Stability Analysis for Electromagnetic Waveguides. Part 2: Non-Homogeneous Waveguides

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Abstract

This paper is a continuation of Melenk et al., "Stability Analysis for Electromagnetic Waveguides. Part 1: Acoustic and Homogeneous Electromagnetic Waveguides" (2023), extending the stability result for homogeneous electromagnetic (EM) waveguides to the non-homogeneous case. The analysis is done using perturbation techniques for self-adjoint operators eigenproblems. We show that the non-homogeneous EM waveguide problem is well-posed with the stability constant scaling linearly with waveguide length L. The results provide a basis for proving convergence of a DPG discretization based on a full envelope ansatz, and the ultraweak variational formulation for the resulting modified system of Maxwell equations.

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1 Introduction

This is the second part of our work devoted to the stability (and well-posedness) analysis of electromagnetic (EM) waveguides; see [5] for an introduction and the motivation for our work. In Part 1 of this work we considered the homogeneous waveguide only, and we showed that the operator corresponding to the first-order system of Maxwell equations is bounded below with a constant scaled inversely with the length L of the waveguide (L is proportional to the number of wavelengths),

$$\|E\|^2 + \|H\|^2 \le CL \left(\|\boldsymbol{\nabla} \times E - i\omega H\|^2 + \|\boldsymbol{\nabla} \times H + i\omega E\|^2 \right),$$

where $i = \sqrt{-1}$, ω denotes the angular frequency of the light, and C is a positive constant. We use the formalism of closed operators; the electric/magnetic field (E, H) pair comes from the domain of the operator. A simple perturbation argument, given at the end of [5], shows that for a sufficiently small perturbation[1] of the dielectric constant (or relative permittivity) $\epsilon = 1 + \delta\epsilon$, the operator remains bounded below but the linear dependence of the stability constant upon L is lost. In fact, the smallness of perturbation $\delta\epsilon$ is expressed in terms of constant CL; hence, the larger the length L, the smaller $\delta\epsilon$ must be.

[1]We use a non-dimensional version of the equations.

Step-index fibers. In this paper, we extend our stability result to non-homogeneous EM waveguides. This case has importance in modeling a large number of EM waveguide applications, such as optical amplifiers which are used to achieve high-power laser outputs very efficiently [6, 8]. A typical optical fiber model is the double-clad step-index fiber—a cylindrical EM waveguide where the cross-section (or transversal domain) consists of a silica-glass fiber core surrounded by a silica-glass inner cladding and an outer polymer cladding (see Figure 1a). The material refractive index n is slightly higher in the core than the inner cladding which enables propagation of core-guided transverse modes. Consequently, the permittivity $\epsilon = \epsilon(x, y, z)$, which depends on the material refractive index $n = n(x, y, z)$, is discontinuous at the core-cladding interface $\partial\Omega_{\text{core}} := \{(x, y, z) : x^2 + y^2 = r_{\text{core}}^2\}$ of a step-index fiber, as illustrated in Figure 1b. Analogously, the material contrast at the inner-outer cladding interface $\partial\Omega_{\text{clad}} := \{(x, y, z) : x^2 + y^2 = r_{\text{clad}}^2\}$ enables propagation of cladding-guided modes by total internal reflection at the glass-polymer interface.

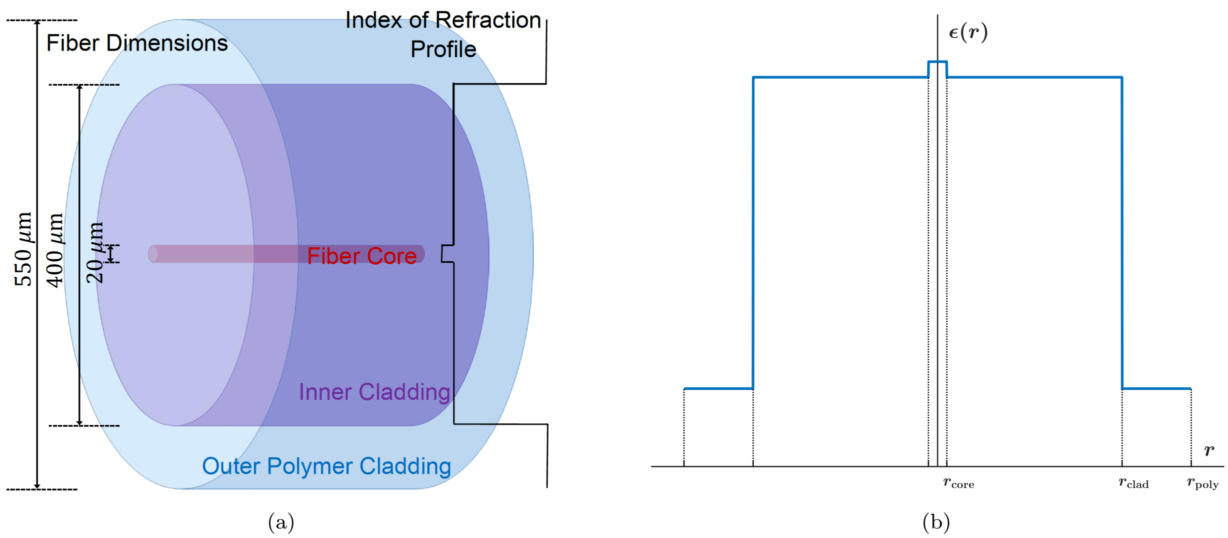

Figure 1: (a) Schematic of a small section of a double-clad step-index fiber, taken from [1]; (b) Transversal profile of the relative permittivity $\epsilon = \epsilon(r)$ in a double-clad step-index fiber.

We note that it is a common assumption in engineering literature to consider ϵ differentiable. Indeed, it is often the case that simplified models of EM waveguide applications (e.g., some beam propagation models) entirely neglect the fact that ϵ is not differentiable. More recently, partly thanks to the increased computing capabilities, it has become possible to numerically solve EM waveguide models of realistic length based directly on the Maxwell equations [3, 2, 4] thereby avoiding such simplifying assumptions. We emphasize that the analysis in this paper considers discontinuous material parameters and is therefore directly applicable to step-index fibers.

Contributions. Extending the stability analysis to the non-homogeneous waveguide problem turns out to be rather non-trivial. We begin by rewriting the Maxwell system in terms of four unknowns: transversal – E_t, H_t, and longitudinal – E_3, H_3 components of electric and magnetic fields. Assuming the exponential ansatz $e^{i\beta z}$ in the (longitudinal) z-direction, we obtain a non-standard eigenvalue problem for propagation constant β. Upon eliminating E_3, H_3, we obtain a more standard system of second-order equations (in x, y) with a non-self-adjoint operator, even for the homogeneous case. Only in the last step, after elimination of H_t (or E_t), we obtain a more standard E-eigenvalue problem for E_t, and the corresponding H-eigenvalue problem for H_t. The operators in the E- and H-eigenvalue problems, for the homogeneous case, turn out to be self-adjoint. This leads to the determination of an orthonormal eigenbasis and corresponding spectral decomposition which, upon the substitution into the original first-order system, decouples the original system into systems of first-order ordinary differential equations (ODEs). A stability analysis for the ODEs and the spectral decomposition argument led to the final result in [5].

In the non-homogeneous case, the operators in the E- and H-eigenvalue problems are not self-adjoint but they represent perturbations of self-adjoint operators. This invites the application of the classical[2] perturbation analysis for self-adjoint operators [7] that we pursue in this paper. The arguments are far from trivial, as we lose the convenient orthonormal basis argument and have to resort to series of non-orthonormal (perturbed) eigenvectors. The decoupling argument then involves adjoint operators which need to be analyzed as well. As always with the perturbation argument, the obtained results are formal, we proceed under the assumption that the non-orthogonal series converge as needed.

Outline. The structure of the paper is as follows. We begin in Section 2 with the derivation of the various eigenvalue problems and relations between them. In Section 3, we develop the classical perturbation argument to compute the perturbed E and H eigenvectors and their counterparts for the adjoint problems. In Section 4, we arrive at our first main result; we reduce the problem to a system of decoupled systems of small subsystems of two ODEs for the coefficients in the spectral representations of E_t and H_t. Upon further reduction to a single second-order ODE, we arrive at essentially the same ODE problem as in the analysis of the homogeneous waveguide. This leads to the final estimates of E_t, H_t (and their curls) in terms of the right-hand side and to our final result presented in Section 5. We finish with short conclusions in Section 6. Finally, Appendix A provides additional algebraic results from the perturbation analysis. In the main body of the paper, we proceed under the customary, simplifying assumption that all perturbed eigenvalues are distinct. In Appendix A though, we provide additional details for the case of multiple eigenvalues as it is in the case of the step-index fiber.

