear problems in structural and solid mechanics. A brief summary of some of these results is given in chapter 4 of this report.

2. Reduced Integration-Penalty Finite Element Method for Constrained Problems in Elasticity and Fluid Flow

Some numerical instabilities encountered early in the project led the investigators into the study of a collection of difficult numerical and theoretical issues connected with the convergence and numerical stability of a variety of popular finite element methods frequently used to study nonlinear problems in solid and fluid mechanics. This particular thrust of the research, which was never anticipated in the original research plan, proved to be one of the most fruitful and important areas on investigation of the entire project.

Particular attention was focused on the idea of using exterior penalty methods and selective reduced integraiton to handle constrained problems in incompressible elasticity, incompressible elastoplasticity, contact problems in elastostatics, and constraints in plate and shell theories. These methods have been in wide use on an international scale in 1976. It was discovered in the present project, however, that many of the more popular methods in use can be dramatically unstable when the finite element mesh is refined and, in fact, lead to divergent approximations of stresses and pressures. These results, it is felt, had a significant impact on this subject as a whole and has changed the thinking on the use of penalty methods throughout this country and abroad. It was discovered that a key condition for the numerical stability of such methods rests in the so-called discrete LBB condiditon (Ladyszhkenskaya, Babuska, Brezzi) which involves a stability parameter, α_h , which governs the stability and convergence characteristics of most of these methods. As noted earlier, it was discovered that many of the more popular finite element methods now in use do not satisfy this condition, and, therefore, can be unstable. The question then arose as to whether or not elements exist which are stable, numerically robust, and which converge at optimal rates of convergence. In the latter phases of the research effort, several such optimal, stable, and convergent methods in this general family have been discovered. These have been completely analyzed from a mathematical point of view and also by numerical experiments. A summary of some of these results is given in Chapter 6 of this report,

3. Contact Problems with Friction

Several classes of contact problems with friction have been analyzed during the course of the project. Numerical schemes and algorithms have been developed for certain special cases, together with proofs of convergence and error estimates. In particular, special problems in which the normal contact pressure is prescribed on surfaces on which frictional forces can be developed have been analyzed in some detail and several papers have been written on this subject. It was concluded during the last year of the project, however, that many of the principal physical and numerical difficulties encountered in this class of problems arose from the inadequacy of the description of friction. Consequently, a completely new line of research was undertaken to study new models of friction. This is summarized briefly in the paragraphs which follow.

4. Non-Classical Friction Laws

During the last four to six months of the project, considerable attention has been given to the study of modelling of static friction between metallic bodies using friction laws which deviate markedly from the classical pointwise law proposed by Coulomb 200 years ago. In particular, on the basis of micro-mechanical mechanisms, some new friction laws have been proposed which feature 1) a nonlocal description of the contact stress in the criteria for sliding and 2) a nonlinear friction law in which the elastic and elastic-plastic response of metallic junctions on the contact surface are taken into account. This has led to some new variational principles for contact problems with friction. At this writing, the full implication of these new theories has not yet been understood, but it is clear that they are sufficiently general to lead to a much better modelling of friction effects in solid and structural components. A summary of some of these ideas is given in chapter 5 of this report.

5. Large Deformation Plasticity and Metal Forming

Considerable effort was spent on the development of finite element simulators of large deformations of elastic-plastic materials. A computer program was developed for the study of large deformation plasticity problems and metal forming which is applicable to a broad class of problems in plane stress, plane strain, and axisymmetric deformation of bodies of revolution. Some impressive numerical results have been obtained from this code, which apparently surpass all existing commerical codes in terms of accuracy, efficiency, and overall applicability. Nevertheless, the convergence and numerical stability characteristics of these methods are still not well understood, and it is clear from some of the computed results that the choice of an appropriate friction law on contact surfaces has a significant impact on computed distributions of residual stresses.

6. Other Areas

Significant research results in a number of other areas were produced during the contract period. These have been discussed in great detail in some of the interim reports submitted to AFOSR over the past 24 months, and therefore are not discussed in detail here. However, some results in this general area are of sufficient importance that they deserve mention.

We first note that new and useful results were obtained on the behavior of finite element methods for singular problems in structural mechanics, particularly for problems with stress singularities. These mathematical results represent the most general that have yet been obtained in this area and provide concrete estimates for singular problems. The results focus on the behavior of finite element models which employ various singular elements, a subject not discussed in any mathematical literature to date. It was shown been written on this subject. It was concluded during the last year of the project, however, that many of the principal physical and numerical difficulties encountered in this class of problems arose from the inadequac of the description of friction. Consequently, a completely new line of research was undertaken to study new models of friction. This is summarize briefly in the paragraphs which follow.

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We also note that new results on viscous incompressible flow problems were also produced. These results were primarily the outcome of studies on reduced integration penalty methods, and on the numerical stability of various finite element techniques which were developed to study problems in finite elasticity, plasticity, and contact problems in elastostatics.

3. SUMMARY OF PUBLISHED RESULTS AND PROJECT PERSONNEL

In this chapter we summarize the publications and oral presentations of results obtatined during the contract period.

3.1 Journal Articles and Scientific Papers

1. Oden, J.T. and Kikuchi, N., "Theory of Variational Inequalities with Applications to Problems of Flow Through Porous Media," International Journal of Engineering Science, Vol. 18, No. 10, pp. 1173-1284, 1980.

2. Oden, J.T. and LeTallec, P., "On the Existence of Hydrostatic Pressure in Regular Finite Deformations of Incompressible Hyperelastic Solids," Nonlinear Partial Differential Equations in Engineering and Applied Science, Marcel Dekker, Inc., pp. 1-8, 1980.

3. Oden, J.T. and Bernadou, M., "Theoreme d'Existence pour une Classe de Problemes Nonlineaires de Coques Peu Profondes," Compte-Rendus-Academy of Science, Paris, (Analyse Numerique), pp. 1025-1028, 1980.

4. Oden, J.T., "Penalty-Finite Element Approximations of Unilateral Problems in Elasticity," Approximation Theory, Edited by W. Cheney, Academic Press, New York, pp. 693-697, 1980.

5. Oden, J.T. and Kikuchi, N., "Use of Variational Methods for the Analysis of Contact Problems in Solid Mechanics," Variational Methods in the Mechanics of Solids, Pergamon Press Ltd., Oxford, pp. 259-264, 1980.

6. Oden, J.T., Ohtake, K., and Kikuchi, N., "Analysis of Certain Unilateral Problems in Von Karman Plate Theory by a Penalty Method Part 1. A Varitional Principle with Penalty," Computer Methods in Applied Mechanics in Engineering, Vol. 24, pp. 187-213, 1980.

7. Oden, J.T., Ohtake, K., and Kikuchi, N., "Analysis of Certain Unilateral Problems in Von Karman Plate Theory by a Penalty Method Part 2. Approximation and Numerical Analysis," Computer Methods in Applied Mechanics and Engineering, Vol. 24, pp. 317-337, 1980.

8. Oden, J.T. and LeTallec, P., "Existence and Characterization of Hydrostatic Pressure in Finite Deformations of Incompressible Elastic Bodies," Journal of Elasticity, Vol. 11, No. 4, pp. 341-357, 1981.

9. Oden, J.T. and Bernadou, M., "An Existence Theorem for a Class of Nonlinear Shallow Shell Problems," Journal de Mathematiques Pures et Appliquees, Vol. 60, No. 3, pp. 285-308, 1981.

10. Oden, J.T. and Kubrusly, R., "Nonlinear Eigenvalue Problems Characterized by Variational Inequalities with Applications to the Postbuckling Analysis of Unilaterally Supported Plates," Journal of Nonlinear Analysis, Vol. 5, No. 12, pp. 1265-1284, 1981.

11. Oden, J.T. and Campos, L., "Some New Results on Finite Element Methods for Contact Problems with Friction," New Concepts in Finite Element Methods, ed. by M. Spilker, T.J.R. Hughes, and D. Gartling, A.S.M.E. Monograph, New York, pp. 1-12, 1981.

12. Oden, J.T. and Carey, G.F., "Variational Inequalities in Finite Element Analysis" New Concepts in Finite Element Methods, ed. by M. Spilker, T.J.R. Hughes, and D. Gartling, A.S.M.E. Monograph, New York, pp. 133-145, 1981.

13. Oden, J.T., "Exterior Penalty Methods for Contact Problems in Elasticity: Nonlinear Finite Element Analysis in Structural Mechanics," Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, pp. 655-665, 1981.

14. Oden, J.T. and Demkowicz, L., "On Some Existence and Uniqueness Results on Contact Problems with Nonlocal Friction," Journal of Nonlinear Analysis, 1982 (to appear)

15. Oden, J.T., Campos, L., and Kikuchi, N., "A Numerical Analysis of a Class of Contact Problems with Friction in Elastostatics," Computer Methods in Applied Mechanics and Engineering, 1982 (to appear)

16. Oden, J.T., "Analysis of a Class of Contact Problems with Friction by Finite Element Methods," The Mathematics of Finite Elements with Applications, ed. by J.R. Whiteman, Academic Press LTD., London, 1982 (to appear)

17. Oden, J.T. and Kim, S.J., "Interior Penalty Methods for Finite Element Approximation of the Signorini Problem in Elastostatics," Computers and Mathematics with Applications, Vol. 8, No.1, pp. 35-56, 1982.

18. Oden, J.T., Song, Y.J., and Kikuchi, N., "Penalty-Finite Element Methods for the Analysis of Stokesian Flows," Computer Methods in Applied Mechanics and Engineering, 1982, (to appear).

19. Oden, J.T. and Kikuchi, N., "Finite Element Methods for Constrained Problems in Elasticity," International Journal for Numerical Methods in Engineering, Vol. 18, pp. 701-725, 1982.

20. Oden, J.T., "Analysis of Galerkin Approximations of a Class of Pseudomonotone Diffusion Problems," SIAM Journal of Mathematical Analysis, Vol. 12, No. 6, pp. 917-930, 1982.

21. Oden, J.T. and Pires, E., "Numerical Analysis of Certain Contact Problems in Elasticity with Non-Classical Friction Laws," Computers and Structures, 1982, (to appear)

22. Oden, J.T. and Whiteman, J.R., "Analysis of Some Finite Element Methods for a Class of Problems in Elasto-Plasticity," International Journal of Engineering Science, Vol. 20, 1982.

23. Oden, J.T., "Penalty Methods for Constrained Problems in Nonlinear

Elasticity," IUTAM Symposium on Finite Elasticity, Martinum-Nijhoff Pub., The Hague, pp. 281-300, 1982.

24. Oden, J.T., "RIP Methods for Stokesian Flows," Finite Elements in Fluids, Vol. IV, ed. R.H. Gallagher, O.C. Zienkiewicz, N. Norrie, John Wiley & Sons, LTD., London, (to appear)

25. Oden, J.T., "Finite Element Methods for Constrained Problems in Elasticity and Fluid Mechanics," Proceedings of Symposium on Finite Element Methods, Hefei, China, 1981, Science Press, (to appear)

26. Oden, J.T., "Mixed Finite Element Approximations via Interior and Exterior Penalties for Contact Problems in Elasticity," Hybrid and Mixed Finite Element Methods, ed. by S. Atluri, John Wiley & Sons Ltd., London, (to appear)

27. Oden, J.T. and Pires, E.B., "Algorithms and Numerical Results for Finite Element Approximations of Contact Problems with Non-Classical Friction Laws," Computers and Structures, (to appear)

28. Oden, J.T. and Jacquotte, O., "Stable Second-Order Accurate Finite Element Scheme for the Analysis of Two-Dimensional Incompressible Viscous Flows," International Conference on Finite Element Methods in Fluids, Tokyo, Japan, (to appear)

29. Oden, J.T. and Pires, E., "On the Analysis of a Class of Contact Problems with Non-local Friction," Proceedings of SECTAM XI, (to appear)

30. Oden, J.T. and Pires, E., "On the Signorini Problem with Non-local Friction," Sixtieth Anniversary Volume in Honor of O.C. Zienkiewicz, John Wiley and Sons, Ltd., London, (to appear)

31. Oden, J.T. and Pires, E., "Nonlocal and Nonlinear Friciton Laws and Variational Principles for Contact Problems in Elasticity," Journal of Applied Mechanics, (to appear)

32. Oden, J.T. and Pires, E., "Error Estimated for the Approximations of a Class of Variational Inequalities Arising in Unilateral Problems with Friction," International Journal of Numerical Functional Analysis and Optimization, (to appear)

33. Oden, J.T. and Campos, L., "Nonquasi-Convex Problems in Nonlinear Elastostatics," (to appear).

3.2 Research Reports

1. Oden, J.T. and Bernadou, M., "An Existence Theorem for a Class of Nonlinear Shallow Shell Problems," TICOM Report 80-4, Austin, 1980.

2. Oden, J.T. and Bernadou, M., "An Existence Theorem for a Class of Nonlinear Shallow Shell Problems," INRIA Report No. 17, Le Chesnay, France, 1980.

3. Oden, J.T., Kikuchi, N., and Song, Y.J., "RIP Methods for Problems in Elasticity," TICOM Report 80-7, Austin, 1980.

4. Oden, J.T., Kikuchi, N., and Song, Y.J., "Reduced Integration and Exterior Penalty Methods for Finite Element Approximations of Contact Problems in Incompressible Elasticity," TICOM Report 80-2, Austin, 1980.

5. Oden, J.T., "RIP Methods for Stokesian Flows," TICOM Report 80-11, Austin, 1980.

6. Oden, J.T. and Kikuchi, N., "Finite Element Methods for Constrained Problems in Elasticity," TICOM Report 81-10, Austin, 1981.

7. Oden, J.T., Kikuchi, N., and Song, Y.J., "Penalty-Finite Element Methods for the Analysis of Stokesian Flows," TICOM Report 81-11, Austin, 1981.

8. Oden, J.T. and Pires, E., "Contact Problems in Elastostatics with Non-local Friction Laws," TICOM Report 81-12, Austin, 1981.

9. Oden, J.T. and Demkowicz, L., "On Some Existence and Uniqueness Results in Contact Problems with Nonlocal Friction," TICOM Report 81-13, Austin, 1981.

10. Oden, J.T. and Pires, E.B., "Nonlocal and Nonlinear Friction Laws and Variational Principles for Contact Problems in Elasticity," TICOM Report 82-3, Austin, 1982.

11. Oden, J.T. and Campos, L.T., "Nonquasi-Convex Problems in Nonlinear Elastostatics," TICOM Report 82-4, Austin, 1982.

3.3 Theses and Dissertations

The following Ph.D. dissertations and Masters of Science these were completed during the project.

1. Campos, Luis, "A Numerical Analysis of a Class of Contact Problems with Friction in Elastostatics," M.S. Thesis, The University of Texas at Austin, January 1981.

2. Kim, S. J., "Interior Penalty Approach to Contact Problems," M.S. Thesis, The University of Texas at Austin, January 1981.

3. Song, Y. J., "Reduced Integration and Exterior Penalty Methods for Finite Element Approximations of Contact Problems in Incompressible Linear Elasticity," Ph.D. dissertation, The University of Texas at Austin, August 1980.

4. Aly, A., " A Finite Element Analysis for Problems of Large Strain and Large Displacement," Ph.D. dissertation, The University of Texas at Austin, May 1981.

5. Kubrusly, R. S., "Variational Methods for Nonlinear Eigenvalue Problems in the Post-Buckling Analysis of Unilaterally Constrained Elastic Structures," Ph.D. dissertation, The Unviersity of Texas at Austin, August 1981.

6. O'Leary, J. T., "An Error Analysis for Singular Finite Elements," Ph.D. dissertation, The University of Texas at Austin, August 1981.

3.4 Oral Presentations

Over 50 oral presentations were given by members of the research team during the contract period on research results. A partial list of these is given as follows:

1. Oden, J.T.,"Penalty Methods and Selective Reduced Integration for Stokesian Flows," Third International Conference on Finite Elements in Flow Problems, Banff Centre, Banff, Alberta, Canada, June 10-13, 1980.

2. Oden, J.T., "Penalty Methods and Reduced Integration in Elasticity Problems," Symposium on Finite Element Methods for Nonlinear and Singular Problems, University of Durham, Durham, England, June 26-July 6, 1980.

3. Oden, J.T., "Analysis of Incompressible Elastic Bodies by Extensive Penalty Methods," U.S./Europe Workshop on Nonlinear Finite Element Analysis in Structural Mechanics, Ruhr University, Bochum, West Germany, July 28-30, 1980.

4. Oden, J.T., "Penalty Methods for Constrained Problems in Nonlinear Elasticity," IUTAM Symposium on Finite Elasticity, LeHigh University, Bethlehem, PA, August 11-15, 1980.

5. Kikuchi, N., "Penalty Finite Element Approximations of a Class of Contact Problems in Linear Elasticity," Presentation made to Exxon Products Research Laboratories, Houston, Texas, August 14, 1980.

6. Oden, J.T., "Penalty Methods for Stokesian Flows," Engineering Mechanics Seminar, The University of Texas at Austin, September 23 and 25, 1980.

7. Oden, J.T., "Constrained Problems in Nonlinear Elasticity," Annual meeting of the Society of Engineering Science, Atlanta, Georgia, December, 1980.

8. Oden, J.T., "Contact Problems with Friction in Elastostatics," Annual meeting of the Society of Engineering Science, Atlanta, Georgia, December, 1980.

9. Oden, J.T., "Finite Element Methods for Contact Problems with Friction," Department of Mathematics, University of Maryland, February 25, 1981.

10. Oden, J.T., "Penalty Methods for Stokesian Flows," Mathematics Seminar, University of Texas at Austin, March 23, 1981.

11. Oden, J.T., "Mixed Finite Element Approximations via Exterior and Interior Penalty Formulations of Contact Problems in Elasticity," Symposium on Hybrid and Mixed Finite Element Methods, Georgia Institute of Technology, Atlanta, Georgia, April 10, 1981.

12. Oden, J.T., "Finite Element Methods for Contact Problems with Friction MAFELAP '81, Fourth Conference on the Mathematics of Finite Elements with Applications, Brunel University, Uxbridge, England, April, 1981.

13. Oden, J.T., "Non-Local Friction Laws in Elastostatics," Department of Civil Engineering, The University of Swansea, Wales, United Kingdom, April 1981.

14. Oden, J.T., "An Introduction to the Exterior Penalty Methods and Reduced Integration and Their Application to Constrained Problems in Elasticity and Fluid Mechanics," Mathematics Seminar, Brunel University, Uxbridge, England, May 13, 1981.

15. Oden, J.T., "Penalty-Finite Element Methods for Constrained Problems in Elasticity," Invitational Symposium on Finite Element Methods, Polytechnical University, Hefei, People's Republic of China, May 19-23, 1981.

16. Oden, J.T., "Some New Results on Finite Element Methods for Contact Problems with Friction," Symposium on New Concepts in Finite Element Methods, Joint ASME/ASCE Mechanics Conference, The University of Colorado, Boulder, June 23, 1981.

17. Oden, J.T., "Variational Inequalities in Finite Element Analysis," Symposium on New Concepts in Finite Element Methods, Joint ASME/ASCE Mechanics Conference, The University of Colorado, Boulder, June 23, 1981.

18. Oden, J.T., "Numerical Analysis of a Class of Contact Problems with Friction in Elastostatics," FENOMECH 1981, Stuttgart, West Germany, August 26-28, 1981.

19. Oden, J.T., "Nonconvex Problems and Phase Transition in Nonlinear Materials," SECTAM, Huntsville, Alabama, April, 1982.

20. Oden, J.T., "Mathematical Theory of Plasticity," EM/TICOM Seminar, The University of Texas at Austin, April 1982.

21. Oden, J.T., "A Stable Second-Order Accurate Finite Element Scheme for the Analysis of Two-Dimensional Incompressible Viscous Flows," 4th International Symposium of Finite Element Methods in Flow Problems, Tokyo, Japan, July 27, 1982.

PROJECT PERSONNEL

The following individuals worked on the project during the contract period:

1) Principal Research Staff:

Dr. J. Tinsley Oden, Principal Investigator

Dr. E. B. Becker, Co-Principal Inviestigator

Dr. N. Kikuchi, Senior Research Associate

Dr. L. Demkowicz, Senior Research Associate

2) Senior Research Assistants:

J. O'Leary	L. Campos
Y.J. Song	E. Pires
S.J. Kim	R. Kubrusly

Messrs. T. Pan, Jerry Fine, R. Chambers and P. Halamek devoted a very small percentage of their time to certain phases of the project.

3) Project Secretary:

The project secretary has been Mrs. Ruth Dye during the past year. During the first year of the project, Ms. Nancy Webster served as project secretary. Some additional part-time secretarial assistance was required occassionally during the project.

PART II

SOME TECHNICAL SUMMARIES

4. THEORY AND APPROXIMATION OF CONTACT PROBLEMS IN ELASTOSTATICS

1. Introduction

Perhaps the most primitive and intrinsic feature of the mechanics of solids is the contact of one body with another. Contact, in fact, is precisely the physical event through which loads are delivered to a structure and by which a structure transmits forces to its supports. Nevertheless, this fundamentally important aspect of structural behavior has, until recently, rarely been taken into account in practical structural analysis and design. The underlying difficulty is that contact problems in solid mechanics are inherently nonlinear: the area of contact is not known prior to the application of loads and complex physical phenomena are experienced on the contact surfaces which often require special mechanical and mathematical considerations.

In recent years, however, significant advances have been made in the study of certain restricted classes of contact problems by finite element methods. These classes of problems include those adequately modelled by the so-called Signorini problem in elastostatics: the behavior of a linearly elastic body in unilateral contact with a rigid frictionless foundation. This particular class of problems can be studied within the framework of the theory of variational inequalities and; consequently, a great deal can be established on the qualitative behavior of the solutions and their finite element approximations. In fact, there are few nonlinear problems in structural mechanics for which a more complete mathematical basis exists.

The present chapter is devoted to a general exposition on finite element methods for contact problems in elastostatics that can be characterized by variational inequalities. In keeping with the general objectives of this project, this class of problems is chosen for consideration because of its practical importance, the richness of the mathematical foundations on which it is based, the level of results one can establish on the behavior (e.g. convergence and numerical stability) of corresponding finite element approximations, and because this class of numerical problems provides the opportunity to examine many new algorithms for treating systems of linear and nonlinear inequalities which are drawn from linear and nonlinear programming theory and optimization.

The account of the subject which follows deals primarily with finite element approximations of the Signorini problem for linearly elastic bodies including, for the sake of completeness, formulations in which friction is taken into account. A lengthy treatise on this and related subjects has been recently completed by Kikuchi and Oden [1983], and this work can be consulted for a complete list of references on the subject and for more details. Following this introductory section, the system of differential equations and boundary conditions for Signorini's problem with friction is derived. In Section 4.3, several variational formulations of contact problems are derived, all of which feature variational inequalities given in terms of an unknown equilibrium displacement field. Existence, uniqueness, and approximation results for the case in which tangential contact pressures are prescribed is taken up in Section 4.4. These problems include as a special case the classical Signorini problem with no friction. Section 4.5 is devoted to contact problems in which the normal contact pressure is prescribed. These classes of problems are shown to be of fundamental importance in the analysis of problems with non-local friction in Chapter 5. The results of several numerical experiments are also discussed in Sections 4.4 and 4.5; concluding comments are collected in a final section.

2. Signorini's Problem with Friction

Consider a deformable body supported unilaterally by a rigid body as shown in Figure 4.1., and suppose that the contact surface of the two bodies is unknown a priori. The problem of finding the deformation and contact force for equilibrium configurations of the body under certain boundary and loading conditions is called Signorini's problem. We note that in most works in theoretical mechanics Signorini's problem is discussed in regard to the special case in which no friction exists on the contact surface. We first formulate Signorini's problem with friction for the case of small deformations in which the displacement field is small enough so that higher order terms of the displacement gradient can be neglected in the equilibrium and strain-displacement equations. The stress tensor $\underset{\sim}{\sigma} = \sigma_{ij}\underset{\sim}{i}_i \otimes \underset{\sim}{i}_j$* is then the Cauchy stress tensor and no distinction between particles or points is necessary in defining its domain.

4.2.1. Equilibrium Equations

Let $\underset{\sim}{u} = u_i\underset{\sim}{i}_i$ be the displacement field that provides equilibrium of the deformable body under given loading and boundary conditions, and let $\underset{\sim}{\varepsilon} = \varepsilon_{ij}\underset{\sim}{i}_i \otimes \underset{\sim}{i}_j$ denote the strain tensor for small deformation defined, for an arbitrary displacement $\underset{\sim}{v}$, by

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*The summation convention is applied throughout this chapter, unless specifically noted otherwise.

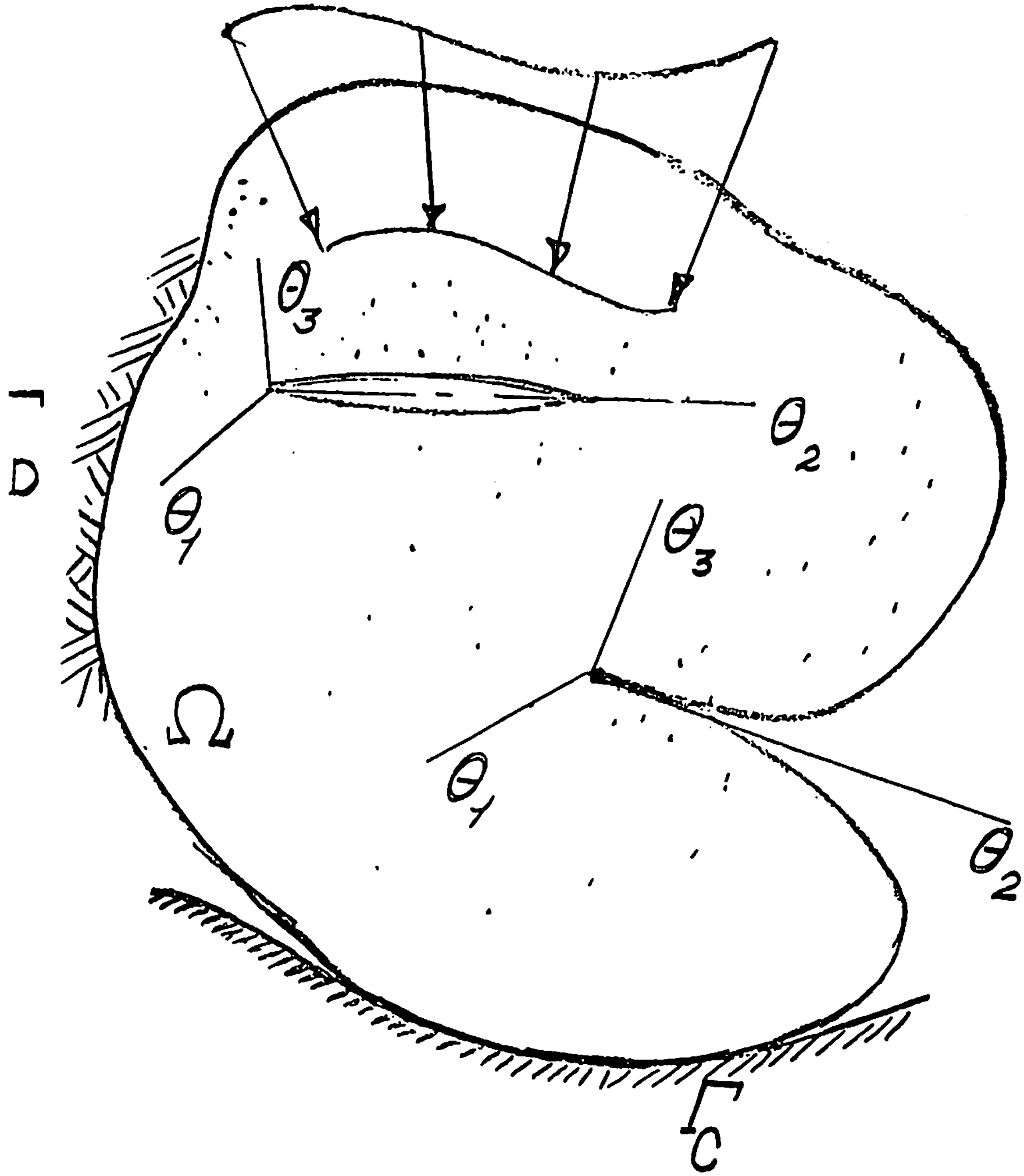


Figure 4.1. Geometry of an elastic body near a rigid foundation.

$$\varepsilon_{ij}(\underset{\sim}{v}) = (v_{i,j} + v_{j,i})/2 \tag{4.2.1}$$

where $v_{i,j}$ denotes the partial derivative of v_i in the j coordinate; i.e., $v_{i,j} = \partial v_i/\partial x_j$. Suppose that the stress-strain relation

$$\sigma_{ij} = E_{ijk\ell}\varepsilon_{k\ell} \tag{4.2.2}$$

is given, wherein the elasticities $E_{ijk\ell}$ of the material satisfy the conditions

$$E_{ijk\ell} = E_{k\ell ij} = E_{jik\ell} \tag{4.2.3}$$

$$m_c > 0, \quad E_{ijk\ell}(\underset{\sim}{x})X_{k\ell}X_{ij} > m_c X_{ij}X_{ij}, \quad \underset{\sim}{x} \quad \Omega, \quad X_{ij} = X_{ji} \tag{4.2.4}$$

$$\underset{\underset{\sim}{x} \quad \Omega}{\text{Max}} \; |E_{ijk\ell}(\underset{\sim}{x})| < M \quad \text{for all } i,j,k, \text{ and } \ell. \tag{4.2.5}$$

Here Ω is an open-bounded domain in R^3 representing the interior of the deformable body.

The condition $E_{ijk\ell} = E_{k\ell ij}$ guarantees the existence of the strain energy function E_0 such that

$$\sigma_{ij} = \partial E_0/\partial\varepsilon_{ij}, \tag{4.2.6}$$

$$E_0 = \frac{1}{2} E_{ijk\ell}\varepsilon_{k\ell}\varepsilon_{ij}. \tag{4.2.7}$$

Condition (4.2.4) is sufficient to imply strict convexity of the strain energy function E_0, and guarantees the coercivity of the total potential energy of the body, as shown later. Condition (4.2.5) merely establishes boundedness of the elasticities $E_{ijk\ell}$; these material parameters may be discontinuous in the domain Ω.

If $\boldsymbol{\sigma}(\mathbf{u})$ denotes the stress field at the equilibrium configuration given by the displacement $\mathbf{u}$, the principles of balance of linear momentum and moment of linear momentum imply the equilibrium equations and symmetry of stress tensor:

$$\left.\begin{aligned} &\rho \frac{\partial^2 u_i}{\partial t^2} - \sigma(\mathbf{u})_{ij,j} = f_i \\ &\sigma_{ij} = \sigma_{ji} \end{aligned}\right\} \quad \text{in } \Omega \qquad (4.2.8)$$

Here ρ is the density of the body, $\mathbf{f} = f_i \mathbf{i}_i$ is the applied body force per unit volume.

4.2.2. Boundary Conditions

Let the boundary Γ of the deformable body be divided into three disjoint parts Γ_{Di}, Γ_{Fi}, and Γ_C for each index $1 < i < 3$. The i component of the displacement is prescribed on the boundary Γ_{Di}, and the i component of the traction is given on Γ_{Fi}. The part Γ_C is identified with the actual contact surface and must be large enough to include the true contact surface that, at this point, is unknown. For simplicity, we assume that there are no other forces except those due to contact on Γ_C.

If $\mathbf{t} = t_i \mathbf{i}_i$ denotes the applied traction, the two standard boundary conditions on Γ_{Di} and Γ_{Fi} are

$$\left.\begin{aligned} &u_i = \bar{u}_i \quad \text{on} \quad \Gamma_{Di} \\ &\sigma(\mathbf{u})_{ij} n_j = t_i \quad \text{on} \quad \Gamma_{Fi} \end{aligned}\right\} \quad , \; 1 < i < 3, \qquad (4.2.9)$$

where $\underset{\sim}{n} = n_i \underset{\sim}{i}_i$ is the unit vector outward normal to the boundary Γ.

Suppose that the boundary Γ_C is separated into two parts, Γ_C^1 and Γ_C^2, as shown in Figure 4.1. On Γ_C^1, the body Ω is in contact only with the rigid foundation, whereas the interaction of two opposite surfaces must be considered on Γ_C^2. Furthermore, let a crack-like slit exist inside the domain Ω, and let its surface be represented by Γ_C^0. Let us construct a coordinate system θ_α, $1 \leq \alpha \leq 3$, on each part of the boundary related to contact, such that two opposite surfaces can be distinguished from each other. Examples of such coordinate systems are given in Figure 4.1. Let u_α be the α component of the displacement $\underset{\sim}{u}$ in the θ_α coordinate system, i.e., it $u_\alpha = \underset{\sim}{u} \cdot \underset{\sim}{i}_\alpha$. Then the kinematical contact conditions due to contact are given by

$$k_{\alpha\beta} u_\beta - g_\alpha \leq 0 \quad \text{on } \Gamma_C^1 \tag{4.2.10}$$

$$k_{\alpha\beta}^1 u_\beta^1 + k_{\alpha\beta}^2 u_\beta^2 - g_\alpha \leq 0 \quad \text{on } \Gamma_C^0 \text{ and } \Gamma_C^2 \tag{4.2.11}$$

where $k_{\alpha\beta}$, $k_{\alpha\beta}^i$, $1 \leq i \leq 2$, and g_α depend on θ_α, $1 \leq \alpha \leq 3$, and the superscripts 1 and 2 on $k_{\alpha\beta}$ and u_β indicate the two opposite surfaces on which these quantities are defined. The functions $k_{\alpha\beta}$ and g_α must be obtained from the geometry of the body and the rigid foundation.

For example, let us consider the part Γ_C^1, and obtain the corresponding expressions for $k_{\alpha\beta}$ and g_α. To do this, suppose that the surfaces of the body and the rigid foundation are given by

$$\theta_3 = \phi(\theta_1, \theta_2), \quad \theta_3 = \psi(\theta_1, \theta_2), \tag{4.2.12}$$

respectively. Since the body cannot penetrate the rigid surface,

$$\theta_3 + u_3 > \psi(\theta_1 + u_1, \theta_2 + u_2), \tag{4.2.13}$$

where $\underset{\sim}{u} = u_\alpha \underset{\sim}{i}_\alpha$. Linearization of the function ψ in u_1 and u_2 yields

$$\theta_3 + u_3 > \psi(\theta_1,\theta_2) + \underset{\sim}{\nabla}_\alpha \psi(\theta_1,\theta_2)\cdot(u_1 \underset{\sim}{i}_1 + u_2 \underset{\sim}{i}_2)$$

where $\underset{\sim}{\nabla}_\alpha = \underset{\sim}{i}_1 \partial/\partial\theta_1 + \underset{\sim}{i}_2 \partial/\partial\theta_2$. Dividing by $\sqrt{1 + |\underset{\sim}{\nabla}_\alpha \psi|^2}$ and noting that

$$\underset{\sim}{N} = N_\alpha \underset{\sim}{i}_\alpha = \frac{1}{\sqrt{1 + |\underset{\sim}{\nabla}_\alpha \psi|^2}} \left(\frac{\partial\psi}{\partial\theta_1} \underset{\sim}{i}_1 + \frac{\partial\psi}{\partial\theta_2} \underset{\sim}{i}_2 - i_3 \right) \tag{4.2.14}$$

is the unit vector inward to the rigid surface $\theta_3 = \psi(\theta_1,\theta_2)$, we have the linearized contact condition

$$N_\beta u_\beta - g < 0, \tag{4.2.15}$$

where

$$g = \frac{\phi - \psi}{\sqrt{1 + |\underset{\sim}{\nabla}_\alpha \psi|^2}} = (\psi - \phi) N_3. \tag{4.2.16}$$

Thus, if

$$k_{\alpha\beta} = \delta_{\alpha\beta} N_\beta, \quad g_1 = g, \quad g_2 = g_3 = +\infty \tag{4.2.17}$$

in the general form (4.2.10), the condition (4.2.10) represents the linearized contact condition (4.2.15) on Γ_C^1. Since $\underset{\sim}{N}$ is the same as the unit outward normal to the surface Γ_C^1 of the deformable body on the contact surface after the deformation, condition (4.2.15) means that the normal component of the displacement to the contact surface must not be larger than the projection of the distance of two surfaces in the normal direction, as shown in Figure 4.2.

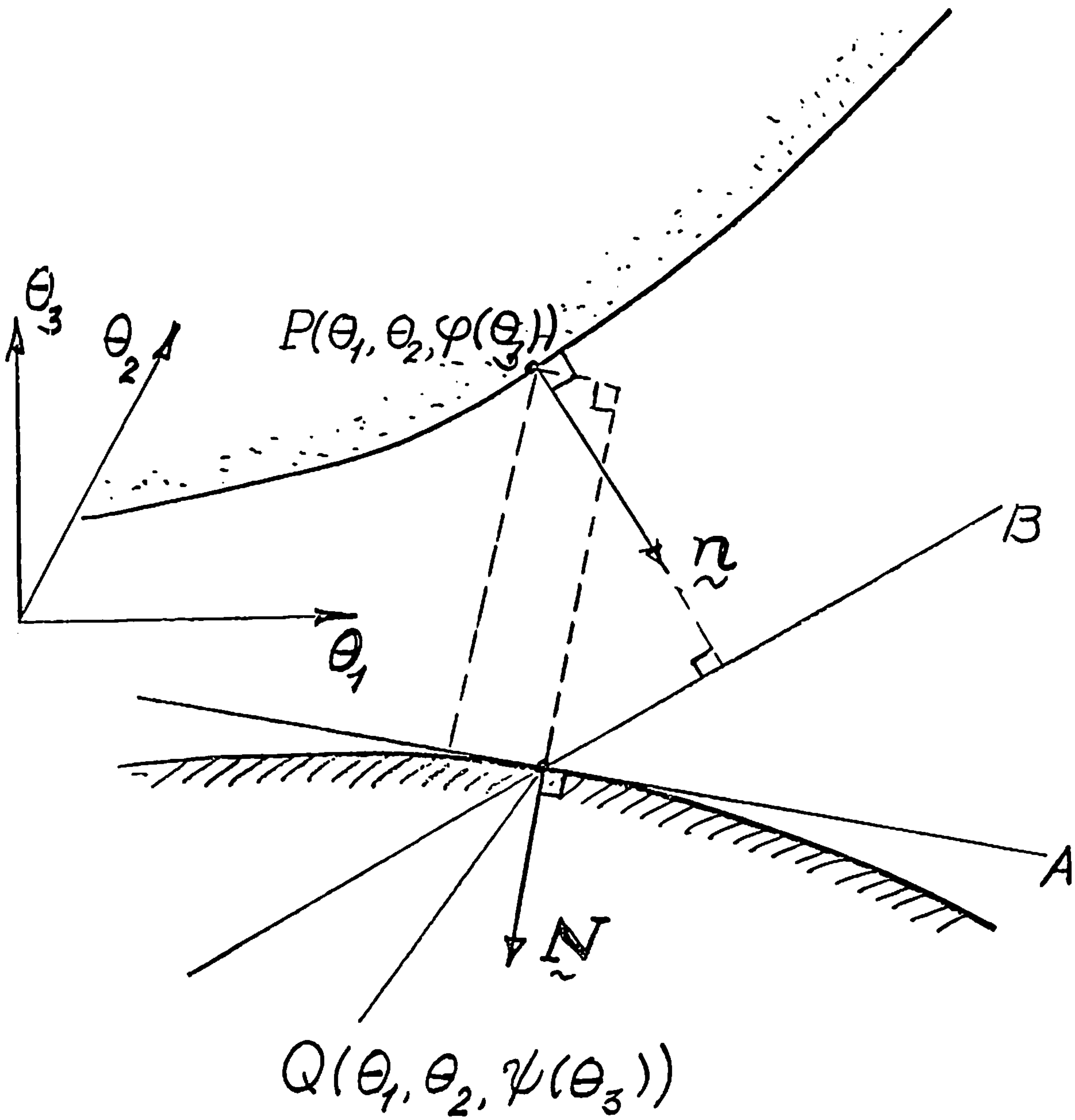


Figure 4.2. Contact surface geometry.