In the end, our main stability result is identical with the one for the homogeneous waveguide, we show the scaling of the stability constant with length L. The (formal) perturbation analysis necessitates the assumption of a small perturbation but only in the L^∞-norm. Nowhere in our analysis do we require the dielectric constant to be differentiable, a common assumption in the engineering literature. The presented analysis thus applies to step-index fibers.

[2] Dating back to Lord Rayleigh.

2 Eigensystems

We will be using the following 2D identities:

$$\begin{aligned} e_z \times (e_z \times E_t) &= -E_t \\ e_z \times (\boldsymbol{\nabla} \times E_3) &= \boldsymbol{\nabla} E_3\,, & e_z \times \boldsymbol{\nabla} E_3 &= -\boldsymbol{\nabla} \times E_3 \\ \operatorname{curl}(e_z \times E_t) &= \operatorname{div} E_t\,, & \operatorname{div}(e_z \times E_t) &= -\operatorname{curl} E_t \end{aligned} \tag{2.1}$$

where E_t, E_3 denote the transversal and longitudinal components of a 3D vector field E. The original system of equations,

$$\boldsymbol{\nabla} \times E - i\omega H = f\,, \qquad \boldsymbol{\nabla} \times H + i\omega\epsilon E = g\,,$$

translates into:

$$\left\{ \begin{aligned} \boldsymbol{\nabla} \times E_3 + e_z \times \tfrac{\partial}{\partial z} E_t - i\omega H_t &= f_t \\ \operatorname{curl} E_t - i\omega H_3 &= f_3 \\ \boldsymbol{\nabla} \times H_3 + e_z \times \tfrac{\partial}{\partial z} H_t + i\omega\epsilon E_t &= g_t \\ \operatorname{curl} H_t + i\omega\epsilon E_3 &= g_3\,. \end{aligned} \right. \tag{2.2}$$

Multiplying the first and third equations by $i\omega\, e_z\times$, we obtain:

$$\left\{ \begin{aligned} \boldsymbol{\nabla} i\omega E_3 - \tfrac{\partial}{\partial z} i\omega E_t + \omega^2\, e_z \times H_t &= i\omega\, e_z \times f_t \\ \operatorname{curl} E_t - i\omega H_3 &= f_3 \\ \boldsymbol{\nabla} i\omega H_3 - \tfrac{\partial}{\partial z} i\omega H_t - \omega^2\, e_z \times \epsilon E_t &= i\omega\, e_z \times g_t \\ \operatorname{curl} H_t + i\omega\epsilon E_3 &= g_3\,. \end{aligned} \right. \tag{2.3}$$

The eigensystem corresponding to the first-order system operator, and $e^{i\beta z}$ ansatz in z, looks as follows:

$$\left\{ \begin{aligned} & E_t \in H_0(\operatorname{curl}, D),\ E_3 \in H_0^1(D) \\ & H_t \in H(\operatorname{curl}, D),\ H_3 \in H^1(D) \\ i\omega \boldsymbol{\nabla} E_3 + \omega^2 e_z \times H_t &= -\omega\beta E_t \\ \operatorname{curl} E_t - i\omega H_3 &= 0 \\ i\omega \boldsymbol{\nabla} H_3 - \omega^2 e_z \times \epsilon E_t &= -\omega\beta H_t \\ \operatorname{curl} H_t + i\omega\epsilon E_3 &= 0\,. \end{aligned} \right. \tag{2.4}$$

And the system corresponding to the adjoint is as follows:

$$\left\{ \begin{aligned} & F_t \in H(\operatorname{div}, D),\ F_3 \in H^1(D) \\ & G_t \in H_0(\operatorname{div}, D),\ G_3 \in H_0^1(D) \\ \boldsymbol{\nabla} \times F_3 + \omega^2 e_z \times \epsilon G_t &= -\omega\beta F_t \\ i\omega(\operatorname{div} F_t - \epsilon G_3) &= 0 \\ \boldsymbol{\nabla} \times G_3 - \omega^2 e_z \times F_t &= -\omega\beta G_t \\ i\omega(\operatorname{div} G_t + F_3) &= 0\,. \end{aligned} \right. \tag{2.5}$$

Eliminating E_3 and H_3 from system (2.4), we obtain a simplified but second-order system for E_t, H_t only.

$$\begin{cases} E_t \in H_0(\mathrm{curl}, D),\ \mathrm{curl}\, E_t \in H^1(D) \\ H_t \in H(\mathrm{curl}, D),\ \frac{1}{\epsilon}\,\mathrm{curl}\, H_t \in H_0^1(D) \\ \qquad -\boldsymbol{\nabla}(\frac{1}{\epsilon}\,\mathrm{curl}\, H_t) + \omega^2 e_z \times H_t \quad = -\omega\beta E_t \\ \qquad \boldsymbol{\nabla}(\mathrm{curl}\, E_t) - \omega^2 e_z \times \epsilon E_t \quad = -\omega\beta H_t\,. \end{cases} \tag{2.6}$$

Similarly, eliminating F_3 and G_3 from system (2.5), we obtain a simplified but second-order system for F_t, G_t only.

$$\begin{cases} F_t \in H(\mathrm{div}, D),\ \frac{1}{\epsilon}\,\mathrm{div}\, F_t \in H_0^1(D) \\ G_t \in H_0(\mathrm{div}, D),\ \mathrm{div}\, G_t \in H^1(D) \\ \qquad -\boldsymbol{\nabla} \times \mathrm{div}\, G_t + \omega^2 e_z \times \epsilon G_t \quad = -\omega\beta F_t \\ \qquad \boldsymbol{\nabla} \times (\frac{1}{\epsilon}\,\mathrm{div}\, F_t) - \omega^2 e_z \times F_t \quad = -\omega\beta G_t\,. \end{cases} \tag{2.7}$$

One can check that the operator in (2.7) corresponds to the adjoint of operator in (2.6). Notice how the boundary conditions (BCs) on E_3, G_3 have been inherited by $\mathrm{curl}\, H_t$ and $\mathrm{div}\, F_t$.

Reduction to single variable eigensystems. Assume $\beta \neq 0$. Solving $(2.6)_2$ for H_t,

$$\begin{aligned} H_t &= -\frac{1}{\omega\beta}[\boldsymbol{\nabla}\,\mathrm{curl}\, E_t - \omega^2 e_z \times \epsilon E_t] \\ \mathrm{curl}\, H_t &= \frac{\omega}{\beta}\,\mathrm{curl}(e_z \times \epsilon E_t) = \frac{\omega}{\beta}\,\mathrm{div}\, \epsilon E_t \end{aligned} \tag{2.8}$$

and substituting it into $(2.6)_1$, we obtain an eigenvalue problem for E_t alone.

$$\begin{cases} E_t \in H_0(\mathrm{curl}, D),\ \mathrm{curl}\, E_t \in H^1(D),\ \frac{1}{\epsilon}\,\mathrm{div}\, \epsilon E_t \in H_0^1(D) \\ \boldsymbol{\nabla} \times \mathrm{curl}\, E_t - \omega^2 \epsilon E_t - \boldsymbol{\nabla}(\frac{1}{\epsilon}\,\mathrm{div}\, \epsilon E_t) = -\beta^2 E_t\,. \end{cases} \tag{2.9}$$

Similarly, Solving $(2.6)_1$ for E_t,

$$\begin{aligned} E_t &= -\frac{1}{\omega\beta}[-\boldsymbol{\nabla}(\frac{1}{\epsilon}\,\mathrm{curl}\, H_t) + \omega^2 e_z \times H_t] \\ \mathrm{curl}\, E_t &= -\frac{\omega}{\beta}\,\mathrm{curl}(e_z \times H_t) = -\frac{\omega}{\beta}\,\mathrm{div}\, H_t \end{aligned} \tag{2.10}$$

and substituting it into $(2.6)_2$, we obtain an eigenvalue problem for H_t alone.

$$\begin{cases} H_t \in H(\mathrm{curl}, D) \cap H_0(\mathrm{div}, D),\ \frac{1}{\epsilon}\,\mathrm{curl}\, H_t \in H_0^1(D),\ \mathrm{div}\, H_t \in H^1(D) \\ \epsilon \boldsymbol{\nabla} \times (\frac{1}{\epsilon}\,\mathrm{curl}\, H_t) - \omega^2 \epsilon H_t - \boldsymbol{\nabla}(\mathrm{div}\, H_t) = -\beta^2 H_t\,. \end{cases} \tag{2.11}$$