That is, the deformation of the point $\underset{\sim}{P}$ is always on or above the tangent plane A at the point Q of the rigid surface $\theta_3 = \psi(\theta_1,\theta_3$.

In almost all literature on contact problems dealing with theoretical aspects of these problems the linearized form (4.2.15) of (4.2.13) is not used. The condition more commonly applied is

$$n_\beta u_\beta - \hat{g} \le 0, \tag{4.2.18}$$

where $\underset{\sim}{n} = n_\beta \underset{\sim}{i}_\beta$ is the unit vector outward normal to the boundary Γ_C^1, and $\hat{g}$ is the projection of $(\psi - \phi)$ in the n_3-direction; $\hat{g} = (\psi - \phi)n_3$. The linearized condition (4.2.18) is represented by the plane B in Figure 4.2. In this case, the deformation of the point $\underset{\sim}{P}$ is always on or above the plane B. Of course, since we are dealing with infinitesimal deformations, the contact surface must be very close to the foundation, and the differences between $\underset{\sim}{n}$ and $\underset{\sim}{N}$ are, accordingly, small. We shall apply (4.2.15) in this chapter.

Following a similar approach, a linearized contact condition on Γ_C^0 and Γ_C^2 can be obtained. Let the particles, the position vectors of which at the initial stage are $\underset{\sim}{\theta}_2^1 = \theta_\alpha^1 \underset{\sim}{i}_\alpha$ and $\underset{\sim}{\theta}^2 = \theta_\alpha^2 \underset{\sim}{i}_\alpha$ where $\theta_3^1 = \phi(\theta_1^1,\theta_2^1)$ and $\theta_3^2 = \psi(\theta_1^1,\theta_2^1)$ be in contact after deformation. That is, let

$$\theta_\alpha^1 + u_\alpha^1 = \theta_\alpha^2 + u_\alpha^2, \quad \alpha = 1,2 \tag{4.2.19}$$

where $\underset{\sim}{u}^1 = u_\alpha^1 \underset{\sim}{i}_\alpha$ and $\underset{\sim}{u}^2 = u_\alpha^2 \underset{\sim}{i}_\alpha$ are the displacement vectors of the particles identified by $\underset{\sim}{\theta}^1$ and $\underset{\sim}{\theta}^2$. Here, the two opposite surfaces are assumed to be represented by the equations

$$\theta_3 = \phi(\theta_1,\theta_2) \quad \text{and} \quad \theta_3 = \psi(\theta_1,\theta_2) \tag{4.2.20}$$

for a proper coordinate system $(\theta_1,\theta_2,\theta_3)$.

The "no penetration" condition is then given by

$$\theta_3^1 + u_3^1 > \theta_3^2 + u_3^2 \tag{4.2.21}$$

Using (4.2.19) we now express the condition (4.2.21) in terms of $\underset{\sim}{\theta}^1$. Neglecting higher order terms, we have

$$\phi(\theta_1^1,\theta_2^1) + u_3^1(\theta_1^1,\theta_2^1,\phi(\theta_1^1,\theta_2^1)) > \psi(\theta_1^1,\theta_2^1)$$

$$+ \frac{\partial\psi}{\partial\theta_1}(\theta_1^1,\theta_2^1)\left(u_1^1(\theta_1^1,\theta_2^1,\phi(\theta_1^1,\theta_2^1)) - u_1^2(\theta_1^1,\theta_2^1,\psi(\theta_1^1,\theta_2^1)) \right)$$

$$+ \frac{\partial\psi}{\partial\theta_2}(\theta_1^1,\theta_2^1)\left(u_2^1(\theta_1^1,\theta_2^1,\phi(\theta_1^1,\theta_2^1)) - u_1^2(\theta_1^1,\theta_2^1,\psi(\theta_1^1,\theta_2^1)) \right)$$

$$+ u_3^2(\theta_1^1,\theta_2^1,\psi(\theta_1^1,\theta_2^1)). \tag{4.2.22}$$

For simplicity, we shall express (4.2.22) as

$$\phi + u_3^1 > \psi + \frac{\partial\psi}{\partial\theta_1}(u_1^1 - u_1^2) + \frac{\partial\psi}{\partial\theta_2}(u_2^1 - u_2^2) + u_3^2$$

Applying the relation (4.2.14) of the unit vector inward normal to the surface $\theta_3 = \psi(\theta_1,\theta_2)$, we can represent (4.2.22) as

$$N_\beta(u_\beta^1 - u_\beta^2) - g_1 < 0, \tag{4.2.23}$$

where

$$g_1 = (\psi - \phi)N_3 \ . \tag{4.2.24}$$

Thus, if

$$k^1_{1\beta} = - k^2_{1\beta} = N_\beta, \quad g_1 = g$$
$$\tag{4.2.25}$$
$$k^1_{\alpha\beta} = k^2_{\alpha\beta} = \delta_{\alpha\beta} \quad \text{for} \quad \alpha = 2,3, \quad g_2 = g_3 = + \infty,$$

then the linearized condition (4.2.23) is given by the generalized form (4.2.11).

As far as the linearized forms are concerned, the contact conditions (4.2.15) and (4.2.23) have a similar form. More precisely, if we replace the displacement $\underset{\sim}{u}$ in (4.2.15) by the relative displacement $\underset{\sim}{u}^R = \underset{\sim}{u}^1 - \underset{\sim}{u}^2$, the condition (4.2.15) can represent (4.2.23) for the boundaries Γ^0_C and Γ^2_C. Therefore, for the mathematical development of contact problems, we need only consider the conditions on boundary Γ^1_C. We thus describe the general kinematic contact condition as

$$k_{\alpha\beta}u_\beta - g_\alpha < 0 \quad \text{on} \quad \Gamma_C, \tag{4.2.26}$$

or more precisely

$$N_\beta u_\beta - g < 0 \quad \text{on } \Gamma_C. \tag{4.2.27}$$

These represent only kinematical requirements. In addition, the stresses must satisfy

$$\sigma_N = \underset{\sim}{\sigma} \cdot \underset{\sim}{N} < 0 \quad \text{if there is contact}$$

$$\sigma_N = 0 \qquad \text{if there is no contact} \tag{4.2.28}$$

where $\underset{\sim}{\sigma} = (\sigma_{ij} n_j)\underset{\sim}{i}_i$ is the traction on the boundary Γ_C. Combining (4.2.27) and (4.2.28) yields

$$u_N - g \leq 0, \quad \sigma_N \leq 0, \quad \sigma_N(u_N - g) = 0, \tag{4.2.29}$$

where $u_N = N_\beta u_\beta = \underset{\sim}{u} \cdot \underset{\sim}{N}$.

If the contact surface is well lubricated, friction might be neglected. In this case, on the boundary Γ_C, the additional boundary condition,

$$\underset{\sim}{\sigma}_T = \underset{\sim}{\sigma} - \sigma_N \underset{\sim}{N} = 0 \tag{4.2.30}$$

must be applied, where $\underset{\sim}{\sigma}$ is again the traction. However, in many cases in practice, friction has an important role to stress analysis. Here we shall simply assume that Coulomb's friction law holds pointwise on the contact surface. More general friction laws have been considered recently by Duvaut [1981], and Oden and Pires [1981] and we explore one such class of friction problems in the next chapter. The tangential velocity is assumed to be governed by the following form of Coulomb's law:

$$|\underset{\sim}{\sigma}_T| < -\mu\sigma_N, \quad \text{then } \dot{\underset{\sim}{u}}_T = 0$$

$$|\underset{\sim}{\sigma}_T| = -\mu\sigma_N, \quad \text{then for some } \lambda \geq 0, \quad \dot{\underset{\sim}{u}}_T = -\lambda\underset{\sim}{\sigma}_T \tag{4.2.31}$$

These conditions hold on the surface in which $\sigma_N < 0$. Here $\dot{\underset{\sim}{u}} = \partial \underset{\sim}{u}/\partial t$ is the velocity and $\dot{\underset{\sim}{u}}_T$ is its tangential component defined by

$$\dot{\underset{\sim}{u}}_T = \dot{\underset{\sim}{u}} - \dot{u}_N \underset{\sim}{N}, \quad \dot{u}_N = \dot{\underset{\sim}{u}} \cdot \underset{\sim}{N}.$$

4.2.3. Boundary-Value Problems

In summary, we have the following initial-boundary value problem

$$\left.\begin{array}{l} \rho \dot{\underset{\sim}{u}} - \operatorname{div} \underset{\approx}{\sigma}(\underset{\sim}{u}) = \underset{\sim}{f} \quad \text{in} \quad \Omega \\[2ex] \underset{\approx}{\sigma}(\underset{\sim}{u}) = \underset{\sim}{t} \quad \text{on} \quad \Gamma_F \\[2ex] \underset{\sim}{u} = \bar{\underset{\sim}{u}} \quad \text{on} \quad \Gamma_D \\[2ex] \left.\begin{array}{l} u_N - g \leq 0, \quad \sigma_N \leq 0, \quad \sigma_N(u_N - g) = 0 \\[2ex] |\underset{\sim}{\sigma}_T| < -\mu\sigma_N, \quad \text{then } \dot{\underset{\sim}{u}}_T = 0 \\[2ex] |\underset{\sim}{\sigma}_T| = -\mu\sigma_N, \quad \text{then } \lambda \geq 0 \text{ s.t. } \dot{\underset{\sim}{u}}_T = -\lambda\underset{\sim}{\sigma}_T \end{array}\right\} \text{on } \Gamma_C \end{array}\right\} \tag{4.2.32}$$

with the initial conditions

$$\underset{\sim}{u} = \underset{\sim}{u}_0 \quad \text{and} \quad \dot{\underset{\sim}{u}} = \underset{\sim}{v}_0 \quad \text{at} \quad t = 0, \tag{4.2.33}$$

for the Signorini problem. Here $\underset{\approx}{\sigma} = \sigma_{ij} \underset{\sim}{i}_i \otimes \underset{\sim}{i}_j$, and div is the divergence operator.

If the motion is slow, then the acceleration is small and the process of the deformation of the body represented by (4.2.32) and (4.2.33) can be approximated by the following incremental form using an artificial monotonically increasing time parameter t. Let $\underset{\sim}{u}^t$ be the deformation at time t, and let $\Delta\underset{\sim}{u}$ be a small increment of $\underset{\sim}{u}$ produced during the time interval Δt, i.e., $\Delta\underset{\sim}{u} = \underset{\sim}{u}^{t+\Delta t} - \underset{\sim}{u}^t$. Suppose that

$$\left.\begin{aligned} -\operatorname{div} \underset{\approx}{\sigma}(\underset{\sim}{u}^t) &= \underset{\sim}{f}^t \quad \text{in} \quad \Omega \\ \underset{\approx}{\sigma}(\underset{\sim}{u}^t) &= \underset{\sim}{t}^t \quad \text{on} \quad \Gamma_F \\ \underset{\sim}{u}^t &= \bar{\underset{\sim}{u}}^t \quad \text{on} \quad \Gamma_D, \end{aligned}\right\} \qquad (4.2.34)$$

where $\underset{\sim}{f}^t$, $\underset{\sim}{t}^t$, and $\bar{\underset{\sim}{u}}^t$ are the values of $\underset{\sim}{f}$, $\underset{\sim}{t}$, and $\bar{\underset{\sim}{u}}$ at time t, respectively. Introducing similar quantities $\Delta\underset{\sim}{f}$, $\Delta\underset{\sim}{t}$, and $\Delta\bar{\underset{\sim}{u}}$ to $\Delta\underset{\sim}{u}$, we arrive at the boundary value problem

$$\left.\begin{aligned} -\operatorname{div} \underset{\approx}{\sigma}(\Delta\underset{\sim}{u}) &= \Delta\underset{\sim}{f} \quad \text{in} \quad \Omega \\ \underset{\approx}{\sigma}(\Delta\underset{\sim}{u}) &= \Delta\underset{\sim}{t} \quad \text{on} \quad \Gamma_F \\ \Delta\underset{\sim}{u} &= \Delta\bar{\underset{\sim}{u}} \quad \text{on} \quad \Gamma_D \\ \Delta u_N - \Delta g < 0, \quad \Delta\sigma_N + \bar{\sigma}_N < 0, &\quad (\Delta\sigma_N + \bar{\sigma}_N)(\Delta u_N - \Delta g) = 0 \\ |\Delta\underset{\sim}{\sigma}_T + \bar{\underset{\sim}{\sigma}}_T| < -\mu(\Delta\sigma_N + \bar{\sigma}_N), &\quad \text{then } \Delta\underset{\sim}{u}_T = 0 \end{aligned}\right\} \qquad (4.2.35)$$

$$
\left.\begin{array}{l}
|\Delta\underset{\sim}{\sigma}_T + \bar{\underset{\sim}{\sigma}}_T| = -\mu(\Delta\sigma_N + \bar{\sigma}_N), \quad \text{then} \quad \hat{\lambda} > 0 \text{ s.t.} \\
\qquad\qquad \Delta\underset{\sim}{u}_T = -\lambda(\Delta\underset{\sim}{\sigma}_T + \bar{\underset{\sim}{\sigma}}_T),
\end{array}\right\} \text{ on } \Gamma_C
$$

during the time interval $[t, t+\Delta t]$. Here

$$
\begin{array}{l}
\Delta g = g - u_N^t, \quad \Delta\sigma_N = \sigma_N(\Delta\underset{\sim}{u}), \quad \bar{\sigma}_N = \sigma_N(\underset{\sim}{u}^t) \\
\Delta\underset{\sim}{\sigma}_T = \underset{\sim}{\sigma}_T(\Delta\underset{\sim}{u}), \quad \bar{\sigma}_T = \underset{\sim}{\sigma}_T(\underset{\sim}{u}^t),
\end{array}
\qquad (4.2.36)
$$

and $\hat{\lambda}$ need not be same as λ in (4.2.32).

We refer to problem (4.2.32), (4.2.33) as the dynamical form of Signorini's problem with Coulomb friction. The system (4.2.35), thus, defines an incremental form of this problem valid for slow deformation or quasi-static conditions.

3. Variational Formulations of Signorini's Problem

As shown in the previous section, Signorini's problem with friction for problems in elastostatics assumes the form of the boundary value problem (4.2.35). We shall consider a variational formulation of the problem that includes the case (4.2.35) as a special case, and shall develop a mathematical theory of such problems subject to a few additional hypotheses.

4.3.1. Abstract Signorini's Problem with Friction

Let us consider the boundary value problem of finding a displacement field $\underset{\sim}{u}$ such that

$$\left.\begin{array}{l} -\operatorname{div} \underset{\sim}{\sigma}(\underset{\sim}{u}) = \underset{\sim}{f} \quad \text{in} \quad \Omega \\[2ex] \underset{\sim}{\sigma}(\underset{\sim}{u}) = \underset{\sim}{t} \quad \text{on} \quad \Gamma_F \\[2ex] \underset{\sim}{u} = \bar{\underset{\sim}{u}} \quad \text{on} \quad \Gamma_D \\[2ex] u_N - g \leq 0, \quad \sigma_N + \bar{\sigma}_N \leq 0, \quad (\sigma_N + \bar{\sigma}_N)(u_N - g) = 0 \\[2ex] |\underset{\sim}{\sigma}_T + \bar{\underset{\sim}{\sigma}}_T| < -\mu(\sigma_N + \bar{\sigma}_N), \text{ then } \underset{\sim}{u}_T = \underset{\sim}{0} \\[2ex] |\underset{\sim}{\sigma}_T + \bar{\underset{\sim}{\sigma}}_T| = -\mu(\sigma_N + \bar{\sigma}_N), \text{ then } \lambda > 0, \text{ s.t. } \underset{\sim}{u}_T = -\lambda(\underset{\sim}{\sigma}_T + \bar{\sigma}_T) \end{array}\right\} \quad (4.3.1)$$

for given functions $\underset{\sim}{f}$, $\underset{\sim}{t}$, $\bar{\underset{\sim}{u}}$, $\bar{\sigma}_N$, $\bar{\underset{\sim}{\sigma}}_T$, and $\underset{\sim}{g}$.

Multiplying the first member of (4.3.1) by an arbitrary smooth test function $\underset{\sim}{v}$ such that $\underset{\sim}{v} = \underset{\sim}{0}$ on Γ_D, and integrating by parts, we have

$$\int_\Omega \sigma(\underset{\sim}{u})_{ij} v_{i,j} dx - \int_\Gamma \sigma(\underset{\sim}{u})_{ij} n_j v_i ds = \int_\Omega f_i v_i dx.$$

Using the relations

$$\underset{\sim}{a} = a_i \underset{\sim}{i}_i = \hat{a}_\alpha \underset{\sim}{i}_\alpha \quad (\hat{a}_\alpha = a_i \underset{\sim}{i}_i \cdot \underset{\sim}{i}_\alpha)$$

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = a_N b_N + \underset{\sim}{a}_T\cdot\underset{\sim}{b}_T, \quad a_N = \underset{\sim}{a}\cdot\underset{\sim}{N}, \quad \underset{\sim}{a}_T = \underset{\sim}{a} - a_N\underset{\sim}{N}, \text{ etc.}$$

for the unit vector $\underset{\sim}{N}$, and applying the boundary conditions $(4.3.1)_2$, and $(4.3.1)_3$, we obtain

$$\int_\Omega \sigma(\underset{\sim}{u})_{ij} v_{i,j} dx = \int_\Omega f_i v_i dx + \int_{\Gamma_F} t_i v_i ds$$

$$+ \int_{\Gamma_C} \left(\sigma(\underset{\sim}{u})_N v_N + \underset{\sim}{\sigma}(\underset{\sim}{u})_T\cdot\underset{\sim}{v}_T\right) ds, \quad \forall \ \underset{\sim}{v} \text{ s.t. } \underset{\sim}{v} = \underset{\sim}{0} \text{ on } \Gamma_D, \tag{4.3.2}$$

where $v_N = \underset{\sim}{v}\cdot\underset{\sim}{N} = \hat{v}_\alpha N_\alpha$, $\hat{v}_\alpha = v_i \underset{\sim}{i}_i\cdot\underset{\sim}{i}_\alpha$, $\underset{\sim}{v}_T = \underset{\sim}{v} - \underset{\sim}{v}_T$, $\sigma(\underset{\sim}{u})_N = \underset{\sim}{\sigma}(\underset{\sim}{u})\cdot\underset{\sim}{N}$, and so on. Let $\underset{\sim}{w}$ be an arbitrary function that satisfies

$$\underset{\sim}{w} = \bar{\underset{\sim}{u}} \text{ on } \Gamma_D, \quad w_N - g \leq 0 \text{ on } \Gamma_C. \tag{4.3.3}$$

It is noted that $(4.3.1)_4$-$(4.3.1)_6$ are equivalent to the two inequalities

$$\sigma(\underset{\sim}{u})_N(w_N - u_N) = \left(\sigma(\underset{\sim}{u})_N + \bar{\sigma}_N\right)\left(w_N - g - u_N + g\right)$$

$$- \bar{\sigma}_N(w_N - u_N) \geq - \bar{\sigma}_N(w_N - u_N),$$

and

$$\underset{\sim}{\sigma}(\underset{\sim}{u})_T\cdot(\underset{\sim}{w}_T - \underset{\sim}{u}_T) \geq \mu\left(\sigma(\underset{\sim}{u})_N + \bar{\sigma}_N\right)\left(|\underset{\sim}{w}_T| - |\underset{\sim}{u}_T|\right) - \bar{\underset{\sim}{\sigma}}_T\cdot(\underset{\sim}{w}_T - \underset{\sim}{u}_T)$$

Thus, (4.3.2) can be written as

$$\int_\Omega \sigma(\underset{\sim}{u})_{ij}(w_{i,j} - u_{i,j}) dx + \int_{\Gamma_C} \{- \mu\left(\sigma(\underset{\sim}{u})_N + \bar{\sigma}_N\right)\left(|\underset{\sim}{w}_T| - |\underset{\sim}{u}_T|\right)\} ds$$

$$\geq \int_\Omega f_i(w_i - u_i)dx + \int_{\Gamma_F} t_i(w_i - u_i)ds$$

$$- \int_{\Gamma_C} \bar{\sigma}_i(w_i - u_i)ds,$$

for every $\underset{\sim}{w}$ satisfying (4.3.3), where the traction vector $\bar{\underset{\sim}{\sigma}}$ is defined by $\bar{\underset{\sim}{\sigma}} = \bar{\sigma}_N \underset{\sim}{N} + \bar{\underset{\sim}{\sigma}}_T$ on Γ_C.

Let us introduce the following standard notation:

$$B(\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega \sigma(\underset{\sim}{u})_{ij}\varepsilon_{ij}(\underset{\sim}{v})dx$$

$$f(\underset{\sim}{v}) = \int_\Omega f_i v_i dx + \int_{\Gamma_F} t_i v_i ds + \int_{\Gamma_C} \bar{\sigma}_i v_i ds \tag{4.3.4}$$

$$j(\underset{\sim}{u};\underset{\sim}{v}) = \int_{\Gamma_C} \{-\mu(\sigma(\underset{\sim}{u})_N + \bar{\sigma}_N)\}|\underset{\sim}{v}_T|ds,$$

Then the problem (4.3.1) yields a variational form

$$B(\underset{\sim}{u},\underset{\sim}{w}-\underset{\sim}{u}) + j(\underset{\sim}{u};\underset{\sim}{w}) - j(\underset{\sim}{u};\underset{\sim}{u}) \geq f(\underset{\sim}{w}-\underset{\sim}{u}), \tag{4.3.5}$$

for every $\underset{\sim}{w}$ satisfying (4.3.3), (since the stress tensor $\underset{\approx}{\sigma}$ is symmetric).

Using (4.2.3) and (4.2.5), we have

$$B(\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega E_{ijk\ell}u_{k,\ell}v_{i,j}dx$$

$$\leq M\|\underset{\sim}{u}\|_1\|\underset{\sim}{v}\|_1 \tag{4.3.6}$$

where

$$\|\underset{\sim}{v}\|_1 = \{\int_\Omega (v_{i,j}v_{i,j} + v_i v_i)dx\}^{1/2}. \qquad (4.3.7)$$

If the given data $\underset{\sim}{f}$, $\underset{\sim}{t}$, and $\underset{\sim}{\bar{\sigma}}$ are smooth enough, i.e., if

$$f_i \in L^2(\Omega), \quad t_i \in L^2(\Gamma_F), \quad \bar{\sigma}_N \in L^2(\Gamma_C), \quad \bar{\sigma}_{Ti} \in L^2(\Gamma_C), \qquad (4.3.8)$$

then

$$f(\underset{\sim}{v}) \leq \left(\|\underset{\sim}{f}\|_0 + \|\underset{\sim}{t}\|_{0,\Gamma_F} + \|\underset{\sim}{\bar{\sigma}}\|_{0,\Gamma_C}\right)\|\underset{\sim}{v}\|_1, \qquad (4.3.9)$$

where

$$\|\underset{\sim}{f}\|_0 = \{\int_\Omega f_i f_i dx\}^{1/2},$$

$$(4.3.10)$$

$$\|\underset{\sim}{t}\|_{0,\Gamma_F} = \{\int_{\Gamma_F} t_i t_i ds\}^{1/2},$$

and $\|\underset{\sim}{\bar{\sigma}}\|_{0,\Gamma_C}$ is defined similarly to $(4.3.10)_2$. Thus the bilinear form $B(\underset{\sim}{u},\underset{\sim}{v})$ represents the virtual work of the elastic body, and the linear functional $f(\underset{\sim}{v})$ of the work done by the virtual displacement $\underset{\sim}{v}$. Both are well-defined on the Sobolev space $\underset{\sim}{H}^1(\Omega)$:

$$\underset{\sim}{H}^1(\Omega) = \{\underset{\sim}{v} = v_i \underset{\sim}{i}_i : v_i \in L^2(\Omega), \ v_{i,j} \in L^2(\Omega), \ 1 \leq i,j \leq 3\}, \qquad (4.3.11)$$

where $v_{i,j}$ is the generalized derivative of v_i. However, the term $j(\underset{\sim}{u};\underset{\sim}{v})$ is not well-defined, since the normal stress $\sigma(\underset{\sim}{u})_N$ cannot be defined for a function $\underset{\sim}{u} \in \underset{\sim}{H}^1(\Omega)$. The contact stress, as it enters the present formulations, cannot be defined in the appropriate space because we cannot infer sufficient smothness of the displacement $\underset{\sim}{u}$. Thus, in this sense, the variational formulation (4.3.5) of

<u>the Signorini problem with friction may not be meaningful in the sense of this variational principle.</u>

For a particular case that the friction coefficient μ is sufficiently small, there are several approaches one can use to show the existence of solutions to the variational formulation (4.3.5) within the context of the Sobolev space $\underset{\sim}{H}^1(\Omega)$. Details of such treatments can be found in e.g., Necas, Haslinger, and Tuisuk [1980], Duvaut [1981], Oden and Demkowicz [1981], Oden and Pires [1981], and others. In the present study, we shall not discuss these mathematical technicalities, rather, we shall address the problem of obtaining a solution of (4.3.5) using finite element methods. To do this the following two particlar cases of (4.3.5) will be studied in detail.

4.3.2. Special Case I: Prescribed Tangential Stress

Suppose that the tangential stress $\underset{\sim}{\sigma}_T$ is known. That is, the tangential stress

$$\underset{\sim}{\sigma}_T = \underset{\sim}{t}_T \quad \text{on} \quad \Gamma_C \tag{4.3.12}$$

is prescribed, and the Coulomb friction law is abandoned on the contact surface Γ_C. Then the variational formulation (4.3.5) reduces to

$$B(\underset{\sim}{u},\underset{\sim}{w}-\underset{\sim}{u}) \geq f_1(\underset{\sim}{w}-\underset{\sim}{u}), \tag{4.3.13}$$

for every $\underset{\sim}{w}$ satisfying (4.3.3), where

$$f_1(\underset{\sim}{v}) = f(\underset{\sim}{v}) + \int_{\Gamma_C} \underset{\sim}{t}_T \cdot \underset{\sim}{v}_T ds. \tag{4.3.14}$$

If

$$t_{Ti} \in L^2(\Gamma_C), \quad 1 \leq i \leq 3, \tag{4.3.15}$$

the linear form f_1 is bounded in $\underset{\sim}{H}^1(\Omega)$, i.e.,

$$f_1(\underset{\sim}{v}) \leq \left(\|\underset{\sim}{f}\|_0 + \|\underset{\sim}{t}\|_{0,\Gamma_F} + \|\bar{\underset{\sim}{\sigma}}\|_{0,\Gamma_C} + \|\underset{\sim}{t}_T\|_{0,\Gamma_C}\right)\|\underset{\sim}{v}\|_1. \tag{4.3.16}$$

In this case, there are no ambiguous terms such as $j(\underset{\sim}{u};\underset{\sim}{v})$ in (4.3.5) in the variational formulation (4.3.13).

4.3.3. Special Case II: Prescribed Normal Stress

Suppose that the normal stress σ_N is known at this time. That is, the normal stress

$$\sigma_N = t_N \in L^2(\Gamma_C) \tag{4.3.17}$$

is prescribed, and only the Coulomb law of friction has to be considered on the contact surface Γ_C, while the unilateral contact condition $(4.3.1)_4$ is abandoned. Then the variational formulation (4.3.5) becomes

$$B(\underset{\sim}{u},\underset{\sim}{w}-\underset{\sim}{u}) + j(\underset{\sim}{w}) - j(\underset{\sim}{u}) \geq \underset{\sim}{f}_2(\underset{\sim}{w}-\underset{\sim}{u}), \tag{4.3.18}$$

for every $\underset{\sim}{w}$ such that

$$w_i = 0 \quad \text{on} \quad \Gamma_{D_i}, \quad 1 \leq i \leq 3, \tag{4.3.19}$$

where

$$j(\underset{\sim}{v}) = \int_{\Gamma_C} \{-\mu(t_N + \bar{\sigma}_N)\}|\underset{\sim}{v}_T|ds, \tag{4.3.20}$$

and

$$f_2(\underset{\sim}{v}) = f(\underset{\sim}{v}) + \int_{\Gamma_C} t_N v_N ds. \tag{4.3.21}$$

Since both $\bar{\sigma}_N$ and t_N are in $L^2(\Gamma_C)$, the convex functional j is bounded in $\underset{\sim}{H}^1(\Omega)$, and is continuous. Indeed,

$$|j(\underset{\sim}{v}) - j(\underset{\sim}{w})| < \int_{\Gamma_C} |-\mu(t_N + \bar{\sigma}_N)|(|\underset{\sim}{v}|_T - |\underset{\sim}{w}_T|)ds$$

$$< \|-\mu(t_N + \bar{\sigma}_N)\|_{0,\Gamma_C} \|\underset{\sim}{v}_T - \underset{\sim}{w}_T\|_{0,\Gamma_C}$$

$$< \|-\mu(t_N + \bar{\sigma}_N)\|_{0,\Gamma_C} \|\underset{\sim}{v} - \underset{\sim}{w}\|_1. \tag{4.3.22}$$

Thus, the variational formulation (4.3.18) is well-defined in the Sobolev space $\underset{\sim}{H}^1(\Omega)$, since f_2 is clearly linear and continuous.

A solution $\underset{\sim}{u}$ to the general problem (4.3.5) might be obtained by the iteration process

(4.3.23)

(i) Solve the Special Case I by assuming the tangential stress $\underset{\sim}{\sigma}_T$. As a result, the normal stress σ_N is computed.

(ii) Using the computed normal stress σ_N, solve the Special (4.3.18) Case II. As a result the tangential stress $\underset{\sim}{\sigma}_T$ is computed.

(iii) Check the convergence of the solution. If convergence is not attained, repeat (i) and (ii).

We know of no convergence proofs for this iteration process, although the process has proved to converge to reasonable approximate solutions to many problems.

4. Special Case I: Prescribed Tangential Stress

As mentioned earlier, we shall solve the Signorini problem (4.3.5) with friction by using the iteration process (4.3.23) that is the sequential approximation of the problem (4.3.5) by the two special cases (4.3.13) and (4.3.18). In this section, the first case (4.3.13), which corresponds to situations in which the tangential stress $\underset{\sim}{\sigma}_T$ (possibly due to friction) is known as a function of $\underset{\sim}{L}^2(\Gamma_C)$, i.e.,

$$\underset{\sim}{\sigma}_T = \underset{\sim}{t}_T, \quad \underset{\sim}{t}_T \in \underset{\sim}{L}^2(\Gamma_C), \tag{4.4.1}$$

where $\underset{\sim}{L}^2(\Gamma_C) = \{\underset{\sim}{t} = t_\beta \underset{\sim}{i}_\beta : t_\beta \in L^2(\Gamma_C)\}$. As shown in (4.3.13), a variational formulation to this special case is governed by

$$\underset{\sim}{u} \in \underset{\sim}{K} : B(\underset{\sim}{u},\underset{\sim}{w}-\underset{\sim}{u}) \geq f_1(\underset{\sim}{w}-\underset{\sim}{u}), \quad \forall \ \underset{\sim}{w} \in \underset{\sim}{K}, \tag{4.4.2}$$

where

$$\underset{\sim}{K} = \{\underset{\sim}{v} \in \underset{\sim}{V} : v_N - g \leq 0, \quad \text{a.e. on } \Gamma_C\}, \tag{4.4.3}$$

$$\underset{\sim}{V} = \{\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) : v_i = \bar{u}_i, \quad \text{a.e. on } \Gamma_{Di}, \ 1 \leq i \leq 3\}, \tag{4.4.4}$$

and $\underset{\sim}{H}^1(\Omega)$ is the Sobolev space defined by (4.3.11). Since the bilinear form $B(\cdot,\cdot)$ and the linear form $f(\cdot)$ are well-defined on $\underset{\sim}{H}^1(\Omega)$, we can define the variational formulation (4.3.13) by using the Sobolev space $\underset{\sim}{H}^1(\Omega)$ as (4.4.2).

The subset $\underset{\sim}{K}$ of $\underset{\sim}{H}^1(\Omega)$ is the so-called admissible set of all possible functions satisfying the given boundary condition on Γ_{Di} and the kinematic contact condition on Γ_C, and producing finite energy. The set $\underset{\sim}{V}$ consists of all functions that satisfy only the boundary condition on Γ_{Di}.

4.4.1. Existence of a Unique Solution

Because of the assumption on the elasticity constant (4.2.4), Korn's inequality yields the property

$$B(\underset{\sim}{v},\underset{\sim}{v}) \geq m\|\underset{\sim}{v}\|_1^2, \qquad \underset{\sim}{v} \in \underset{\sim}{V} \tag{4.4.5}$$

where m is a constant independent of $\underset{\sim}{v}$ $\underset{\sim}{V}$. Details of Korn's inequality can be found in e.g., Hlavacek and Necas [1970]. We also recall that (see (4.3.6) and (4.3.16))

$$B(\underset{\sim}{v},\underset{\sim}{w}) \leq M\|\underset{\sim}{v}\|_1\|\underset{\sim}{w}\|_1, \qquad \underset{\sim}{v},\ \underset{\sim}{w} \in \underset{\sim}{H}^1(\Omega) \tag{4.4.6}$$

and

$$|f_1(\underset{\sim}{v})| \leq \|f_1\|_1^*\|\underset{\sim}{v}\|_1, \qquad \underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega), \tag{4.4.7}$$

where $\|f_1\|_1^* = \|\underset{\sim}{f}\|_0 + \|\underset{\sim}{t}\|_{0,\Gamma_F} + \|\bar{\underset{\sim}{g}}\|_{0,\Gamma_C} + \|\underset{\sim}{t}_T\|_{0,\Gamma_C}$.

These properties are sufficient to guarantee the existence of a unique solution $\underset{\sim}{u} \in \underset{\sim}{K}$ to the problem (4.4.2) since the set $\underset{\sim}{K}$ is closed and convex in $\underset{\sim}{H}^1(\Omega)$ under the assumption of the Lipschitz domain Ω.

Theorem 4.1. Let the domain Ω is Lipschitzian, and let (4.2.3)-(4.2.5) hold. Suppose that (4.4.1) holds. Then there exists a unique solution $\underset{\sim}{u} \in \underset{\sim}{K}$ to the variational inequality (4.4.2):

$$\underset{\sim}{u} \in \underset{\sim}{K} : B(\underset{\sim}{u}, \underset{\sim}{w}-\underset{\sim}{u}) \geq f_1(\underset{\sim}{w}-\underset{\sim}{u}), \qquad \underset{\sim}{w} \in \underset{\sim}{K}$$

Proof. See e.g. Lions and Stampacchia [1967]. □

4.4.2. Penalty Resolution of the Contact Condition

In the formulation of (4.4.2), there is an inequality constraint $v_N - g \leq 0$ a.e. on Γ_C, that leads to the inequality form (4.4.2) of the variational formulation. We shall resolve the condition $v_N - g \leq 0$ by using exterior penalty methods. Introducing penalty parameter $\varepsilon > 0$ such that $\varepsilon \to 0$, the variational inequality (4.4.2) is approximated by

$$\underset{\sim}{u}_\varepsilon \in \underset{\sim}{V} : B(\underset{\sim}{u}_\varepsilon, \underset{\sim}{v}) + \frac{1}{\varepsilon}\left(\beta(u_{\varepsilon N}), v_N\right) = f_1(\underset{\sim}{v}), \qquad \underset{\sim}{v} \in \underset{\sim}{V}_0, \tag{4.4.8}$$

where

$$\beta(\psi) = (\psi - g)^+, \quad \phi^+(\underset{\sim}{x}) = \sup\{\phi(\underset{\sim}{x}), 0\}, \quad \text{a.e. on } \Gamma_C, \tag{4.4.9}$$

$(\cdot,\cdot)$ is the inner product of $L^2(\Gamma_C)$ such that

$$(\phi,\psi) = \int_{\Gamma_C} \phi\psi d\Gamma, \qquad \phi,\psi \in L^2(\Gamma_C) \tag{4.4.10}$$

and $\underset{\sim}{V}_0$ is a subspace of $\underset{\sim}{H}^1(\Omega)$ defined by

$$\underset{\sim}{V}_0 = \{\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) : v_i = 0 \quad \text{on } \Gamma_{Di}, \quad 1 \leq i \leq 3\}. \tag{4.4.11}$$

We expect that $\underset{\sim}{u}_\varepsilon$ converges to $\underset{\sim}{u}$ as $\varepsilon \to 0$.

The physical idea behind the penalty approximation (4.4.8) is that the rigid support can be approximated by the Winkler foundation consisting of continuously distributed springs.

Theorem 4.2. Under the same conditions of Theorem 4.1, the sequence $\underset{\sim}{u}_\varepsilon$ of the solutions to (4.4.8) converges to the solution of the variational inequality (4.4.2), as $\varepsilon \to 0$.

Proof. We first note monotonicity of the operator β. Indeed,

$$\big(\beta(a) - \beta(b),\ a-b\big) = \big(\beta(a) - \beta(b),\ (a-g) - (b-g)\big)$$

$$= \int_{\Gamma_C} \{(a-g)^+ - (b-g)^+\}\{(a-g) - (b-g)\}ds$$

$$\geq \int_{\Gamma_C} \{(a-g)^+ - (b-g)^+\}\{(a-g)^+ - (b-g)^+\}ds,$$

i.e.,

$$\big(\beta(a) - \beta(b),\ a-b\big) \geq \|(a-g)^+ - (b-g)^+\|^2_{0,\Gamma_C} \geq 0 \qquad (4.4.12)$$

for every $a, b \in L^2(\Gamma_C)$. Here it has been applied that $\phi = \phi^+ - \phi^-$, $\phi^- = \sup\{-\phi, 0\}$, $\phi^+\phi^- = 0$, and $\phi^+\psi^- \geq 0$, for every $\phi, \psi \in R$.

Since $\beta(w_N) = 0$ for any $\underset{\sim}{w} \in \underset{\sim}{K}$, (4.4.8) implies

$$B(\underset{\sim}{u}_\varepsilon, \underset{\sim}{w}-\underset{\sim}{u}_\varepsilon) + \frac{1}{\varepsilon}\big(\beta(u_{\varepsilon N}), w_N - u_{\varepsilon N}\big) = f_1(\underset{\sim}{w}-\underset{\sim}{u}_\varepsilon)$$

and

$$B(\underset{\sim}{u}_\varepsilon,\underset{\sim}{w}-\underset{\sim}{u}_\varepsilon) - \frac{1}{\varepsilon}\left(\beta(w_N) - \beta(u_{\varepsilon N}), w_N - u_{\varepsilon N}\right) = f_1(\underset{\sim}{w}-\underset{\sim}{u}_\varepsilon). \qquad (4.4.13)$$

Applying the monotonicity of the operator β yields

$$B(\underset{\sim}{u}_\varepsilon,\underset{\sim}{w}-\underset{\sim}{u}) \geq f_1(\underset{\sim}{w}-\underset{\sim}{u}_\varepsilon), \qquad \underset{\sim}{w} \in \underset{\sim}{K}.$$

Applying (4.4.5)-(4.4.7), we have

$$m\|\underset{\sim}{u}_\varepsilon\|_1^2 \leq M\|\underset{\sim}{u}_\varepsilon\|_1\|\underset{\sim}{w}\|_1 + \|f_1\|_1^*(\|\underset{\sim}{w}\|_1 + \|\underset{\sim}{u}_\varepsilon\|_1)$$

Using Young's inequality

$$\phi\psi \leq \delta\phi^2 + \frac{1}{4\delta}\psi^2, \qquad \delta > 0, \qquad (4.4.14)$$

it can be concluded that

$$\|\underset{\sim}{u}_\varepsilon\|_1 \leq C(m, M, \|f_1\|_1^*, \|\underset{\sim}{w}\|_1) < +\infty.$$

In other words, the sequence $\{\underset{\sim}{u}_\varepsilon\}$ of the solutions of (4.4.8) is uniformly bounded in $\underset{\sim}{H}^1(\Omega)$ in ε. Thus, sequential compactness of the Sobolev space $\underset{\sim}{H}^1(\Omega)$ implies the existence of at least one subsequence of $\{\underset{\sim}{u}_\varepsilon\}$, still denoted by $\{\underset{\sim}{u}_\varepsilon\}$, that converges weakly to a limit, say $\underset{\sim}{u} \in \underset{\sim}{H}^1(\Omega)$.

Now we recall that convexity and Gâteaux differentiability of the functional $\underset{\sim}{v} \to B(\underset{\sim}{v},\underset{\sim}{v})$ in $\underset{\sim}{H}^1(\Omega)$ yields weak lower semicontinuity of the functional. Thus, to such a weak weak convergent subsequence $\{\underset{\sim}{u}_\varepsilon\}$, we have

$$B(\underset{\sim}{u},\underset{\sim}{w}-\underset{\sim}{u}) \geq \limsup B(\underset{\sim}{u}_\varepsilon,\underset{\sim}{w}-\underset{\sim}{u}_\varepsilon) \geq f_1(\underset{\sim}{w}-\underset{\sim}{u}),$$

i.e.,

$$B(\underset{\sim}{u},\underset{\sim}{w}-\underset{\sim}{u}) \geq f_1(\underset{\sim}{w}-\underset{\sim}{u}), \qquad \underset{\sim}{w} \in \underset{\sim}{K}. \tag{4.4.15}$$

On the other hand, relation (4.4.12) and equation (4.4.13) imply

$$\|(u_{\varepsilon N}-g)^+\|^2_{0,\Gamma_C} \leq \varepsilon\{B(\underset{\sim}{u}_\varepsilon,\underset{\sim}{w}) - f(\underset{\sim}{w}-\underset{\sim}{u}_\varepsilon)\}. \tag{4.4.16}$$

for any $\underset{\sim}{w} \in \underset{\sim}{K}$. Continuity of the norm and uniform boundedness of $\|\underset{\sim}{u}_\varepsilon\|_1$ yield

$$\|(u_N-g)^+\|_{0,\Gamma_C} = 0, \text{ i.e., } u_N - g \leq 0 \text{ a.e. on } \Gamma_C.$$

Since $\underset{\sim}{u}_\varepsilon$ $\underset{\sim}{V}$ and since $\underset{\sim}{V}$ is closed in $\underset{\sim}{H}^1(\Omega)$, the limit $\underset{\sim}{u}$ of $\{\underset{\sim}{u}_\varepsilon\}$ belongs to $\underset{\sim}{K}$. Thus, the limit of a subsequence $\{\underset{\sim}{u}_\varepsilon\}$ is a solution of (4.4.2).

Because of the uniqueness of the solution to (4.4.2), every convergent sequence must have a common limit. Therefore the sequence $\{\underset{\sim}{u}_\varepsilon\}$ itself as well as convergent subsequences has to converge to the solution $\underset{\sim}{u} \in \underset{\sim}{K}$ of (4.4.2). □

Let the approximation of the contact pressure (normal stress) $\sigma_{\varepsilon N}$ be defined by

$$\sigma_{\varepsilon N} = -\frac{1}{\varepsilon}\,\beta(u_{\varepsilon N}) \text{ in } L^2(\Gamma_C). \tag{4.4.17}$$

It can be shown that (4.4.17) is equivalent to the variational inequality form

$$\sigma_{\varepsilon N} \in M_0 : (\tau_N - \sigma_{\varepsilon N}, \varepsilon\sigma_{\varepsilon N} + u_{\varepsilon N} - g) \geq 0, \qquad \tau_N \in M_0, \tag{4.4.18}$$

where

$$M_0 = \{\tau_N \in L^2(\Gamma_C) : \tau_N \leq 0 \text{ a.e. on } \Gamma_C\}. \qquad (4.4.19)$$

To show (4.4.18), it suffices to note that for x,y, and f $\in$ R, the solution x $\in$ R to the problem

$$x \leq 0 : (\varepsilon x + f)(y - x) \geq 0, \qquad y \leq 0$$

is given by $x = \frac{1}{\varepsilon} f^+$.

Thus the exterior penalty formulation (4.4.8) of the variational inequality (4.4.2) is equivalent to the following perturbed Lagrangian multiplier formulation:

$$(\underset{\sim}{u}_\varepsilon, \sigma_{\varepsilon N}) \in \underset{\sim}{V} \times M_0 :$$

$$B(\underset{\sim}{u}_\varepsilon, \underset{\sim}{v}) - (\sigma_{\varepsilon N}, v_N) = f_1(\underset{\sim}{v}), \qquad \underset{\sim}{v} \in \underset{\sim}{V}_0 \qquad (4.4.20)$$

$$(\tau_N - \sigma_{\varepsilon N}, \varepsilon\sigma_{\varepsilon N} + u_{\varepsilon N} - g) \geq 0, \qquad \tau_N \in M_0$$

Since there are no guarantees that contact pressures belong to $L^2(\Gamma_C)$, the convergence argument of (4.4.20) in $\underset{\sim}{H}^1(\Omega) \times L^2(\Gamma_C)$ does not have any sense here. Indeed if a flat rigid punch is indented into an elastic foundation, the contact pressure σ_N is generally not in $L^2(\Gamma_C)$. Thus, we need to consider convergence of $(\underset{\sim}{u}_\varepsilon, \sigma_{\varepsilon N})$ in larger spaces.

If the fractional Sobolev space $\underset{\sim}{H}^S(\Gamma)$, $0 < S < 1$, is defined by the subspace of $\underset{\sim}{L}^2(\Gamma)$ equipped with the norm

$$\|\underset{\sim}{\phi}\|_{S,\Gamma} = \left\{ \|\underset{\sim}{\phi}\|_{0,\Gamma}^2 + \int_\Gamma \int_\Gamma \frac{|\underset{\sim}{\phi}(\underset{\sim}{x})-\underset{\sim}{\phi}(\underset{\sim}{y})|^2}{|\underset{\sim}{x}-\underset{\sim}{y}|^{N+2S-1}} \, d\Gamma d\Gamma \right\}^{1/2} \tag{4.4.21}$$

where $\Omega \subset R^N$ is Lipschizian, the extension of a function $\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega)$ to the boundar belongs to $\underset{\sim}{H}^{1/2}(\Gamma)$ by the trace theorem, see Necas [1967, Chapter 2]. suppose that the surface of the rigid foundation is polygonal. That is, assume that

$$\underset{\sim}{N} \in \underset{\sim}{H}^{1/2-\delta}(\Gamma_C), \quad \delta > 0 \tag{4.4.22}$$

for a small positive number δ. Then the normal displacement v_N of $\underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega)$ is in the space

$$Q = H^{1/2-\delta}(\Gamma_C), \quad \delta > 0. \tag{4.4.23}$$

Let Q' be the topological dual of Q, and let $[\cdot,\cdot]$ be the duality pairing on $Q' \times Q$. The partial ordering in Q' is defined by

$$\tau \in Q', \quad \tau \leq 0 \text{ if and only if } \quad \tau_n \in M_0 \text{ s.t. } \tau_n \to \tau \text{ in } Q' \tag{4.4.24}$$

Then the convex subset

$$M = \{\tau \in Q' : \tau \leq 0\} \tag{4.4.25}$$

is closed in Q'. Note that since Q is densely imbedded in $L^2(\Gamma_C)$, the duality pairing $[\cdot,\cdot]$ has the property that

$$[\rho,\phi] = (\tau,\phi) = \int_{\Gamma_C} \tau\phi ds, \qquad \tau \in L^2(\Gamma_C), \ \phi \in Q.$$

thus the $L^2(\Gamma_C)$-inner produce $(\cdot,\cdot)$ in (4.4.20) can be replaced by $[\cdot,\cdot]$ as long as $g \in Q$.

Theorem 4.3. In addition to the conditions in Theorem 4.1, suppose that (4.4.22) holds. Then the sequence $\{(\underset{\sim}{u}_\varepsilon, \sigma_{\varepsilon N})\} \in \underset{\sim}{V} \times M_0$ of the solutions to (4.4.20) converges to the solution $(\underset{\sim}{u}, \sigma_N) \in \underset{\sim}{V} \times M$ of the problem

$$\left.\begin{aligned} &B(\underset{\sim}{u},\underset{\sim}{v}) - [\sigma_N, v_N] = f_1(\underset{\sim}{v}), \qquad \underset{\sim}{v} \in \underset{\sim}{V}_0 \\ &[\tau_N - \sigma_N,\ u_N - g] \geq 0, \qquad \tau_N \in M \end{aligned}\right\} \tag{4.4.26}$$

as $\varepsilon \to 0$.

Proof. It suffices to show that $\|\sigma_{\varepsilon N}\|_{Q'}$ is uniformly bounded in Q' and in ε. To do this, we note that since the trace map $\gamma : \underset{\sim}{H}^1(\Omega) \to \underset{\sim}{H}^{1/2}(\Gamma)$ is linear continuous and surjective, and since the map $\underset{\sim}{\phi} \to \phi_N = \underset{\sim}{\phi}\cdot\underset{\sim}{N}$ from $\underset{\sim}{H}^{1/2}(\Gamma_C)$ into $H^{1/2-\delta}(\Gamma_C)$ is linear continuous and surjective, it can be found an element $\underset{\sim}{v} \in \underset{\sim}{V}_0$ for every $\phi \in H^{1/2-\delta}(\Gamma_C)$ such that

$$v_N = \phi \quad \text{and} \quad \|\underset{\sim}{v}\|_1 \leq C\|\phi\|_Q, \tag{4.4.27}$$

for some constant $C > 0$ independent of the choice ϕ.

Then

$$\|\sigma_{\varepsilon N}\|_{Q'} = \sup_{\phi \in Q} \frac{[\sigma_{\varepsilon N}, \phi]}{\|\phi\|_Q} \leq C \sup_{\underset{\sim}{V}_Q} \frac{[\sigma_{\varepsilon N}, v_N]}{\|\underset{\sim}{v}\|_1},$$

where $\underset{\sim}{V}_Q$ is the set of $\underset{\sim}{v}$ satisfying (4.4.27). Applying $(4.4.20)_1$, we have

$$\|\sigma_{\varepsilon N}\|_{Q'} < C\{M\|\underset{\sim}{u}_{\varepsilon}\|_1 + \|f_1\|_1^*\}.$$

Since the uniform boundedness of $\|\underset{\sim}{u}_{\varepsilon}\|_1$ is obtained similarly to Theorem 4.2, the uniform boundedness of $\|\sigma_{\varepsilon N}\|_{Q'}$ is assured. □

4.4.3. Finite Element Approximations

There are at least three kinds of finite element formulations of the variational problem (4.4.2). If aproximation is performed through the variational inequality (4.4.2) itself, a finite element formulation of the so-called displacement type can be obtained. This method, however, does not involve the contact pressure σ_N and requires an auxiliary scheme to compute it with sufficient accuracy. A second approach is to obtain a mixed finite element approximation through the formulation (4.4.26) using the Lagrangian multiplier method. In this case, the contact pressure σ_N itself becomes an additional unknown to be calculated along with the nodal displacements. Naturally, information on the deformation and the contact pressure can be easily obtained by this method. This approach, of course, involves the approximation of two independent fields, ($\underset{\sim}{u}$ and σ_N) and may be computationally inefficient. A third approach is to use penalty methods to handle the contact constraint. Then again only one unknown field appears in the variational problem; the penalized displacement field $\underset{\sim}{u}_{\varepsilon}$. An approximate contact pressure σ_N can be obtained later using (4.4.17) after solving the displacement. Thus, in this case, the contact pressure is computed by a simple post-processing procedure. In the present study, the penalty method will be applied to solve the problem (4.4.2).

The following finite element analysis will be performed for the case that the domain Ω is a bounded polygon in R^2. That is, all deformations of the three dimensional body are projected in the plane, and the domain Ω can be covered by finite elements that have straight edges. Extension to three-dimensional and curved finite elements is straightforward, although technically very complex.

Let the domain Ω be covered by E-finite elements, i.e.,

$$\bar{\Omega} = \bigcup_{e=1}^{E} \bar{\Omega}e.$$

As usual, we suppose that each element Ω_e contains N_e nodal points on which shape functions $\{\psi_i^e\}_{i=1}^{N_e}$ are assigned so that

$$\psi_i^e(\underset{\sim}{x}_j) = \delta_{ij}, \quad \sum_{i=1}^{N_e} \psi_i^e(\underset{\sim}{x}) = 1 \qquad \underset{\sim}{x} \in \Omega_e$$

where $\{\underset{\sim}{x}_j\}_{j=1}^{N_e}$ is the set of nodal points in Ω_e. The functions $\underset{\sim}{u}_\varepsilon$ and $\underset{\sim}{v}$ in (4.4.8) are approximated by piecewise polynomials $\underset{\sim}{u}_\varepsilon^h$ and $\underset{\sim}{v}^h$ defined by

$$u_{\varepsilon i}^h|_{\Omega_e} = u_{\varepsilon i}^j \psi_j^e(\underset{\sim}{x}), \quad v_j^h|_{\Omega_e} = v_j^k \psi_k^e(\underset{\sim}{x}) \tag{4.4.28}$$

in each element Ω_e, respectively. Here $u_{\varepsilon i}^h|_{\Omega_e}$ means the restriction of the i component of $\underset{\sim}{u}_\varepsilon^h$ into the subdomain Ω_e. Approximation (4.4.28) leads to the following finite element approximation of $\underset{\sim}{V}$ and $\underset{\sim}{V}_0$:

$$\underset{\sim}{V}^h = \{\underset{\sim}{v}^h : \underset{\sim}{v}^h = v_i^h \underset{\sim}{i}_i, \quad v_i^h \in C^0(\bar{\Omega}), \quad v_i^h|_{\Omega_e} = v_i^k \psi_k^e(\underset{\sim}{x}),$$

$$v_i^k = \bar{u}_i(\underset{\sim}{x}_k) \quad \text{on } \Gamma_{Di}, \quad 1 < i < 2\}, \tag{4.4.29}$$

and $\underset{\sim}{V}_0^h$ is a subspace of $\underset{\sim}{V}^h$ containin functions such that $\bar{u}_i^h(\underset{\sim}{x}_k) = 0$.

It is worthwhile to note that the term $\beta(u^R_{\varepsilon N})$ may not be a piecewise polynomial even though $\underset{\sim}{u}^h_\varepsilon$ is, because of the special operation $\phi \to \phi^+$. Then the penalty term cannot be integrated as the other terms when the stiffness matrix and the generalized load vector are computed.

To resolve this difficulty, let us introduce a quadrature rule to evaluate the penalty term (recall Chapter 3 of Volume II):

$$\left.\begin{aligned}
&\left(\beta(u^h_{\varepsilon N}), v^h_N\right) \doteq I\left(\beta(u^h_{\varepsilon N})v^h_N\right) \\
&I\left(\beta(u^h_{\varepsilon N})v^h_N\right) = \sum_{e=1}^{E'} I_e\left(\beta(u^h_{\varepsilon N})v^h_N\right) \\
&I_e\left(\beta(u^h_{\varepsilon N})v^h_N\right) = \sum_{J=1}^{G} w_J \beta(u^h_{\varepsilon N})(\underset{\sim}{x}_J) v^h_N(\underset{\sim}{x}_J).
\end{aligned}\right\} \tag{4.4.30}$$

Here $\{w_J\}$ and $\{\underset{\sim}{x}_J\}$ are the sets of weight and integration points, G the number of integration points, and E' is the number of finite element edges covering the contact boundary Γ_C.

The penalty formulation (4.4.8) is thus approximated by the following reduced-integration penalty finite element method:

$$\underset{\sim}{u}^h_\varepsilon \in \underset{\sim}{V}^h : B(\underset{\sim}{u}^h_\varepsilon, \underset{\sim}{v}^h) + \frac{1}{\varepsilon} I\left(\beta(u^h_{\varepsilon N})v^h_N\right) = f(\underset{\sim}{v}^h),$$

$$\underset{\sim}{v}^h \in \underset{\sim}{V}^h_0 \tag{4.4.31}$$

Because of the term $\beta(u^h_{\varepsilon N})$, problem (4.4.31) is nonlinear, and the stiffness matrix due to the penalty term is not readily computed. The actual computation depends upon the method we use to deal with the nonlinear term.

To illustrate one alternative, suppose that ${}^{i}\underset{\sim}{u}_{\varepsilon}^{h}$ is the approximation of the i step, and let

$$\beta_i({}^{i}u_{\varepsilon N}^{h}) = \begin{cases} {}^{i}u_{\varepsilon N}^{h} - g & \text{if} \quad {}^{i-1}u_{\varepsilon N}^{h} - g > 0 \\ \\ 0 & \text{if} \quad {}^{i-1}u_{\varepsilon N}^{h} - g < 0. \end{cases} \tag{4.4.32}$$

wherein i is not summed. Then the nonlinear equation (4.4.31) can be solved by the successive iteration scheme

$${}^{i}\underset{\sim}{u}_{\varepsilon}^{h} \in \underset{\sim}{V}^{h} : B({}^{i}\underset{\sim}{u}_{\varepsilon}^{h}, \underset{\sim}{v}^{h}) + \frac{1}{\varepsilon} I(\beta_i({}^{i}u_{\varepsilon N}^{h})v_N^h) = f(\underset{\sim}{v}^{h}), \qquad \underset{\sim}{v}^{h} \in \underset{\sim}{V}_0^h \tag{4.4.33}$$

for $i=1,2,3,\ldots,$ and the initial assumption $\underset{\sim}{u}_{\varepsilon}^{h}$. Since (4.4.33) is linear for each i, the stiffness matrix and load vector can be obtained by standard finite element procedures for linear problems discussed in great detail in Vols. I and III. Indeed, for each finite element Ω_e, we have

$$\left.\begin{aligned} B|_{\Omega_e}(\underset{\sim}{u}^{h}, \underset{\sim}{v}^{h}) &= v_i^{\alpha} K_{i\alpha j\beta}^{e} u_j^{\beta} \\ \\ I|_{\Omega_e}\big(\beta_t(u_N^h)v_N^h\big) &= v_i^{\alpha}\big({}^{t}N_{i\alpha j\beta}^{e} u_j^{\beta} - {}^{t}g_{i\alpha}^{e}\big) \\ \\ f|_{\Omega_e}(\underset{\sim}{v}^{h}) &= v_i^{\alpha} f_{i\alpha}^{e} \end{aligned}\right\} \tag{4.4.34}$$

where

$$K_{i\alpha j\beta}^{e} = \int_{\Omega_e} E_{ikj\ell}\, \psi_{\alpha,k}^{e}\, \psi_{\beta,\ell}^{e}\, d\Omega,$$

$$f^e_{i\alpha} = \int_{\Omega_e} f_i \psi^e_\alpha d\Omega + \int_{\Gamma^e_{Fi}} t_i \psi^e_\alpha ds + \int_{\Gamma^e_C} {}^t T_j(\delta_{ji} - N_j N_i)\psi^e_\alpha ds \qquad (4.4.35)$$

$${}^t N^e_{i\alpha j\beta} = \sum_{J=1}^{G} w^t_J (N_i \psi^e_\alpha N_j \psi^e_\beta)(\underset{\sim}{x}_J)$$

and

$${}^t g^e_{i\alpha} = \sum_{J=1}^{G} w^t_J (g N_i \psi^e_\alpha)(\underset{\sim}{x}_J).$$

Here $\Gamma^e_C = \Gamma_C \cap \partial\Omega_e$, $\Gamma^e_{Fi} = \Gamma_{Fi} \cap \partial\Omega_e$, $1 < i < 2$, and the "weights" $\{w^t_J\}$ are defined by

$$w^t_J = \begin{cases} w_J & \text{if } \; ({}^{t-1}u^h_{\varepsilon N} - g)(\underset{\sim}{x}_J) > 0 \\ 0 & \text{if } \; ({}^{t-1}u^h_{\varepsilon N} - g)(\underset{\sim}{x}_J) < 0. \end{cases} \qquad (4.4.36)$$

The terms ${}^t N^e_{i\alpha j\beta}$ and ${}^t g^e_{i\alpha}$ will be evaluated only on Γ^e_C. Using the above notation the variational form (4.4.33) yields the system of linear equations for each iteration step t=1,2,..., such that

$$\sum_{e=1}^{E} (K^e_{i\alpha j\beta} + \frac{1}{\varepsilon}\, {}^t N^e_{i\alpha j\beta})(u^\alpha_{\varepsilon i}) = \sum_{e=1}^{E} (f^e_{i\alpha} + \frac{1}{\varepsilon}\, {}^t g^e_{i\alpha}) \qquad (4.4.37)$$

The assembly and solution of these linear equations is now done following any of the standard procedures described in Vol. III.

Since the coercivity of the bilinear form B(•,•) and continuity of B(•,•) and the linear form f(•) are independent of the choice of the penalty parameter $\varepsilon > 0$ and the step i of the iteration (4.4.33), arguments similar to those in Theorem 4.2 yield the following result.

Theorem 4.4. Under the same conditions in Theorem 4.1., if the weights $\{w_J\}$ of the quadrature rule (4.4.30) are all positive, then the sequence $\{{}^i\underset{\sim}{u}_\varepsilon^h\}$ of the solutions to (4.4.33) converges to the solution $\underset{\sim}{u}_\varepsilon^h$ of (4.4.31) as $i \to +\infty$. In addition, the sequence $\{\underset{\sim}{u}_\varepsilon^h\}$ of limits obtained for each $\varepsilon > 0$ converges as ε tends to zero to the solution $\underset{\sim}{u}^h \in \underset{\sim}{K}^h$ of the variational inequality

$$\underset{\sim}{u}^h \in \underset{\sim}{K}^h : B(\underset{\sim}{u}^h, \underset{\sim}{v}^h - \underset{\sim}{u}^h) \geq f(\underset{\sim}{v}^h - \underset{\sim}{u}^h), \qquad \underset{\sim}{v}^h \in \underset{\sim}{K}^h \tag{4.4.38}$$

where

$$\underset{\sim}{K}^h = \{\underset{\sim}{v}^h \in \underset{\sim}{V}^h : (v_N^h - g)(\underset{\sim}{x}_J) \leq 0 \text{ on } \Gamma_C\}. \tag{4.4.39}$$

□

It is noted that the iteration process finding $\underset{\sim}{u}_\varepsilon^h$ for each $\varepsilon > 0$ converges rather quickly for example problems solved later, although the results in Theorem 4.4 imply only the convergence of the sequence $\{{}^i\underset{\sim}{u}_\varepsilon^h\}$ as $i \to \infty$.

4.4.4 Convergence of the Penalty/Finite Element Method

We shall define an approximation of the contact pressure $\sigma_{\varepsilon N}^h$ as in (4.4.17):

$$\sigma_{\varepsilon N}^h \in Q_h' \text{ and } \sigma_{\varepsilon N}^h(\underset{\sim}{x}_J) = -\frac{1}{\varepsilon}\beta(u_{\varepsilon N}^h)(\underset{\sim}{x}_J),$$

$$J = 1, \ldots, G, \text{ on } \Gamma_C^e, \; e = 1, 2, \ldots, E', \tag{4.4.40}$$

where

$$Q_h' = \{\tau^h : \tau^h|_{\Gamma_C^e} = \tau_\beta M_\beta, \; 1 \leq e \leq E'\} \tag{4.4.41}$$

and $\{M_\beta\}$ is the set of shape functions associated to the integration points of the quadrature rule. Each function τ^h in Q_h' might not be continuous along the boundary Γ_C. Continuity of τ^h depends upon the choice of the quadrature rule.

Applying (4.4.40) to the equation (4.4.31) yields

$$(\underset{\sim}{u}^h_\varepsilon,\ \sigma^h_{\varepsilon N}) \in \underset{\sim}{V}^h \times M^h:$$

$$B(\underset{\sim}{u}^h_\varepsilon,\ \underset{\sim}{v}^h) = I(\sigma^h_{\varepsilon N}\ v^h_N) - f\ (\underset{\sim}{v}^h)\ ,\ \forall\ \underset{\sim}{v}^h \in \underset{\sim}{V}^h \tag{4.4.42}$$

$$I\left((\tau^h_N - \sigma^h_{\varepsilon N})\quad (\varepsilon\sigma^h_{\varepsilon N} + u^h_{\varepsilon N} - g)\right) \geq 0,\ \forall\ \tau^h_N \in M^h,$$

where M^h is a subset of Q'_h defined by

$$M^h = \{\tau^h \in Q'_h : \tau^h\ (\underset{\sim}{x}_J) \leq 0,\ 1 \leq J \leq G \text{ on } \Gamma^e_C\ ;\ 1 \leq e \leq E'\}. \tag{4.4.43}$$

<u>Lemma 4.1</u> Let $(\underset{\sim}{u},\ \sigma_N)\ \ \underset{\sim}{V} \times M$ and $(\underset{\sim}{u}^h_\varepsilon,\ \sigma^h_{\varepsilon N}) \in \underset{\sim}{V}^h \times M^h$ be the solution of (4.4.26) and (4.4.42), respectively. Let E_I be the quadrature error of the numerical integration introduced in (4.4.30):

$$E_I\ (f,g) = (f,g) - I(fg) \tag{4.4.44}$$

for properly defined functions f and g so that the quadrature rule (4.4.30) makes sense. Then for every $\underset{\sim}{v}^h \in \underset{\sim}{K}^h$ and $\tau^h_N \in M^h$

$$B(\underset{\sim}{u} - \underset{\sim}{u}^h_\varepsilon,\ \underset{\sim}{u} - \underset{\sim}{u}^h_\varepsilon) \leq B(\underset{\sim}{u} - \underset{\sim}{u}^h_\varepsilon,\ u - v^h) + [\sigma_N,\ v^h_N - u_N]$$

$$+ [\sigma_N - \tau^h_N,\ u_N - u^h_{\varepsilon N}] + [\tau^h_N - \sigma_N,\ u_N - g]$$

$$- E_I\left(\tau^h_N\ (u^h_{\varepsilon N} - g)\right) + I\ \left((\tau^h_N - \tau^h_{\varepsilon N})\quad \tau^h\right)\ \varepsilon, \tag{4.4.45}$$

under the assumption that the "gap" function g is well defined so that the quadrature rule I is also meaningful for g.

Proof from (4.4.26) and (4.4.42), we have

$$B(\underset{\sim}{u} - \underset{\sim}{u}^h_{\varepsilon}, \underset{\sim}{u} - \underset{\sim}{u}^h_{\varepsilon}) = B(\underset{\sim}{u} - \underset{\sim}{u}^h_{\varepsilon}, \underset{\sim}{u} - \underset{\sim}{v}^h) + B(\underset{\sim}{u} - \underset{\sim}{u}^h_{\varepsilon}, \underset{\sim}{v}^h - \underset{\sim}{u}^h_{\varepsilon})$$

$$= B(\underset{\sim}{u}-\underset{\sim}{u}^h_{\varepsilon}, \underset{\sim}{u}-\underset{\sim}{v}^h) + [\sigma_N, v^h_N - u^h_{\varepsilon N}] - I\left(\sigma^h_{\varepsilon N}(v^h_N - u^h_{\varepsilon N})\right).$$

The last two terms become

$$[\sigma_N, v^h_N - u^h_{\varepsilon N}] - I\left(\sigma^h_{\varepsilon N}(v^h_N - u^h_{\varepsilon N})\right)$$

$$= [\sigma_N, v^h_N - u_N] + [\sigma_N - \tau^h_N, u_N - u^h_{\varepsilon N}] + [\tau^h_N, u_N - g]$$

$$-[\tau^h_N, u^h_{\varepsilon N} - g] - I\left(\tau^h_{\varepsilon N}(v^h_N - g)\right) + I\left(\sigma^h_{\varepsilon N}(u^h_{\varepsilon N} - g)\right)$$

$$\leq [\sigma_N, v^h_N - u_N] + [\sigma_N - \tau^h_N, u_N - u^h_{\varepsilon N}] + [\tau^h_N - \sigma_N, u_N - g]$$

$$- E_I\ (\tau^h_N, u^h_{\varepsilon N} - g) - I\left(\tau^h_N\ (u^h_{\varepsilon N} - g)\right) + I\left(\sigma^h_{\varepsilon N}\ (u^h_{\varepsilon N} - g)\right)$$

$$\leq [\sigma_N, v^h_N - u_N] + [\sigma_N - \tau^h_N, u_N - u^h_{\varepsilon N}] + [\tau^h_N - \sigma_N, u_N - g]$$

$$- E_I\ (\tau^h_N, u^h_{\varepsilon N} - g) + I\left((\sigma^h_{\varepsilon N} - \tau^h_N)\ (u^h_{\varepsilon N} - g)^+\right).$$

Then the estimate (4.4.45) follows from

$$I\left((\sigma^h_{\varepsilon N} - \tau^h_N)\,(u^h_{\varepsilon N} - g)^+\right) = I\left((\sigma^h_{\varepsilon N} - \tau^h_N)\,(-\varepsilon\sigma^h_{\varepsilon N})\right)$$

$$\leq - I\left((\sigma^h_\varepsilon - \tau^h_N)\,\tau^h_N\right)\varepsilon\ . \quad \square$$

It is noted that the estimate (4.4.45) is obtained within the restricted sets $\underset{\sim}{K}^h$ and M^h. The first four terms of the right hand side of (4.4.45) are related to the interpolation error of function by finite element methods, the fifth term comes from the integration error for the penalty term, and the last one is from the method of penalty. It is expected that if $\varepsilon \to o$, the last term goes to zero. We shall study this more precisely.

Toward this end, let us introduce an approximation Λ_h of the normal trace operator $\underset{\sim}{v} \to v_n$ on $\underset{\sim}{H}^1(\Omega)$:

$$[\tau^h_N, \Lambda_h\,(\underset{\sim}{v}^h)] = \left(\tau^h_N,\ \Lambda_h\,(\underset{\sim}{v}^h)\right) = I\,(\tau^h\, v^h_N) \qquad (4.4.46)$$

for every $\underset{\sim}{v}^h \in \underset{\sim}{V}^h$.

<u>Lemma 4.2</u> Let $(\underset{\sim}{u},\ \sigma_n) \in \underset{\sim}{V} \times M$ and $(\underset{\sim}{u}^h_\varepsilon,\ \sigma^h_{\varepsilon N}) \in \underset{\sim}{V}^h \times M^h$ be the solutions of (4.4.26) and (4.4.42) respectively. Suppose that there exist a positive number α_h independent of τ^h_N and an element $\underset{\sim}{v}^h \in V_h$ such that

$$\Lambda_h (\underset{\sim}{v}^h) = \tau_N^h, \quad \alpha_h \|\underset{\sim}{v}^h\|_1 \leq \|\tau_N^h\|_{0,\Gamma_C} \tag{4.4.47}$$

for a given $\tau_N^h \in M^h \cap R_g (\Lambda_h)$, where $R_g (\Lambda_h)$ is the range of the operator Λ_h. Then, for every $\tau_N^h \in M^h$,

$$\left| I \left((\tau_N^h - \sigma_{\varepsilon N}^h)^2 \right) \right| \leq M \|\underset{\sim}{u} - \underset{\sim}{u}_\varepsilon^h\|_1 \|\tau_N^h - \sigma_{\varepsilon N}^h\|_{0,\Gamma_C} / \alpha_h$$

$$+ \|\tau_N^h - \sigma_N\|_{0,\Gamma_C} \|\tau_N^h - \sigma_{\varepsilon N}^h\|_{0,\Gamma_C} + \left| E_I \left(\tau_N^h (\tau^h - \sigma_{\varepsilon N}^h) \right) \right| \tag{4.4.48}$$

Proof. From (4.4.26) and (4.4.42),

$$I \left((\tau_N^h - \sigma_{\varepsilon N}^h) v_N^h \right) = I \left((\sigma_N - \sigma_{\varepsilon N}^h) v_N^h \right) + I \left((\tau_N^h - \sigma_N) v_N^h \right)$$

$$+ [\sigma_N, v_N^h] - I(\tau_N^h \ v_N^h) + I(\sigma_N \ v_N^h)$$

$$- [\sigma_N, v_N^h] + I \left((\tau_N^h - \sigma_N) \ v_N^h \right)$$

$$= B(\underset{\sim}{u} - \underset{\sim}{u}_\varepsilon^h, \underset{\sim}{v}^h) - [\sigma_N, v_N^h] + I (\tau_N^h \ v_N^h)$$

$$= B(\underset{\sim}{u} - \underset{\sim}{u}_\varepsilon^h, \underset{\sim}{v}^h) + [\tau_N^h - \sigma_N, v_N^h] - E_I (\tau_N^h, v_N^h)$$

Applying (4.4.47) and taking absolute value yield the result. □

Let us now apply the interpolation results of conforming finite element approximations on the restricted sets K^h and M^h following Falk [1974].

- there is a $\underset{\sim}{v}^h \in \underset{\sim}{K}^h$ such that

$$\| \underset{\sim}{v}^h - \underset{\sim}{v} \|_r \leq C_1 h^{\mu_1} \| v \|_s$$

for every $\underset{\sim}{v} \in \underset{\sim}{K} \cap \underset{\sim}{H}^s(\Omega)$, $r \leq s$, and $s \geq 2$, where

$$\mu_1 = \text{Min } \{k + 1 - r, s - r\},$$

and k is the order of the complete piecewise polynomials contained in finite element approximations of $\underset{\sim}{v}$, (4.4.49)

- there is a $\tau_N^h \in M^h$ such that

$$\| \tau_N^h - \tau_N \|_{p,\Gamma_C} \leq C_2 h^{\mu_2} \| \tau_N \|_{q,\Gamma_C}$$

for every $\tau_N \in M \cap H^q(\Gamma_C)$, $p \leq q$, and $q \geq \frac{1}{2} + \delta$, $\delta > 0$, where

$$\mu_2 = \text{Min } \{t + \frac{1}{2} - p, q - p\}$$

and t is the order of complete picewise polynomials contained finite element approximations of τ_N. (4.4.50)

We shall suppose that the parameters r and p in (4.4.49) and (4.4.50), respectively, can be negative numbers (see Babuska and Aziz [1972, p.95]) and that M^h need not be conforming.

Theorem 4.5. Suppose that the quadrature rule I satisfies for some positive constants C_i, $i = 3, \ldots, 6$, and parameters λ_1 and λ_2, the inequalities

$$I({v_N^h}^2) \geq C_3 \| v_N^h \|^2_{0,\Gamma_C} \quad , \quad I(\tau_N^h \, v_N^h) \leq C_4 \| \tau_N^h \|_{0,\Gamma_C} \| v_N^h \|_{0,\Gamma_C} \tag{4.4.51}$$

and

$$\left.\begin{aligned} |E_I(\tau_N^h, v_N^h)| &\leq C_5 h^{\lambda_1} \| \tau_N^h \|_{q,\Gamma_C} \| v_N^h \|_{s-1/2,\Gamma_C} , \\ |E_I(\tau_N^h, \hat{\tau}_N^h)| &\leq C_6 \, h^{\lambda_2} \| \tau_N^h \|_{q,\Gamma_C} \| \hat{\tau}_N^h \|_{0,\Gamma_C} , \\ |E_I(\tau_N^h, g)| &\leq C_7 h^{\lambda_3} \| \tau_N^h \|_{q,\Gamma_C} \| g \|_{s-1/2,\Gamma_C} \end{aligned}\right\} \tag{4.4.52}$$

where q and s are the parameters in (4.4.49) and (4.4.50).

Then under the regularity assumption

$$\underset{\sim}{u} \in \underset{\sim}{H}^s(\Omega), \; s \geq 2, \; \sigma_N \in H^{s-3/2}(\Gamma_C), \text{ and } g \in H^{s-1/2}(\Gamma_C) \tag{4.4.53}$$

the following error estimates are obtained:

$$\left.\begin{aligned} \| \underset{\sim}{u} - \underset{\sim\varepsilon}{u}^h \|_1 &\leq C\,(\| \underset{\sim}{u} \|_s, \| \sigma_N \|_{s-3/2,\Gamma_C}, \| g \|_{s-1/2,\Gamma_C})\,(h^{\mu_3} + \varepsilon/\alpha_h + \varepsilon h^{\mu_4}) \\ \| \sigma_N - \sigma_{\varepsilon N}^h \|_{0,\Gamma_C} &\leq \bar{C}(\| \sigma_N \|_{s-3/2,\Gamma_C})\,(h^{\mu_4} + \| \underset{\sim}{u} - \underset{\sim\varepsilon}{u}^h \|_1/\alpha_h) \end{aligned}\right\} \quad (4.4.54)$$

where

$$\left.\begin{aligned} \mu_3 &= \min\,\{k,\ s-1,\ (k + s-1)/2,\ t,\ (t + s)/2,\ \lambda_1/2,\ \lambda_3/2\} \\ \mu_4 &= \min\,\{t + 1/2,\ s-3/2,\ \lambda_2\}\ . \end{aligned}\right\} \quad (4.4.55)$$

<u>Proof</u>. Applying (4.4.51) and (4.4.52) into (4.4.48) yields

$$C_3 \| \tau_N^h - \sigma_{\varepsilon N}^h \|_{0,\Gamma_C} \leq M \| \underset{\sim}{u} - \underset{\sim\varepsilon}{u}^h \|_1/\alpha_h + \| \tau_N^h - \sigma_N \|_{0,\Gamma_C} + C_6\, h^{\lambda_2} \| \tau_N^h \|_{q,\Gamma_C}$$

Applying the interpolation theorem (4.4.50), we have

$$\| \tau_N^h - \sigma_{\varepsilon N}^h \|_{0,\Gamma_C} \leq C\,(\| \underset{\sim}{u} - \underset{\sim\varepsilon}{u}^h \|_1/\alpha_h + h^{\min\,\{t + 1/2, q, \lambda_2\}} \| \tau_N^h \|_{q,\Gamma_C}) \quad (4.4.56)$$

On the other hand, from the estimate (4.4.45), we obtain

$$m\| \underset{\sim}{u} - \underset{\sim\varepsilon}{u}^h \|_1^2 \leq M \| \underset{\sim}{u} - \underset{\sim\varepsilon}{u}^h \|_1 \| \underset{\sim}{u} - \underset{\sim}{v}^h \|_1 + \| \sigma_N \|_{q,\Gamma_C} \| u_N - v_N^h \|_{\left(H^q(\Gamma_C)\right)^1}$$

$$+ \| \sigma_N - \tau_N^h \|_{\left(H^{1/2}(\Gamma_C)\right)'} \| u_N - u_{\varepsilon N}^h \|_{1/2,\Gamma_C} +$$

$$\| \sigma_N - \tau_N^h \|_{\left(H^{s-1/2}(\Gamma_C)\right)'} \| u_N - g \|_{s-1/2,\Gamma_C} + C_5 h^{\lambda_1} \| \tau_N^h \|_{q,\Gamma_C} \| u_{\varepsilon N}^h \|_{s-1/2,\Gamma_C}$$

$$+ \; C_7 h^{\lambda_3} \| \tau_N^h \|_{q,\Gamma_C} \| g \|_{s-1/2,\Gamma_C}$$

$$+ C_4 \; \varepsilon \| \tau_N^h - \sigma_{\varepsilon N}^h \|_{0,\Gamma_C} \| \tau_N^h \|_{0,\Gamma_C}$$

Substitution of (4.4.56) and applying Young's inequality imply

$$\| \underset{\sim}{u} - \underset{\sim\varepsilon}{u}^h \|_1^2 \leq C \Big\{ \| \underset{\sim}{u} - \underset{\sim}{v}^h \|_1^2 \| + \| \sigma_N \|_{q,\Gamma_C} \| u_N - v_N^h \|_{\left(H^q(\Gamma_C)\right)'}$$

$$+ \| \sigma_N - \tau_N^h \|^2_{\left(H^{1/2}(\Gamma_C)\right)'} + \| u_N - g \|_{s-1/2,\Gamma_C} \| \sigma_N - \tau_N^h \|_{\left(H^{s-1/2}(\Gamma_C)\right)'}$$

$$+ C_5 h^{\lambda_1} \| \tau_N^h \|_{q,\Gamma_C} \| u_{\varepsilon N}^h \|_{s-1/2,\Gamma_C} + C_7 h^{\lambda_3} \| \tau_N^h \|_{q,\Gamma_C} \| g \|_{s-1/2,\Gamma_C}$$

$$+ C_8^2 \, (\varepsilon/\alpha_h)^2 \| \tau_N^h \|^2_{0,\Gamma_C} + C_9^2 \, \varepsilon^2 \, h^{2\{\mathrm{Min}\ t + 1/w,\ g, \lambda_2\}} \| \tau_N^h \|^2_{q,\Gamma_C}$$

$$\times \| \tau_N^h \|^2_{0,\Gamma_C} \Big\}$$

putting $q = s - 3/2$, we arrive at the bound

$$\| \underset{\sim}{u} - \underset{\sim}{u}^h \|_1 \leq C\Big(\| \underset{\sim}{u} \|_s, \| \sigma_N \|_{s-3/2,\Gamma_C}, \| g \|_{s-1/2,\Gamma_C} \Big) \Big\{ h^{\min\{k,s-1\}}$$

$$+ h^{\min\{k+s-1,2s-2\}/2} + h^{\min\{t,s-1\}} + h^{\min\{t+s,2s-2\}/2}$$

$$+ h^{\lambda_1/2} + h^{\lambda_3/2} + (\varepsilon/\alpha h) + \varepsilon h^{\min\{t+1/2,s-3/2,\lambda_2\}} \Big\}$$

i.e.,

$$\| \underset{\sim}{u} - \underset{\sim}{u}^h_\varepsilon \|_1 \leq C\Big(\| \underset{\sim}{u} \|_s, \| \sigma_N \|_{s-3/2,\Gamma_c}, \| g \|_{5-1/2,\Gamma_c} \Big) \Big(h^{\mu_3}$$

$$+ \varepsilon/\alpha_h + \varepsilon h^{\mu_4} \Big),$$

where

$$\mu_3 = \min\Big\{ k, s-1, (k+s-1)/2, t, (t+s)/2, \lambda_1/2, \lambda_3/2 \Big\}$$

and

$$\mu_4 = \min\Big\{ t+1/2, s-3/2, \lambda_2 \Big\}.$$

Now an estimate for the approximate contact pressure is derived immediately from (4.4.56):

$$\|\sigma_N - \sigma_{\varepsilon N}^h\|_{0,\Gamma_c} \leq \|\sigma_N - \tau_N^h\|_{0,\Gamma_c} + \|\tau_N^h - \sigma_{\varepsilon N}^h\|_{0,\Gamma_c}$$

$$\leq \bar{C}\left(\|\sigma_N\|_{s-3/2,\Gamma_c}\right)\left(h^{\mu_4} + \|\underset{\sim}{u} - \underset{\sim\varepsilon}{u}^h\|_1/\alpha_h\right). \quad \square$$

It is clear from these results that three kinds of estimates of the error introduced by the numerical integration I are needed as well as an estimate of the parameter α_h of (4.4.47) for each given choice of integration rule and finite elements basis. We shall discuss these in the subsequent subsections.