Note that BC $n \times E_t = 0$ implies BC: $n \cdot H_t = 0$. We proceed in the same way with the adjoint. Solving $(2.7)_2$ for G_t,

$$\begin{aligned} G_t &= -\frac{1}{\omega\beta}[\boldsymbol{\nabla} \times (\frac{1}{\epsilon}\,\mathrm{div}\, F_t) - \omega^2 e_z \times F_t] \\ \mathrm{div}\, G_t &= \frac{\omega}{\beta}\,\mathrm{div}(e_z \times F_t) = -\frac{\omega}{\beta}\,\mathrm{curl}\, \epsilon F_t \end{aligned} \tag{2.12}$$

and substituting it into $(2.7)_1$, we obtain an eigenvalue problem for F_t alone.

$$\left\{ \begin{array}{l} F_t \in H_0(\mathrm{curl}, D) \cap H(\mathrm{div}, D),\ \frac{1}{\epsilon}\,\mathrm{div}\, F_t \in H_0^1(D),\ \mathrm{curl}\, F_t \in H^1(D) \\ \boldsymbol{\nabla} \times \mathrm{curl}\, F_t - \omega^2 \epsilon F_t - \epsilon \boldsymbol{\nabla}(\frac{1}{\epsilon}\,\mathrm{div}\, F_t) = -\gamma^2 F_t\,. \end{array} \right. \tag{2.13}$$

Note that BC: $n \cdot G_t = 0$ implies BC: $n \times F_t = 0$. Similarly, Solving $(2.7)_1$ for F_t,

$$\begin{aligned} F_t &= -\frac{1}{\omega\beta}[-\boldsymbol{\nabla} \times \mathrm{div}\, G_t) + \omega^2 e_z \times \epsilon G_t] \\ \mathrm{div}\, F_t &= -\frac{\omega}{\beta}\,\mathrm{div}(e_z \times \epsilon G_t) = -\frac{\omega}{\beta}\,\mathrm{curl}\, \epsilon G_t \end{aligned} \tag{2.14}$$

and substituting it into $(2.7)_2$, we obtain an eigenvalue problem for G_t alone.

$$\left\{ \begin{array}{l} G_t \in H_0(\mathrm{div}, D),\ \mathrm{div}\, G_t \in H^1(D),\ \frac{1}{\epsilon}\,\mathrm{curl}\, \epsilon G_t \in H_0^1(D) \\ \boldsymbol{\nabla} \times (\frac{1}{\epsilon}\,\mathrm{curl}\, \epsilon G_t) - \omega^2 \epsilon G_t - \boldsymbol{\nabla}(\mathrm{div}\, G_t) = -\gamma^2 G_t\,. \end{array} \right. \tag{2.15}$$

Lemma 1

(a) *Let $((E_t, H_t), -\omega\beta)$ be an eigenpair for system (2.6). Then $(E_t, -\beta^2)$ solves (2.9), and $(H_t, -\beta^2)$ solves (2.11).*

(b) *Conversely, if $(E_t, -\beta^2)$ is an eigenpair for (2.9), and we define H_t by:*

$$H_t = \frac{1}{\omega(\pm\beta)}\left(-\boldsymbol{\nabla}\,\mathrm{curl}\, E_t + \omega^2 e_z \times \epsilon E_t\right)$$

then $(E_t, H_t), -\omega(\pm\beta))$ is an eigenpair for system (2.6). Each eigenpair for (2.9) generates two eigenpairs for (2.6).

(c) *Similarly, if $(H_t, -\beta^2)$ is an eigenpair for (2.11), and we define E_t by:*

$$E_t = \frac{1}{\omega(\pm\beta)}\left(\boldsymbol{\nabla}(\frac{1}{\epsilon}\,\mathrm{curl}\, H_t) - \omega^2 e_z \times H_t\right)$$

then $(E_t, H_t), -\omega(\pm\beta))$ is an eigenpair for system (2.6). Each eigenpair for (2.11) generates two eigenpairs for (2.6).

$\blacksquare$

Proof: We have already proved (a). To prove (b), check that the formula for H_t and (2.9) imply (just algebra) equation $(2.6)_1$. Same procedure to prove (c). $\blacksquare$

In particular, Lemma 1 implies that eigenproblems (2.9) and (2.11) have the same eigenvalues β^2.

Lemma 2

(a) Let $((F_t, G_t), \omega\gamma)$ be an eigenpair for system (2.7). Then $(G_t, -\gamma^2)$ solves (2.15) and $(F_t, -\gamma^2)$ solves (2.13).

(b) Conversely, if $(F_t, -\gamma^2)$ is an eigenpair for (2.13), and we define G_t by:

$$G_t = \frac{1}{\omega(\pm\gamma)} \left(-\nabla \times (\frac{1}{\epsilon} \operatorname{div} F_t) + \omega^2 e_z \times F_t \right)$$

then $(F_t, G_t), \omega(\pm\gamma))$ is an eigenpair for system (2.7). Each eigenpair for (2.13) generates two eigenpairs for (2.7).

(c) Similarly, if $(G_t, -\gamma^2)$ is an eigenpair for (2.15), and we define F_t by:

$$F_t = \frac{1}{\omega(\pm\gamma)} \left(\nabla \times \operatorname{div} G_t - \omega^2 e_z \times \epsilon G_t \right)$$

then $(F_t, G_t), \omega(\pm\gamma))$ is an eigenpair for system (2.7). Each eigenpair for (2.15) generates two eigenpairs for (2.7).

$\blacksquare$

In particular, Lemma 2 implies that eigenproblems (2.13) and (2.15) have the same eigenvalues γ^2.

Lemma 3
$(E_t, -\beta^2)$ is an eigenpair for problem (2.9) if an only if $(e_z \times E_t, -\beta^2)$ is an eigenpair for (2.15). Similarly, $(H_t, -\beta^2)$ is an eigenpair for problem (2.11) if and only if $(e_z \times H_t, -\beta^2)$ is an eigenpair for (2.13). In particular, this implies that all four individual eigenproblems share the same eigenvalues. $\blacksquare$

Proof: Use identities (2.1). $\blacksquare$

3 A Perturbation Analysis

In this section, we will use the classical perturbation theory for self-adjoint operators to analyze two eigenvalue problems:

- the electric eigenvalue problem (2.9):

$$\nabla \times \operatorname{curl} E_t - \omega^2 \epsilon E_t - \nabla(\frac{1}{\epsilon} \operatorname{div} \epsilon E_t) \quad = -\beta^2 E_t \quad (E \text{ problem}) \tag{3.1}$$

- and the magnetic problem (2.11):

$$\epsilon \nabla \times (\frac{1}{\epsilon} \operatorname{curl} H_t) - \omega^2 \epsilon H_t - \nabla(\operatorname{div} H_t) \quad = -\gamma^2 H_t \quad (H \text{ problem})\,. \tag{3.2}$$

We have already argued that the problems share the same eigenvalues. Problem (3.1) is a perturbation of a self-adjoint eigenvalue problem for the electric field representing the homogeneous waveguide,

$$\underbrace{\boldsymbol{\nabla} \times \operatorname{curl} E - \omega^2 E - \boldsymbol{\nabla}(\operatorname{div} E)}_{=:AE} = -\beta^2 E \tag{3.3}$$

where $E = E_t$. We have learned in [5] that the problem admits two families of eigenvectors:

$$\begin{aligned} E_i &= \boldsymbol{\nabla} \times \psi_i & \beta_i^2 &= \omega^2 - \mu_i \\ E_j &= \boldsymbol{\nabla} \phi_j & \beta_j^2 &= \omega^2 - \lambda_j \end{aligned} \tag{3.4}$$

where (μ_i, ψ_i) and (λ_j, ϕ_j) are Neumann and Dirichlet eigenpairs for the Laplace operator. We will consistently use indices i and j to denote the two families. Problem (3.2) is a perturbation of a self-adjoint eigenvalue problem for the magnetic field representing the homogeneous waveguide,

$$\underbrace{\boldsymbol{\nabla} \times \operatorname{curl} H - \omega^2 H - \boldsymbol{\nabla}(\operatorname{div} H)}_{=:BE} = -\gamma^2 H \tag{3.5}$$

where $H = H_t = G_t$. The problem admits two families of eigenvectors:

$$\begin{aligned} H_i &= \boldsymbol{\nabla} \psi_i & \beta_i^2 &= \omega^2 - \mu_i \\ H_j &= \boldsymbol{\nabla} \times \phi_j & \beta_j^2 &= \omega^2 - \lambda_j \end{aligned} \tag{3.6}$$

where (μ_i, ψ_i) and (λ_j, ϕ_j) denote again the Neumann and Dirichlet eigenpairs for the Laplace operator. We will consistently use indices i and j to denote the two families as well. The two unperturbed problems look the same but they differ with the boundary conditions. The corresponding perturbed eigenpairs are:

$$(-\beta^2 - \delta\beta^2, E + \delta E), \quad (-\gamma^2 - \delta\gamma^2, H + \delta H)\,.$$