4-Node Isoparametric Element and Trapezoid Rule

One of the simplest finite elements to the contact problem is the 4-node isoparametric quadrilateral element. This element yields piecewise linear approximation of the displacement on the boundary. Moreover, the edges are just straight lines. For the integration rule "I" , let us apply the trapezoid formula on each boundary element that correpsonds to an edge of 4-node element. The "normal" component v_N^h of $\underset{\sim}{v}^h$ on the boundary element is constructed by taking the same interpolation to $\underset{\sim}{v}^h$ by using the value $\underset{\sim}{v}^h \cdot \underset{\sim}{N}$ at each nodal points on the boundary. That is, v_N^h is a piecewise linear polynominal on Γ_C which is the same degree as $\underset{\sim}{v}^h$. On the other hand, the trapezoid rule on each boundary element yields the same approximation of the contact pressure τ_N^h of the "normal" displacement v_N^h. That is, both functions v_N^h and τ_N^h are linear polynomials on each boundary element Γ_e.

<u>Lemma 4.3</u>. For the functions v_N^h and τ_N^h defined in above, we have

$$\left.\begin{aligned} & I\left({v_N^h}^2\right) \geq \|v_N^h\|_{0,\Gamma_C}^2 \\ & I\left(\tau_N^h\, v_N^h\right) \leq 9\, \|\tau_N^h\|_{0,\Gamma_C} \|v_N^h\|_{0,\Gamma_C} \end{aligned}\right\} \tag{4.4.57}$$

<u>Proof</u>. Noting that $\|v_N^h\|_{0,\Gamma_C}^2 = \sum_{e=1}^{E'} \|v_N^h\|_{0,\Gamma_e}^2$,

$\Gamma_e = \partial\Omega_e \cap \Gamma_C$, we shall derive the inequalities of (4.4.57) on each boundary element Γ_e. Since v_N^h and τ_N^h are linear polynomials on Γ_e, we can express these as

$$v_N^h = a + bs \quad \text{and} \quad \tau_N^h = c + ds \ , \tag{4.4.58}$$

where s is the local coordinate along the boundary element, and $\{a,b,c,d\}$ is the set of numbers defined by the values of v_N^h and τ_N^h at nodes and the length of the element Γ_e. Then we have

$$\|v_N^h\|_{0,\Gamma_e}^2 = \int_0^{h_e} (a + bs)^2 ds = h\left(a^2 + abh + \frac{b^2}{3} h^2\right)$$

$$I\left(v_N^{h2}\right) = h\left(a^2 + abh + \frac{b^2}{2} h^2\right) \geq \|v_N^h\|^2_{0,\Gamma_C}$$

$$I\left(\tau_N^h v_N^h\right) \leq \left\{I(\tau_N^{h^2})\right\}^{1/2} \left\{I(v_N^{h^2})\right\}^{1/2} \leq 3 \|\tau_N^h\|_{0,\Gamma_C} \|v_N^h\|_{0,\Gamma_C} \quad \square$$

We now show error of the quadrature rule I for the choice of the 4-node element and the trapezoid rule to the penalty term.

Lemma 4.4. For the functions τ_N^h , v_N^h , and $\hat{\tau}_N^h$, we have the following estimates:

$$|E_I\left(\tau_N^h , v_N^h\right)| \leq C_1 h^2 \|\tau_N^h\|_{1/2,\Gamma_C} \|v_N^h\|_{3/2,\Gamma_C}$$

$$|E_I\left(\tau_N^h, \hat{\tau}_N^h\right)| \leq C_2 h \|\tau_N^h\|_{1/2,\Gamma_C} \hat{\tau}_N^h \|_{1/2,\Gamma_C} \tag{4.4.59}$$

$$|E_I\left(\tau_N^h , g\right)| \leq C_3 h^2 \|\tau_N^h\|_{1/2,\Gamma_C} \|g\|_{3/2,\Gamma_C}$$

where C_1, C_2, and C_3 are positive constant independent of the mesh size h.

Proof. Noting that τ_N^h and v_N^h are piecewise linear on the boundary element,

$$|E_I\left(\tau_N^h, v_N^h\right)| \leq \sum_{e=1}^{E'} \left| \frac{1}{b} \left(\tau_N^h\right)' \left(v_N^h\right)' h^3 \right| \leq \frac{1}{6} h^2 \|\left(\tau_N^h\right)'\|_{(H^{1/2}(\Gamma_C)}$$

$$\|\left(v_N^h\right)' \|_{1/2,\Gamma_C} \leq C_1 h^2 \|\tau_N^h\|_{1/2,\Gamma_C} \|v_N^h\|_{3/2,\Gamma_C} .$$

Similarly,

$$|E_I(\tau_N^h,\hat{\tau}_N^h)| \leq \sum_{e=1}^{E'} |\frac{1}{-}(\tau_h^N)'(\tau_h^N)'h^3| \leq \hat{C}_2 h^2 \|\tau_h^N\|_{1,\Gamma_C} \|\hat{\tau}_h^N\|_{1,\Gamma_C}$$

Applying the inverse inequality

$$\|\tau_N^h\|_{1,\Gamma_C} \leq Ch^{-1/2} \|\tau_N^h\|_{1/2,\Gamma_C} , \tag{4.4.60}$$

we obtain

$$|E_I(\tau_N^h,\hat{\tau}_N^h)| \leq C_2 h \|\tau_N^h\|_{1/2,\Gamma_C} \|\hat{\tau}_N^h\|_{1/2,\Gamma_C}$$

The last inequality in (4.4.59) is obtained by the following triangular inequality

$$|E_I(\tau_N^h ,g)| \leq |E_I(\tau_N^h , g^h)| + |E_I(\tau_N^h , g-g^h)|$$

by using the piecewise linear interpolation of g. The first part in the right hand side is estimated by

$$|E_I(\tau_N^h , g^h)| \leq \hat{C}_3 h^2 \|\tau_N^h\|_{1/2,\Gamma_C} \|g^h\|_{3/2,\Gamma_C} \tag{4.4.61}$$

similarly to the first inequality of (4.4.59). The second part is estimated by using the result of interpolation:

$$|E_I(\tau_N^h , g-g^h)| \leq C \|\tau_N^h\|_{1/2,\Gamma_C} \|g-g^h\|_{(H^{1/2}(\Gamma_C))'}$$

$$\leq \hat{C}_3 h^2 \|\tau_N^h\|_{1/2,\Gamma_C} \|g\|_{3/2,\Gamma_C} . \qquad (4.4.62)$$

Thus combining (4.4.61) and (4.4.62) yields the inequality that we need. □

The last preliminary to the application of the general result (4.4.54) to specific the error estimates for the penalty finite element approximation is to find the stability constant α_h of the approximate contact pressure stated by (4.4.47).

Lemma 4.5. For the choice of 4-node isoparametric elements and the trapizoid rule, we have $\alpha_h = \alpha_o h^{1/2}$ and $v^h \in \underset{\sim}{V}^h$ for a given τ_N^h such that

$$\Lambda_h(\underset{\sim}{v}^h) = \tau_N^h , \ \alpha_h \|\underset{\sim}{v}^h\|_1 \leq \|\tau_N^h\|_{0,\Gamma_C} \qquad (4.4.63)$$

where α_o is a positive constant independent of mesh size and τ_N^h .

Proof. For the case $\tau_N^h \equiv 0$, the inequality (4.4.63) is obvious for the choice $\underset{\sim}{v}^h \equiv 0$. Thus we need to consider the case $\tau_N^h \not\equiv 0$, i.e., $\tau_N^h \in \ker(\sigma_N)$, where $\ker(\sigma_N)$ is the kernel of the normal trace operator such that $\sigma_N \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{N}$ on the boundary for a function $\underset{\sim}{v} \in \underset{\sim}{C}(\Omega)$. Applying the Banach inverse theorem to the continuous linear operator σ_N, we may condlude that there exists a positive constant $C>0$ such that

$$\|\underset{\sim}{v}\|_1 \leq C \|v_N^h\|_{1/2,\Gamma_C} \qquad (4.4.64)$$

for $v_N^h \notin \ker(\sigma_N) \cap \underset{\sim}{V}^h$, where C is independent of h.

Now, taking the special case that $v_N^h = \tau_N^h$ at each integration point, we have

$$I\left(\tau_N^h v_N^h\right) = \left\{I({\tau_N^h}^2)\right\}^{1/2} \left\{I({v_N^h}^2)\right\}^{1/2} .$$

Applying the results in Lemma 4.3 implies

$$I\left(\tau_N^h v_N^h\right) \geq \| \tau_N^h \|_{0,\Gamma_C} \| v_N^h \|_{0,\Gamma_C} .$$

The following inverse inequality holds under the assumptions of the

$$Ch^{1/2} \| v_N^h \|_{1/2,\Gamma_C} \leq \| v_N^h \|_{0,\Gamma_C} , \tag{4.4.65}$$

Thus, we have

$$I\left(\tau_N^h v_N^h\right) \geq Ch^{1/2} \| \tau_N^h \|_{0,\Gamma_C} \| v_N^h \|_{1/2,\Gamma_C} .$$

Finally, applying (4.4.64) yields

$$I\left(\tau_N^h v_N^h\right) \geq Ch^{1/2} \| \tau_N^h \|_{0,\Gamma_C} \| \underset{\sim}{v}^h \|_1 .$$

The inequalities in (4.4.63) now follow from this result. □

Using the all results in Lemma 4.3, 4.4, and 4.5 and Theorem 4.5, we can obtain an error estimate for the penalty/finite element approximation (4.4.31) for 4-node quadrilateral isoparametric elements and the trapezoid rule.

Theorem 4.6. Let the hypotheses of Lemmas 4.4-4.5 and Theorem 4.5 hold and suppose that the solution $(\underset{\sim}{u},\sigma_N)$ to the original problem is smooth enough so that

$$\underset{\sim}{u}\in \underset{\sim}{H}^2(\Omega) \quad \text{and} \quad \sigma_N \in H^{1/2}(\Gamma_C) \ . \tag{4.4.66}$$

Then, for $g\in H^{3/2}(\Gamma_C)$ we have

$$\left.\begin{aligned} \|\underset{\sim}{u} - \underset{\sim}{u}^h\|_1 &\le C_1\left(h^1 + h^{-1/2}\varepsilon + h^{1/2}\varepsilon\right) \\ \|\sigma_N - \sigma_N^\varepsilon\|_{0,\Gamma_C} &\le C_2\left(h^{1/2} + h^{-1}\varepsilon + \varepsilon\right) \end{aligned}\right\} \tag{4.4.67}$$

where C_1 and C_2 are positive constants independent of h. □

4.4.5 Numerical Examples. The results of numerical experiments on two example problems are given to demonstrate the performance of the penalty/finite element approximation (4.4.31). The first example is a classical Hertzian contact problem in which two identical circular cylinders are pushed into each other by applied line forces P on the exterior surfaces. Symmetry of this problem yields a Signorini-type problem since the deformation is the same as that of a circular cylinder at rest on frictionless flat rigid surface and is subjected to the force P applied on the top. If the circular cylinder is long enough, the problem can be considered as a plane strain problem on a circular domain.

As a further simplification, we shall solve the problem only on the quadrant applying a uniformly distributed force instead of the "point" force

P on the top. Forty-four elements are used for the discretization of the quadrant of the cross section of the cylinder as shown in Fig. 4.3. As an example, let the radius R = 8 cm, Young's modulus 200 kg/cm^2, Poisson's ratio ν = 0.3, and the applied force P = 156 kg on the top. The iteration scheme (4.4.37) converges at the 3rd iteration within 10^{-4} tolerance. The computed deformed configuration is shown in Fig. 4.4, and the process of convergence of the iterative method (4.4.37) is described in Fig. 4.5 using the contact pressure p_ε^h obtained by (4.4.40).

The second example is a rigid punch problem in which a rigid circular cylinder is indented into an elastic foundation, as shown in Fig. 4.6. The size of the punch is R = 8 cm, Young's modulus E = 1000 kg/cm^2, Poisson's ratio ν = 0.3, and the depth of indentation is 0.6 cm. The width of the elastic foundation is 8 cm, and its height is 4 cm. We again assume plane strain. For this problem the relative errors of the total strain energy defined by

$$E_\varepsilon = \frac{1}{2}\, a\left(\underset{\sim}{u}_\varepsilon^{\hat{h}},\ \underset{\sim}{u}_\varepsilon^{\hat{h}}\right) - \frac{1}{2}\, a\left(\underset{\sim}{u}_{\hat{\varepsilon}}^{\hat{h}},\ \underset{\sim}{u}_{\hat{\varepsilon}}^{\hat{h}}\right)$$

$$E_h = \frac{1}{2}\, a\left(\underset{\sim}{u}_{\hat{\varepsilon}}^{h},\ \underset{\sim}{u}_{\hat{\varepsilon}}^{h}\right) - \frac{1}{2}\, a\left(\underset{\sim}{u}_{\hat{\varepsilon}}^{\hat{h}},\ \underset{\sim}{u}_{\hat{\varepsilon}}^{\hat{h}}\right)$$

are computed, where "^" indicates fixed values. For E_ε, we take $\hat{h}$ = 0.8 cm and $\hat{\varepsilon} = 10^{-4}$, and for E_h we take $\hat{\varepsilon} = 10^{-4}$ and $\hat{h}$ = 0.8cm. Figures 4.7 and 4.8 contain the computed result values of E_ε and E_h. Since the error estimates (4.4.67) indicate $E_\varepsilon = O(\varepsilon)$ and $E_h = O(h^2)$, we have a slight gap in numerical and theoretical results.

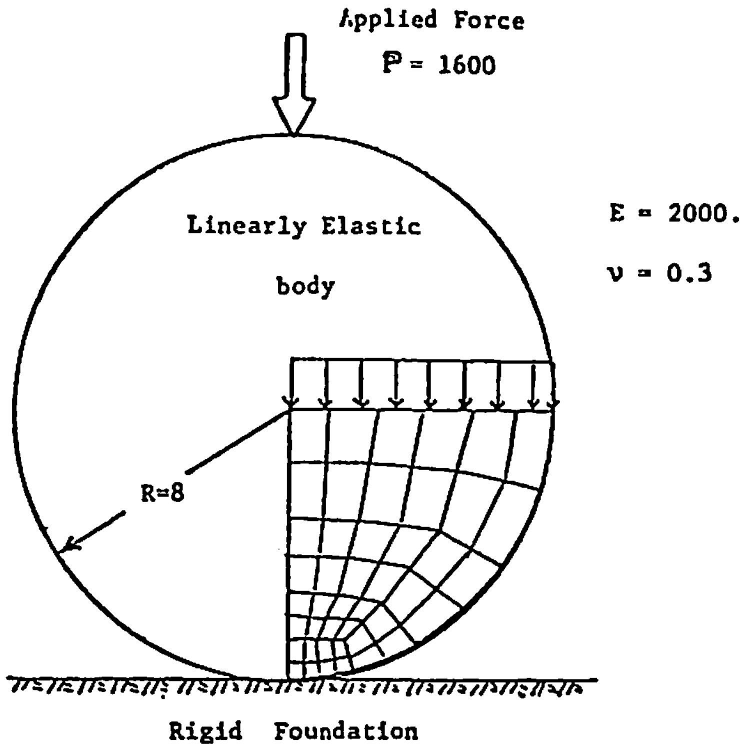


Figure 4.3 Physical Model for Example 1

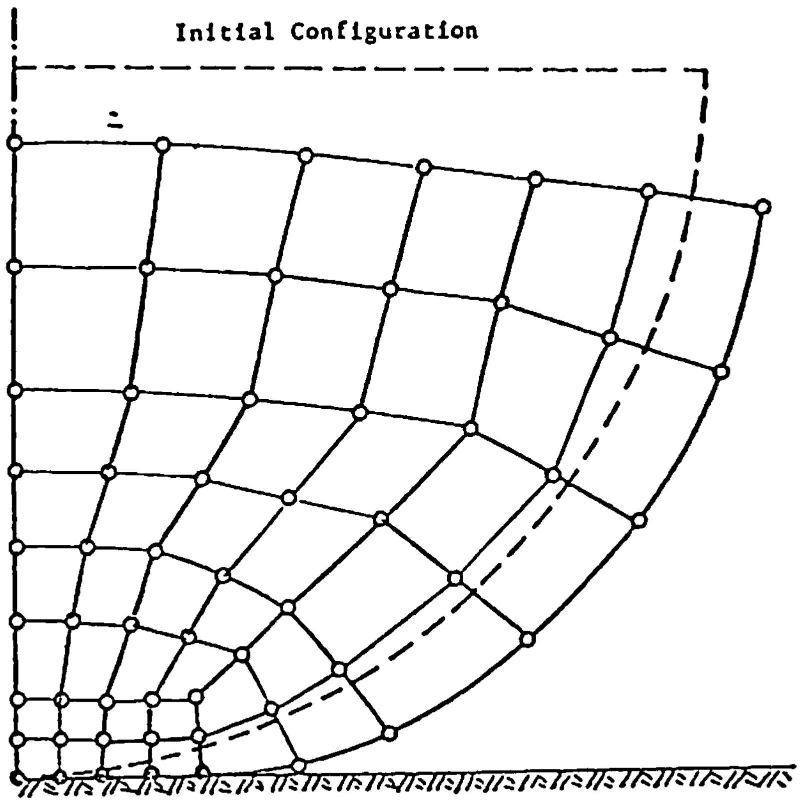


Figure 4.4 Deformed Configuration

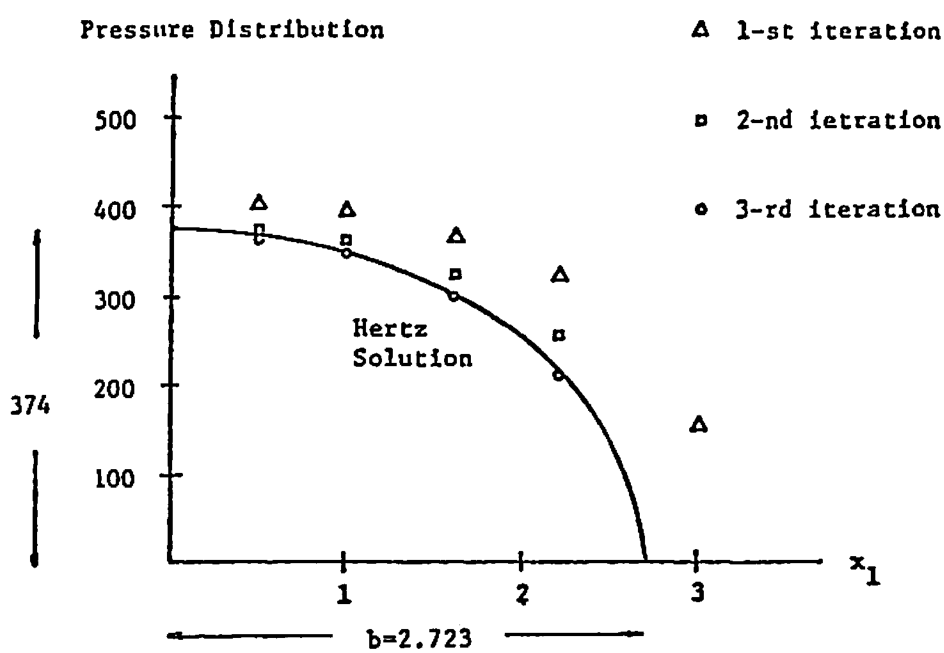


Figure 4.5 Convergence Procedure of the Iteration Scheme

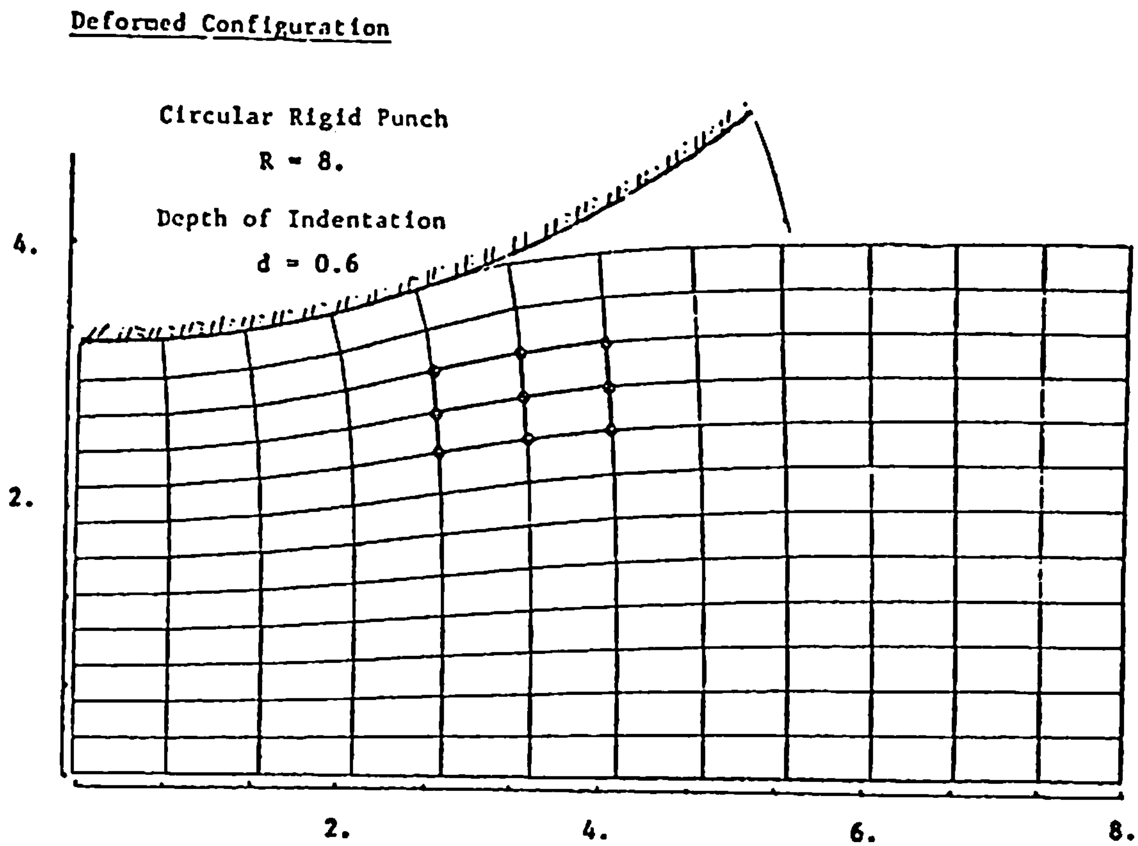


Figure 4.6 Indentation Problem for Example 2

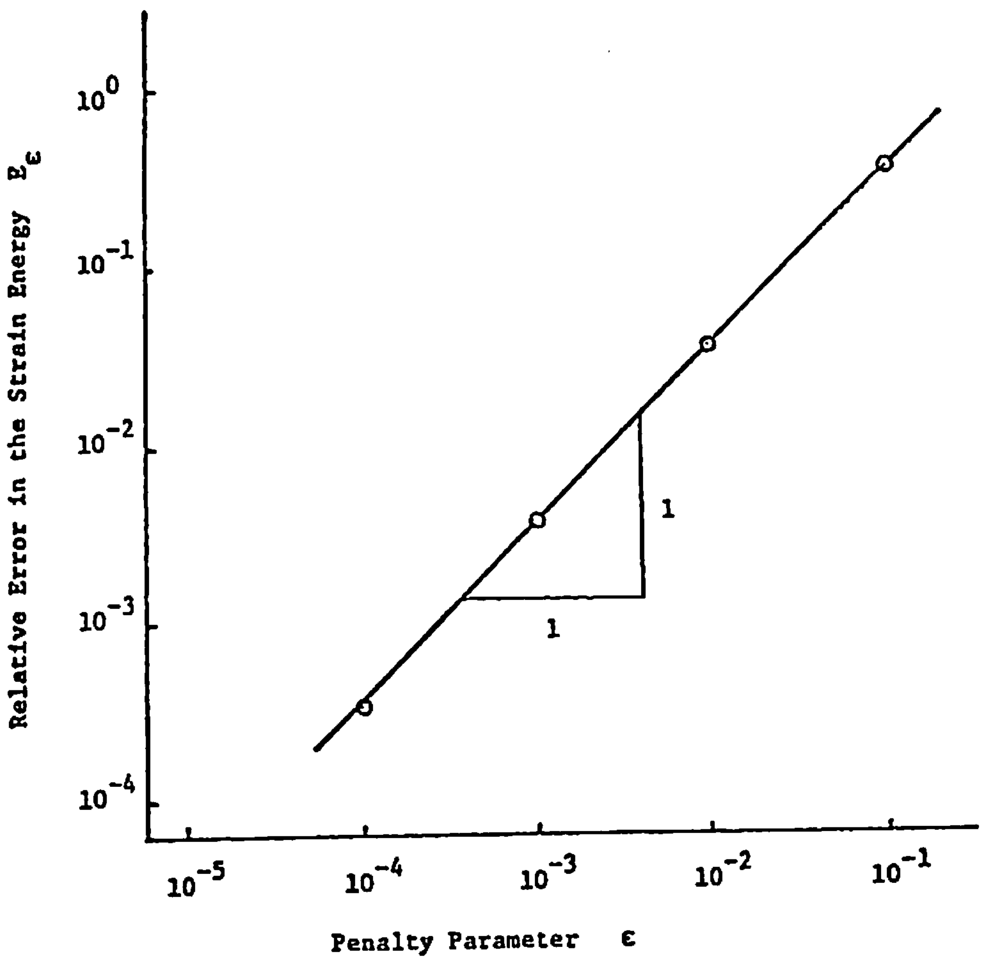


Figure 4.7 Convergence of the Penalty/Finite Element Method in ε

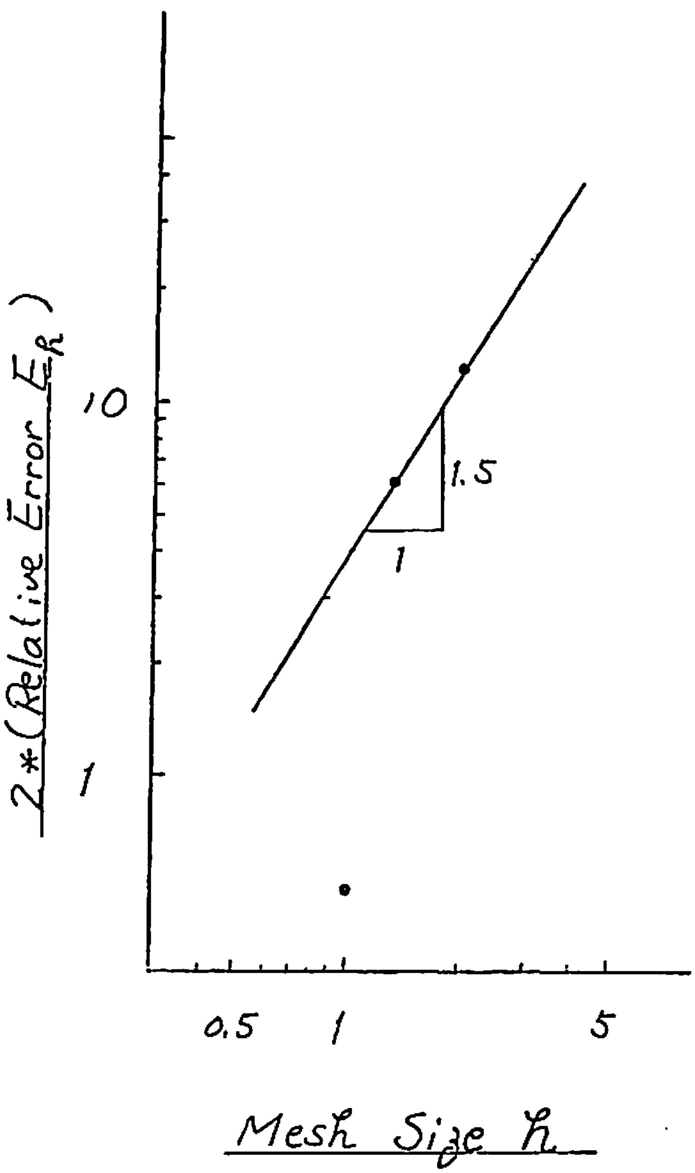


Figure 4.8 Convergence Test for h

4.5. Special Case II. Prescribed Normal Stress. As shown in the previous section, unilateral contact problems can be solved by the penalty method if the tangential stress along the candidate contact surface Γ_C is known. In this section, we shall study the reverse situation: we shall assume that the normal stress σ_n and, therefore, the actual contact surface Γ_C are known a priori.

Let

$$\sigma_N = t_N \in L^2(\Gamma_C) \tag{4.5.1}$$

be given on the contact boundary Γ_C. Then, as shown in Section 4.3, a variational formulation of the corresponding equilibrium problem assumes the form

$$\underset{\sim}{u} \in \underset{\sim}{V} : a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) + j(\underset{\sim}{v}) - j(\underset{\sim}{u}) \geq f_2(\underset{\sim}{v}-\underset{\sim}{u}), \quad \forall \underset{\sim}{v} \in \underset{\sim}{V} \tag{4.5.2}$$

where

$$j(\underset{\sim}{v}) = \int_{\Gamma_C} (-\mu\bar{t}_N)\ |\underset{\sim}{v}_T|\ d\Gamma \qquad \text{with} \quad \bar{t}_N = t_N + \bar{\sigma}_N, \tag{4.5.3}$$

and the space $\underset{\sim}{V}$ is defined in (4.4.4). Because of the assumption (4.5.1), the functional $j(\cdot)$ is well defined. The fact that $j(\cdot)$ is continuous on $\underset{\sim}{H}^1(\Omega)$ follows from (4.3.22). The functional $j(\cdot)$ is also convex. To see this, we need only recall that the R^N-euclidean norm is convex,

$$|(1-\theta)\underset{\sim}{v}_T + \theta\underset{\sim}{w}_T| \leq (1-\theta)\ |\underset{\sim}{v}_T| + \theta|\underset{\sim}{w}_T|\ . \tag{4.5.4}$$

Thus, a direct application of well-known results from BREZIS [1968] on variational inequalities leads immediately to the following existence and uniqueness theorem.

Theorem 4.7. Let the domain Ω be Lipschitzian, and let (4.3.8), (4.4.5), (4.4.6), and (4.5.1) hold. Then there exists a unique solution $u \in V$ to the variational inequality (4.5.2).

4.5.1 Regularization of the Functional $j(\cdot)$ The particular class of problems characterized by (4.5.2) may seem to be of limited practical interest since it depicts a rather rare situation in which normal stresses are prescribed on a surface with unknown frictional stresses. However, we shall now describe a regularization of this problem which, interestingly enough, can depict in a much more realistic way certain features on friction mechanisms observed in experiments with metals.

We begin by observing that the functional $j(\cdot)$ in (4.5.3) is not Gâteaux differentiable at the origin. This is not surprising because the Euclidean norm $|\cdot|$ in $\mathbb{R}^N$ is not differentiable at the origin, nor is the function $x \to |x|$ in $\mathbb{R}^1$. However, the source of the non-differentiability of $j(\cdot)$ is the fact that this particular model of friction depicts the separation of the sliding and full adhesion portions of the contact surface as a point or line. Physical experiments on friction on metallic surfaces show that no such line of separation of sliding and full adhesion exists; rather, there is a boundary-layer between regions of full "stick" and "slip." This, in turn, suggests that an approximation of $j(\cdot)$ differentiable at the origin can be designed which leads to a representation of such a boundary layer but which, at the same time, can be made arbitrarily close to the functional $j(\cdot)$ in some sense.

Toward the construction of such an approximation, we first consider smooth approximations of the function $x \to |x|$. As examples, consider the functions,

$$\phi_\varepsilon^1(x) = \begin{cases} x - \frac{1}{2}\varepsilon \\ \frac{1}{2\varepsilon}\, x^2 \\ -x + \frac{1}{2}\varepsilon \end{cases} ; \quad \frac{d\phi_\varepsilon}{dx}(x) = \begin{cases} 1 & \text{if } x \geqslant \varepsilon \\ \frac{1}{\varepsilon}\, x & \text{if } |x| < \varepsilon \\ -1 & \text{if } x \leqslant -\varepsilon \end{cases} \tag{4.5.5}$$

$$\phi_\varepsilon^2(x) = \frac{\varepsilon}{\varepsilon+1}\left(\frac{|x|}{\varepsilon}\right)^{\varepsilon+1} ; \frac{d\phi_\varepsilon}{dx}(x) = \begin{cases} \left(\frac{x}{\varepsilon}\right)^{\varepsilon} & \text{if } x > 0 \\ -\left(\frac{-x}{\varepsilon}\right)^{\varepsilon} & \text{if } x < 0 \end{cases} \tag{4.5.6}$$

and

$$\phi_\varepsilon^3(x) = \frac{2}{\pi}\left(x \tan^{-1}\frac{x}{\varepsilon} - \frac{\varepsilon}{2}\,\ell n\,(x^2+\varepsilon^2)\right); \frac{d\phi_\varepsilon}{dx}(x) = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{\varepsilon}\right). \tag{4.5.7}$$

It is easily verified that all of these approximations converge to the absolute value function as $\varepsilon \to 0$. The most popular approximation is the first one which exhibits a piecewise linear first derivative. However, the other two approximations have nonlinear first derivatives.

It is also worth mentioning at this point that an analogy of the friction problem to the perfect plasticity can be realized by examining at the graph of the first derivatives of ϕ_ε^1 , ϕ_ε^2 , and ϕ_ε^3 . The first resembles the stress-strain curve of an elastic-perfectly plastic material whereas the second resembles that of a material with strain hardening.

Another approximation which is reminiscent of plasticity is the function

$$\phi_\varepsilon(x) = \begin{cases} x + \frac{\hat{\varepsilon}}{2}x^2 - \frac{1}{2}(\varepsilon + \hat{\varepsilon}\varepsilon^2) & \text{if } x > \varepsilon \\ \frac{1}{2\varepsilon}x^2 & \text{if } |x| < \varepsilon \\ -x + \frac{\hat{\varepsilon}}{2}x^2 - \frac{1}{2}(\varepsilon + \hat{\varepsilon}\varepsilon^2) & \text{if } x < -\varepsilon \end{cases} \tag{4.5.8}$$

with

$$\frac{d\phi_\varepsilon}{dx}(x) = \begin{cases} 1 + \hat{\varepsilon}x & \text{if } x > \varepsilon \\ \frac{1}{\varepsilon}x & \text{if } |x| < \varepsilon \\ -1 + \hat{\varepsilon}x & \text{if } x < -\varepsilon \ , \end{cases} \tag{4.5.9}$$

Here ε and $\hat{\varepsilon}$ are two positive parameters such that $\phi_\varepsilon(x) \to |x|$ as $\varepsilon \to 0$ and $\hat{\varepsilon} \to 0$. In this case, we have a sort of strain hardening effect after the slipping (i.e., yielding).

To introduce similar approximations for the functional $j(\cdot)$ in friction problems, we first introduce a regularization of the function $\phi(\underset{\sim}{x}) = |\underset{\sim}{x}|$, where $|\underset{\sim}{x}|$ is the Euclidean norm for $\mathbb{R}^n$: $|\underset{\sim}{x}| = \sqrt{\underset{\sim}{x}\cdot\underset{\sim}{x}}$. Then its approximation ϕ_ε similar to (4.5.8) and (4.5.9) and is given by

$$\phi_\varepsilon(\underset{\sim}{x}) = \begin{cases} |\underset{\sim}{x}| + \frac{\mathcal{E}}{2}\underset{\sim}{x}\cdot\underset{\sim}{x} - \frac{1}{2}(\varepsilon + \mathcal{E}\varepsilon^2) & \text{if } |\underset{\sim}{x}| > \varepsilon \\ \frac{1}{2\varepsilon}\,\underset{\sim}{x}\cdot\underset{\sim}{x} & \text{if } |x| < \varepsilon\ , \end{cases} \tag{4.5.10}$$

where

$$\frac{\partial\phi_\varepsilon}{\partial \underset{\sim}{x}}(\underset{\sim}{x})\cdot\underset{\sim}{y} = \begin{cases} \left(\frac{\underset{\sim}{x}}{|\underset{\sim}{x}|} + \mathcal{E}\underset{\sim}{x}\right)\cdot\underset{\sim}{y} & \text{if } |\underset{\sim}{x}| > \varepsilon \\ \frac{1}{\varepsilon}\underset{\sim}{x}\cdot\underset{\sim}{y} & \text{if } |\underset{\sim}{x}| < \varepsilon\ . \end{cases} \tag{4.5.11}$$

Let us now make an estimate of the difference of ϕ_ε and ϕ . Note that

$$|\phi_\varepsilon(\underset{\sim}{x}) - \phi(\underset{\sim}{x})| = \begin{cases} |\frac{\ell}{2}\underset{\sim}{x}\cdot\underset{\sim}{x} - \frac{1}{2}(\varepsilon + \ell\varepsilon^2)| & \text{if } |\underset{\sim}{x}| > \varepsilon \\ |\frac{1}{2\varepsilon}\underset{\sim}{x}\cdot\underset{\sim}{x} - |\underset{\sim}{x}|| & \text{if } |\underset{\sim}{x}| < \varepsilon \end{cases}$$

$$< \begin{cases} \frac{1}{2}(\ell|\underset{\sim}{x}|^2 + \varepsilon) & \text{if } |\underset{\sim}{x}| > \varepsilon \\ \frac{1}{2}\varepsilon & \text{if } |\underset{\sim}{x}| < \varepsilon \end{cases}$$

$$< \frac{1}{2}(\ell|\underset{\sim}{x}|^2 + \varepsilon) \qquad (4.5.12)$$

i.e,

$$|\phi_\varepsilon(\underset{\sim}{x}) - \phi(\underset{\sim}{x})| < \frac{1}{2}(|\underset{\sim}{x}|^2\ \ell + \varepsilon). \qquad (4.5.13)$$

Thus, as the parameters ℓ and ε tend to zero, the approximation ϕ_ε of the nondifferentiable function $\phi(\underset{\sim}{x}) = |\underset{\sim}{x}|$ converges to ϕ at a rate $O(\varepsilon + \ell)$. The differentiable function ϕ_ε is called a <u>regularization</u> of the nondifferentiable function ϕ. As shown above, there are infinitely many choices of such regularizations. However, for definiteness, we shall consider only the particular choice (4.5.10) in the following discussions. Other choices of regularization and their quality in computation are studied in e.g., KIKUCHI [1982].

We now return to the work done $j(\cdot)$ by the friction force and its regularization. In view of (4.5.13), we take as a regularization of this functional,

$$j_\varepsilon(\underset{\sim}{v}) = \int_{\Gamma_C} (-\mu \bar{t}_N)\ \phi_\varepsilon(\underset{\sim}{v}_T) ds \qquad (4.5.14)$$

We easily verify that with this approximation

$$|j_\varepsilon(\underset{\sim}{v}) - j(\underset{\sim}{v})| < \frac{1}{2}\|-\mu\bar{t}_N\|_{0,\Gamma_C}(\|\underset{\sim}{v}\|^2_{0,4,\Gamma_C}\mathcal{E} + (\text{mes } \Gamma_C)\varepsilon), \qquad (4.5.15)$$

where mes Γ_C is the measure of the boundary Γ_C, and $\|\cdot\|_{0,4,\Gamma_C}$ is the norm of $L^4(\Gamma_C)$ defined by

$$\|\underset{\sim}{v}\|_{0,4,\Gamma_C} = \left\{\int_{\Gamma_C}(\underset{\sim}{v}\cdot\underset{\sim}{v})^2 ds\right\}^{1/4} . \qquad (4.5.16)$$

Note that we have used here the facts that $v_i \in L^4(\Gamma_C)$ for $v_i \in \underset{\sim}{H}^1(\Omega)$ if Ω is Lipschitzian for $1<i<n$, and that

$$\underset{\sim}{v} \cdot \underset{\sim}{v} = \underset{\sim}{v}_T \cdot \underset{\sim}{v}_T + v_N^2 .$$

<u>Theorem</u> <u>4.8</u>. Let Ω be Lipschitzian and let $\bar{t}_N \in L^2(\Gamma_C)$. Suppose that the conditions (4.3.8), (4.4.5), (4.4.6), and (4.5.1) hold. Then there is a unique solution $\underset{\sim}{u}_\varepsilon$ to the regularized problem

$$\underset{\sim}{u}_\varepsilon \in \underset{\sim}{V} : a(\underset{\sim}{u}_\varepsilon, \underset{\sim}{v}) + \langle Dj_\varepsilon(\underset{\sim}{u}_\varepsilon), \underset{\sim}{v}\rangle = f_2(\underset{\sim}{v}) , \quad \forall\, \underset{\sim}{v} \in V_0 \qquad (4.5.17)$$

for a given pair $(\varepsilon,\mathcal{E})$, where V_0 is the space defined in (4.4.11), and

$$\langle Dj_\varepsilon(\underset{\sim}{u}_\varepsilon),\ \underset{\sim}{v}\rangle = \int_{\Gamma_C}(-\mu\bar{t}_N)\frac{\partial\phi_\varepsilon}{\partial\underset{\sim}{u}_{\varepsilon T}}(\underset{\sim}{u}_{\varepsilon T})\cdot\underset{\sim}{v}_T ds\ . \tag{4.5.18}$$

Furthermore, the sequence of the solutions $\{\underset{\sim}{u}_\varepsilon\}$ to the regularized problem (4.5.17) converges to the solution $\underset{\sim}{u}$ of the friction problem (4.5.2) as ε and $\mathcal{E}$ tend to zero. Indeed, there exists a positive constant C, independent of ε and $\mathcal{E}$, such that

$$\|\underset{\sim}{u}_\varepsilon - \underset{\sim}{u}\|_1 \leq C(\sqrt{\varepsilon} + \sqrt{\mathcal{E}}), \tag{4.5.19}$$

□

Before proving the above assertion, let us note that the regularized problem (4.5.17) is equivalent to the problem

$$\underset{\sim}{u}_\varepsilon \in \underset{\sim}{V} : a(\underset{\sim}{u}_\varepsilon,\ \underset{\sim}{v} - \underset{\sim}{u}_\varepsilon) + j_\varepsilon(\underset{\sim}{v}) - j_\varepsilon(\underset{\sim}{u}_\varepsilon) \geq f_2(\underset{\sim}{v}-\underset{\sim}{u}_\varepsilon),$$
$$\forall\underset{\sim}{v} \in \underset{\sim}{V}\ , \tag{4.5.20}$$

since $j_\varepsilon(\cdot)$ is a convex continuous and differentiable functional on $\underset{\sim}{V}$.

<u>Proof of Theorem 4.8</u>. Because of the equivalence of (4.5.17) to (4.5.20) the existence of a unique solution $\underset{\sim}{u}_\varepsilon$ follows from reasoning similar to that used to derive (4.5.2). Thus, it suffices to merely verify the inequality (4.5.19).

Applying (4.5.2) and (4.5.20), we have

$$a(\underset{\sim}{u}-\underset{\sim}{u}_\varepsilon,\ \underset{\sim}{u}-\underset{\sim}{u}_\varepsilon) \leq j_\varepsilon(\underset{\sim}{u}) - j(\underset{\sim}{u}) - j_\varepsilon(\underset{\sim}{u}_\varepsilon) + j(\underset{\sim}{u}_\varepsilon)$$
$$\leq \hat{C}\|-\mu\bar{t}_N\|_{0,\Gamma_C}\ ((\|\underset{\sim}{u}\|^2_{0,4,\Gamma_C} + \|\underset{\sim}{u}_\varepsilon\|^2_{0,4,\Gamma_C})\mathcal{E} + (\text{mes}\ \Gamma_C)\varepsilon)$$

Because of (4.4.5), the solution $\|\underset{\sim}{u}_\varepsilon\|_{0,4,\Gamma_C}$ is uniformly bounded in $\hat{\varepsilon}$ and ε. We can thus conclude that (4.5.19) holds for sufficiently small $\hat{\varepsilon}$ and ε. □

Let us now investigate in more detail the physical meaning of the regularization method. Toward this end, the variational form (4.5.17) is considered. If the generalized Green theorem introduced by Aubin [1979, Chapter 13] is applied, and if the normal and tangent stresses $\sigma_{\varepsilon N}$ and $\underset{\sim}{\sigma}_{\varepsilon T}$ resulting from the displacement field $\underset{\sim}{u}_\varepsilon$ are well-defined in, e.g., $L^2(\Gamma_C)$, then we have the following characterization of the solution $\underset{\sim}{u}_\varepsilon$ of (4.5.17):

$$- \frac{\partial}{\partial x_i} (\sigma_{ij}(\underset{\sim}{u}_\varepsilon)) = f_i \text{ in } \Omega, \tag{4.5.21}$$

in the sense of distributions,

$$\underset{\sim}{\sigma}_\varepsilon = \underset{\sim}{t} \text{ on } \Gamma_F , \; \sigma_{\varepsilon N} = t_N + \bar{\sigma}_N \text{ on } \Gamma_C , \tag{4.5.22}$$

and

$$\underset{\sim}{\sigma}_{\varepsilon T} = \mu \bar{t}_N \frac{\partial \phi_\varepsilon}{\partial \underset{\sim}{u}_{\varepsilon T}} (\underset{\sim}{u}_{\varepsilon T}) \text{ on } \Gamma_C . \tag{4.5.23}$$

Here $\sigma_{\varepsilon i} = \sigma_{ij}(\underset{\sim}{u}_\varepsilon) n_j$, $\sigma_{\varepsilon N} = \underset{\sim}{\sigma}_\varepsilon \cdot \underset{\sim}{N}$, and $\underset{\sim}{\sigma}_{\varepsilon T} = \underset{\sim}{\sigma}_\varepsilon - \sigma_{\varepsilon N}\underset{\sim}{N}$.

Consider the equilibrium of the tangential stress on the contact surface Γ_C. If the relation (4.5.11) is introduced into (4.5.23), we have the friction stresses

$$\underset{\sim}{\sigma}_{\varepsilon T} = \begin{cases} \mu \bar{t}_N \left(-\dfrac{\underset{\sim}{u}_{\varepsilon T}}{|\underset{\sim}{u}_{\varepsilon T}|} + \mathcal{E} \underset{\sim}{u}_{\varepsilon T} \right) & \text{if } |\underset{\sim}{u}_{\varepsilon T}| > \varepsilon \\ \\ \mu \bar{t}_N \dfrac{1}{\varepsilon} \underset{\sim}{u}_{\varepsilon T} & \text{if } |\underset{\sim}{u}_{\varepsilon T}| < \varepsilon \end{cases} \tag{4.5.24}$$

on the surface Γ_C .

It is clear that the inequality

$$|\underset{\sim}{\sigma}_{\varepsilon T}| < \begin{cases} -\mu \bar{t}_N (1 + \mathcal{E} |\underset{\sim}{u}_{\varepsilon T}|) & \text{if } |\underset{\sim}{u}_{\varepsilon T}| > \varepsilon \\ \\ -\mu \bar{t}_N & \text{if } |\underset{\sim}{u}_{\varepsilon T}| < \varepsilon \end{cases}$$

holds. This means that if $\mathcal{E} \to 0$, the friction stress never exceeds the value $-\mu \bar{t}_N$ which indicates the Coulomb law of friction. Furthermore, (4.5.24) yields

$$\underset{\sim}{u}_{\varepsilon T} = \varepsilon \, \mu \bar{t}_N \underset{\sim}{\sigma}_{\varepsilon T} \quad \text{for} \quad |\underset{\sim}{u}_{\varepsilon T}| < \varepsilon \ .$$

Thus, by passing to the limit $\varepsilon \to 0$, the stick portion of the contact surface is identified with the set $S_\varepsilon = \{\underset{\sim}{x} \in \Gamma_C : |\underset{\sim}{u}_{\varepsilon T}(\underset{\sim}{x})| < \varepsilon\}$.

A simple spring model of this particular friction mechanism is depicted in Fig. 4.9. Note that the initial tangential stiffness at the contact surface is $1/\varepsilon$ and that some tangential displacement δ is reached before the spring of stiffness $\mathcal{E}$, representing a hardening effect, is activated.

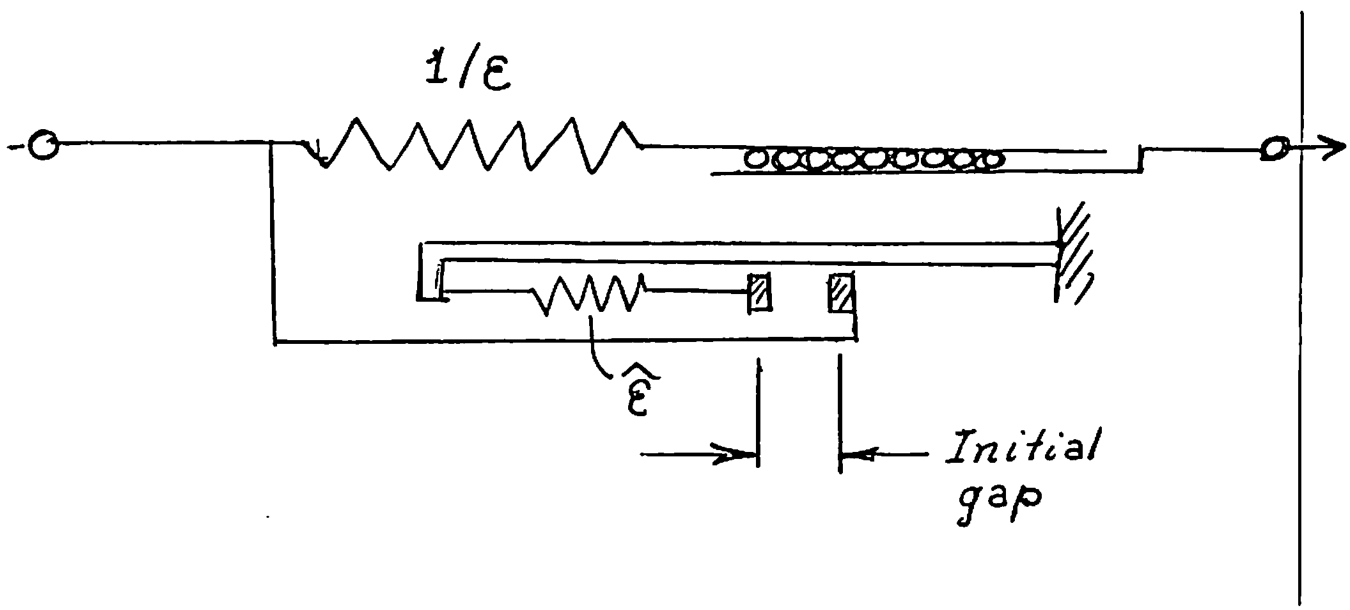


Figure 4.9 Model of friction mechanism

4.5.2 Finite Element Approximations. We shall now examine approximations of the regularized problem (4.5.17) by finite element methods. Applying the notations and conventions of the previous section (recall (4.4.28) - (4.4.37)), the regularized form (4.5.17) is approximated by

$$\underset{\sim}{u}_\varepsilon^h \in V^h : a(\underset{\sim}{u}_\varepsilon^h, \underset{\sim}{v}^h) + I(Dj_\varepsilon(\underset{\sim}{u}_\varepsilon^h)\cdot \underset{\sim}{v}_T^h) = f_2(\underset{\sim}{v}^h),$$

$$\forall \underset{\sim}{v}^h \in V_0^h , \qquad (4.5.25)$$

where the term $I(Dj_\varepsilon(\underset{\sim}{u}_\varepsilon^h)\cdot \underset{\sim}{v}_T^h)$ is the quadrature rule for the friction term $\langle Dj_\varepsilon(\underset{\sim}{u}_\varepsilon^h), \underset{\sim}{v}^h\rangle$ similar to the penalty term in (4.4.31). It is clear that the finite element equation resulting from (4.5.25) is nonlinear because of the gradient $Dj_\varepsilon(\underset{\sim}{u}_\varepsilon^h)$. Thus, to solve (4.5.25) we must generally resort to some successive iterative scheme. Because of the nature of the nonlinearity in $Dj_\varepsilon(\underset{\sim}{u}_\varepsilon^h)$, it is natural to consider a sequence of linearized problems which employ the recurrence formula,

$$Dj_\varepsilon^i({}^i\underset{\sim}{u}_\varepsilon^h) = \begin{cases} \mu \bar{t}_N(\varepsilon\, {}^i\underset{\sim}{u}_{\varepsilon T}^h + \dfrac{{}^{i-1}\underset{\sim}{u}_{\varepsilon T}^h}{|{}^{i-1}\underset{\sim}{u}_{\varepsilon T}^h|}) \text{ if } |{}^{i-1}\underset{\sim}{u}_{\varepsilon T}^h| > \varepsilon \\ \\ \mu \bar{t}_N \dfrac{1}{\varepsilon} {}^i\underset{\sim}{u}_{\varepsilon T} \qquad \text{if } |{}^{i-1}\underset{\sim}{u}_{\varepsilon T}^h| < \varepsilon \end{cases} \qquad (4.5.26)$$

Then the solution $\underset{\sim}{u}_\varepsilon^h$ in (4.5.25) is achieved as the limit of the sequence $\{{}^i\underset{\sim}{u}_\varepsilon^h\}$ such that

$$ {}^{i}\underset{\sim}{u}_{\varepsilon}^{h} \in \underset{\sim}{V}^{h} : a({}^{i}\underset{\sim}{u}_{\varepsilon}^{h}, \underset{\sim}{v}^{h}) + I(Dj_{\varepsilon}^{i}({}^{i}\underset{\sim}{u}_{\varepsilon}^{h})\cdot \underset{\sim}{v}_{T}^{h}) $$

$$ = f_2(\underset{\sim}{v}^{h}), \qquad \forall \underset{\sim}{v}^{h} \in \underset{\sim}{V}_{0}^{h}, \tag{4.5.27} $$

for $i = 1,2,\ldots,m$, which evolves from a proper initial guess ${}^{o}\underset{\sim}{u}_{\varepsilon}^{h}$.

The convergence of the iterative scheme (4.5.27) is more delicate than the case for the penalty formulation (4.4.33) since the term $\mu\bar{t}_N {}^{i-1}\underset{\sim}{u}_{\varepsilon T}^{h} / |{}^{i-1}\underset{\sim}{u}_{\varepsilon T}^{h}|$ destroys the monotonicity of the form (4.5.27), unless the parameter $\mathcal{E}$ is sufficiently large. Since j_ε is convex, the operator constructed by the quasilinear form $a(\cdot,\cdot) + \langle Dj_\varepsilon(\cdot),\cdot\rangle$ is strongly monotone on $\underset{\sim}{V} \times \underset{\sim}{V}_0$. Thus, it is possible to establish the existence of a convergent subsequence. However, this does not imply that the form $a(\cdot,\cdot) + \langle Dj_\varepsilon^{i}(\cdot),\cdot\rangle$ is monotone. This possibly lead to a sort of oscillation about the solution as $i \to \infty$. Such oscillations might be reduced or eliminated by taking a larger value of ε(or $\mathcal{E}$) that results in the reduction of the effect of the term $\mu\bar{t}_N {}^{i-1}\underset{\sim}{u}_{\varepsilon T}^{h}/|{}^{i-1}\underset{\sim}{u}_{\varepsilon T}^{h}|$ during the iteration process.

4.5.3 Convergence of the Finite Element Method. Suppose that the unique solution $\underset{\sim}{u}_{\varepsilon}^{h}$ of (4.5.25) is obtained by the iterative scheme (4.5.27). It is, however, noted that while the existence of a unique solution to (4.5.25) is guaranteed, it might not be realized by this iterative scheme (4.5.27). We shall verify the convergence of the regularization finite element approximation by taking the following steps: an estimate of $\|\underset{\sim}{u}_{\varepsilon}^{h} - \underset{\sim}{\hat{u}}_{\varepsilon}^{h}\|_1$ is first obtained, where $\underset{\sim}{\hat{u}}_{\varepsilon}^{h}$ is the solution of the problem

$$ \underset{\sim}{\hat{u}}_{\varepsilon}^{h} \in \underset{\sim}{V}^{h} : a(\underset{\sim}{\hat{u}}_{\varepsilon}^{h},\underset{\sim}{v}^{h}) + \langle Dj_\varepsilon(\underset{\sim}{\hat{u}}_{\varepsilon}^{h}),\underset{\sim}{v}^{h}\rangle = f_2(\underset{\sim}{v}^{h}), $$

$$\forall \underset{\sim}{v}^h \in \underset{\sim}{V}_0^h \ . \qquad (4.5.28)$$

Because of the application of the numerical integration to evaluate the quantity

$$\langle Dj_\varepsilon(\hat{\underset{\sim}{u}}_\varepsilon^h), \underset{\sim}{v}^h \rangle = \int_{\Gamma_C} \mu \bar{t}_N \frac{\partial \phi_\varepsilon(\hat{\underset{\sim}{u}}_{\varepsilon T}^h)}{\partial \hat{\underset{\sim}{u}}_{\varepsilon T}^h} \cdot \underset{\sim}{v}_T^h \, d\ddot{s} \ ,$$

the quadrature error must be considered at first. Then applying the result in Theorem 4.8 yields the estimate $\|\hat{\underset{\sim}{u}}_\varepsilon^h - \underset{\sim}{u}^h\|_1$ in terms of ε and $\mathcal{E}$, where $\underset{\sim}{u}^h$ is the solution of the problem

$$\underset{\sim}{u}^h \in \underset{\sim}{V}^h : a(\underset{\sim}{u}^h, \underset{\sim}{v}^h) + j(\underset{\sim}{v}^h) - j(\underset{\sim}{u}^h) \geq f_2(\underset{\sim}{v}^h - \underset{\sim}{u}^h),$$
$$\forall \underset{\sim}{v}^h \in \underset{\sim}{V}^h \ . \qquad (4.5.29)$$

The final step is the estimate of $\|\underset{\sim}{u}^h - \underset{\sim}{u}\|_1$ of the solution $\underset{\sim}{u}$ of (4.5.2).

To accomplish the first step, let us note that the form (4.5.25) is equivalent to

$$\underset{\sim}{u}_\varepsilon^h \in \underset{\sim}{V}^h : a(\underset{\sim}{u}_\varepsilon^h, \underset{\sim}{v}^h) + I(\mu t_N \phi_\varepsilon(\underset{\sim}{v}_T^h)) - I(\mu t_N \phi_\varepsilon(\underset{\sim}{u}_{\varepsilon T}^h))$$
$$\geq f_2(\underset{\sim}{v}^h - \underset{\sim}{u}_\varepsilon^h), \quad \forall \underset{\sim}{v}^h \in \underset{\sim}{V}^h \ . \qquad (4.5.30)$$

Similarly, (4.5.28) is equivalent to

$$\hat{\underset{\sim}{u}}_\varepsilon^h \in \underset{\sim}{V}^h : a(\hat{\underset{\sim}{u}}_\varepsilon^h, \underset{\sim}{v}^h) + j_\varepsilon(\underset{\sim}{v}^h) - j_\varepsilon(\hat{\underset{\sim}{u}}_\varepsilon^h) \geq f_2(\underset{\sim}{v}^h - \hat{\underset{\sim}{u}}_\varepsilon^h),$$
$$\forall \underset{\sim}{v}^h \in \underset{\sim}{V}^h \ . \qquad (4.5.31)$$

Lemma 4.6. Suppose that the quadrature rule I satisfies the condition

$$|E_I(f^h g^h)| \le C_\delta h^{2(s-1)} \|f^h\|_{s-3/2,\Gamma_C} \|g^h\|_{s-1/2,4,\Gamma_C} \qquad (4.5.32)$$

for piecewise polynomials contained in the trace of the functions in $\underset{\sim}{V}^h$ on the boundary Γ. Then there is a positive constant C independent of the mesh size h and the parameters $\hat{\varepsilon}$ and ε such that

$$\|\underset{\sim}{u}_\varepsilon^h - \underset{\sim}{\hat{u}}_\varepsilon^h\|_1 \le (1+\hat{\varepsilon})\, C(\|\underset{\sim}{u}_\varepsilon^h\|_s, \|\underset{\sim}{\hat{u}}_\varepsilon^h\|_s, \|\mu\bar{t}_N\|_{s-3/2})h^{s-1} \qquad (4.5.33)$$

for $s \ge 1$.

Proof. From (4.5.30) and (4.5.31), we have

$$a(\underset{\sim}{u}_\varepsilon^h - \underset{\sim}{\hat{u}}_\varepsilon^h, \underset{\sim}{u}_\varepsilon^h - \underset{\sim}{\hat{u}}_\varepsilon^h) \le |E_I(\mu\bar{t}_N \phi_\varepsilon(\underset{\sim}{u}_{\varepsilon T}^h))|$$
$$+ |E_I(\mu\bar{t}_N \phi_\varepsilon(\underset{\sim}{\hat{u}}_{\varepsilon T}^h))|.$$

Without loss of the rate in the estimate, we can assume that μt_N is also a piecewise polynomial (see the procedure to (4.4.62)). Applying the assumption (4.5.32), it follows from (4.4.5) that

$$m\|\underset{\sim}{u}_\varepsilon^h - \underset{\sim}{\hat{u}}_\varepsilon^h\|_1^2 \le \hat{C}\|\mu t_N\|_{s-3/2}(1+\hat{\varepsilon}\|\underset{\sim}{\hat{u}}_\varepsilon^h\|_{s-1/2,4,\Gamma_C}^2$$
$$+ \hat{\varepsilon}\|\underset{\sim}{u}_\varepsilon^h\|_{s-1/2,4,\Gamma_C}^2)h^{2(s-1)}.$$

Thus (4.5.33) follows. □

The second estimate,

$$\|\tilde{u}^h_\varepsilon - u^h\|_1 \le C(\sqrt{\varepsilon} + \sqrt{\bar{\varepsilon}}) \qquad (4.5.34)$$

follows easily from the same arguments used in the proof of Theorem 4.8.

Lemma 4.7. Suppose that the interpolation estimate (4.4.49) holds for the function on Ω and on Γ. Suppose that

$$u \in H^s(\Omega),\ u|_{\Gamma_C} \in H^{s-1/2}(\Gamma_C),\ \sigma_T(u) \in H^{s-3/2}(\Gamma_C),$$
$$\text{and } \mu\bar{t}_N \in H^{s-3/2}(\Gamma_C). \qquad (4.5.35)$$

Then for the solutions u^h of (4.5.29) and u of (4.5.2), we have the estimate

$$\|u^h - u\|_1 \le C(\|u\|_s)h^{s-1}, \qquad (4.5.36)$$

where C is a constant independent of the mesh size.

Proof. Using (4.5.2) and (4.5.29) we obtain the estimate

$$\begin{aligned} a(u-u^h,u-u^h) \le\ & a(u-u^h,u-v^h) + a(u,v-u^h) \\ & + j(v)-j(u^h) - f_2(v-u^h) + a(u,v^h-u) \\ & + j(v^h)-j(u) - f_2(v^h-u), \qquad v^h \in V^h,\ v \in V. \end{aligned}$$

Taking $v = u^h$ yields

$$a(\underset{\sim}{u}-\underset{\sim}{u}^h,\underset{\sim}{u}-\underset{\sim}{u}^h) \leq a(\underset{\sim}{u}-\underset{\sim}{u}^h,\underset{\sim}{u}-\underset{\sim}{v}^h) + a(\underset{\sim}{u},\underset{\sim}{v}^h-\underset{\sim}{u})$$

$$+ j(\underset{\sim}{v}^h)-j(\underset{\sim}{u})-f_2(\underset{\sim}{v}^h-\underset{\sim}{u}).$$

The characterization of the solution $\underset{\sim}{u}$ of (4.5.2) after applying the generalized Green theorem leads to the inequality

$$a(\underset{\sim}{u}-\underset{\sim}{u}^h,\underset{\sim}{u}-\underset{\sim}{u}^h) \leq a(\underset{\sim}{u}-\underset{\sim}{u}^h,\underset{\sim}{u}-\underset{\sim}{v}^h) + \int_{\Gamma_C}\sigma_T(\underset{\sim}{u})(\underset{\sim}{v}_T^h-\underset{\sim}{u}_T)ds$$

$$+ \int_{\Gamma_C}\mu\bar{t}_N(|\underset{\sim}{v}_T^h| - |\underset{\sim}{u}_T|)ds$$

Under the assumption (4.5.35), we have

$$m\|\underset{\sim}{u}-\underset{\sim}{u}^h\|_1^2 \leq M\|\underset{\sim}{u}-\underset{\sim}{u}^h\|_1 \|\underset{\sim}{u}-\underset{\sim}{v}^h\|_1$$

$$+ \|\underset{\sim}{\sigma}_T(\underset{\sim}{u})\|_{s-3/2,\Gamma_C} \|\underset{\sim}{u}_T-\underset{\sim}{v}_T^h\|_{-s+3/2,\Gamma_C}$$

$$+ \|\mu\bar{t}_N\|_{s-3/2,\Gamma_C} \|\underset{\sim}{u}_T-\underset{\sim}{v}_T^h\|_{-s+3/2,\Gamma_C}$$

Here we have used the fact that $|\underset{\sim}{v}_T^h|-|\underset{\sim}{u}_T| \leq |\underset{\sim}{v}_T^h-\underset{\sim}{u}_T|$. Applying the interpolation estimates (4.4.49), we obtain

$$m\|\underset{\sim}{u}-\underset{\sim}{u}^h\|_1^2 \leq CM\|\underset{\sim}{u}-\underset{\sim}{u}^h\|_1 h^{s-1}\|\underset{\sim}{u}\|_s$$

$$+(\|\underset{\sim}{\sigma}_T(\underset{\sim}{u})\|_{s-3/2,\Gamma_C} + \|\mu\bar{t}_N\|_{s-3/2,\Gamma_C})Ch^{2(s-1)}\|\underset{\sim}{u}\|_{s-1/2,\Gamma_C}$$

The estimate (4.5.35) is then obtained by using Young's inequality. □

Combining the above results we arrive at the estimate of the approximation of the regularization finite element method (4.5.25)

Theorem 4.9. Under the assumptions (4.4.5), (4.4.6), (4.4.49), (4.5.32), and (4.5.35), we have

$$\|\underset{\sim}{u}_{\varepsilon}^{h}-\underset{\sim}{u}\|_1 < C(\sqrt{\varepsilon} + \sqrt{\ell} + h^{s-1})\|\underset{\sim}{u}\|_s \tag{4.5.37}$$

where C is a proper positive constant independent of h, ℓ, an ε. □

If the four node quadrilateral element with the trapezoid rule for the quadrature "I" is applied, an analysis similar to that leading to Lemma 4.4 shows that the estimate (4.5.32) reduces to

$$|E_I(\mu\bar{t}_N^I|\underset{\sim}{u}_{\varepsilon T}^{h}|^2)| < Ch^2\|\mu\bar{t}_N^I\|_{1/2,\Gamma_C}\|\underset{\sim}{u}_{\varepsilon T}^{h}\|^2_{3/2,4,\Gamma_C} \tag{4.5.38}$$

where $\mu\bar{t}_N^I$ is the interpolation of the $\mu\bar{t}_N$ on Γ_C. Thus, if $s=2$, the error estimate becomes

$$\|\underset{\sim}{u}_{\varepsilon}^{h}-\underset{\sim}{u}\|_1 < C(\sqrt{\varepsilon} + \sqrt{\ell} + h)\|\underset{\sim}{u}\|_2. \tag{4.5.39}$$

4.5.4 Numerical Examples. We shall describe three example problems for friction contact problems. The first is a footing problem in foundation engineering depicted in Figure 4.10.

Suppose that the foundation is composed of an isotropic elastic material with Young's modulus $E=5\times10^5 gK/m^2$ and Poisson's ratio 0.3 for the shaded area, the remaining portion begin a second elastic material which has a Young's modulu of $E=2\times10^6 gK/m^2$ and Poisson's ratio of 0.3 for the other part. The size of the foundation is 8m x 16m, and the friction coefficient μ between the footing and the foundation is assumed to be $\mu=0.3$.

A finite element model is obtained by 64 four - node elements as shown in Figure 4.10 for the case in which the footing is pushed and rotated so that the bottom line of the footing lies along the plane $y = -0.1 + 0.001(x - 6)$. That is, the left edge of the footing

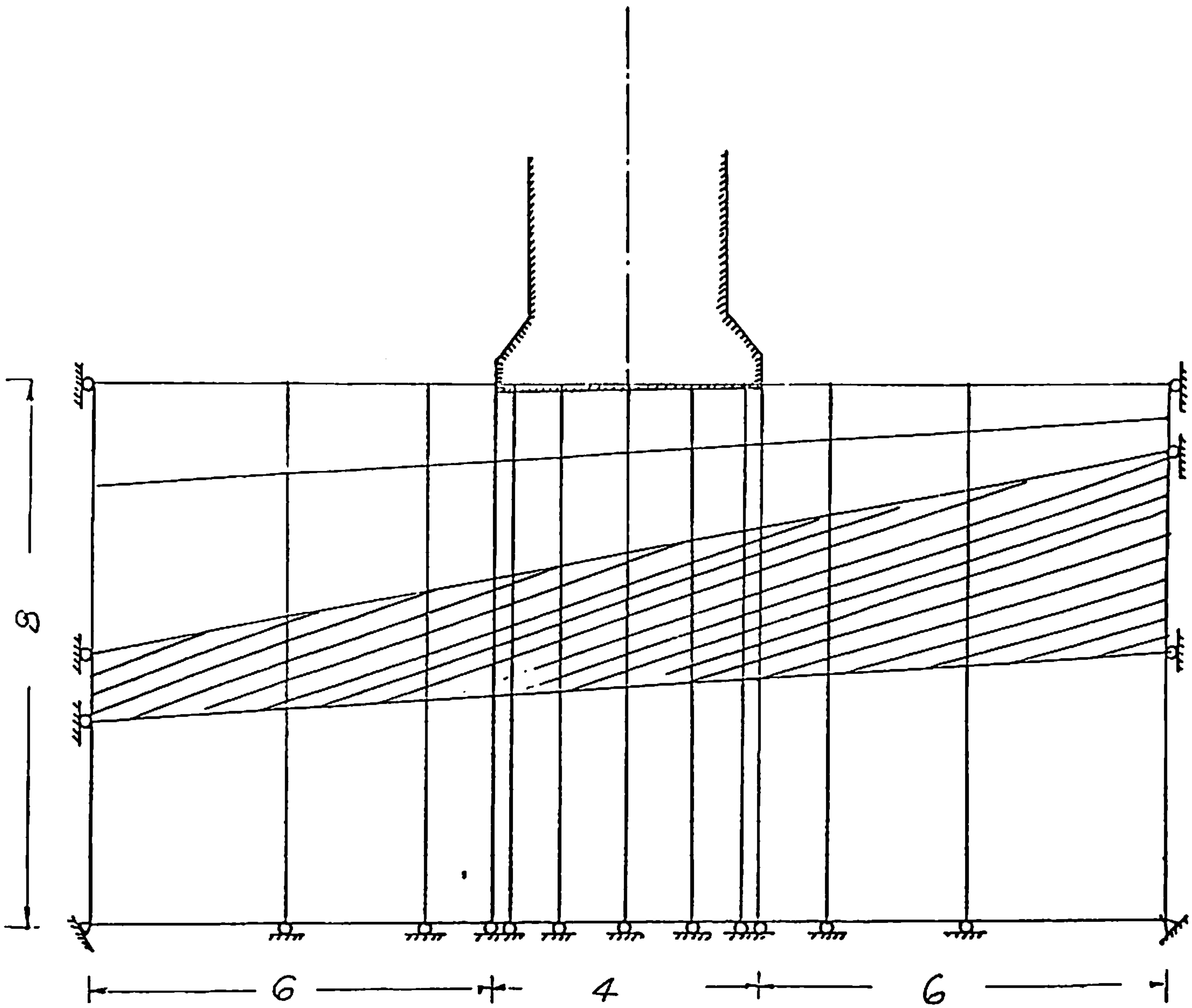


Figure 4.10 Footing Problem on a Non-homogeneous Elastic Foundation with Friction

is indented 10 cm into the foundation, and the tangent is 0.001. For the choice of $\varepsilon = 10^{-4}/E$ and $\hat{\varepsilon}=0$, we can obtain the result shown in Fig. 4.11 in which three nodal points from the right edge of the footing are sliding. Because of the singularities on both edges, the contact pressure is very high on those points. It is noted that no oscillatory tangential stress along the bottom of the footing is observed because of the relatively course finite element mesh used.

The second example is a Hertzian problem with friction. A circular rigid cylinder is idented into the deformable body whose Young's modulus E is 1000 gK/cm^2 and Poisson's ratio ν is 0.3. Under the assumption of plane strain, the problem is solved by 36 four-node elements with the depth of indentation d=0.4 cm. The radius of the cylinder is taken to be R=4 cm, and the size of the deformable body is 8 cm x 4 cm. Using the symmetry of the problem, only half portion of the body is analyzed. Computed results for the case $\mu=0.3$, $\varepsilon=2.6 \times 10^{-6}$, and $\hat{\varepsilon}=0$, are shown in Fig. 4.12. The deformed configuration is also given in Figure 4.12. Two nodes under the circular rigid punch are sliding, and the computed contact pressure and the friction stress are plotted in Figure 4.13.