3.1 Perturbation Analysis for the E Eigenvalue Problem

We will present now in detail the analysis for the first perturbed problem. Operator A is self-adjoint in $L^2(D)$, so the eigenvalues are real and the eigenvectors form an L^2-orthonormal basis. Consider now a perturbation,

$$\epsilon := 1 + \delta\epsilon, \qquad E := E + \delta E, \quad \beta^2 := \beta^2 + \delta\beta^2\,.$$

Plugging the perturbations into (3.1) and linearizing, we obtain the corresponding linearized problem:

$$A(\delta E_t) + \beta^2 \delta E_t \ = \omega^2 \delta\epsilon E \ \ -\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} E) \ \ +\boldsymbol{\nabla} \operatorname{div}(\delta\epsilon E) \ \ -\delta\beta^2 E \quad (\delta E \text{ problem}) \tag{3.7}$$

Consider now problem (3.3) and (3.7) for a specific eigenpair $(-\beta_i^2, E_i)$. Representing the perturbation in eigenbasis E_j, we have:

$$\begin{aligned} \delta E_i &= \sum_j (\delta E_i, E_j) E_j \\ A(\delta E_i) &= \sum_j (\delta E_i, E_j)(-\beta_j^2) E_j \\ (A(\delta E_i), E_k) &= \sum_j (-\beta_j^2)(\delta E_i, E_j) \underbrace{(E_j, E_k)}_{=\delta_{jk}} = (-\beta_k^2)(\delta E_i, E_k)\,. \end{aligned}$$

Taking the L^2-product of (3.7) with E_k, we obtain:

$$(-\beta_k^2)(\delta E_i, E_k) = -\beta_i^2(\delta E_i, E_k) - \delta\beta_i^2 \underbrace{(E_i, E_k)}_{=\delta_{ik}} + \omega^2(\delta\epsilon E_i, E_k) - (\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} E_i), E_k) + (\boldsymbol{\nabla} \operatorname{div}(\delta\epsilon E_i), E_k)\,,$$

or,

$$(\beta_i^2 - \beta_k^2)(\delta E_i, E_k) + \delta\beta_i^2 \delta_{ik} = \omega^2(\delta\epsilon E_i, E_k) - (\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} E_i), E_k) + (\boldsymbol{\nabla} \operatorname{div}(\delta\epsilon E_i), E_k)\,. \tag{3.8}$$

Assumption A: We assume now that the eigenvalues are *distinct* (simple). This is a customary assumption in the perturbation argument to simplify the presentation. The case of multiple eigenvalues is more complicated and it is discussed in Appendix A.

Under the assumption of distinct (simple) eigenvalues, for $k = i$, we get a formula for perturbation $\delta\beta_i^2$,

$$\delta\beta_i^2 = \omega^2(\delta\epsilon E_i, E_i) + (\delta\epsilon \operatorname{div} E_i, \operatorname{div} E_i) - (\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_i)\,. \tag{3.9}$$

For $k \neq i$, formula (3.8) allows to compute perturbation δE_i; the i-th component of δE_i comes from a normalization argument.

Assumption B: We assume:

$$(\delta E_i, E_i) = 0\,. \tag{3.10}$$

The assumption implies that the perturbed eigenvector $E_i + \delta E_i$ is (approximately) of length one, see the discussion in Section 4.

We have:

$$(\beta_i^2 - \beta_k^2)(\delta E_i, E_k) = \omega^2(\delta\epsilon E_i, E_k) + (\delta\epsilon \operatorname{div} E_i, \operatorname{div} E_k) - (\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_k)\,.$$

Linearized mass matrices. We shall now compute linearized mass matrices for the E-eigenproblem, and the two families of eigenvectors. Table 1 presents results for the $(\delta E_i, E_j)$ term.

Table 1: Mass term $(\delta E, E)$ for different families of eigenvectors.

$(\delta E, E)$	$E_k = \boldsymbol{\nabla} \times \psi_k$	$E_l = \boldsymbol{\nabla}\phi_l$
$\delta E_i = \delta(\boldsymbol{\nabla} \times \psi_i)$	$\dfrac{\omega^2(\delta\epsilon E_i, E_k)}{\mu_k - \mu_i}$	$\dfrac{(\omega^2 - \lambda_l)(\delta\epsilon E_i, E_l)}{\lambda_l - \mu_i}$
$\delta E_j = \delta(\boldsymbol{\nabla}\phi_j)$	$\dfrac{\omega^2(\delta\epsilon E_j, E_k)}{\mu_k - \lambda_j}$	$\dfrac{(\omega^2 - \lambda_l)(\delta\epsilon E_j, E_l) + \lambda_j\lambda_l(\delta\epsilon\phi_j, \phi_l)}{\lambda_l - \lambda_j}$

We can now compute the linearized mass matrix:

$$(\delta E_i, E_j) + (E_i, \delta E_j) = (\delta E_i, E_j) + \overline{(\delta E_j, E_i)}\,.$$

The second term is obtained by swapping indices in Table 1, and changing the order of arguments in the L^2-inner products to account for conjugation. For instance, for the first term,

$$\frac{\omega^2(\delta\epsilon E_i, E_k)}{\mu_k - \mu_i} \quad \rightarrow \quad \frac{\omega^2(\delta\epsilon E_k, E_i)}{\mu_i - \mu_k} \quad \rightarrow \quad \frac{\omega^2(\delta\epsilon E_i, E_k)}{\mu_i - \mu_k}\,.$$

Table 2 presents selected (those that we will need) elements of the linearized mass matrix.

Table 2: Linearized mass matrix $(\delta E_i, E_k) + (E_i, \delta E_k)$ for different families of eigenvectors.

$(\delta E, E) + (E, \delta E)$	$\delta E_k = \delta(\boldsymbol{\nabla} \times \psi_k)$	$\delta E_l = \delta(\boldsymbol{\nabla}\phi_l)$
$\delta E_i = \delta(\boldsymbol{\nabla} \times \psi_i)$	0	
$\delta E_j = \delta(\boldsymbol{\nabla}\phi_j)$		$-(\delta\epsilon E_j, E_l)$

Curl-curl coupling. Let $E_i + \delta E_i$ be the perturbed eigenvectors for the electric eigenproblem. We will now investigate the linearized curl-curl mass matrix:

$$(\operatorname{curl}\delta E_i, \operatorname{curl} E_j) + (\operatorname{curl} E_i, \operatorname{curl}\delta E_j)\,.$$

We have:

$$\begin{aligned} \delta E_i &= \sum_k (\delta E_i, E_k) E_k && \text{(summation over both curls and grads)} \\ \operatorname{curl}\delta E_i &= \sum_k (\delta E_i, \boldsymbol{\nabla} \times \psi_k)\mu_k \psi_k && \text{(summation over curls only.)} \end{aligned}$$

Hence,

$$(\operatorname{curl}\delta E_i, \operatorname{curl} E_j) = \Big(\sum_k (\delta E_i, \boldsymbol{\nabla}\times\psi_k)\mu_k\psi_k, \operatorname{curl} E_j\Big) = \sum_k (\delta E_i, \boldsymbol{\nabla}\times\psi_k)(\mu_k\psi_k, \mu_j\psi_j) = (\delta E_i, \boldsymbol{\nabla}\times\psi_j)\mu_j$$

is non-zero only if E_j is a curl, $E_j = \boldsymbol{\nabla} \times \psi_j$.

Consequently, the linearized curl-curl mass matrix is equal to:

$$(\delta E_i, E_j)\mu_j + (E_i, \delta E_j)\mu_i = \mu_j \frac{\omega^2(\delta\epsilon E_i, E_j)}{\mu_j - \mu_i} + \mu_i \frac{\omega^2(\delta\epsilon E_i, E_j)}{\mu_i - \mu_j} = \omega^2(\delta\epsilon E_i, E_j)$$

for $E_i = \boldsymbol{\nabla} \times \psi_i, E_j = \boldsymbol{\nabla} \times \psi_j$.