The last example is an axisymmetric problem for an annular punch. The computed results are compared with the analytical solution of Shibuya, Koizumi, and Nakahara [1930] for the case of full adhesion. The deformed configuration and the contact stresses are shown in Fig. 4.14 and 4.15, for the case that E=1000 g Kg/cm^2, $\nu=0.3$, and $\mu=0.3$. The size of the domain $\Omega=2$ cm x 1 cm by applying the axisymmetry. The domain is discretized by 200 four-node elements. We here need not to

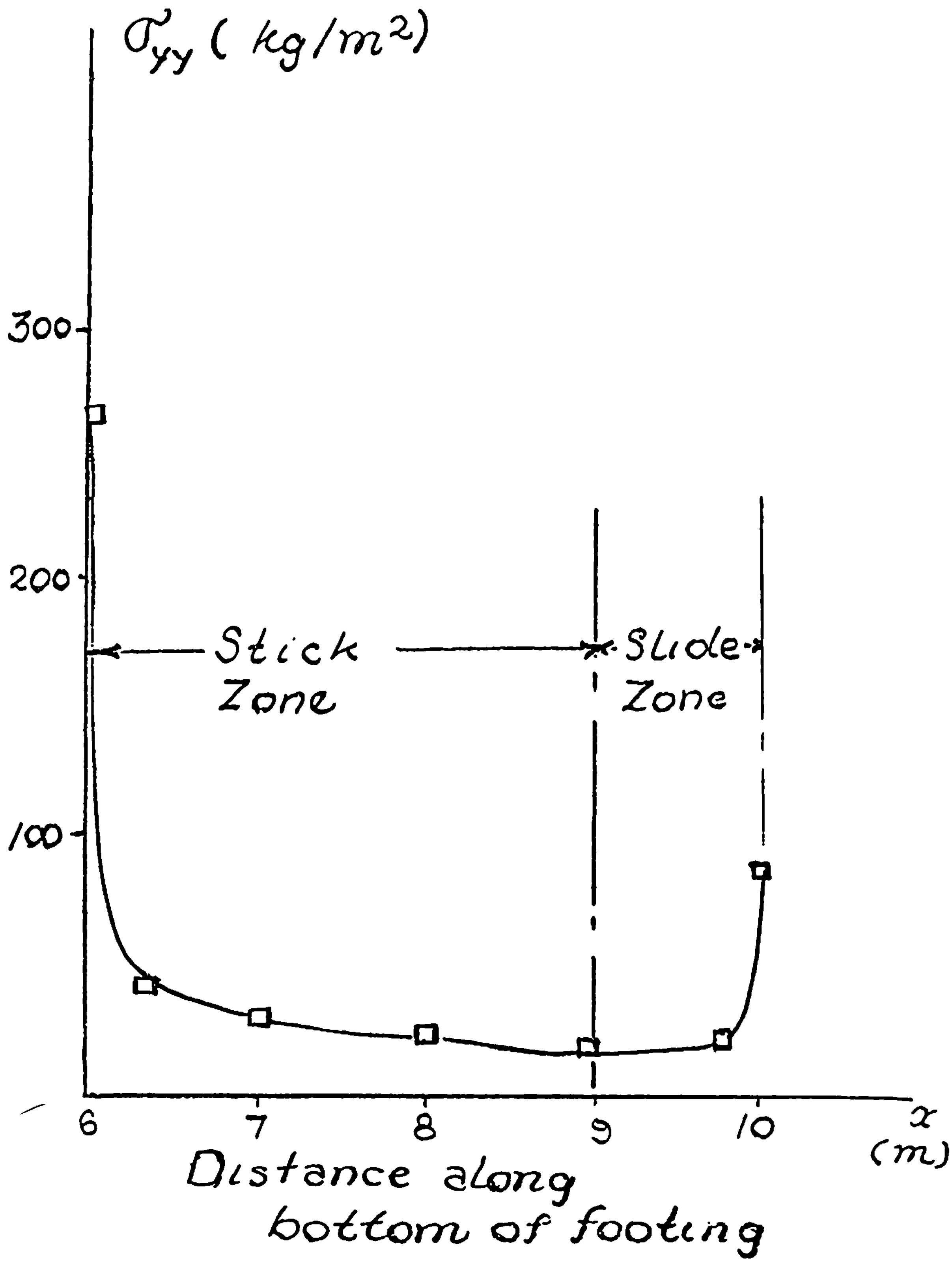


Figure 4.11 Normal Stress on the Contact Surface (Pressure Field on the Bottom of the Footing)

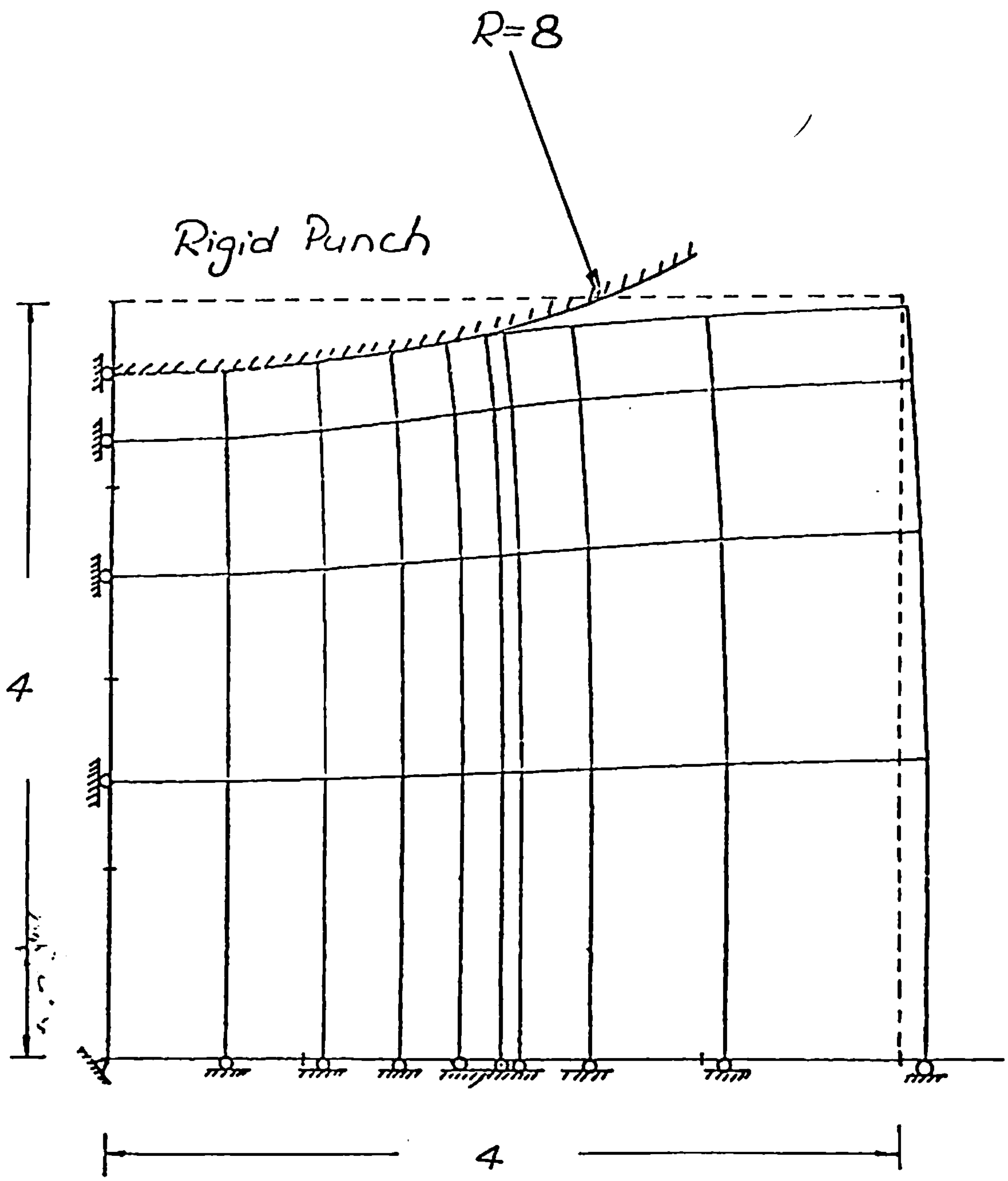


Figure 4.12 Finite Element Model and the Deformed Configuration (with Friction)

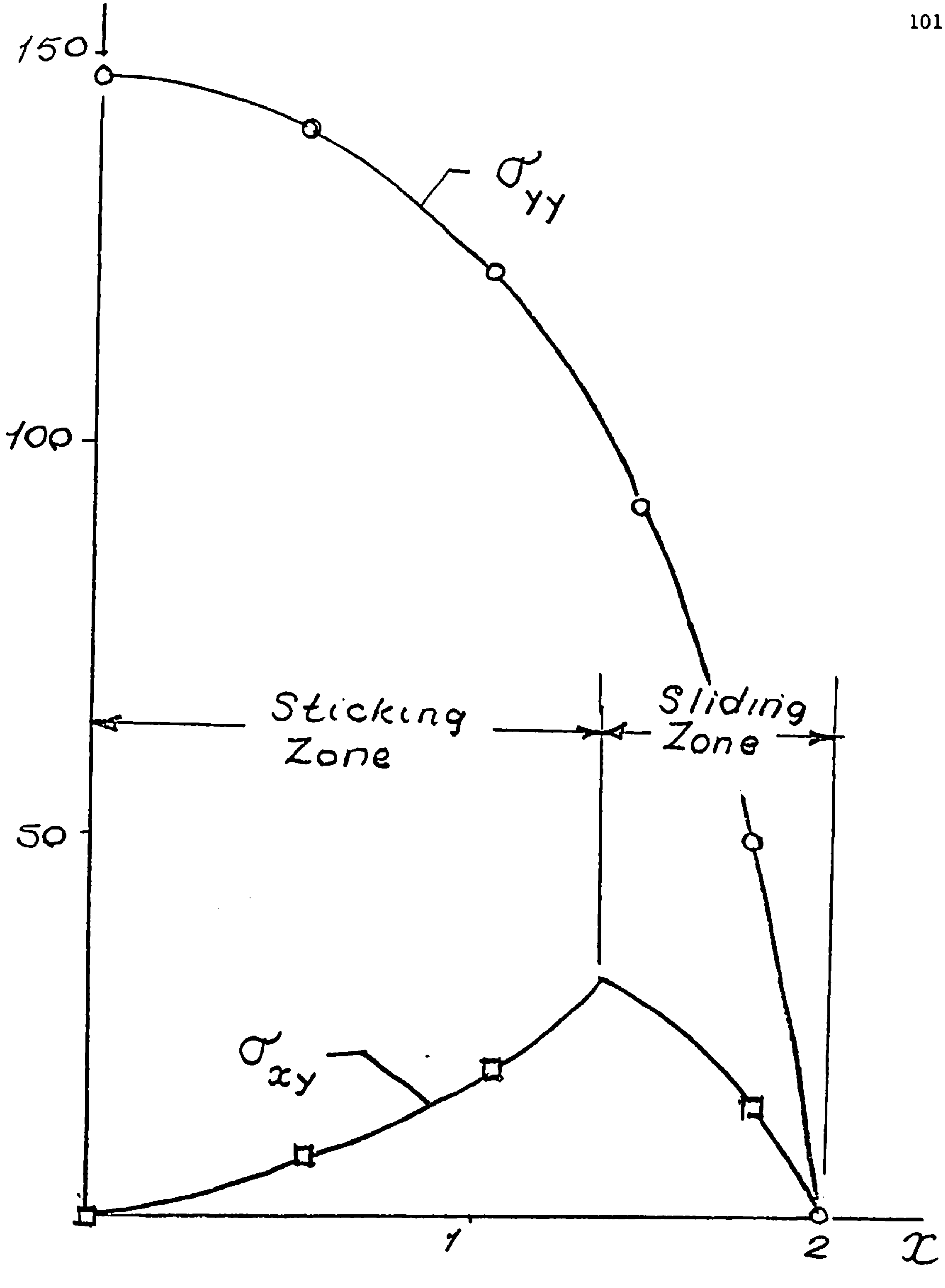


Figure 4.13 Contact Normal and Tangential Stresses under the Rigid Punch

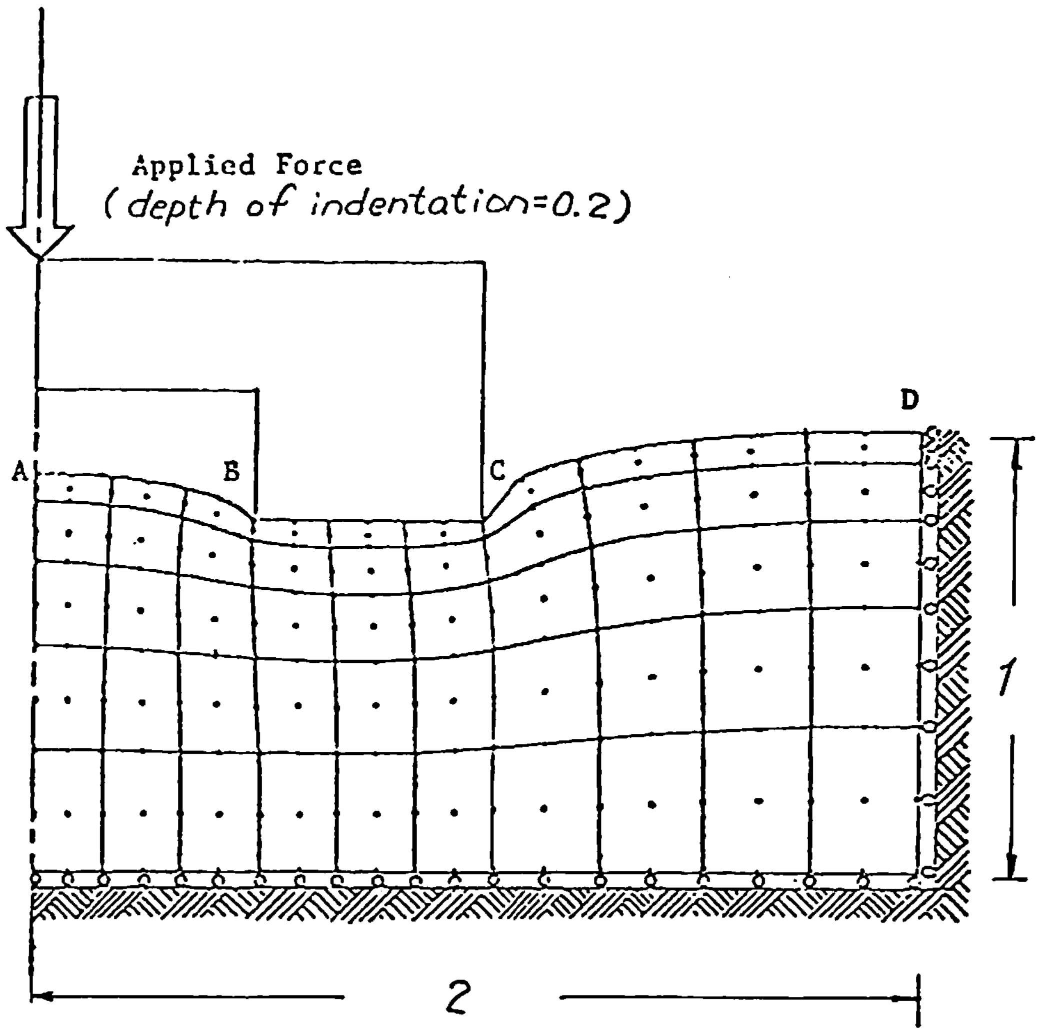


Figure 4.14 Computed Deformed Configuration

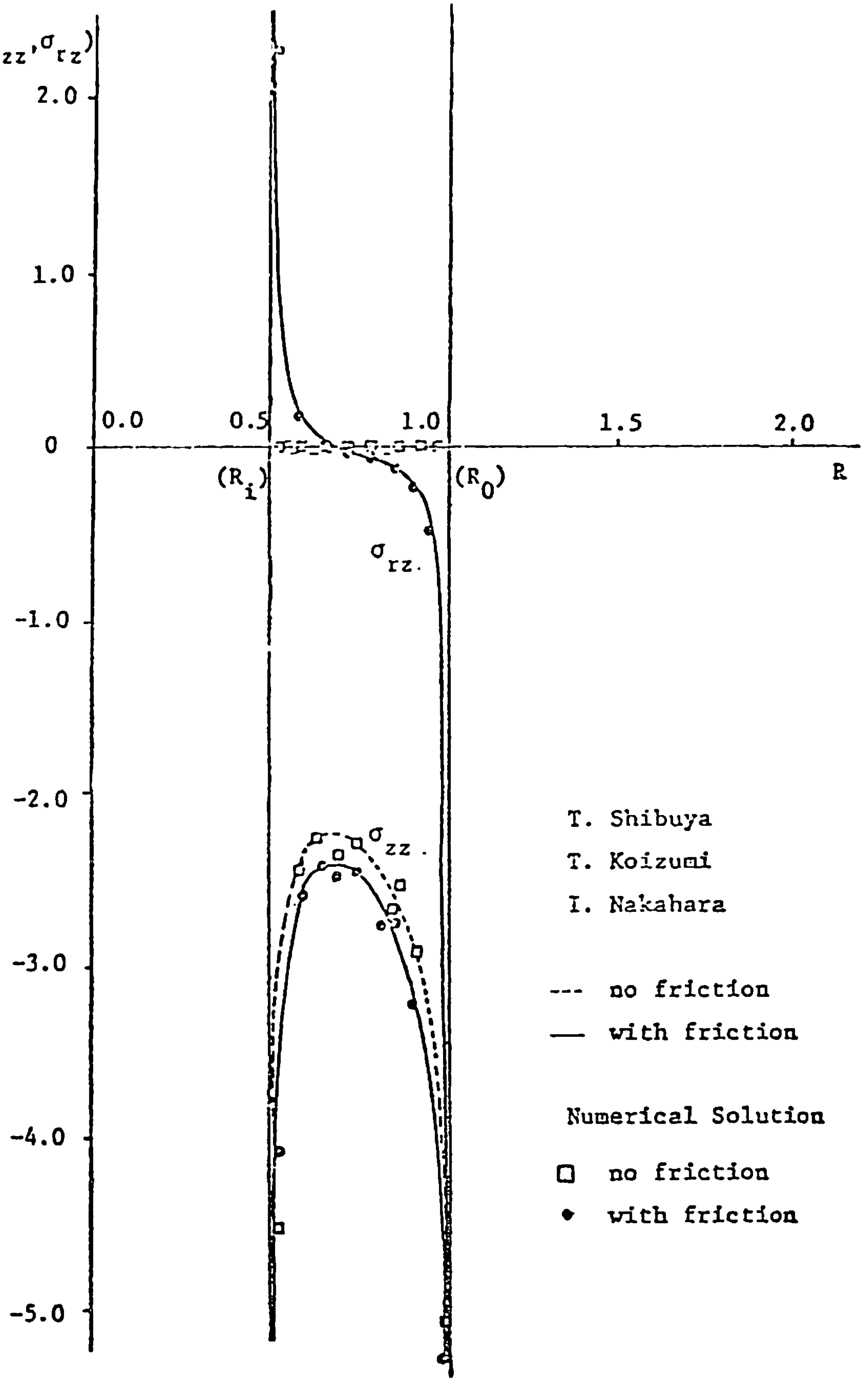


Figure 4.15 Computed Stresses and Comparison with Analytic Solutions

use the non-zero regularity parameter ε. If $\varepsilon = 10^{-5}/E$ is taken, the iterative scheme is stable, and converges monotonically.

4.6. CONCLUDING REMARKS

Starting from the contact conditions for both large and small deformations, the mathematical description (4.2.32) of the friction contact problem is derived using the Coulomb friction law. Under the assumptions of small deformation and quasi-static motion, the set of dynamical equations (4.2.32) is reduced to the incremental form (4.2.35) in the pseudo-time t and its increment Δt. The variational statement corresponding to the incremental form (4.2.35) is derived by using a quasi-variational inequality (4.3.5), the existence of solution of which is proved only for the case of small friction coefficient μ.

In order to obtain an iterative scheme to solve the quasi-variational inequality that represents the incremental form of friction contact problems, two special cases are studied in detail. The first case is the one with the prescribed tangential stress that is possibly caused by friction. Nonlinearity then arises from the unilateral contact in the normal direction, and is resolved by applying the exterior penalty method (4.4.8). Conditions for the convergence of (4.4.8) as the penalty parameter ε tends to zero to the corresponding Lagrange multiplier formulation (4.4.26) are given in Theorem 4.3. The finite element approximation of the exterior penalty formulation is obtained using four-node quadrilateral isoparametric elements, and its convergence analysis in Theorem 4.5 is established for both parameters h and ε, where h is the representative mesh size of the finite element model.

The second special case corresponds to problems with prescribed normal stresses on the known contact surface. In this case, if the normal stress is smooth enough, the variational formulation is given by a variational inequality of the first kind as in (4.5.2). Because of the non-differentiability of the functional j, that represents the work done by friction forces, a regularization existence of a unique solution to the regularized problem (4.5.20) is shown, and the convergence of the regularization is proved in Theorem 8. It is noted that the convergence of the tangential stress $\underset{\sim}{\sigma}_{\varepsilon T}$ has not been analyzed in this study. The regularized problem (4.5.20) is approximated by four-node elements. Convergence of the finite element approximation (4.5.30) is established in Theorem 4.9, under the assumption that the iterative scheme (4.5.27) gives a convergent result.

5. NON-CLASSICAL FRICTION LAWS

The use of the classical Coulomb law of friction in the formulation of contact problems in elasticity leads to both physical and mathematical difficulties; the former arising from the fact that this law provides a poor model of frictional stresses at points on metallic surfaces in contact and latter due to the fact that the existence of solutions of the governing equations can be proved only for very special situations. In the present article, non-classical friction laws are proposed in an attempt to overcome both of these difficulties. We consider a class of contact problems involving the equilibrium of linearly elastic bodies in contact on surfaces on which nonlocal and nonlinear friction laws are assumed to hold. The physics of friction between metallic bodies in contact is discussed and arguments in support of the theory are presented. Variational principles for boundary value problems in elasticity in which such nonlinear nonlocal laws hold are then developed. A brief discussion of the questions of existence and uniqueness of solutions to the nonlocal and nonlinear problems is given.

5.1 Introduction

In 1781, the French engineer C. A. Coulomb published his "Théorie des Machines Simples" in which he presented his celebrated law of friction. This work earned him a double prize from the French Royal Academy of Sciences in 1785. The classical Coulomb law of static dry friction, of course, asserts that *relative sliding between two bodies in contact along plane surfaces will occur when the net shear force parallel to the plane reaches a critical value proportional to the net normal force* *pressing the two bodies together. The constant of proportionality is called the coefficient of friction.*

It can be argued that as a basis for contact problems in the theory of elasticity, Coulomb's law is not acceptable from either a physical or a mathematical point of view. From the purely physical side, it has been recognized for many years that Coulomb's law is capable of describing only friction effects between effectively rigid bodies and gross sliding of one body relative to another. Indeed, it is clear that Coulomb himself never inteded that his law be applied pointwise in boundary-value problems in elasticity; the foundarions of continuum mechanics, particularly the concept of stress and the equations of linear elastostatics, were only fully developed many decades after Coulomb proposed his law, and the first successful formulation of a contact problem in elasticity came over a full century after Coulomb's work. From the mathematical point of view, it is known (see DUVAUT [1980] and also DUVAUT and LIONS [1976]) that if Coulomb's law is applied pointwise in contact problems involving linearly elastic bodies, then the contact stress σ_n developed normal to the contact surface is ill-defined. Except for some very special cases (e.g., NEČAS, JARUŠEK and HASLINGER [1980]) the fundamental question of existence of solutions of the friction problem is open (for other related open questions, see DUVAUT and LIONS [1976]).

There are several aspects of actual friction phenomena between metallic bodies that suggest alternative friction laws which represent a marked departure from the classical formulations. First, we mention the obvious nonlocal character of the mechanism by which normal stresses are distributed on machined contact surfaces. These stresses are transmitted over junctions formed by deformed asperities and are not concentrated at isolated points on the contact surface. Second, we note that upon the application of loads, experiments show that there always exists a small tangential displacement of points on the contact surfaces due to the elastic and elasto-plastic deformation of these junctions; "sliding" occurs when these junctions are actually fractured. Since these junctions can be recovered upon a quasi-static reversal of loads, the actual "adhesion-sliding" friction mechanism is highly nonlinear and depends upon the elasto-plastic properties of the metal oxide and contaminant film on the contact surfaces.

Independent of the nonlinear character of local friction phenomena, there are also mathematical reasons to expect that a nonlocal friction law might lead to a more tractable theory. Recently, DUVAUT [1980] published a brief note in which he observed that the source of difficulties in establishing an existence theory for Signorini's problem with Coulomb friction was the lack of smoothness of the normal contact pressure σ_n. Therefore, by considering, instead, a proper mollification of the normal stresses on the contact boundary he was able to state that the principal obstacles in the way of deriving an existence and uniqueness theory for contact problems with friction could be overcome. One interpretation of such a smoothing of the contact pressure is as result of nonlocal effects arising from micromechanical phenomena taking place on the contact regions.

In the present study, we propose nonlinear, nonlocal friction laws for contact problems involving linearly elastic bodies and we present variational principles for contact problems in elastostatics in which these laws hold. Roughly speaking, a nonlocal friction law proposes that *impending motion at a point of contact between two deformable bodies will occur when the shear stress at that point reaches a value proportional to a weighted measure of the normal stresses in a neighborhood of the point.* The character of the effective local neighborhood and the manner in which neighborhood stresses contribute to the sliding condition depends upon features of the microstructure of the materials involved.

While such nonlocal laws do lead to a mathematically tractable theory, they still do not capture the effects of the tangential elastic-plastic deformations of the contact junctions mentioned earlier. To accomodate such effects, we present a further amendment which provides for small but nonzero elastic tangential displacements at the contact surface for tangential stresses below a certain critical level. For shear stresses at or near this critical level, substantially larger motions can occur which effectively represent large tangential motions such as sliding. This critical value may be proportional to a weighted measure of the normal stresses in a neighborhood of the point on the contact surface.

An interesting feature of our results is that these non-conventional friction laws are given in terms of three positive material parameters: ν, ρ, and ε. The parameter ν is the coefficient of friction, although its actual interpretation is somewhat more complex than that of classical mechanics. The parameter ρ quantifies the nonlocal character of the response - for $\rho = 0$ a fully local law is obtained. Finally, ε is a measure of the tangential stiffness of the elastic-plastic junctions on the contact surface; the case $\varepsilon = 0$ corresponds to a fully rigid response full adhesion or full sliding of contact surfaces. Thus, by allowing ρ and ε to approach zero, we can recover the classical, local, pointwise formulation of contact problems in elastistatics based on Coulomb's law.

Following this introduction, we give a brief account of the physics of friction as well as a justification of specific friction models. Several variational principles for boundary value problems in elasticity in which nonlocal and nonlinear laws are assumed to hold are derived in Section 3. The results of some theoretical studies of these principles are summarized in Section 4. Our results include conditions sufficient to guarantee the existence and uniqueness of solutions to the variational problems as well as results which establish the asymptotic behavior of solutions as parameters ρ and ε tend to zero.

5.2 A Basis for Nonlocal and Nonlinear Friction Laws

5.2.1 Micromechanics of Friction. Friction phenomena have been the subject of considerable experimental and theoretical research over the last 30 years and its study is a popular and important aspect of modern mechanical engineering design. A standard reference is the monumental two-volume treatise by BOWDEN and TABOR [1950 and 1964]; more concise accounts can be found in standard texts on the subject (e.g. RABINOWICZ [1965]).

To understand friction, one must first appreciate the role of the microstructure of the materials involved. Consider an experiment in which two metallic bodies are placed in contact along two apparently machined flat surfaces. At microscopic levels, specifically at magnifications of 1000x to 5000x, machined metal surfaces are seen to be not smooth homogeneous planes, but rough contours with numerous irregularities which are large compared with molecular dimensions. We refer to these deviations from the plane as *asperities*.

When we press together two surfaces, actual contact initially occurs only at the peaks or summits of the asperities. Large areas of the surfaces are separated by a distance which is large compared with the range of molecular action, so that these gaps in the surfaces are completely separated and have no interaction with one another. The load is, therefore, initially supported at the tips of the asperities; the area of contact is extremely small, and the pressure at the points of contact, even for lightly loaded surfaces, is high. Plastic deformation of the tip of the asperities occurs at small loads while the bulk of the underlying metal deforms elastically. As the normal load is further increased, the asperities deform and fracture with the result that the local load is distributed over an area surrounding each deformed asperity. At this stage, each asperity has been flattened and the local contact forces are distributed over a neighborhood of the asperity. The *real area of contact* A_r (as opposed to the *apparent* area) is therefore the sum of the areas of all the surface irregularities which are touching and which support the load.

It is often assumed that the local plastic yield pressure p_0 is nearly constant and is comparable to the indentation hardness of the metal. Under these circumstances, the real area of contact for any one asperity bearing a load N_i is $A_i = N_i/p_0$; for the assembly of the asperities, the real contact area is

$$A_r = A_1 + A_2 + \dots = \frac{N_1}{p_0} + \frac{N_2}{p_0} \dots = \frac{N}{p_0} \tag{5.2.1}$$

where N is the total normal force pressing the surfaces together. The real area of contact is, thus, proportional to the load and independent of the size of the surfaces. Over these regions where intimate contact occurs, strong adhesion and welding of the metal surfaces takes place and the specimens become, in effect, a continuous solid. We refer to these regions as *junctions*.

Under most working conditions, metal surfaces are covered by a thin film of oxide, water vapor and other absorbed impurities. The shear strength of these junctions can be strongly dependent upon the shear strength of these surface films. In particular, it is the shear strength of this layer of oxide and impurities that determines the coefficient of friction and not, in general, the shear strength of the parent metals.

The application of a tangential force T creates a tendency for the two bodies to slide relative to each other. The contact pressure must then decrease (since it was near or equal the plastic yield stress). Some microscopic motion will occur (junction growth) even when T is small. If T is steadily increased, a value of sufficient magnitude to fracture the contaminant films is eventually reached and gross sliding occurs. It is customary to set the ratio of the magnitude of T at which sliding occurs to the net normal force N equal to the coefficient of friction ν. If s is the average shear strength of the interface, it follows that approximately

$$T = A_r s \tag{5.2.2}$$

i.e., T is independent of the apparent area of contact (since A_r is). Then, substituting for A_r, we obtain

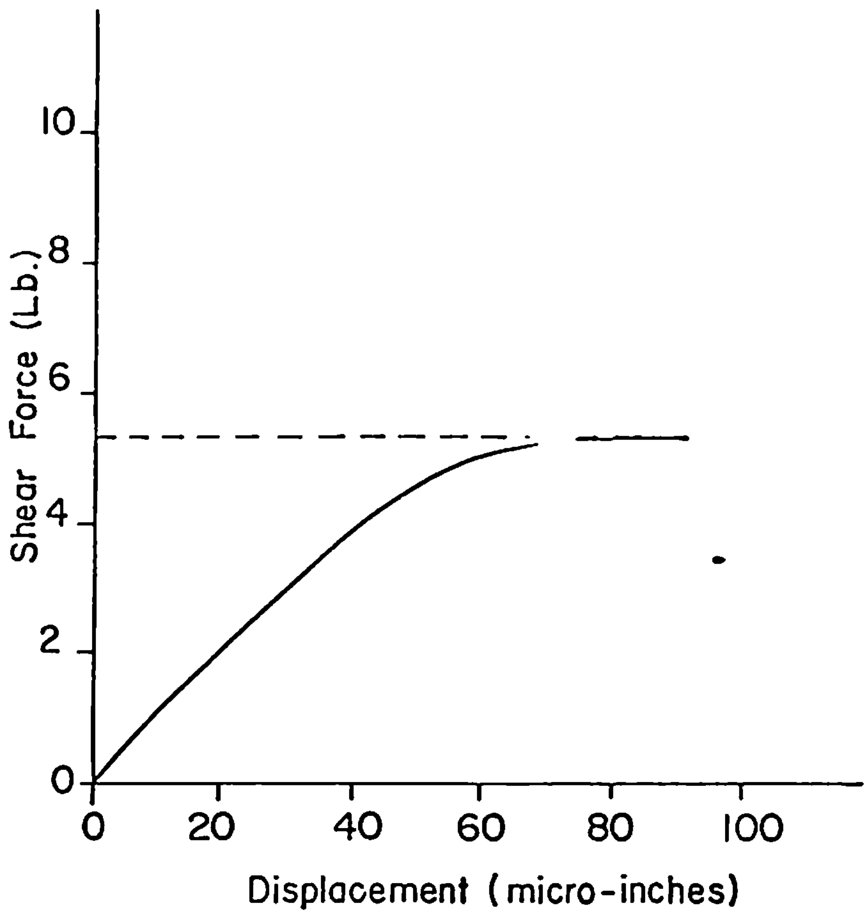
$$T = \frac{s}{p_0} N \qquad \text{or} \qquad \nu = \frac{s}{p_0} \tag{5.2.3}$$

i.e., T is directly proportional to the load (or ν is independent of the load).

As an idealization of the contact surfaces, one may assume that the asperities are superposed upon the surface of spherical protuberances with a larger radius of curvature. It is then possible to consider that although each individual asperity at the interface will deform plastically, the deformation of each spherical protuberance will be elastic. This idealization has two purposes: first, the area of real contact still remains approximately proportional to the load, although the overall deformation is elastic (see ARCHARD [1957]); second, it allows us to treat each region of contact as being roughly circular (see microphotographs in BOWDEN and TABOR [1964, p. 71]) and to regard the contact pressure as being essentially symmetric, attaining its maximum magnitude at the center of the circle of contact, in a manner consistent with the well known analysis of MINDLIN [1949].

We emphasize that the junctions through which loads are transmitted from one body to another are not rigid; indeed, they are composed of a deformable composite of metal, metal oxide, and surface contaminant that, for our purposes, can be assumed to be elasto-plastic or nonlinearly elastic. Several researchers have actually measured the tangential micro-displacements that occur, in friction experiments on metals, prior to gross sliding of the surfaces, and we mention as examples the works of JOHNSON [1955], BOWDEN and TABOR [1964] and RABINOWICZ [1965]. Figure 1 reproduces a typical results of static displacement tests of JOHNSON [1955] which involved the contact of hard steel balls with the flat end of a hard steel roller. Mirco-displacements are produced by an applied shear force varying progressively from zero to the value necessary to produce slip.

5.2.2 A Nonlocal Friction Law. In order to develop a basis for a nonlocal friction law we consider here the two simple physical models shown in Figs. 2 and 3. Fig. 2, a thin weightless strip A of length 2ℓ



Diameter of the ball = 0.375 in.
Normal load = 10.5 lbs.

Figure 5.1. Measured shear-tangential displacement variations after the experiments of Johnson [1955].

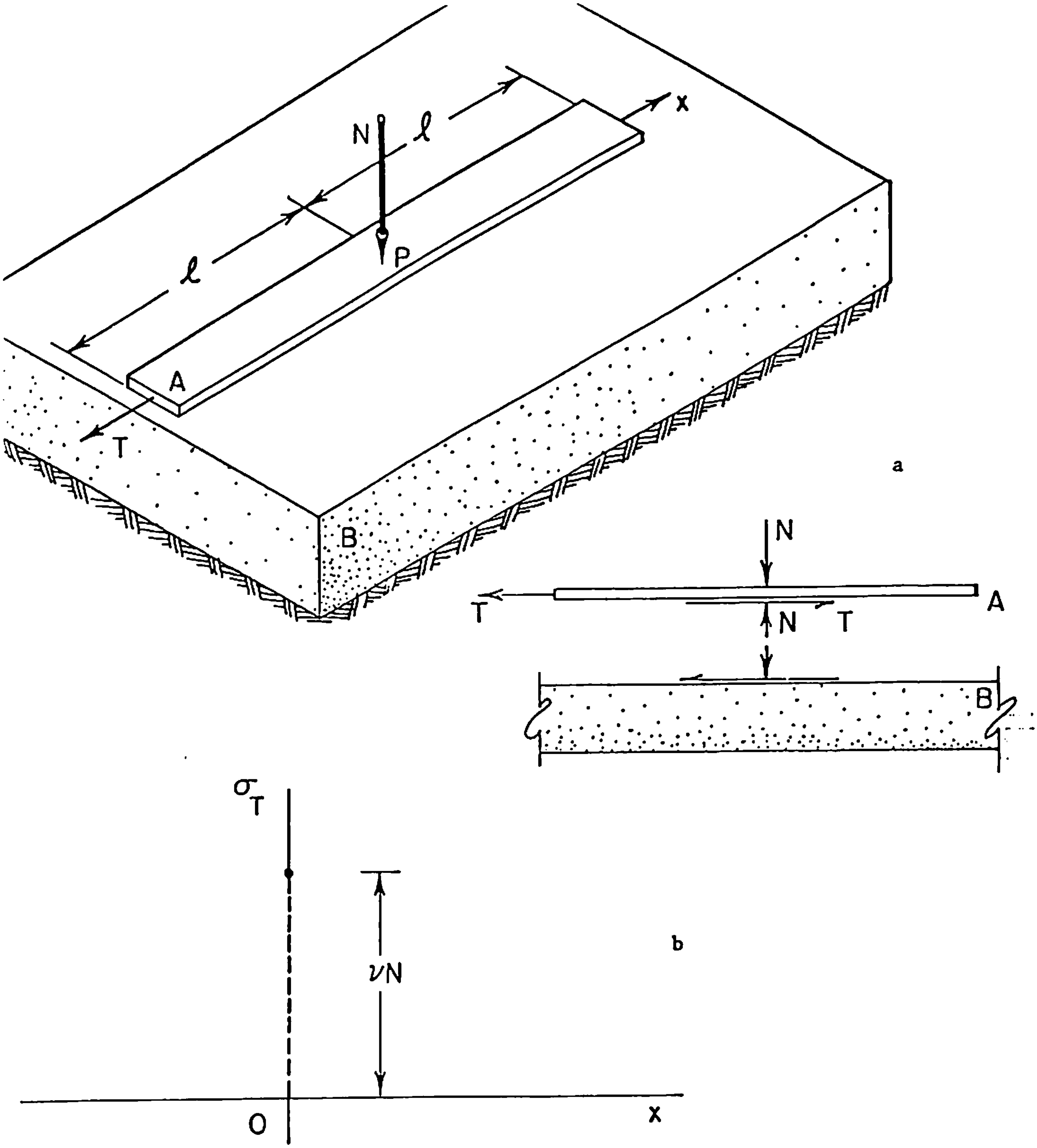
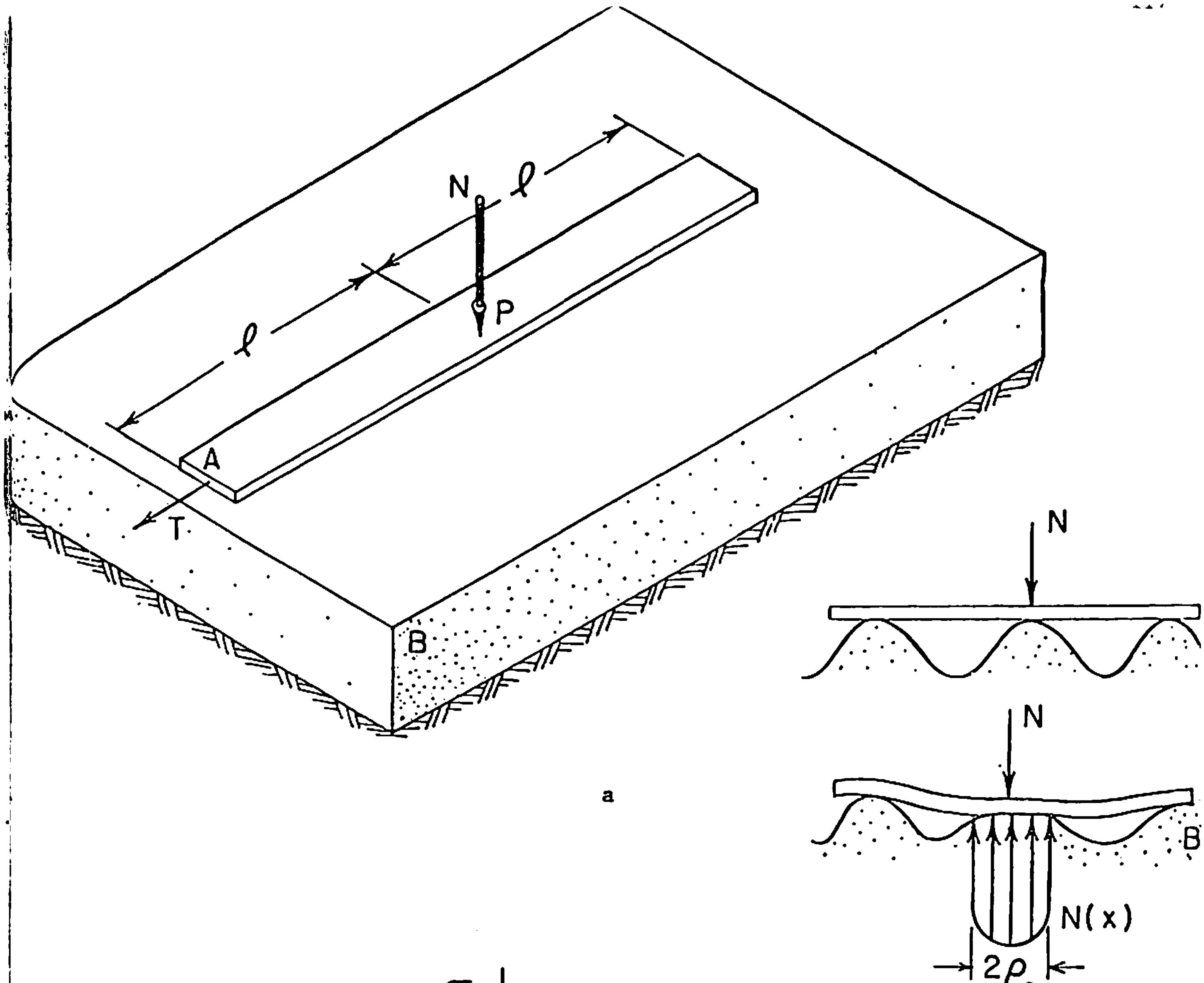
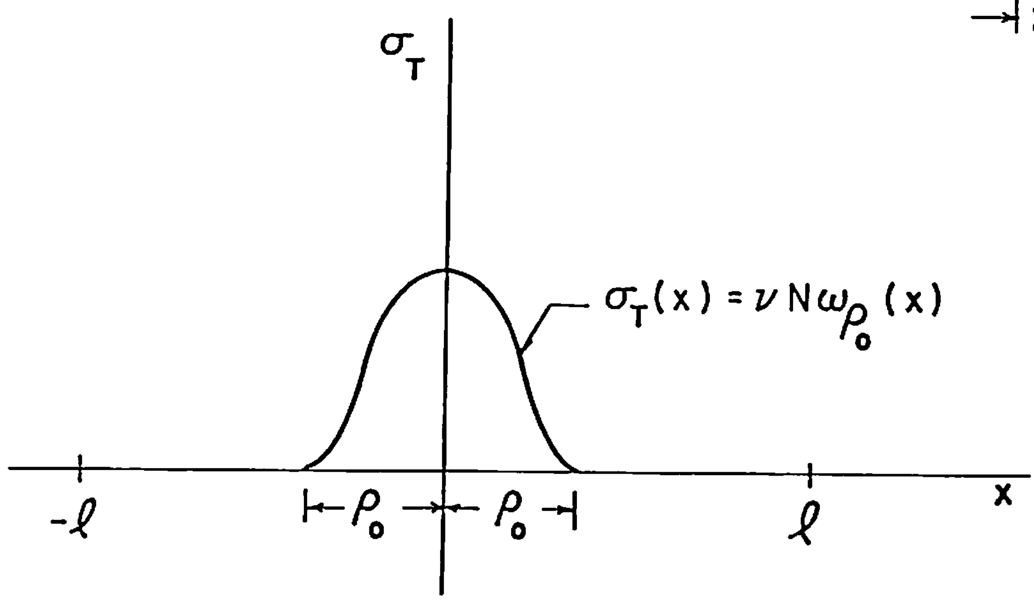


Figure 5.2. a) A thin strip A pressed on an elastic block B ,
b) shear stress distribution (symbolic) assuming a pointwise Coulomb law of friction



a



b

Figure 5.3. a) A thin strip A pressed onto an elastic block B with

is pressed against a fixed elastic block B by a concentrated normal force N applied at its midpoint P ; then a force T is applied to the strip and is increased slowly in magnitude until motion (sliding) of the strip relative to the block occurs. For this idealized situation, designed to emphasize the local character of the classical Coulomb law of friction, we are interested in calculating the distribution of frictional (shear) stresses between the strip and the block assuming that Coulomb's law holds and that the coefficient of friction ν is given.

We consider the origin of a coordinate axis x along the length of the strip to coincide with the midpoint P of the strip. Then, the only point on the contact surfaces between bodies A and B at which a resistive force can be developed is at an isolated point beneath P. Thus, impending sliding is reached when the resisting shear σ_T is formally given by

$$\sigma_T(x) = \nu N \delta(x) \; ; \quad -\ell < x < \ell \tag{5.2.4}$$

where δ is the Dirac delta corresponding to the point source at the origin. Of course, is merely the symbolic representation of the distribution

$$\langle \sigma_T, \phi \rangle = \nu N \langle \delta, \phi \rangle = \nu N \phi(0) \tag{5.2.5}$$

for all test functions ϕ in the class $\mathcal{D}(-\ell,\ell)$ of infinitely differentiable functions with support in the interval $(-\ell,\ell)$, where $\langle \cdot , \cdot \rangle$ denotes duality pairing on distributions and test functions (i.e., the action of

a distribution q on a test function ϕ is denoted $q(\phi) \equiv \langle q,\phi \rangle$).
Alternatively, δ can be interpreted as the limit of a δ-sequence, $\{\omega_\rho\}_{0<\rho}$, ω_ρ in $\mathcal{D}(-\ell,\ell)$:

$$\phi(0) = \delta(\phi) = \lim_{\rho \to 0} \int_{-\ell}^{\ell} \omega_\rho \, \phi \, dx \qquad \text{for all } \phi \text{ in } \mathcal{D}(-\ell,\ell) \tag{5.2.6}$$

Then we have, instead of (5.2.5)

$$\langle \sigma_T,\phi \rangle = \nu \lim_{\rho \to 0} \langle N\,\omega_\rho,\phi \rangle \qquad \text{for all } \phi \text{ in } \mathcal{D}(-\ell,\ell) \tag{5.2.7}$$

We see that the classical pointwise version of Coulomb's law must be interpreted in the sense of distributions for this situation. As a typical δ-sequence, we mention:

$$\omega_\rho(x) = \begin{cases} c \exp[\rho^2/(x^2 - \rho^2)], & |x| \leq \rho \\ 0 & , |x| \geq \rho \end{cases} \tag{5.2.8}$$

A more realistic model of friction from the physical point of view is obtained if we take into account the microscopic aspects of the physics of friction described earlier. Specifically, the contact surface of body B will present asperities deviating from a smooth plane. As the normal force N is gradually applied, these asperities are gradually deformed and broken down until equilibrium of normal forces is reached. The normal force reaching body B through the strip A must then be distributed over the contact area of the deformed asperity as indicated in Fig. 3. We shall now assume that the asperity's finite transmission area is accounted for by using the δ-sequence $\{\omega_\rho\}$ of 5.2.8 keeping $\rho = \rho_0$, ρ_0 being the radius of the contact area of the deformed asperity. Since $N = N(x)$ is now a function, we have, instead of (5.2.7),

$$\sigma_T(x) = \nu N(y) * \omega_{\rho_0}(x - y) \tag{5.2.9}$$

where * denotes the convolution of the two functions. Thus, we have arrived at a friction law in which impending motion occurs at a point x on the contact surface when the shear stress at that point reaches a value proportional to the weighed average of the normal stress in a neighborhood of the point. If ω_{ρ_0} is used to characterize this weighting function, then the neighborhood is a circular disc of radius ρ_0 centered at x, the maximum weight is given to the stress intensity at the center of the disc (the contact area of the deformed asperity) and exponentially decreasing weights are assigned to stress intensities as one moves from the center of the neighborhood outward to the periphery of the disc.

We can now generalize these results to the three-dimensional case: let $\underset{\sim}{u}_T$ denote the relative tangential component of displacement of a point $\underset{\sim}{x} = (x_1, x_2, x_3)$ on the contact surface between two deformable bodies and let $\sigma_n(\underset{\sim}{u})$ and $\underset{\sim}{\sigma}_T(\underset{\sim}{u})$ denote the normal and tangential stresses on the contact surface corresponding to the displacement field $\underset{\sim}{u}$. Then

$$\left. \begin{array}{ll} |\sigma_T(u)| < \nu S_{\rho_0}(\sigma_n(u)) & \text{implies } u_T = 0 \\ |\sigma_T(u)| = \nu S_{\rho_0}(\sigma_n(u)) & \text{implies that there exists} \\ & \lambda \geq 0 \text{ such that } u_T = -\lambda\sigma_T \end{array} \right\} \qquad (5.2.10)$$

where S_{ρ_0} is an operator mollifying the normal stress distribution; e.g.

$$S_{\rho_0}(\sigma_n(u))(x) = \int_{\Gamma_C} \omega_{\rho_0}(|x - y|)(-\sigma_n(u(y)))dy \qquad (5.2.11)$$

where x and y are points on the contact surface Γ_C.

We mention that nonlocal theories for other classes of problems in solid and fluid mechanics have been put forth by ERINGEN and EDELEN [1972]; a detailed account of this work can be found in ERINGEN [1976].

5.2.3 Model of Nonlinear Friction. Both Coulomb's law and the nonlocal law given in 5.2.10 depict perfect rigid-adhesion-sliding conditions on the contact surface: they assert that there is absolutely no motion of points of one body relative to those of another if the tangential stress on the contact surface remains below some critical value τ ; but when this limit is reached, unbounded motions can occur, the ensuing tangential displacement being directed opposite to the tangential stress vector. It was pointed out in the Introduction and in Section 5.2 that in physical experiments on contact, tangential displacements are produced by any nonzero tangential force developed on the contact surface since elastic-plastic deformations of the junctions will always accompany the application of tangential forces (recall Fig. 1).

To model this phenomenon, we shall consider a family of nonlinear friction laws of the form

$$\underset{\sim}{\sigma}_T(\underset{\sim}{u}) = -\tau\ \phi_\varepsilon(|\underset{\sim}{u}_T|)\ \frac{\underset{\sim}{u}_T}{|\underset{\sim}{u}_T|} \tag{5.2.12}$$

where the function $\phi_\varepsilon(\cdot)$ satisfies the following conditions:

$$\left.\begin{array}{l} \text{i) } \phi_\varepsilon \text{ is a continuous, monotone, real-valued function of} \\ \quad \text{the non-negative real numbers } r \text{ , depending on a para-} \\ \quad \text{meter } \varepsilon > 0, \text{ such that} \\ \qquad\qquad 0 \le \phi_\varepsilon(r) \le 1\ ,\quad r \ge 0 \\ \text{ii) } \lim\limits_{\varepsilon \to 0} \phi_\varepsilon(r) = 1 \text{ for all } r > 0 \\ \text{iii) } \lim\limits_{r \to \infty} \phi_\varepsilon(r) = 1 \text{ for all } \varepsilon > 0 \end{array}\right\} \tag{5.2.13}$$

In 5.2.12, τ is a non-negative function of the displacement vector $\underset{\sim}{u}$ representing the critical value that the tangential stress cannot exceed, i.e., $\tau = \nu|\sigma_n(\underset{\sim}{u})|$ for the local case and $\tau = \nu S_\rho(\sigma_n(\underset{\sim}{u}))$ for the non-local case. It will also be of interest, as it will be seen later, to consider the case in which τ is a given (known) function of the position vector $\underset{\sim}{x}$, thus no longer dependent upon the displacement $\underset{\sim}{u}$.

As specific examples, we select for ϕ_ε the following two functions:

$$\text{A)}\quad \phi_\varepsilon(|u_T|) = \begin{cases} 1 & \text{if } |\underset{\sim}{u}_T| > \varepsilon \\ |\underset{\sim}{u}_T|/\varepsilon & \text{if } |\underset{\sim}{u}_T| \le \varepsilon \end{cases} \tag{5.2.14}$$

$$\text{B)}\quad \phi_\varepsilon(|\underset{\sim}{u}_T|) = \tanh \frac{|\underset{\sim}{u}_T|}{\varepsilon} \tag{5.2.15}$$

It is readily seen that both of these functions satisfy 5.2.13. The first example represents an "elastic-perfectly plastic" type response in which slipping can occur only after a tangential displacement $|\underset{\sim}{u}_T| > \varepsilon$. On the other hand, the second example describes a situation in which the critical stress is approximated asymptotically as $|\underset{\sim}{u}_T| \to \infty$. Both curves are depicted in Fig. 4. We notice that these are not the only possible choices and that several others can also be considered (e.g. arctan, etc.). Also, we observe that the slope of ϕ_ε at the origin, i.e., the derivative of ϕ_ε at zero displacement equals $1/\varepsilon$. Thus ε provides a measure of the rigidity or stiffness of the elastic-plastic or nonlinear elastic junctions.

Finally, we wish to comment on the combination of a nonlocal law with a nonlinear law. A nonlocal-nonlinear law of friction will be of the form (5.2.12) with τ replaced by $\nu S_\rho(\sigma_n(\underset{\sim}{u}))$, i.e.,

$$\underset{\sim}{\sigma}_T(\underset{\sim}{u}) = -\nu S_\rho(\sigma_n(\underset{\sim}{u}))\ \phi_\varepsilon(|\underset{\sim}{u}_T|)\ \frac{\underset{\sim}{u}_T}{|\underset{\sim}{u}_T|} \tag{5.2.16}$$

where S_ρ is of the form given in (5.2.11). If we allow $\rho \to 0$ maintaining $\varepsilon > 0$ fixed we obtain a local nonlinear law, i.e.,

$$\underset{\sim}{\sigma}_T(\underset{\sim}{u}) = -\nu\ |\sigma_n(\underset{\sim}{u})|\ \phi_\varepsilon(|\underset{\sim}{u}_T|)\ \frac{\underset{\sim}{u}_T}{|\underset{\sim}{u}_T|} \tag{5.2.17}$$

On the other hand, if we allow $\varepsilon \to 0$ for a given positive ρ we recover the perfect rigid-adhesion-sliding nonlocal law given in (5.2.10). Finally if both $\varepsilon \to 0$ and $\rho \to 0$ the static Coulomb's law (local) for unilateral contact in the contact surface Γ_C is obtained:

$$\left.\begin{array}{ll} |\underset{\sim}{\sigma}_T(\underset{\sim}{u})| < \nu\ |\sigma_n(\underset{\sim}{u})| & \text{implies } \underset{\sim}{u}_T = \underset{\sim}{0} \\ |\sigma_T(\underset{\sim}{u})| = \nu\ |\sigma_n(\underset{\sim}{u})| & \text{implies the existence of } \lambda \geq 0 \\ & \text{such that } \underset{\sim}{u}_T = -\lambda \underset{\sim}{\sigma}_T \end{array}\right\} \tag{5.2.18}$$

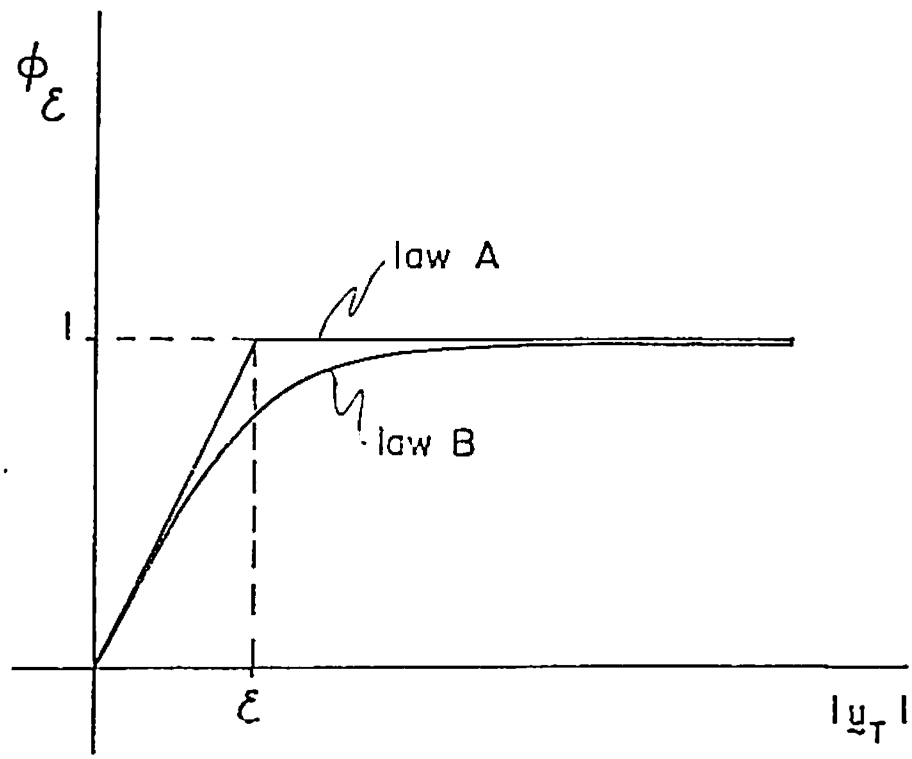


Figure 5.4. Graphs of functions (2.14) and (2.15)

5.3 Variational Principles for Nonlocal Nonlinear Friction

5.3.1 Signorini Problems with Nonlocal Nonlinear Friction. We consider here the Signorini problem of contact of a linearly elastic body with a rigid foundation on which nonlocal nonlinear friction laws hold.

We begin our analysis by considering a linearly elastic body the particles of which occupy a smooth bounded domain $\overline{\Omega}$ in $\mathbb{R}^N$, $N = 2,3$, with open interior Ω. The boundary Γ of the body is assumed to consist of three disjoint parts, Γ_D, Γ_F and Γ_C, where Γ_D and Γ_F are the portions of the boundary on which the displacements and forces (tractions) are prescribed respectively and Γ_C is the candidate contact area; i.e. Γ_C is a portion of the boundary which contains the material surface which comes in unilateral contact with a rigid foundation F upon the application of loads (see Fig. 5). The external forces on the body consist of a prescribed body force field of intensity $\underset{\sim}{f}$ per unit volume and of surface tractions of intensity $\underset{\sim}{t}$ per unit surface area.

We shall assume that Γ_D is perfectly fixed, so that

$$\underset{\sim}{u} = \underset{\sim}{0} \quad \text{on} \quad \Gamma_D$$

$\underset{\sim}{u}$ being the displacement field. On Γ_F we will have

$$\sigma_{ij}(\underset{\sim}{u})n_i = t_j$$

where $\sigma_{ij}(\underset{\sim}{u})$ is the stress produced by $\underset{\sim}{u}$ and n_i are the components of the unit outward normal $\underset{\sim}{n}$ to Γ. Here and throughout our presentation Cartesian index notation and the summation convention are employed. Since the body is assumed to be linearly elastic, Hooke's law holds so that

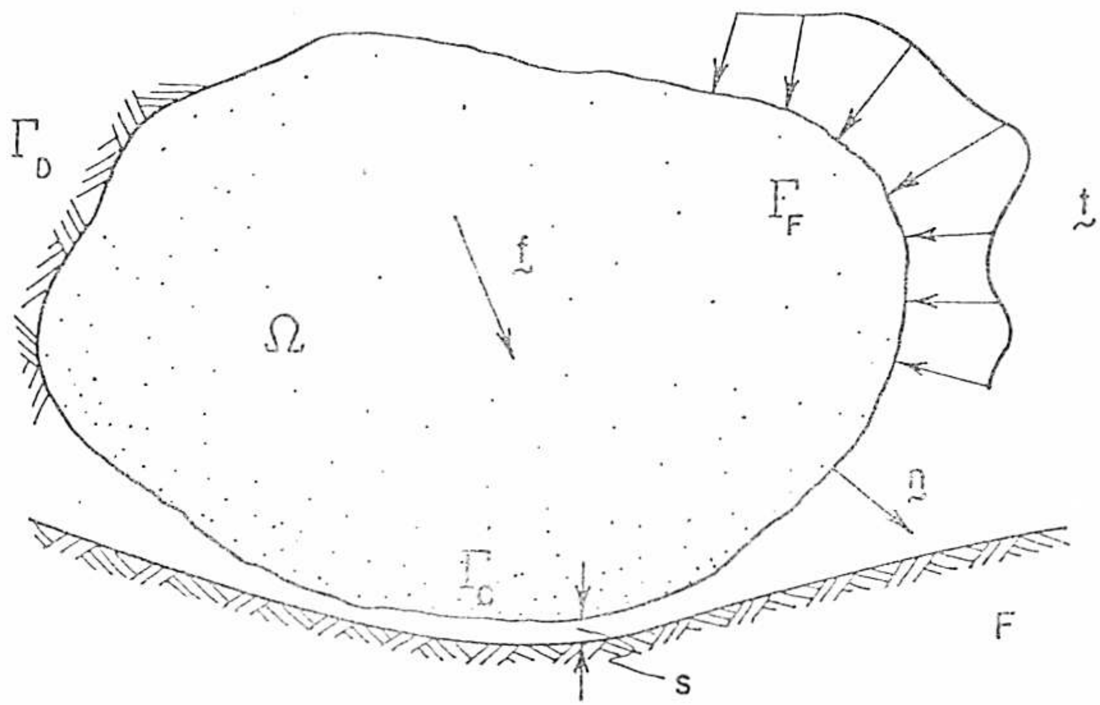


Figure 5.5. An elastic body in contact with a rigid rough foundation

$$\sigma_{ij}(\underset{\sim}{u}) = E_{ijk\ell}\, u_{k,\ell}$$

where $E_{ijk\ell}$ are the usual elastic constants of the material, $u_{k,\ell} \equiv \partial u_k/\partial x_\ell$ and

$$E_{ijk\ell} = E_{jik\ell} = E_{k\ell ij} = E_{ij\ell k}$$

$$1 \leq i,j,k,\ell \leq N$$

Of course, particles within Ω are assumed to be in static equilibrium so that

$$\sigma_{ij}(\underset{\sim}{u})_{,j} + f_i = 0 \quad \text{in } \Omega$$

The unilateral motion of particles of the body on the material surface Γ_C is constrained by the presence of a rigid foundation which is a given distance s from the body prior to the application of loads. Mathematically, this constraint is represented by the requirement that the normal displacement of boundary points cannot exceed s :

$$\underset{\sim}{u} \cdot \underset{\sim}{n} \leq s \qquad \text{on } \Gamma_C \tag{5.3.1}$$

If $\underset{\sim}{u} \cdot \underset{\sim}{n} = s$ then contact is established, while $\underset{\sim}{u} \cdot \underset{\sim}{n} < s$ indicates the existence of a gap between the support and the body. Thus 5.3.1 represents a non-penetration condition. If contact is not made (i.e., if $\underset{\sim}{u} \cdot \underset{\sim}{n} < s$) then the normal contact pressure $\sigma_n(\underset{\sim}{u}) = 0$, where

$$\sigma_n(\underset{\sim}{u}) = E_{ijk\ell}\, u_{k,\ell}\, n_i\, n_j$$

Alternatively, if $\underset{\sim}{u} \cdot \underset{\sim}{n} = s$ at a point on Γ_C , then σ_n must be non-positive:

$$\underset{\sim}{u} \cdot \underset{\sim}{n} = s \quad \text{implies} \quad \sigma_n(\underset{\sim}{u}) \leq 0$$

Thus, the unilateral contact conditions on the contact surface Γ_C are:

$$\left.\begin{array}{r} \underset{\sim}{u} \cdot \underset{\sim}{n} - s \leq 0 \ , \quad \sigma_n(\underset{\sim}{u}) \leq 0 \\ \sigma_n(\underset{\sim}{u})(\underset{\sim}{u} \cdot \underset{\sim}{n} - s) = 0 \end{array}\right\} \quad \text{on } \Gamma_C$$

Condition $\sigma_n(\underset{\sim}{u})\ (\underset{\sim}{u} \cdot \underset{\sim}{n} - s) = 0$ signifies that the pressure can only be non-zero when there is contact.

There remains the characterization of friction on the contact surface Γ_C. In agreement with the nonlinear and nonlocal friction laws described in the previous Section, we will have

$$\left.\begin{array}{l} \text{If } \underset{\sim}{u} \cdot \underset{\sim}{n} < s \ , \quad \underset{\sim}{\sigma}_T(\underset{\sim}{u}) = \underset{\sim}{0} \\ \text{If } \underset{\sim}{u} \cdot \underset{\sim}{n} = s \ , \\ \quad \underset{\sim}{\sigma}_T(\underset{\sim}{u}) = -\nu S_\rho(\sigma_n(\underset{\sim}{u}))\ \phi_\varepsilon(|\underset{\sim}{u}_T|)\ \dfrac{\underset{\sim}{u}_T}{|\underset{\sim}{u}_T|} \end{array}\right\} \quad \text{on } \Gamma_C \qquad (5.3.2)$$

where S_ρ is of the form 5.3.11 and ϕ_ε is a nonlinear function of the type in 5.2.13 or 5.2.14. It is important to note that always

$$|\underset{\sim}{\sigma}_T(\underset{\sim}{u})| \leq \nu S_\rho(\sigma_n(\underset{\sim}{u}))$$

since $\phi_\varepsilon(|\underset{\sim}{u}_T|) \leq 1$ for every $\varepsilon > 0$ and any $\underset{\sim}{u}_T(\underset{\sim}{x})$, $\underset{\sim}{x}$ in Γ_C . By computing the inner product of both sides of the second equation (5.3.2) with $\underset{\sim}{u}_T$, we can write, equivalently,

$$\underset{\sim}{\sigma}_T(\underset{\sim}{u}) \cdot \underset{\sim}{u}_T + \nu S_\rho(\sigma_n(\underset{\sim}{u}))\ \phi_\varepsilon(|\underset{\sim}{u}_T|)\ |\underset{\sim}{u}_T| = 0$$

Summarizing, the Signorini problem with nonlocal-nonlinear friction consists of seeking a displacement field $\underset{\sim}{u}$, which satisfies the following system of equations and inequalities:

1) Equilibrium equations

$$(E_{ijk\ell}\, u_{k,\ell})_{,j} + f_i = 0 \quad \text{in} \quad \Omega \tag{5.3.3}$$

2) Boundary conditions

a) Prescribed displacements

$$u_i = 0 \quad \text{on} \quad \Gamma_D \tag{5.3.4}$$

b) Prescribed traction

$$E_{ijk\ell}\, u_{k,\ell}\, n_j = t_i \quad \text{on} \quad \Gamma_F \tag{5.3.5}$$

c) Unilateral constraint

$$\begin{gathered} \underset{\sim}{u} \cdot \underset{\sim}{n} \le s\ , \quad \sigma_n(\underset{\sim}{u}) \le 0\ , \\ \sigma_n(\underset{\sim}{u})(\underset{\sim}{u} \cdot \underset{\sim}{n} - s) = 0 \qquad \text{on} \quad \Gamma_C \end{gathered} \tag{5.3.6}$$

d) Friction condition

$$\underset{\sim}{\sigma}_T(\underset{\sim}{u}) = -\nu S_\rho(\sigma_n(\underset{\sim}{u}))\ \phi_\varepsilon(|\underset{\sim}{u}_T|)\ \frac{\underset{\sim}{u}_T}{|\underset{\sim}{u}_T|} \quad \text{on} \quad \Gamma_C \tag{5.3.7}$$

where

$$S_\rho(\sigma_n(\underset{\sim}{u}))(\underset{\sim}{x}) = \int_{\Gamma_C} \omega_\rho(|\underset{\sim}{x} - \underset{\sim}{y}|)(-\sigma_n(\underset{\sim}{u}(\underset{\sim}{y})))ds$$

and ω_ρ and ϕ_ε are given, respectively, by 5.2.8 and 5.2.14 or 5.2.15.

5.3.2 Variational Principles for Nonlocal Nonlinear Friction. Relationship with the Classical Problems. We now introduce variational principles for the nonlocal-nonlinear Signorini problem and establish the relationship between the variatonal and the classical formulations.

Using the notation introduced previously, we define

V = space of admissible displacements. A displacement vector $\underset{\sim}{v}$ will belong to V if and only if

1) $\underset{\sim}{v} = \underset{\sim}{0}$ on Γ_D

2) $\underset{\sim}{v}$ produces finite (normalized) strain energy in the sense that the norm

$$\|\underset{\sim}{v}\|_V = \{\int_\Omega v_{i,j}\, v_{i,j}\, dx\}^{1/2} \qquad (5.3.8)$$

is finite, where $dx = dx_1 dx_2 \ldots dx_N$.

K = subset of V consisting of all admissible displacements $\underset{\sim}{v}$ in V for which $\underset{\sim}{v} \cdot \underset{\sim}{n} \leq s$ at all points on the contact surface Γ_C.

$a(\underset{\sim}{u},\underset{\sim}{v})$ = virtual work produced by the action of stresses $\sigma_{ij}(u)$ on strains caused by the displacement $\underset{\sim}{v}$

$$= \int_\Omega E_{ijk\ell}\, u_{k,\ell}\, v_{i,j}\, dx$$

$j_{\rho,\varepsilon}(\underset{\sim}{u},\underset{\sim}{v})$ = virtual work done by the frictional forces on the displacement $\underset{\sim}{v}$

$$= \int_{\Gamma_C} \nu S_\rho(\sigma_n(\underset{\sim}{u}))\, \psi_\varepsilon(|\underset{\sim}{v}_T|)ds$$

Here ds is an element of surface area on Γ_C and ψ_ε is the primitive of ϕ_ε : $\phi_\varepsilon = \psi'_\varepsilon$. For example,

$$\psi_\varepsilon(|\underset{\sim}{v}_T|) = \begin{cases} |\underset{\sim}{v}_T| - \varepsilon/2 & \text{if } |\underset{\sim}{v}_T| > \varepsilon \\ |\underset{\sim}{v}_T|^2/2\varepsilon & \text{if } |\underset{\sim}{v}_T| \le \varepsilon \end{cases} \tag{5.3.9}$$

when $\phi_\varepsilon(|\underset{\sim}{v}_T|)$ is given by (5.2.14) or

$$\psi_\varepsilon(|\underset{\sim}{v}_T|) = \varepsilon \ \ell n \cosh \frac{|\underset{\sim}{v}_T|}{\varepsilon} \tag{5.3.10}$$

when $\phi_\varepsilon(|\underset{\sim}{v}_T|)$ is given by (5.2.15).

$f(\underset{\sim}{v})$ = virtual work done by the external forces on the displacement $\underset{\sim}{v}$

$$= \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v} \, dx + \int_{\Gamma_F} \underset{\sim}{t} \cdot \underset{\sim}{v} \, ds$$

With the above definitions and notations now established, we consider the following variational boundary-value problem:

Find an admissible displacement vector $\underset{\sim}{u}$ in the set K such that

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) + j_{\rho,\varepsilon}(\underset{\sim}{u},\underset{\sim}{v}) - j_{\rho,\varepsilon}(\underset{\sim}{u},\underset{\sim}{u}) \geq f(\underset{\sim}{v}-\underset{\sim}{u}) \tag{5.3.11}$$

for all admissible displacements $\underset{\sim}{v}$ in K .

Inequality 5.3.11 is a statement of the principle of virtual work for an elastic body restrained by frictional forces of the type in 5.3.2 Note that this characterizes equilbrium configurations by an inequality rather than an equality because of the presence of the unilateral contact constraint $\underset{\sim}{u} \cdot \underset{\sim}{n} \leq s$ on Γ_C . We also notice that the actual contact surface depends upon the solution $\underset{\sim}{u}$ and is, therefore, not known in advance.

Our first major result is stated in the following proposition, the proof of which is given in the Appendix:

Proposition 3.1: Let $\underset{\sim}{u}$ be a sufficiently smooth solution of the Signorini problem with nonlocal nonlinear friction 5.3.3 - 5.3.7. Then $\underset{\sim}{u}$ is also a solution of the variational inequality 5.3.11. Conversely, if $\underset{\sim}{u}$ is a solution of 5.3.11, then $\underset{\sim}{u}$ also satisfies the system 5.3.3 - 5.3.7 if these relations are interpreted in a weak or distributional sense. □

5.3.3 Other Related Friction Problems. Several friction problems can be obtained as special cases of the nonlocal nonlinear problem discussed in the previous Sections by allowing $\rho \to 0$ or $\varepsilon \to 0$ or both $\rho, \varepsilon \to 0$ or by restricting the dependence of the friction functional to its second variable. Results similar to those established in Proposition 3.1 can also be stated. Moreover, if $\nu S_\rho(\sigma_n(\underset{\sim}{u}))$ is prescribed on Γ_C, and, hence, is independent of the displacement $\underset{\sim}{u}$, the static friction problem thus obtained becomes equivalent to a constrained minimization problem involving a functional representing the associated potential energy. Thus, we have the following cases:

Case I ($\rho = 0$). The friction law, which is now of a local type, is given by 5.2.17. Thus the system 5.3.3 - 5.3.6 together with condition 5.2.17 will produce a Signorini problem with local-nonlinear friction which can be shown to be equivalent to the variational principle 5.3.11 with the functional $j_{\rho,\varepsilon}(\cdot,\cdot)$ replaced by

$$j_\varepsilon(\underset{\sim}{u},\underset{\sim}{v}) = \int_{\Gamma_C} \nu|\sigma_n(\underset{\sim}{u})|\ \psi_\varepsilon(|\underset{\sim}{v}_T|)ds$$

Case II ($\varepsilon = 0$). For this case, the friction law is given by 5.2.10 If we add this condition to the system 5.3.3 - 5.3.6 we will obtain a Signorini problem with nonlocal friction which may be seen to be equivalent to the variational inequality 5.3.11 if we replace $j_{\rho,\varepsilon}(\cdot,\cdot)$ in 5.3.11 by the functional

$$j_\rho(\underset{\sim}{u},\underset{\sim}{v}) = \int_{\Gamma_C} \nu S_\rho(\sigma_n(\underset{\sim}{u}))\ |\underset{\sim}{v}_T|ds$$

The proof of this equivalence differs in some aspects from the one given in the Appendix; it can be found in the unpublished report by ODEN and PIRES [1981].

Case III ($\rho = \varepsilon = 0$). When both $\rho = 0$ and $\varepsilon = 0$, the corresponding friction law is given by the conditions 5.2.18. The system 5.3.3 - 5.3.6 and 5.2.18 will then correspond to the Signorini problem with Coulomb friction. DUVAUT and LIONS [1976] derived a variatonal principle characterizing this problem which is given by 5.3.11 if we replace the functional $j_{\rho,\varepsilon}(\cdot,\cdot)$ by the functional

$$j(\underset{\sim}{u},\underset{\sim}{v}) = \int_{\Gamma_C} \nu\ |\sigma_n(\underset{\sim}{u})|\ |\underset{\sim}{v}_T|ds$$

Finally we mention as a last special case, an auxiliary problem that proves to be useful in the next Section when we establish the conditions for the existence of solutions to the nonlocal-nonlinear friction problem. This auxiliary problem involves a friction law for which the critical or limiting value of the tangential stress is prescribed rather than being determined by the equilibrium displacement field $\underset{\sim}{u}$. We will then have in the nonlinear case ($\varepsilon > 0$) , a law of friction of type 5.2.12

this friction law assumes the form:

$$\left.\begin{array}{ll} |\sigma_T(u)| < \tau & \text{implies } u_T = 0 \\ |\sigma_T(u)| = \tau & \text{implies the existence of } \lambda \geq 0 \\ & \text{such that } u_T = -\lambda\sigma_T \end{array}\right\} \text{ on } \Gamma_C \qquad (5.3.12)$$

Thus, depending on ε being strictly positive or zero, we have

Case IV (τ fixed, $\varepsilon > 0$). The corresponding Signorini problem now consists of 5.3.3 - 5.3.6 together with the friction condition 5.2.12 (τ given on Γ_C). The equivalent variational formulation of this problem is :

Find a displacement field u in K such that

$$a(u,v-u) + j_{0_\varepsilon}(v) - j_{0_\varepsilon}(u) \geq f(v-u) \qquad (5.3.13)$$

for all v in K

where

$$j_{0_\varepsilon} = \int_{\Gamma_C} \tau\, \psi_\varepsilon(|v_T|)ds \qquad (5.3.14)$$

Also, it is not difficult to show that in this case (5.3.13) is equivalent to the constrained minimization problem of finding u in K such that

$$I_\varepsilon(u) \leq I_\varepsilon(v) \qquad (5.3.15)$$

for all v in K, where the energy functional $I_\varepsilon(\cdot)$ is defined by

$$I_\varepsilon(v) = \frac{1}{2} a(v,v) - f(v) + j_{0_\varepsilon}(v) \qquad (5.3.16)$$

Case V (τ fixed, $\varepsilon = 0$). The law of friction is now of the form (5.3.12) which, when added to the system 5.3.3 -5.3.6 , gives the corresponding Signorini problem. We emphasize here that condition 5.3.12 represents a law of friction different from Coulomb's law. It is easy to establish the equivalence between 5.3.3 - 5.3.6 together with 5.3.12 and 5.3.13 if we replace in 5.3.13 $j_{0_\varepsilon}(\cdot)$ by the functional

$$j_0(v) = \int_{\Gamma_C} \tau |\underset{\sim}{v}_T| ds \tag{5.3.17}$$

The energy functional for this case is defined by

$$I(\underset{\sim}{v}) = \frac{1}{2} a(\underset{\sim}{v},\underset{\sim}{v}) - f(\underset{\sim}{v}) + j_0(\underset{\sim}{v}) \ , \qquad \underset{\sim}{v} \text{ in } K \tag{5.3.18}$$

Then problem with $j_{0_\varepsilon}(\cdot)$ replaced by $j_0(\cdot)$ is equivalent to the problem of seeking minimizers of the functional in 5.3.18 which satisfies the unilateral constraints.

Remark 3.1: It is interesting to note that when $\varepsilon = 0$ the functional $j_0(\cdot)$ defined in 5.3.16 is non-differentiable while the nonlinear functional $j_{0_\varepsilon}(\cdot)$ given by 5.3.14 is differentiable on all of V . Its derivative $Dj_{0_\varepsilon}(\cdot)$ at the point $\underset{\sim}{u}$, in the direction $\underset{\sim}{v}$ is given by

$$Dj_{0_\varepsilon}(\underset{\sim}{u}) \cdot \underset{\sim}{v} = \int_{\Gamma_C} \tau \, \phi_\varepsilon(|\underset{\sim}{u}_T|) \frac{\underset{\sim}{u}_T \cdot \underset{\sim}{v}_T}{|\underset{\sim}{u}_T|} ds \tag{5.3.19}$$

It is then possible to show that the variational inequality 5.3.13 is equivalent to the variational inequality

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) + Dj_{0_\varepsilon}(\underset{\sim}{u}) \cdot (\underset{\sim}{v}-\underset{\sim}{u}) \geq f(\underset{\sim}{v}-\underset{\sim}{u}) \ , \text{ for all } \underset{\sim}{v} \text{ in } K \tag{5.3.20}$$

The differentiability of the functional $j_{0_\varepsilon}(\cdot)$ has important implications when we wish to consider finite element approximations of the friction problems discussed so far. In fact, the direct approximation of the variational problem considered in Case V ($\varepsilon = 0$) by finite elements through the minimization of the functional defined in 5.3.18 leads to a discrete system for which the most popular methods for solving nonlinear variational inequalities do not apply, owing to the non-differentiability of $j_0(\cdot)$.□

<u>5.3.4 Estimate of the Difference Between the Solutions of the Friction Problems with $\varepsilon > 0$ (Nonlinear) and $\varepsilon = 0$.</u> We wish to record here an estimate of the difference between the solutions of problem 5.3.11 and of the problem considered in Case II ($\varepsilon = 0$) of the previous Section, as a function of ε . The first result is concerned with the approximation of the functional $j_\rho(\cdot,\cdot)$ by the functional $j_{\rho,\varepsilon}(\cdot,\cdot)$.

<u>Proposition 3.2</u>: For a given element $\underset{\sim}{u}$ in K for which $\sigma_n(\underset{\sim}{u})$ is well defined there exists a constant $c > 0$ independent of ε , such that

$$|j_{\rho,\varepsilon}(\underset{\sim}{u},\underset{\sim}{v}) - j_\rho(\underset{\sim}{u},\underset{\sim}{v})| \leq c\ \varepsilon \tag{5.3.21}$$

for all $\underset{\sim}{v}$ in K . Thus, $j_{\rho,\varepsilon}$ approximates j_ρ arbitrarily closely as $\varepsilon \to 0$.□

This result constitutes the basis for the proof of the following estimate:

Proposition 3.3: Let $\underset{\sim}{u}_\varepsilon$ denote a solution of 5.3.11 for fixed $\varepsilon > 0$ and let $\underset{\sim}{u}$ be a solution of the corresponding variational inequality obtained by setting $\varepsilon = 0$ in 5.3.11. Then, for a sufficiently small coefficient of friction ν, there exists a constant $k > 0$, independent of ε, such that

$$\|\underset{\sim}{u}_\varepsilon - \underset{\sim}{u}\| \le k\sqrt{\varepsilon} \tag{5.3.22}$$

where $\|\cdot\|$ is the norm given in 5.3.8.□

Results similar to 5.3.21 and 5.3.22 can be easily derived for the cases $\rho = 0$ and τ fixed (given) on Γ_C.

5.4 Existence and Uniqueness of Solutions to the Nonlocal Nonlinear Friction Problem

We shall now establish conditions sufficient to guarantee the existence of solutions to the nonlocal nonlinear problem 5.3.11 as well as additional requirements which provide for uniqueness of solutions.

We begin by considering an important preliminary result concerning the auxiliary problem 5.3.13 introduced in the previous Section.

Proposition 4.1: Given $\tau \ge 0$ on Γ_C, τ smooth enough,

(i) there exists a unique solution $\underset{\sim}{u}$ to 5.3.13;

(ii) the correspondence that gives for each τ the corresponding solution $\underset{\sim}{u}$ of 5.3.13 defines a continuous, nonlinear map B, $B(\tau) = \underset{\sim}{u}$;

(iii) the normal contact stress produced by the displacement $\underset{\sim}{u}$,

$\sigma_n(\underset{\sim}{u}) = E_{ijk\ell}\, u_{k,\ell}\, n_i n_j$ is well defined, and is continuous as a function of $\underset{\sim}{u}$. □

We next state our second major result which concerns the existence of solutions to the nonlocal nonlinear friction problem:

Proposition 4.2: For the smoothing operator S_ρ defined in 5.2.11 which transforms the normal contact stresses σ_n into regularized ones, there exists at least one solution $\underset{\sim}{u}$ in K of the nonlocal nonlinear friction problem 5.3.11 for each choice of smooth enough data $\underset{\sim}{f}$ and $\underset{\sim}{t}$. □

One method of proof of this proposition was suggested by DUVAUT [1980]. A complete proof for the nonlocal friction problem ($\varepsilon = 0$) is given in ODEN and PIRES [1981]. Only the general structure of the proof is of interest here since it suggests a means for actually calculating solutions of the Signorini problem. The key steps in the proof are outlined as follows:

1. Pick an arbitrary smooth enough $\tau \geq 0$ on Γ_C .
2. By Proposition 4.1, for each such τ , there exists a unique solution to

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) + \int_{\Gamma_C} \tau\, (\psi_\varepsilon(|\underset{\sim}{v}_T|) - \psi_\varepsilon(|\underset{\sim}{u}_T|))ds \geq f(\underset{\sim}{v}-\underset{\sim}{u}) \tag{5.4.1}$$

   for every $\underset{\sim}{v}$ in K .
3. Let $\underset{\sim}{u} = B(\tau)$, where B is defined in Proposition 4.1.
4. Compute $\sigma_n(\underset{\sim}{u})$.
5. Calculate $\nu S_\rho(\sigma_n(B(\tau)))$ and check if it is equal to τ . If so, we can write

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) + \int_{\Gamma_C} \nu S_\rho(\sigma_n(\underset{\sim}{u}))\ (\psi_\varepsilon(|\underset{\sim}{v}_T|) - \psi_\varepsilon(|\underset{\sim}{u}_T|))ds \geq f(\underset{\sim}{v}-\underset{\sim}{u})$$

for every $\underset{\sim}{v}$ in K. If not, we go back to step 2 and replace in 5.4.1 τ by the new value $\nu S_\rho(\sigma_n(B(\tau)))$. An iterative scheme is therefore obtained by repeating in this way, steps 2 through 5.