3.2 Perturbation Analysis for the $\boldsymbol{H}$ Eigenvalue Problem

The linearized problem is:

$$B\delta H_t + \gamma^2 \delta H_t = -\delta\epsilon\, \boldsymbol{\nabla} \times \operatorname{curl} H_t - \boldsymbol{\nabla} \times (\delta\epsilon \operatorname{curl} H_t) + \omega^2 \delta\epsilon\, H_t - \delta\gamma^2\, H_t$$

where operator B is formally the same as operator A for the E problem (BCs are different). Performing the same analysis as for the E problem, we get:

$$(\gamma_i^2 - \gamma_k^2)(\delta H_i, H_k) + \delta\gamma_i^2 \delta_{ik} = \omega^2(\delta\epsilon H_i, H_k) - (\delta\epsilon \boldsymbol{\nabla} \times \operatorname{curl} H_i, H_k) + (\boldsymbol{\nabla} \times (\delta\epsilon \operatorname{curl} H_i), H_k)\,. \quad (3.11)$$

Under **Assumption A** of distinct (simple) eigenvalues, for $k = i$, we get a formula for perturbation $\delta\gamma_i^2$,

$$\delta\gamma_i^2 = \omega^2(\delta\epsilon H_i, H_i) - (\delta\epsilon \boldsymbol{\nabla} \times \operatorname{curl} H_i, H_i) + (\boldsymbol{\nabla} \times (\delta\epsilon \operatorname{curl} H_i), H_i)\,. \quad (3.12)$$

For $k \neq i$, formula (3.11) allows to compute perturbation δH_i; the i-th component of δH_i comes from a normalization assumption.

Assumption C: We assume:

$$(\delta H_i, H_i) = 0\,. \quad (3.13)$$

The assumption implies that the perturbed eigenvector $H_i + \delta H_i$ is (approximately) of length one, see the discussion in Section 4.

After integrating the last term by parts, we obtain:

$$(\gamma_i^2 - \gamma_k^2)(\delta H_i, H_k) = \omega^2(\delta\epsilon\, H_i, H_k) - (\delta\epsilon \boldsymbol{\nabla} \times \operatorname{curl} H_i, H_k) + (\delta\epsilon\, \operatorname{curl} H_i, \operatorname{curl} H_k)\,.$$

Table 3 presents selected (those that we need) elements of the linearized mass matrix.

Table 3: Linearized mass matrix $(\delta H_i, H_k) + (H_i, \delta H_k)$ for different families of eigenvectors.

$(\delta H, H) + (H, \delta H)$	$\delta H_k = \delta(\boldsymbol{\nabla}\psi_k)$	$\delta H_l = \delta(\boldsymbol{\nabla} \times \phi_l)$
$\delta H_i = \delta(\boldsymbol{\nabla}\psi_i)$	0	
$\delta H_j = \delta(\boldsymbol{\nabla} \times \phi_j)$		$(\delta\epsilon H_j, H_l)$

Finally, the curl-curl linearized mass matrix for the grad eigenvectors vanishes, and for the curl eigenvectors looks as follows:

$$\begin{aligned}
&\big(\operatorname{curl}(\boldsymbol{\nabla} \times \phi_i) + \operatorname{curl}\delta(\boldsymbol{\nabla} \times \phi_i), \operatorname{curl}(\boldsymbol{\nabla} \times \phi_j) + \operatorname{curl}\delta(\boldsymbol{\nabla} \times \phi_j)\big) \\
&\approx \big(\operatorname{curl}(\boldsymbol{\nabla} \times \phi_i), \operatorname{curl}(\boldsymbol{\nabla} \times \phi_j)\big) + \big(\operatorname{curl}\delta(\boldsymbol{\nabla} \times \phi_i), \operatorname{curl}(\boldsymbol{\nabla} \times \phi_j)\big) + \big(\operatorname{curl}(\boldsymbol{\nabla} \times \phi_i), \operatorname{curl}\delta(\boldsymbol{\nabla} \times \phi_j)\big) \\
&= \lambda_i\lambda_j(\phi_i, \phi_j) + \omega^2(\delta\epsilon\, \boldsymbol{\nabla} \times \phi_i, \boldsymbol{\nabla} \times \phi_j) + \lambda_i\lambda_j(\delta\epsilon\, \phi_i, \phi_j) \\
&= \lambda_i^{\frac{1}{2}}\lambda_j^{\frac{1}{2}}\big((1 + \delta\epsilon)\lambda_i^{\frac{1}{2}}\phi_i, \lambda_j^{\frac{1}{2}}\phi_j\big) + \omega^2(\delta\epsilon\, \boldsymbol{\nabla}\phi_i, \boldsymbol{\nabla}\phi_j)\,.
\end{aligned}$$

4 Stability Analysis

We return to system (2.3). We test the first equation with F_t, and the third equation with $G_t, n\cdot G_t = 0$ on ∂D, to obtain:

$$\begin{cases} -(i\omega E_3, \operatorname{div} F_t) + \omega^2(e_z \times H_t, F_t) - \frac{\partial}{\partial z} i\omega(E_t, F_t) &= i\omega\,(e_z \times f_t, F_t) \\ \operatorname{curl} E_t - i\omega H_3 &= f_3 \\ -(i\omega H_3, \operatorname{div} G_t) - \omega^2(e_z \times \epsilon E_t, G_t) - \frac{\partial}{\partial z} i\omega(H_t, G_t) &= i\omega\,(e_z \times g_t, G_t) \\ \operatorname{curl} H_t + i\omega\epsilon E_3 &= g_3\,. \end{cases}$$

Note that, when integrating by parts the first terms, we have used the fact that $E_3 = 0$ and $n \cdot G_t = 0$ on ∂D. Solving the second and fourth equations in (2.2) for E_3 and H_3,

$$E_3 = \frac{1}{i\omega\epsilon} g_3 - \frac{1}{i\omega\epsilon} \operatorname{curl} H_t\,, \qquad H_3 = -\frac{1}{i\omega} f_3 + \frac{1}{i\omega} \operatorname{curl} E_t\,,$$

and substituting into the first and the third equations, we obtain a system of two variational equations for E_t, H_t :

$$\begin{cases} (\frac{1}{\epsilon} \operatorname{curl} H_t, \operatorname{div} F_t) + \omega^2(e_z \times H_t, F_t) - \frac{\partial}{\partial z} i\omega(E_t, F_t) &= i\omega\,(e_z \times f_t, F_t) + (\frac{1}{\epsilon} g_3, \operatorname{div} F_t) \\ -(\operatorname{curl} E_t, \operatorname{div} G_t) - \omega^2(e_z \times \epsilon E_t, G_t) - \frac{\partial}{\partial z} i\omega(H_t, G_t) &= i\omega\,(e_z \times g_t, G_t) - (f_3, \operatorname{div} G_t)\,. \end{cases} \tag{4.1}$$

Variational eigenvalue problem:

$$\begin{cases} E_t \in H_0(\operatorname{curl}, D), H_t \in H(\operatorname{curl}, D) \\ (\frac{1}{\epsilon} \operatorname{curl} H_t, \operatorname{div} F_t) + \omega^2(e_z \times H_t, F_t) = -\omega\beta(E_t, F_t) \\ -(\operatorname{curl} E_t, \operatorname{div} G_t) - \omega^2(e_z \times \epsilon E_t, G_t) = -\omega\beta(H_t, G_t) \\ \qquad F_t \in H(\operatorname{div}, D),\ G_t \in H_0(\operatorname{div}, D)\,, \end{cases}$$

is equivalent to eigenproblem (2.6). Similarly, switching the role of (E_t, H_t) and (F_t, G_t) above, we obtain the adjoint variational eigenvalue problem equivalent to (2.7).

Consider now system (4.1). We expand the unknowns into series of the perturbed eigenvectors:

$$\begin{aligned} E_t &= \sum_i \alpha_i E_{t1,i} + \sum_j \tau_j E_{t2,j} \\ H_t &= \sum_i \delta_i H_{t1,i} + \sum_j \eta_j H_{t2,j} \end{aligned}$$

where $\alpha_i, \tau_j, \delta_i, \eta_j$ are functions of z, and

$$E_{t1,i} = \boldsymbol{\nabla} \times \psi_i + \delta E_{t1,i} \qquad E_{t2,j} = \boldsymbol{\nabla}\phi_j + \delta E_{t2,j}$$

$$H_{t1,i} = \boldsymbol{\nabla}\psi_i + \delta H_{t1,i} \qquad H_{t2,j} = \boldsymbol{\nabla} \times \phi_j + \delta H_{t2,j}$$

are the two E and H families of (perturbed) eigenvectors. Let

$$F_{t1,i} = \boldsymbol{\nabla} \times \psi_i + \delta F_{t1,i} \qquad F_{t2,j} = \boldsymbol{\nabla}\phi_j + \delta F_{t2,j}$$

$$G_{t1,i} = \boldsymbol{\nabla}\psi_i + \delta G_{t1,i} \qquad G_{t2,j} = \boldsymbol{\nabla} \times \phi_j + \delta G_{t2,j}$$

be the corresponding families of perturbed adjoint eigenvectors.