6. Obviously, step 5 describes a fixed point problem for the operator

$$T = \nu S_\rho \circ \sigma_n \circ B$$

We must therefore show that there exists at least one element τ^* such that $T(\tau^*) = \tau^*$. Then $\underset{\sim}{u}^* \equiv B(\tau^*)$ will be a solution of the contact problem with nonlocal nonlinear friction.

For small ν, the composition T defined above, becomes a contraction and the fixed point is unique. Hence, we can state the uniqueness result:

Proposition 4.3: If the coefficient of friction is sufficiently small, the nonlocal nonlinear friction problem 5.3.11 possesses a unique solution. □

Future Work. Numerical solutions of the nonlocal-nonlinear friction problems considered in this paper are currently under investigation. These include studies of behavior of the solution for various values of the major parameters; the coefficient of friction ν, the nonlocal contact parameter ρ, and the tangential stiffness of the junctions ε. This work is to be the subject of a forthcoming paper.

APPENDIX

PROOF OF PROPOSITION 1

Let $\underset{\sim}{u}$ be a sufficiently smooth solution of the Signorini problem with nonlocal nonlinear friction (3.3) - (3.7) . Then, the following inequality holds for every admissible displacement $\underset{\sim}{v}$:

$$\underset{\sim}{\sigma}_T(\underset{\sim}{u}) \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T) + \nu S_\rho(\sigma_n(\underset{\sim}{u}))(\psi_\varepsilon(|\underset{\sim}{v}_T|) - \psi_\varepsilon(|\underset{\sim}{u}_T|) \geq 0 \text{ on } \Gamma_C \qquad (A.1)$$

In fact, since the function $\psi_\varepsilon(\cdot)$ is convex and differentiable,

$$\psi'_\varepsilon(|\underset{\sim}{u}_T|) \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T) \leq \psi_\varepsilon(|\underset{\sim}{v}_T|) - \psi_\varepsilon(|\underset{\sim}{u}_T|)$$

or

$$\psi_\varepsilon(|\underset{\sim}{u}_T|) \frac{\underset{\sim}{u}_T}{|\underset{\sim}{u}_T|} \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T) \leq \psi_\varepsilon(|\underset{\sim}{v}_T|) - \psi_\varepsilon(|\underset{\sim}{u}_T|)$$

Hence, since $\underset{\sim}{u}$ is such that (3.7) holds,

$$\underset{\sim}{\sigma}_T(\underset{\sim}{u}) \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T) = -\nu S_\rho(\sigma_n(\underset{\sim}{u}))\ \phi_\varepsilon(|\underset{\sim}{u}_T|) \frac{\underset{\sim}{u}_T}{|\underset{\sim}{u}_T|} \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T)$$

$$\geq -[\ \psi_\varepsilon(|\underset{\sim}{v}_T|) - \psi_\varepsilon(|\underset{\sim}{u}_T|)]$$

or

$$\underset{\sim}{\sigma}_T(\underset{\sim}{u}) \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T) + \psi_\varepsilon(|\underset{\sim}{v}_T|) - \psi_\varepsilon(|\underset{\sim}{u}_T|) \geq 0$$

Since we assume $\underset{\sim}{u}$ to be sufficiently regular (e.g. $\underset{\sim}{u} \in \underset{\sim}{H}^2(\Omega) \cap K^\dagger$),

---

$^\dagger H^2(\Omega) \cap K$ = intersection of the Sobolev space $\underset{\sim}{H}^2(\Omega) = (H^2(\Omega))^N = H^2(\Omega) \times H^2(\Omega) \times \cdots \times H^2(\Omega)$ with the set K. For the definition of the space $H^2(\Omega)$ see ADAMS, R. A., Sobolev Spaces, Academic Press, New York, 1975.

the following Green's formula holds for every $\underset{\sim}{v}$ in K :

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) \equiv \int_{\Omega} \sigma_{ij}(\underset{\sim}{u})(v_{i,j} - u_{i,j})dx$$

$$= -\int_{\Omega} \sigma_{ij}(\underset{\sim}{u})_{,j}(v_i - u_i)dx$$

$$+\int_{\Gamma} \sigma_{ij}(\underset{\sim}{u})n_j(v_i - u_i)ds$$

But $\sigma_{ij}(\underset{\sim}{u})_{,j} = -f_i$ in Ω by (3.3) and

$$\int_{\Gamma} \sigma_{ij}(\underset{\sim}{u})n_j(v_i - u_i)ds = \int_{\Gamma_F} t_i(v_i - u_i)ds +$$

$$+\int_{\Gamma_C} \sigma_{ij}(\underset{\sim}{u})n_j(v_i - u_i)\text{is}$$

for any $\underset{\sim}{v}$ in K since $\underset{\sim}{u} = \underset{\sim}{0}$ on Γ_D by (3.4) and $\sigma_{ij}(\underset{\sim}{u})n_j = t_i$ on Γ_F by (3.5). Therefore

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) - f(\underset{\sim}{v}-\underset{\sim}{u}) = \int_{\Gamma_C} \sigma_{ij}(\underset{\sim}{u})n_j(v_i - u_i)ds$$

$$= \int_{\Gamma_C} [\underset{\sim}{\sigma}_T(\underset{\sim}{u}) \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T) + \sigma_n(\underset{\sim}{u})(v_n - u_n)]ds$$

Here $v_n = \underset{\sim}{v} \cdot \underset{\sim}{n}$ and similarly for u_n . Hence for every $\underset{\sim}{v}$ in K

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) + \int_{\Gamma_C} \nu S_\rho(\sigma_n(\underset{\sim}{u}))(\psi_\varepsilon(|\underset{\sim}{v}_T|) - \psi_\varepsilon(|\underset{\sim}{u}_T|))ds - f(\underset{\sim}{v}-\underset{\sim}{u}) =$$

$$= \int_{\Gamma_C} [\underset{\sim}{\sigma}_T(\underset{\sim}{u}) \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T) + \nu S_\rho(\sigma_n(\underset{\sim}{u}))(\psi_\varepsilon(|\underset{\sim}{v}_T|) -$$

$$-\psi_\varepsilon(|\underset{\sim}{u}_T|)) + \sigma_n(\underset{\sim}{u})(v_n - u_n)]ds$$

But

$$\sigma_n(\underset{\sim}{u})(v_n - u_n) = \sigma_n(\underset{\sim}{u})[v_n - s - (u_n - s)]$$
$$= \sigma_n(\underset{\sim}{u})(v_n - s)$$
$$\geq 0$$

by (3.6) and the definition of the set K . Hence applying inequality (A.1) we finally obtain

$$a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) + j_{\rho,\varepsilon}(\underset{\sim}{u},\underset{\sim}{v}) - j_{\rho,\varepsilon}(\underset{\sim}{u},\underset{\sim}{u}) \geq f(\underset{\sim}{v}-\underset{\sim}{u})$$

for every $\underset{\sim}{v}$ in K , i.e., variational inequality (3.11) and the first part of the proposition is proved.

Conversely, let $\underset{\sim}{u}$ in K be a solution of the variational inequality (3.11). Following the proof given by ODEN and PIRES [1981, p. 20-22] for the case $\varepsilon = 0$ and omitting algebraic manipulations, we are led to conditions (3.3) - (3.6) if we interpret them in the sense of distributions. Then, variational inequality (3.11) will be reduced to

$$\int_{\Gamma_C} [\sigma_n(\underset{\sim}{u}) \cdot (\underset{\sim}{v}_T - \underset{\sim}{u}_T) + \nu S_\rho(\sigma_n(\underset{\sim}{u}))(\psi_\varepsilon(|\underset{\sim}{v}_T|) -$$
$$-\psi_\varepsilon(|\underset{\sim}{u}_T|))]ds \geq 0$$

for every $\underset{\sim}{v}$ in K . If we let $\underset{\sim}{v}_T$ to be of the form

$$\underset{\sim}{v}_T = \underset{\sim}{u}_T + \theta(\underset{\sim}{w}_T - \underset{\sim}{u}_T)$$

where θ belongs to the open interval (0,1) and $\underset{\sim}{w}_T$ is the tangential component on Γ_C of an arbitrary displacement $\underset{\sim}{w}$ in K , we obtain

$$\int_{\Gamma_C} [\theta \sigma_T(u) \cdot (w_T - u_T) + \nu S_\rho(\sigma_n(u))(\psi_\varepsilon(|u_T + \theta(w_T - u_T)|) - \psi_\varepsilon(|u_T|))]ds \geq 0$$

Dividing through by θ, taking the limit as $\theta \to 0$ and noticing that ψ_ε is differentiable gives

$$\int_{\Gamma_C} [\sigma_T(u) \cdot (w_T - u_T) + \nu S_\rho(\sigma_n(u)) \; \phi_\varepsilon(|u_T|) \frac{u_T}{|u_T|} \cdot (w_T - u_T)]ds \geq 0$$

for every w in K. Finally taking first $w_T = -u_T$ and then $w_T = 2u_T$ produces

$$\sigma_T(u) \cdot u_T + \nu S_\rho(\sigma_n(u)) \; \phi_\varepsilon(|u_T|) \; |u_T| = 0 \quad \text{on} \quad \Gamma_C$$

which we have seen in Section 3.1 to be equivalent to the law of friction (3.7). □

6. SUMMARY OF STABILITY RESULTS FOR REDUCED-INTEGRATION-PENALTY METHODS

A discussion of the accuracy and numerical stability of several reduced-integration-penalty methods for the analysis of Stokesian flow in two dimensions is presented. A summary of results on analytical studies of the LBB condition is recorded. Recommendations on which elements provide good accuracy and stability for use in computational fluid dynamics are given.

6.1 Introduction

In this communication, some of the numerical and theoretical results we have obtained over the last several years on reduced-integration-penalty (RIP) methods shall be summarized. Complete proofs and more detailed discussions can be found in references ODEN et al [1980, 1981, 1982].

The basic problem to be considered here is Stokes' problem for steady confined flow of a viscous incompressible fluid, which can be characterized by the following variational boundary-value problem:

Find a velocity field $\underset{\sim}{u} \in V$ and a hydrostatic pressure field $p \in Q$ such that

$$\left.\begin{aligned} a(\underset{\sim}{u},\underset{\sim}{v}) - (p, \operatorname{div} \underset{\sim}{v}) &= f(\underset{\sim}{v}) \\ (q, \operatorname{div} \underset{\sim}{u}) &= 0 \end{aligned}\right\} \quad (6.1.1)$$

for all admissible velocities $\underset{\sim}{v}$ in V and all admissible pressures q in Q

Here,

V = space of admissible velocities

$= \{\underset{\sim}{v} = (v_1, v_2) \mid v_i \in H_0^1(\Omega)$, Ω a regular open bounded domain in $\mathbb{R}^2\}$

$a(\underset{\sim}{u},\underset{\sim}{v})$ = the virtual work, a continuous bilinear form on V

$$= \nu \int_\Omega \text{grad } \underset{\sim}{u} : \text{ grad } \underset{\sim}{v} \, dx$$

$$= \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx; \; dx = dx_1 dx_2 \quad ,$$

ν being the viscosity ($\nu > 0$)

$$(p,q) = \int_\Omega pq dx = L^2(\Omega)\text{-inner product}$$

Q = space of hydrostatic pressures = $L^2(\Omega)$

= space of Lagrange multipliers corresponding with the constraint "div $\underset{\sim}{u} = 0$ in Ω "

$f(\underset{\sim}{v})$ = virtual work of body forces $\underset{\sim}{f} = (f_1, f_2)$

$= \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v} \, dx$, $\underset{\sim}{v}$ an arbitrary "virtual" velocity in V .

We endow V with the energy norm,

$$\| \underset{\sim}{v} \|_V^2 = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx$$

and Q with the usual L^2-norm,

$$\operatorname{div} \underset{\sim}{u} = 0$$

In fact, many ideas of convex optimization surface when we consider the fact that problem (4) is equivalent to the problem of minimizing the functional

$$J : V \to \mathbb{R} \ ; \ J(\underset{\sim}{v}) = \frac{1}{2} a(\underset{\sim}{v},\underset{\sim}{v}) - f(\underset{\sim}{v})$$

on the linear subspace

$$K = \{\underset{\sim}{v} \text{ in } V \mid \operatorname{div} \underset{\sim}{v} = 0 \text{ (in } Q)\}$$

One can <u>relax</u> such constraints by appending to J a convex, differentiable penalty term $P(\underset{\sim}{v}) = (2\varepsilon)^{-1}(\operatorname{div} \underset{\sim}{v}, \operatorname{div} \underset{\sim}{v})$ so as to produce the penalized functional, for $\varepsilon > 0$, given by

$$J_\varepsilon : V \to \mathbb{R} \ ; \ J_\varepsilon(\underset{\sim}{v}) = J(\underset{\sim}{v}) + \frac{1}{2\varepsilon} \|\operatorname{div} \underset{\sim}{v}\|_0^2 \qquad (6.1.5)$$

Minimizers $\underset{\sim}{u}_\varepsilon$ of J_ε are characterized by,

$$\underset{\sim}{u}_\varepsilon \in V : a(\underset{\sim}{u}_\varepsilon,\underset{\sim}{v}) + \varepsilon^{-1}(\operatorname{div} \underset{\sim}{u}_\varepsilon, \operatorname{div} \underset{\sim}{v}) = f(v)$$

$$\text{for all } \underset{\sim}{v} \text{ in } V \qquad (6.1.6)$$

It is informative to note that an alternative way at arriving at the same formulation (6.1.6) is to use the so-called perturbed Lagrangian method. The Lagrangian associated with J and the incompressibility constraint is

$$L : V \times Q \to \mathbb{R} \; ; \; L(\underset{\sim}{v},q) = J(\underset{\sim}{v}) - (q, \operatorname{div} \underset{\sim}{v})$$

The perturbed Lagrangian is defined by

$$L_\varepsilon(\underset{\sim}{v},q) = L(\underset{\sim}{v},q) - \frac{\varepsilon}{2}(q,q)$$

and its saddle points $(\underset{\sim}{u}_\varepsilon, p_\varepsilon)$ are characterized by the system

$$\left.\begin{aligned} a(\underset{\sim}{u}_\varepsilon,\underset{\sim}{v}) - (p_\varepsilon, \operatorname{div} \underset{\sim}{v}) &= f(\underset{\sim}{v}) \quad \text{for all } \underset{\sim}{v} \text{ in } V \\ (\varepsilon p_\varepsilon + \operatorname{div} \underset{\sim}{u}_\varepsilon, q) &= 0 \quad \text{for all } q \text{ in } Q \end{aligned}\right\} \qquad (6.1.7)$$

Then one immediately has

$$p_\varepsilon = -\varepsilon^{-1} \operatorname{div} \underset{\sim}{u}_\varepsilon \quad \text{in } Q \qquad (6.1.8)$$

and, hence, the first member of (6.1.7) reduces to

$$a(\underset{\sim}{u}_\varepsilon,\underset{\sim}{v}) + \varepsilon^{-1}(\operatorname{div} \underset{\sim}{u}_\varepsilon, \operatorname{div} \underset{\sim}{v}) = f(\underset{\sim}{v}) \quad \text{for all } \underset{\sim}{v} \text{ in } V$$

which is precisely (6.1.6). Hence (6.1.7) and (6.1.6) are equivalent formations, but (6.1.8) provides a method for also calculating approximations of the hydrostatic pressure from the penalty approximation of the velocity field.

6.2 Full-Integration Penalty Methods

To construct a finite element approximation of problem (1) [or, equivalently, of (4)], we proceed in the usual fashion by replacing Ω by a mesh Ω_h consisting of a collection of E finite elements.

Continuous piecewise polynomial approximations of the velocities over Ω_h are denoted $\underset{\sim}{v}_h$. If Ω is polygonal, we can usually construct Ω_h so that the approximate velocities $\underset{\sim}{v}_h$ be in a finite-dimensional subspace V^h of the space of admissible velocities V . We generally denote by h the mesh parameter

$$h = \max_{1 \leq e \leq E} h_e \ ; \ h_e = \text{dia}(\Omega_e)$$

where $\bar{\Omega}_e$ is a finite element in $\bar{\Omega}_h$, and by regular refinements of the mesh we generate a family $\{V^h\}_{h>0}$ of subspaces, the union of which is everywhere dense in V .

It is, of course, also possible to introduce a space $Q^h \subset Q$ of approximate hydrostatic pressures q_h defined over the mesh Ω_h , these being piecewise polynomials not necessarily continuous across interelement boundaries. But the spaces Q^h are not generally explicit in a penalty approximation of (6.1.1). We shall show below that, in fact, these spaces are always inherently defined by the manner in which one approximates the penalty functional.

The first method that one uses to approximate (6.1.1) is as follows:

$$\left.\begin{aligned} &\text{For given } \varepsilon > 0 \ , \ \text{ find } \underset{\sim}{u}_h^\varepsilon \in V^h \text{ such that} \\ &a(\underset{\sim}{u}_h^\varepsilon, \underset{\sim}{v}_h) + \varepsilon^{-1}(\text{div } \underset{\sim}{u}_h^\varepsilon, \text{ div } \underset{\sim}{v}_h) = f(\underset{\sim}{v}_h) \\ &\qquad\qquad\qquad \text{for all } \underset{\sim}{v}_h \text{ in } V^h \end{aligned}\right\} \qquad (6.2.1)$$

Most engineers seem to think method (6.2.1) "will not work" and that it leads to a "locked solution" $(u_h^\varepsilon \to 0$ as $\varepsilon \to 0)$, but this is not the case. The problem is that if the terms in (6.2.1) are integrated exactly, then the stability of the method is conditional, depending on the relationship between ε and h . For instance, Falk [1975] and Falk and King [1975] have considered finite element approximations of the type (6.2.1) with a penalty parameter of the form

$$\varepsilon = \gamma_0 h^\sigma$$

where γ_0 and σ are positive constants, with σ to be chosen so that an optimal rate of convergence is obtained. It is interesting to note that no such σ exists and that post processing using an extrapolation technique is needed to achieve the optimal rate. However, without extrapolation, the best choice of σ for their method is $\sigma = \frac{2}{3}(s-1)$, where $s = \min(\beta,k)$ and the body force data $\underset{\sim}{f}$ is in $(H^{\beta-2}(\Omega))^2$, $\beta > 2$, and k is the degree of the complete polynomial approximation of the velocities. The point is that these methods can converge, but only if ε is taken as a special function of h and even then at a suboptimal rate.

The problem with method (6.2.1) is that, from a practical point of view, it is not satisfactory because, for a reasonably fine mesh, ε must be taken so large to produce a non-degenerate solution that the incompressibility constraint is not adequately satisfied. The mesh sizes needed to make this method attractive are so small that the computational effort needed to extract a solution is prohibitive.

6.3 Reduced Integration

To overcome the difficulties mentioned above, it has become common practice to use a numerical quadrature scheme for evaluating the penalty integral $\int_\Omega \operatorname{div} \underset{\sim}{u}_\varepsilon \operatorname{div} \underset{\sim}{v}\, dx$ which is of lower order than that required to integrate this term exactly. Let

$$
\begin{aligned}
I(fg) &= \sum_{e=1}^{E} I_e(fg) \qquad (fg \in C^\circ(\bar{\Omega})) \\
I_e(fg) &= \sum_{j=1}^{G} W_j^e\, f(\underset{\sim}{\xi}_j^e)\, g(\underset{\sim}{\xi}_j^e)
\end{aligned}
\tag{6.3.1}
$$

denote a numerical quadrature rule for integrating the product of· piecewise continuous functions fg over the mesh, where $W_j^e > 0$ are the quadrature weights for element e and $\underset{\sim}{\xi}_j^e$ are the quadrature points in element e. Suppose that a number $\hat{G}$ of such points must be used to integrate the functions $\operatorname{div} \underset{\sim}{u}_h \operatorname{div} \underset{\sim}{v}_h$ $(\underset{\sim}{u}_h, \underset{\sim}{v}_h \in V^h)$ exactly and that $G < \hat{G}$. Then a reduced-integration-penalty approximation of problem (6.1.1) consists of solving, instead of (6.2.1), the following discrete problem:

$$
\left.
\begin{aligned}
&\text{For } \varepsilon > 0, \text{ find } \underset{\sim}{u}_h^\varepsilon \text{ in } V^h \text{ such that} \\
&a(\underset{\sim}{u}_h^\varepsilon, \underset{\sim}{v}_h) + \varepsilon^{-1} I(\operatorname{div} \underset{\sim}{u}_h^\varepsilon \operatorname{div} \underset{\sim}{v}_h) = f(\underset{\sim}{v}_h) \\
&\qquad\qquad \text{for all } \underset{\sim}{v}_h \text{ in } V^h
\end{aligned}
\right\}
\tag{6.3.2}
$$

Corresponding to each choice of V^h and $I(\cdot)$ there is uniquely defined by (6.3.2) a finite-dimensional space $Q^h \subset Q$ of hydrostatic pressures q_h. The unique element p_h in Q^h defined by

$$I((\varepsilon p_h^\varepsilon + \text{div } \underset{\sim}{u}_h^\varepsilon) q_h) = 0 \quad \text{for all } q_h \text{ in } Q_h \tag{6.3.3}$$

or, equivalently,

$$p_h^\varepsilon(\underset{\sim}{\xi}_j^e) = -\varepsilon^{-1} \text{div } \underset{\sim}{u}_h^\varepsilon(\underset{\sim}{\xi}_j^e) \tag{6.3.4}$$

is the corresponding approximation of the pressure p. In fact, (6.3.2) and (6.3.3) correspond to (6.1.7) with $(\cdot,\cdot)$ replaced by $I(\cdot)$. For instance, if V^h consists of piecewise polynomials on rectangular elements and $I(\cdot)$ is 2 x 2 Gaussian quadrature, Q^h then is spanned by piecewise bilinear functions, discontinuous across interelement boundaries, with nodes at the four Gaussian quadrature points. If $I(\cdot)$ is one point integration, Q^h is then spanned by piecewise constraints, etc.

The key issue is whether or not solutions of (6.3.2) exist and, if so, how they behave as h tends to zero. Let Div_h and ∇_h be discrete divergence and gradient operators defined by

$$\left.\begin{aligned} &\text{Div}_h : V^h \to Q^h \; ; \; \nabla_h : Q^{h'} \to V^{h'} \\ &\langle q_h , \text{Div}_h \underset{\sim}{v}_h \rangle = I(q_h \text{ div } \underset{\sim}{v}_h) = -[\nabla_h \underset{\sim}{v}_h , q_h] \end{aligned}\right\} \tag{6.3.5}$$

for all $\underset{\sim}{v}_h$ in V^h and q_h in Q^h. Then, in view of the results listed in Section 1, a unique solution to (6.3.2), (6.3.3) exists whenever (6.1.2) holds for all $\underset{\sim}{u}_h, \underset{\sim}{v}_h$ in V^h (which is always true if (6.1.2) holds and

and $V^h \subset V$) and if, in analogy with (6.1.3), the following discrete Babuska-Brezzi condition holds:

$$\left.\begin{array}{l}\text{There exists a constant } \beta_h > 0 \text{ such that for} \\ \text{all hydrostatic pressures } q_h \text{ in } Q^h , \\ \beta_h \|q_h\|_{Q/\ker \nabla_h} \leq \sup \dfrac{|I(q_h \operatorname{div} \underset{\sim}{v}_h)|}{\|\underset{\sim}{v}_h\|_1}\end{array}\right\} \quad (6.3.6)$$

Here

$$\ker \nabla_h = \{q_h \text{ in } Q^h \mid I(q_h \operatorname{div} \underset{\sim}{v}_h) = 0 \quad \text{for all } \underset{\sim}{v}_h \text{ in } V^h\} \quad (6.3.7)$$

Error estimates can be obtained in certain cases. Suppose that, as $\varepsilon \to 0$, (6.3.2) and (6.3.3) lead to the mixed finite element problem

$$\begin{aligned} a(\underset{\sim}{u}_h, \underset{\sim}{v}_h) - I(p_h \operatorname{div} \underset{\sim}{v}_h) &= f(\underset{\sim}{v}_h) \\ I(q_h \operatorname{div} \underset{\sim}{u}_h) &= 0 \end{aligned} \quad (6.3.8)$$

for arbitrary $\underset{\sim}{v}_h$ and q_h . Let E_I denote a generic integration error defined by

$$E_I = E_I(p_h \operatorname{div} v_h) = (p_h, \operatorname{div} \underset{\sim}{v}_h) - I(p_h \operatorname{div} \underset{\sim}{v}_h) \quad (6.3.9)$$

Setting $\underset{\sim}{v} - \underset{\sim}{v}_h$ in (6.1.1) and substracting (6.3.7) gives

$$a(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{v}_h) - (p - p_h, \text{div } \underset{\sim}{v}_h) - E_I = 0 \qquad (6.3.10)$$

Using (6.1.1) and (6.3.10), we have

$$\alpha \| \underset{\sim}{u} - \underset{\sim}{u}_h \|_V^2 \leq a(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{u} - \underset{\sim}{u}_h)$$

$$= a(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{u} - \underset{\sim}{v}_h) + a(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{v}_h - \underset{\sim}{u}_h)$$

$$\leq M \| \underset{\sim}{u} - \underset{\sim}{u}_h \|_V \| \underset{\sim}{u} - \underset{\sim}{v}_h \|_V + (p - p_h, \text{div}(\underset{\sim}{v}_h - \underset{\sim}{u}_h) + E_I$$

$$\leq \frac{\alpha}{2} \| \underset{\sim}{u} - \underset{\sim}{u}_h \|_V^2 + \frac{M^2}{2\alpha} \| \underset{\sim}{u}_h - \underset{\sim}{v}_h \|_V^2$$

$$+ \| p - p_h \|_Q [\| \underset{\sim}{u} - \underset{\sim}{u}_h \|_V + \| \underset{\sim}{u} - \underset{\sim}{v}_h \|_V] + E_I$$

Thus,

$$\| u - u_h \|_V \leq C\{ \inf_{\underset{\sim}{v}_h \text{ in } V^h} \| \underset{\sim}{u} - \underset{\sim}{v}_h \|_V + \| p - p_h \|_Q \} + E_I \qquad (6.3.11)$$

But

$$\| p - p_h \|_Q \leq \inf_{q_h \text{ in } Q^h} \{ \| p - q_h \|_Q + \| p_h - q_h \|_Q \}$$

$$(6.3.12)$$

$$\leq \inf_{q_h \text{ in } Q^h} \| p - q_h \|_Q + \| p_h - q_h \|_{Q/\ker \Delta_h}$$

$$\|p_h - q_h\|_{Q/\ker \nabla_h} \leq \frac{1}{\beta_h} \sup_{V^h} \frac{|a(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{v}_h) + E_I|}{\|\underset{\sim}{v}_h\|_V}$$

$$\leq \frac{M}{\beta_h} \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_V$$

$$+ \frac{1}{\beta_h} \sup_{V^h} \frac{|E_I\ (p_h, \mathrm{div}\ \underset{\sim}{v}_h)}{\|\underset{\sim}{v}_h\|_V} \qquad (6.3.13)$$

etc. Again we note that the convergence and stability of the method are strongly tied to the constant β_h, the characterization of $\ker \nabla_h$, and the interpolation properties of Q^h and V^h.

6.4 Summary of Some Stability Results

A mathematical analysis of the discrete Babuska-Brezzi condition (6.3.6) has been made by Oden and Kikuchi [1982], Oden, Kikuchi and Song [1981] and Oden, Jacquotte [1982] for several finite elements for a model two-dimensional Stokes' problem on a uniform mesh. We shall summarize these results here which pertain to the behavior of the "LBB-constant" β_h and the stability of the pressure calculations. We use the notations

P_k = space of complete piecewise polynomials of degree k over an element

Q_k = space of tensor products of complete polynomials of degree k

I8 = the eight-node isoparametric element

Results are summarized in Table 1. In this table, figures 1, 2, and 7 "lock" at small values of the penalty parameter ε . This means that for a given mesh size h , ε cannot be taken arbitrarily small, as noted earlier. Of course, for an acceptable ε for reasonable mesh sizes, ε is so large that the constraint of incompressibility is not adequately satisfied. Hence these elements should generally be avoided. Elements 2, 4, 5, 8, 11, and 14 are unstable since $\beta_h = O(h)$. Remarkably, these instabilities frequently are not observed on uniform meshes when the solution is very smooth. Mild irregularities in the solution or small perturbations in the mesh may, however, produce violent oscillations in computed pressures the magnitudes of which increase without bound as h tends to zero. In many cases, however, these oscillations disappear upon "filtering" the pressure solutions (i.e. upon averaging the pressures over one or more elements). In the case of elements 2 and 14 it has been proved mathematically (by N. Kikuchi and the author) that certain filtering schemes will produce a stable and convergent method. However, it is not known if filtering can be used to stabilize and salvage the remaining unstable elements.

Elements 6 and 10 lead to stable and convergent schemes and are quite robust in the sense that they are insensitive to singularities in the solution. However, they are not too accurate and converge at a suboptimal rate.

Elements 5 and 9 are calculated using the perturbed Lagrangian ideas discussed in Section 1: a piecewise linear approximation of the regularized pressure p_ε is computed over each element. Then (6.1.7) leads to a discrete system for each element of the form

$$\begin{aligned} \underset{\sim}{K}\underset{\sim}{u}_\varepsilon - \underset{\sim}{B}^T \underset{\sim}{p}_\varepsilon &= \underset{\sim}{f} + \underset{\sim}{\sigma} \\ \varepsilon \underset{\sim}{M}\underset{\sim}{p} + \underset{\sim}{B}\underset{\sim}{u}_\varepsilon &= \underset{\sim}{0} \end{aligned} \tag{6.4.1}$$

where $\underset{\sim}{K}$ is the element stiffness matrix, $\underset{\sim}{B}$ the non-rectangular constraint matrix, $\underset{\sim}{f}$ the load vector, $\underset{\sim}{\sigma}$ the "connecting" vector (which sums to zero upon connecting elements together to form the mesh), and $\underset{\sim}{M}$ is the Gram matrix corresponding to the linear shape functions for p_ε. Thus,

$$\underset{\sim}{p}_\varepsilon = -\varepsilon^{-1}\underset{\sim}{M}^{-1}\underset{\sim}{B}^T\underset{\sim}{u}_\varepsilon \tag{6.4.2}$$

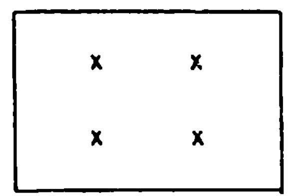
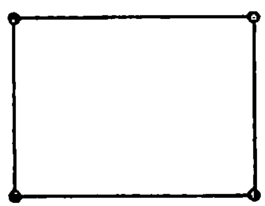
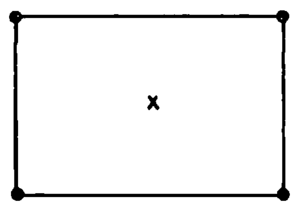
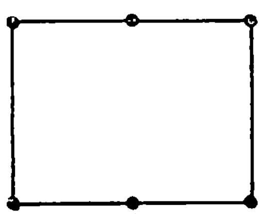
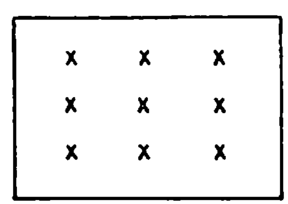
i.e., the fact that p_h^ε is discontinuous across interelement boundaries makes it possible to eliminate the pressure <u>at the element</u> level by (6.4.2). Then th penalty approximation over an element is characterized by

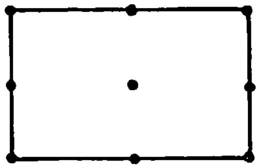
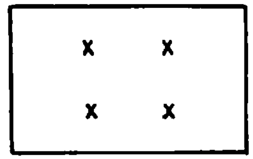
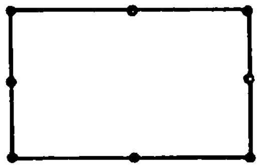
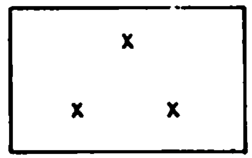
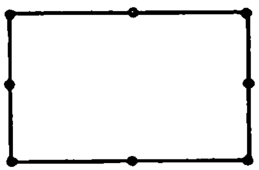
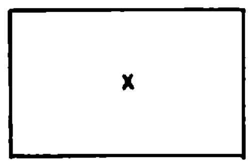
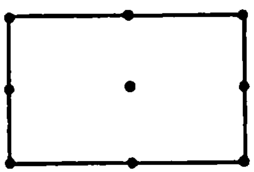
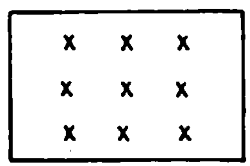
$$(\underset{\sim}{K} + \varepsilon^{-1}\, \underset{\sim}{B}^T\underset{\sim}{M}\, \underset{\sim}{B})\underset{\sim}{u}_\varepsilon = \underset{\sim}{f} + \underset{\sim}{\sigma} \tag{6.4.3}$$

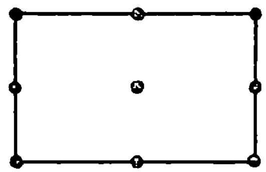
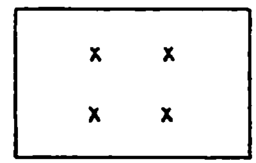
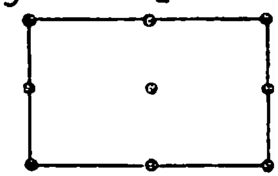
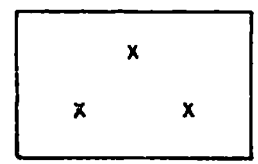
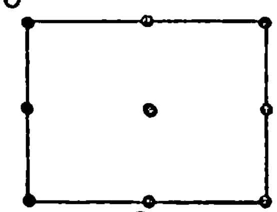
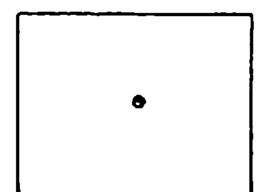
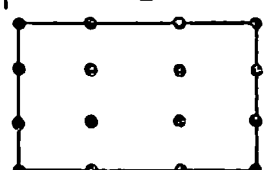
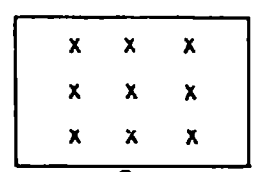
Thus, no reduced integration rule is actually used in constructing elements 5 and 9.

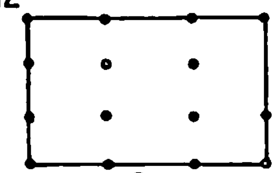
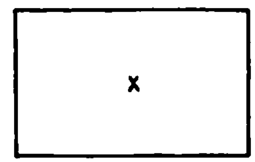
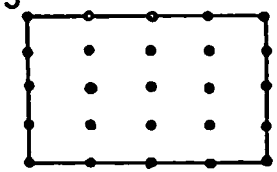
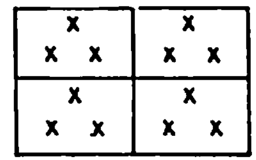
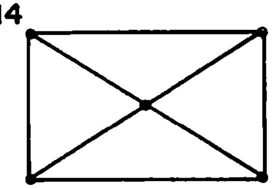
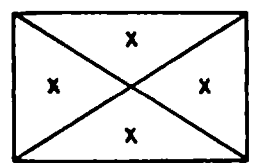
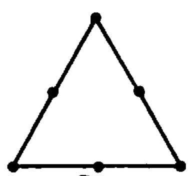
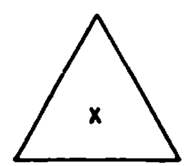
Element 9 is clearly the superior of any listed: it is unconditionally stable, it provides both velocity and pressure approximations which converge at the optimal rate, and

TABLE 6.1

Velocity Approx. V^h	Quadrature Rule (Pressure Approx. Q^h)	β_h	Rate of Convergence
1  Q_1	 Q_2	O(1)	Locks for small ε; ε must be taken as dependent on h.
2  Q_1	 Q_0	O(h)	Unstable Pressure
3  IB	 Q_2	O(1)	Locks for small ε; ε must be taken as dependent on h.

V_h	Q^h	β_h	Rate of Convergence
4  I8	 Q_1	$O(h)$	Unstable Pressure
5  I8	 P_1	$O(h)$	Unstable Pressure
6  I8	 Q_0	$O(1)$	Suboptimal ($O(h)$) in velocity error in energy norm
7  Q_2	 Q_2	$O(1)$	Locks for small ε; ε must be taken as dependent on h

V^h	Q^h	β_h	Rate of Convergence
8  Q_2	 Q_1	O(h)	Unstable Pressure
9  Q_2	 P_1	O(1)	Optimal: $\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_V = O(h^2)$ $\|p - p_h\|_Q = O(h^2)$
10  Q_2	 Q_0	O(1)	Suboptimal (O(h)) in velocity error in energy norm
11  Q_3	 Q_2	O(h)	Unstable Pressure*

	V^h	Q^h	β_h	Rate of Convergence
12	 Q_3	 Q_0	O(1)	Suboptimal (O(h)) in velocity error in energy norm*
13	 Composite $Q_2/I8$	 Composite $4P_1$	O(1)	Optimal
14	 Composite $4P_1$	 Composite $4P_0$	O(h)	Unstable Pressures
	 P_2	 P_0	O(1)	Suboptimal (O(h)) in velocity error in energy norm

$$\ker \nabla_h = \ker \nabla$$

Element 13 is somewhat of a novelty. While element 5 yields unstable pressure approximations, ODEN and JACQUOTTE [4] have shown that a composite of three Q_2/P_1 elements (no. 9) and one $I8/P_1$ element (no. 5) is stable.

The behavior of elements 11 and 12, marked with an asterik, is only conjectured here and has not been rigorously proven.

Extensions of these results to three-dimensional elements are straightforward.

7. SUGGESTED AREAS FOR FUTURE RESEARCH

During the course of this project, two areas have emerged as representing important research subjects needing significant amount of additional study. These are: 1) the study of nonlinear and non-classical friction laws in contact phenomena in solids and structures and 2) a continued study of models and numerical techniques for the analysis of finite elastoplastic deformations of the type characteristic of metal forming processes.

The idea that new descriptions of friction are necessary to adequately describe the phenomena of contact, impact, and wear of deformable bodies will have a significant impact on broad areas of applied mechanics in engineering. It will mean that a variety of new models and results will need to be developed to adequately describe such phenomena as load reversal on contact surfaces, heat generation, abrasion and wear, impact, dynamical friction effects, even fracture initiation and growth. The multitude of this phenomena in which very crude friction models have been used in the past, must ultimately be re-examined in some detail. This will represent a research effort of very large proportion, but should ultimately have a significant pay-off in terms of the liability of mathematical models and numerical techniques for simulating the nonlinear behavior of complex structures.

The reliability of most of the numerical simulators of metal forming processes is very much in doubt. Recent results seem to indicate that most of the popular methods may be marginally stable and produce stress approximations which are very sensitive to perturbations in the mesh material properties. However, if true, this would be a very undesirable situation, since these factors play a fundamental role in the prediction of residual stresses in machine parts and structures. Therefore, a careful mathematical and numerical study of mathematical models and numerical methods for handling these

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classes of problems seems to be in order and to represent an important are for future research.

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