Scaling of the eigenvectors. The unperturbed E eigenvectors are scaled to provide an orthonormal basis, i.e., $\|\boldsymbol{\nabla} \times \psi_i\| = \|\boldsymbol{\nabla}\phi_j\| = 1$. This implies that the unperturbed H eigenvectors are also unit vectors as $\|\boldsymbol{\nabla}\psi_i\| = \|\boldsymbol{\nabla} \times \psi_i\|$, etc. The unperturbed F and G eigenvectors coincide with the E and H eigenvectors. We learned in Section 3 that the perturbations $\delta E_{t1,i}$ are scaled by the condition $(\delta E_{t1,i}, \boldsymbol{\nabla} \times \psi_i) = 0$. This implies that the perturbed eigenvector is, up to the linearization, a unit vector as well,

$$(\boldsymbol{\nabla} \times \psi_i + \delta E_{t1,i}, \boldsymbol{\nabla} \times \psi_i + \delta E_{t1,i}) \approx (\boldsymbol{\nabla} \times \psi_i, \boldsymbol{\nabla} \times \psi_i) + (\boldsymbol{\nabla} \times \psi_i, \delta E_{t1,i}) + (\delta E_{t1,i}, \boldsymbol{\nabla} \times \psi_i) = 1\,.$$

The same comment applies to all remaining perturbed eigenvectors. Note additionally that the bi-orthogonality condition is also (approximately) satisfied,

$$(\boldsymbol{\nabla} \times \psi_i + \delta E_{t1,i}, \boldsymbol{\nabla} \times \psi_i + \delta F_{t1,i}) \approx (\boldsymbol{\nabla} \times \psi_i, \boldsymbol{\nabla} \times \psi_i) + (\boldsymbol{\nabla} \times \psi_i, \delta F_{t1,i}) + (\delta E_{t1,i}, \boldsymbol{\nabla} \times \psi_i) = 1\,.$$

Decoupling the equations. Let $-\beta^2$ be an eigenvalue for eigenproblems (2.9) and (2.11) with the corresponding eigenvectors E_t, H_t scaled as discussed above. In order to invoke Lemma 1(b), we have to replace H_t with cH_t where constant c is computed by comparing eigenvector cH_t with H_t given by relation (2.8),

$$cH_t = \frac{1}{\omega\beta}[-\boldsymbol{\nabla}\,\mathrm{curl}\,E_t + \omega^2 e_z \times \epsilon E_t]\,.$$

Pair (E_t, cH_t) constitutes then an eigenvector for system (2.6) corresponding to root β of β^2 selected in such a way that $e^{i\beta z}$ represents an outgoing wave.[3] We proceed similarly with the adjoint eigenvectors. Let $-\gamma^2$ be an eigenvalue for problems (2.13) and (2.15) with the corresponding eigenvectors F_t, H_t. After scaling the second component, pair (F_t, dG_t) constitutes an eigenvector for system (2.7) corresponding to a root γ of γ^2. Constant d is obtained[4] by comparing dG_t with G_t given by (2.12), comp. Lemma 2,

$$dG_t = \frac{1}{\omega\beta}[-\boldsymbol{\nabla} \times (\frac{1}{\epsilon}\,\mathrm{div}\,F_t) + \omega^2 e_z \times F_t]\,.$$

Case: $\beta^2 \neq \gamma^2$ and, therefore, $\beta \neq \gamma$. Multiplying system (2.6) with pair (F_t, dG_t), we obtain the bi-orthogonality condition,

$$c(BH_t, F_t) + d(CE_t, G_t) = 0$$

where B and C denote the operators on the left-hand side of (2.6). But, testing with the adjoint eigenpair $(F_t, -dH_t)$ (corresponding to eigenvalue $-\gamma \neq \beta$, we obtain also

$$c(BH_t, F_t) - d(CE_t, G_t) = 0$$

Consequently, we have,

$$(BH_t, F_t) = 0 \qquad \text{and} \qquad (CE_t, G_t) = 0\,.$$

[3]The choice depends upon the ansatz in time.

[4]We learn in Appendix A that d is real.

Case: $\beta^2 = \gamma^2$ and $\beta = \gamma$. Multiplying system (2.6) with pair (F_t, G_t), we obtain:

$$c(BH_t, F_t) + d(CE_t, G_t) = -\omega\beta[1 + cd]$$

But, testing with the adjoint eigenpair $(F_t, -dG_t)$ (corresponding to eigenvalue $-\gamma \neq \beta$), we obtain also

$$c(BH_t, F_t) - d(CE_t, G_t) = 0$$

Consequently, we have,

$$\theta := (BH_t, F_t) = -\frac{\omega\beta}{2c}[1 + cd] \qquad \text{and} \qquad \nu := (CE_t, G_t) = -\frac{\omega\beta}{2d}[1 + cd]\,.$$

THEOREM 1
Testing in (4.1) with $(F_{t1,j}, dG_{t1,j})$ and with $(F_{t2,j}, dG_{t2,j})$, we obtain a decoupled system of ODEs for the coefficients α_j, δ_j:

$$\begin{cases} \theta_1\delta_j - i\omega\alpha_j' & = r_1(z) := (i\omega\, e_z \times f_t, F_{t1,j}) + (\frac{1}{\epsilon}g_3, \operatorname{div} F_{t1,j}) \\ \nu_1\alpha_j - i\omega\delta_j' & = r_2(z) := (i\omega\, e_z \times g_t, G_{t1,j}) - (f_3, \operatorname{div} G_{t1,j}) \end{cases} \tag{4.2}$$

and τ_j, η_j:

$$\begin{cases} \theta_2\eta_j - i\omega\tau_j' & = s_1(z) := (i\omega\, e_z \times f_t, F_{t2,j}) + (\frac{1}{\epsilon}g_3, \operatorname{div} F_{t2,j}) \\ \nu_2\tau_j - i\omega\eta_j' & = s_2(z) := (i\omega\, e_z \times g_t G_{t2,j}) - (f_3, \operatorname{div} G_{t2,j}) \end{cases} \tag{4.3}$$

where θ_i, ν_i, $i = 1, 2$, are the values of coefficients θ, ν for the two families of eigenvectors. ▮

We refer to Appendix A, for the computation of constants c, d, θ, ν using the perturbation analysis, and the final values of θ, ν listed in Table 4. The constants take different values for the two families of E and F eigenvectors. For the homogeneous waveguide, the systems reduce to the ones in [5].

REMARK 1 While we use the perturbation analysis to evaluate constants c, d, θ, ν, the decoupling result in Theorem 1 is general, and valid for arbitrary ϵ. ▮

5 Estimation of $\boldsymbol{E_t}, \boldsymbol{H_t}$, and $\operatorname{curl} \boldsymbol{E_t}, \operatorname{curl} \boldsymbol{H_t}$

Recall that $i\omega H_3 = \operatorname{curl} E_t - f_3$. An estimate for $\operatorname{curl} E_t$ is thus equivalent to an estimate for H_3. Similarly, $i\omega\epsilon E_3 = -\operatorname{curl} H_t - g_3$, an estimate for $\operatorname{curl} H_t$ is equivalent to an estimate for E_3.

5.1 Estimation of E_t and H_t with Their Spectral Components

If the L^2 mass matrix corresponding to the perturbed eigenvectors represents a bounded operator in L^2, then we can bound the L^2-norm of E_t with the sum of its spectral components. More precisely,

$$\begin{aligned}
\|E_t\|^2 &\leq 2\left[\|\sum_{i=1}^{\infty}\alpha_i E_{t1,i}\|^2 + \|\sum_{j=1}^{\infty}\tau_j E_{t1,j}\|^2\right] \\
&= 2\lim_{N\to\infty}\left[(\sum_{i=1}^{N}\alpha_i E_{t1,i}, \sum_{k=1}^{N}\alpha_k E_{t1,k}) + (\sum_{j=1}^{N}\tau_j E_{t2,j}, \sum_{l=1}^{N}\tau_l E_{t2,l})\right] \\
&= 2\lim_{N\to\infty}\left[\sum_{i,k=1}^{N}\alpha_i\overline{\alpha_k}(E_{t1,i}, E_{t1,k}) + \sum_{j,l=1}^{N}\tau_j\overline{\tau_l}(E_{t2,j}, E_{t2,l})\right] \\
&\leq \lim_{N\to\infty} 2C\left[\sum_{i=1}^{N}|\alpha_i|^2 + \sum_{j=1}^{N}|\tau_j|^2\right] \\
&= 2C\left[\sum_{i=1}^{\infty}|\alpha_i|^2 + \sum_{j=1}^{\infty}|\tau_j|^2\right]
\end{aligned}$$

where C is assumed to be independent of N. Note that we do not need any information about the off-diagonal terms $(E_{t1,i}, E_{t2,j})$. According to the results from the previous section $C = 1 + \|\delta\epsilon\|_{L^\infty(D)}$.

Similarly,

$$\begin{aligned}
\|H_t\|^2 &\leq 2\left[\|\sum_{i=1}^{\infty}\delta_i H_{t1,i}\|^2 + \|\sum_{j=1}^{\infty}\eta_j H_{t2,j}\|^2\right] \\
&\leq 2C\left[\sum_{i=1}^{\infty}|\delta_i|^2 + \sum_{j=1}^{\infty}|\eta_j|^2\right]
\end{aligned}$$

where, by the results from Section 3.2, $C = 1 + \|\delta\epsilon\|_{L^\infty(D)}$ as well.

After integrating in z, we get

$$\begin{aligned}
\int_0^L \|E_t\|^2\,dz &\leq 2C\left[\sum_{i=1}^{\infty}\int_0^L |\alpha_i|^2\,dz + \sum_{j=1}^{\infty}\int_0^L |\tau_j|^2\,dz\right] \\
\int_0^L \|H_t\|^2\,dz &\leq 2C\left[\sum_{i=1}^{\infty}\int_0^L |\delta_i|^2\,dz + \sum_{j=1}^{\infty}\int_0^L |\eta_j|^2\,dz\right].
\end{aligned} \tag{5.1}$$

5.2 Estimation of $\operatorname{curl} \boldsymbol{E}_t, \operatorname{curl} \boldsymbol{H}_t$ with Spectral Components of $\boldsymbol{E}_t, \boldsymbol{H}_t$

By the same token,

$$
\begin{aligned}
\| \operatorname{curl} E_t \|^2 &\leq 2 \left[\| \sum_{i=1}^{\infty} \alpha_i \operatorname{curl} E_{t1,i} \|^2 + \| \sum_{j=1}^{\infty} \tau_j \operatorname{curl} E_{t2,j} \|^2 \right] \\
&= 2 \lim_{N\to\infty} \left[(\sum_{i=1}^{N} \alpha_i \operatorname{curl} E_{t1,i}, \sum_{k=1}^{N} \alpha_k \operatorname{curl} E_{t1,k}) + (\sum_{j=1}^{N} \tau_j \operatorname{curl} E_{t2,j}, \sum_{l=1}^{N} \tau_l \operatorname{curl} E_{t2,l}) \right] \\
&= 2 \lim_{N\to\infty} \left[\sum_{i,k=1}^{N} \alpha_i \overline{\alpha_k} (\operatorname{curl} E_{t1,i}, \operatorname{curl} E_{t1,k}) + \sum_{j,l=1}^{N} \tau_j \overline{\tau_l} (\operatorname{curl} E_{t2,j}, \operatorname{curl} E_{t2,l}) \right] \\
&\approx 2 \sum_{i=1}^{\infty} (\mu_i + \omega^2 \|\delta\epsilon\|_{L^\infty(D)}) |\alpha_i|^2 \, .
\end{aligned}
$$

Note that, like for the homogeneous case, the perturbed gradients do not contribute (the linearized perturbed curl mass matrix is zero).

Similarly, using results from Section 3.2,

$$
\begin{aligned}
\| \operatorname{curl} H_t \|^2 &\leq 2 \left[\| \sum_{i=1}^{\infty} \alpha_i \operatorname{curl} H_{t1,i} \|^2 + \| \sum_{j=1}^{\infty} \tau_j \operatorname{curl} H_{t2,j} \|^2 \right] \\
&\lesssim 2 \sum_{i=1}^{\infty} (\lambda_i \|\epsilon\|_{L^\infty(D)} + \omega^2 \|\delta\epsilon\|_{L^\infty(D)}) |\eta_i|^2 \, .
\end{aligned}
$$

Note again that the perturbed gradients do not contribute.

5.3 Estimation of Spectral Components $\boldsymbol{\alpha_i}, \boldsymbol{\delta_i}$

We focus now on the ODE boundary-value problem for coefficients α and δ,

$$
\left\{
\begin{aligned}
& \alpha(0) = 0, \ \delta(L) = \sqrt{\tfrac{\nu}{\theta}} \alpha(L) \\
& \theta \delta - i\omega \alpha' = r_1 \\
& \nu \alpha - i\omega \delta' = r_2
\end{aligned}
\right.
\tag{5.2}
$$

where $\theta = \theta_1$ and $\nu = \nu_1$ are the coefficient values for the first family of eigenvectors. Testing the second equation with $\delta\alpha, \delta\alpha(0) = 0$, integrating the derivative term by parts, and utilizing the impedance BC, we obtain:

$$
i\omega(\delta, \delta\alpha') = -\nu(\alpha, \delta\alpha) + i\omega\alpha(L)\delta\alpha(L) + (r_2, \delta\alpha) \, .
$$

Testing the first equation with $\delta\alpha'$ and using the formula above, we obtain the ultimate variational problem for coefficient α:

$$
\left\{
\begin{aligned}
& \alpha(0) = 0 \\
& (\alpha', \delta\alpha') + \kappa^2(\alpha, \delta\alpha) + \kappa\alpha(L)\delta\alpha(L) = \tfrac{i}{\omega}(r_1, \delta\alpha') - \tfrac{\theta}{\omega^2}(r_2, \delta\alpha) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \delta\alpha \, : \, \delta\alpha(0) = 0
\end{aligned}
\right.
\tag{5.3}
$$

where $\kappa = i\frac{\sqrt{\theta\nu}}{\omega}$. For the homogeneous waveguide, $\kappa = i\beta$ and the equation coincides with that derived in [5]. For the non-homogeneous waveguide,

$$\kappa = i\sqrt{\beta^2 + \omega^2(\delta\epsilon\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi)}\,.$$

The perturbed κ is still of order β. As $\theta = \theta_1 = -\omega^2$, the right-hand side reduces to:

$$\frac{i}{\omega}(r_1, \delta\alpha') + (r_2, \delta\alpha)\,. \tag{5.4}$$

The following lemma was proved in [5].

Lemma 4
Let $I = (0, L)$. Consider two problems: Find $q_1, q_2 \in H^1_{(0}(I) := \{v \in H^1(I) \,:\, v(0) = 0\}$ such that:

$$\begin{aligned}(q_1', v') + \lambda^2(q_1, v) + \lambda q_1(L)v(L) &= (f, v) \quad v \in H^1_{(0}(I)\\ (q_2', v') + \lambda^2(q_2, v) + \lambda q_2(L)v(L) &= (f, v') \quad v \in H^1_{(0}(I)\end{aligned}$$

where $f \in L^2(I)$. Then, denoting $\|q\|^2_{1,\beta} := \|q'\|^2 + \beta^2\|q\|^2$, we have:

- *Case (i): $\lambda = i\beta, \beta > 0$. There exists a constant $C > 0$, depending only on a lower bound for $L\beta$ such that*
$$\begin{aligned}\|q_1\|^2_{1,\beta} &\le CL^2\|f\|^2\\ \|q_2\|^2_{1,\beta} &\le CL^2\beta^2\|f\|^2\end{aligned}$$
- *Case (ii): $\lambda = \beta, \beta > 0$. There exists a constant $C > 0$, depending only on a lower bound for $L\beta$ such that*
$$\begin{aligned}\|q_1\|^2_{1,\beta} &\le C\beta^{-2}\|f\|^2 &\Rightarrow\quad \|q_1\|^2 &\le C\beta^{-4}\|f\|^2\\ \|q_2\|^2_{1,\beta} &\le C\|f\|^2 &\Rightarrow\quad \|q_2\|^2 &\le C\beta^{-2}\|f\|^2\,.\end{aligned}$$

∎

We will use Lemma 4 to estimate the L^2-norms of coefficients α_j, δ_j by the L^2-norms of the right-hand sides $r_{1,j}, r_{2,j}$ and, in turn, the $L^2(0, L)$-norms of $r_{1,j}, r_{2,j}$ by the L^2-norms of f_t, f_3, g_t, g_3. While the stability of propagating modes (Case (i) in Lemma 4) implies the linear dependence of stability constant upon L, the stability of evanescent modes (Case (ii) in Lemma 4) provides the desired asymptotic scaling properties in terms of eigenvalues $|\beta_i| \approx \mu_i^{\frac{1}{2}}, \lambda_i^{\frac{1}{2}}$. Note that, since the number of propagating modes is finite, their stability does not affect the asymptotic scaling properties with $|\beta_i|$. We will skip the dependence of stability constants C upon L but keep track of the dependence upon the eigenvalues $|\beta_i| \approx \mu_i^{\frac{1}{2}}, \lambda_i^{\frac{1}{2}}$.

Estimation of $\boldsymbol{\alpha_j}$. We will consider the four terms contributing to the right-hand side (5.4) and estimate the corresponding solutions α_j, one at the time. By linearity, this will imply the estimate for the ultimate coefficients α_j.

Term 1: $i\omega(e_z \times f_t, F_{t1,j})$ contributing to r_1. Skipping factor $i\omega$, we have:

$$
\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2 \, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(e_z \times f_t, F_{t1,j} + \delta F_{t1,j})|^2 && \text{(Lemma (ii)}_2\text{)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} [|(e_z \times f_t, F_{t1,j})|^2 + |(e_z \times f_t, \delta F_{t1,j})|^2] && \text{(Young's inequality)} \\
&\lesssim 2 \sum_j \int_0^L |(e_z \times f_t, F_{t1,j})|^2 && (\text{linearization}, \beta_j^{-2} \lesssim 1) \\
&\le 2 \int_0^L \|e_z \times f_t\|^2 \, dz \\
&= 2 \int_0^L \|f_t\|^2 \, dz \, .
\end{aligned}
$$

REMARK 2 Note that the application of Young's inequality and neglection of the second-order terms reduces the estimation of coefficients α_j to the case of the homogeneous waveguide. The ODE systems (4.2) and (4.3) are identical with those for the homogeneous waveguide except for the values of θ_i, ν_i which are different but of the same order as for the homogeneous system. Hence the estimation of coefficients $\alpha_i, \delta_i, \tau_j, \eta_j$ in the perturbed case is identical with the estimation for the homogeneous waveguide. For the reader's convenience, we estimate explicitly each term, repeating arguments from [5]. $\blacksquare$

Term 2: $(\frac{1}{\epsilon} g_3, \operatorname{div} F_{t1,j})$ contributing to r_1.

$$
\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2 \, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j} + \delta F_{t1,j}))|^2 && \text{(Lemma (ii)}_2\text{)} \\
&\le 2 \sum_j \int_0^L \beta_j^{-2} [|(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j}))|^2 + |(\frac{1}{\epsilon} g_3, \operatorname{div}(\delta F_{t1,j}))|^2] && \text{(Young's lemma)} \\
&\lesssim 2 \sum_j \int_0^L |(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j}))|^2 && (\text{linearization}, \beta_j^{-2} \lesssim 1) \\
&\lesssim 0 && (\operatorname{div} F_{t1,j} = 0)
\end{aligned}
$$

Term 3: $i\omega(e_z \times g_t, G_{t1,j})$ contributing to r_2. We follow exactly the same reasoning as for Term 1. Note that Lemma $(\text{ii})_1$ gives us even a better factor β^{-4}.

Term 4: $(f_3, \operatorname{div} G_{t1,j})$ contributing to r_2.

$$\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2 \, dz &\lesssim \sum_j \int_0^L \beta_j^{-4} |(f_3, \operatorname{div}(G_{t1,j} + \delta G_{t1,j}))|^2 && \text{(Lemma (ii)}_1\text{)} \\
&\leq 2 \sum_j \int_0^L \beta_j^{-4} [|(f_3, \operatorname{div}(G_{t1,j}))|^2 + |(f_3, \operatorname{div}(\delta G_{t1,j}))|^2] && \text{(Young's lemma)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-4} |(f_3, \operatorname{div}(G_{t1,j}))|^2 && \text{(linearization)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-4} \mu_j |(f_3, \mu_j^{1/2} \psi_j)|^2 \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} |(f_3, \mu_j^{1/2} \psi_j)|^2 && (\beta_j^{-2} \mu_j \approx O(1)) \\
&\lesssim 2 \sum_j \int_0^L |(f_3, \mu_j^{1/2} \psi_j)|^2 && (\beta_j^{-2} \lesssim 1) \\
&\lesssim 2 \sum_j |(f_3, \mu_j^{1/2} \psi_j)|^2 = 2 \|f_3\|^2 \, .
\end{aligned}$$

Estimation of curl $\boldsymbol{E_t}$. In the estimation of $\operatorname{curl} E_t$, we need to estimate:

$$\sum_i \int_0^L \underbrace{(\mu_i + \|\delta\epsilon\|_{L^\infty(D)})}_{\sim \beta_i^2} |\alpha_i|^2 \, dz \, .$$

We follow exactly the same strategy as above. In all cases, we can accommodate the extra $\mu_i \approx \beta_i^2$ factor.

Estimation of $\boldsymbol{\delta_j}$. The first equation of system (5.2) implies:

$$\omega^2 \|\delta\|_{L^2(I)} \leq \omega \|\alpha'\|_{L^2(I))} + \|r_1\|_{L^2(I)} \, .$$

Estimation of the derivatives $\sum_i \|\alpha_i'\|^2_{L^2(I))}\|$ is done in exactly the same way as for α_i's, except that Lemma (ii) delivers now less by a factor of β_j^{-2}. However, we can spare it in all of the four discussed cases. It remains to estimate r_1. We proceed in the same way as before.

Term 1: $(e_z \times f_t, F_{t1,j})$ contributing to r_1. We have:

$$\begin{aligned}
&\sum_j \int_0^L |(e_z \times f_t, F_{t1,j} + \delta F_{t1,j})|^2 \\
&\quad \lesssim 2 \sum_j \int_0^L [|(e_z \times f_t, F_{t1,j})|^2 + |(e_z \times f_t, \delta F_{t1,j})|^2] && \text{(Young's inequality)} \\
&\quad \lesssim 2 \sum_j \int_0^L |(e_z \times f_t, F_{t1,j})|^2 && \text{(linearization)} \\
&\quad \leq 2 \int_0^L \|e_z \times f_t\|^2 \, dz = 2 \int_0^L \|f_t\|^2 \, dz \, .
\end{aligned}$$

Term 2: $(\frac{1}{\epsilon}g_3, \operatorname{div} G_{t1,j})$ contributing to r_1.

$$
\begin{aligned}
&\sum_j \int_0^L |(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t1,j} + \delta F_{t1,j}))|^2 \\
&\quad \leq 2\sum_j \int_0^L [|(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t1,j}))|^2 + |(\frac{1}{\epsilon}g_3, \operatorname{div}(\delta F_{t1,j}))|^2] \quad \text{(Young's lemma)} \\
&\quad \lesssim 2\sum_j \int_0^L |(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t1,j}))|^2 \quad \text{(linearization)} \\
&\quad \lesssim 0 \quad (\operatorname{div} F_{t1,j} = 0)
\end{aligned}
$$

5.4 Estimation of Spectral Components τ_i, η_i

We use exactly the same techniques to estimate the spectral components corresponding to the second families of E and H eigenvectors. We will point out only to the differences between the two cases. The ODE boundary-value problem for coefficients τ and η takes the same form as system (5.2),

$$
\begin{cases}
\beta(0) = 0,\ \eta(L) = \sqrt{\frac{\theta_2}{\theta_1}}\beta(L) \\
\theta\eta - i\omega\beta' = s_1 \\
\nu\beta - i\omega\eta' = s_2
\end{cases} \tag{5.5}
$$

where

$$
\begin{aligned}
s_1 &= (i\omega\, e_z \times f_t, F_{t2,j}) + (\frac{1}{\epsilon}g_3, \operatorname{div} F_{t2,j}) \\
s_2 &= (i\omega\, e_z \times g_t G_{t2,j}) - (f_3, \operatorname{div} G_{t2,j})\,.
\end{aligned}
$$

The coefficients θ, ν are now different. By the results from Appendix A.1,

$$
\theta = \theta_2 = \beta^2 + \lambda(\delta\epsilon\, \lambda^{\frac{1}{2}}\phi, \lambda^{\frac{1}{2}}\phi), \qquad \nu = \nu_2 = \omega^2 + \omega^2(\delta\epsilon\, \boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\phi)\,.
$$

This gives for the homogeneous waveguide $\kappa = i\beta$, and for the non-homogeneous case,

$$
\kappa = i\frac{\sqrt{(\beta^2 + \lambda(\delta\epsilon\, \lambda^{\frac{1}{2}}\phi, \lambda^{\frac{1}{2}}\phi))(\omega^2 + \omega^2(\delta\epsilon\, \boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\phi))}}{\omega}
$$

which is still of order $i\beta$.

The first essential difference is the scaling in the right-hand side of the second-order problem (5.3),

$$
\frac{i}{\omega}(s_1, \delta\alpha') - \frac{\theta}{\omega^2}(s_2, \delta\alpha) = \frac{i}{\omega}(s_1, \delta\alpha') - \frac{(\beta^2 + \lambda(\delta\epsilon\, \lambda^{\frac{1}{2}}\phi, \lambda^{\frac{1}{2}}\phi))}{\omega^2}(s_2, \delta\alpha)\,.
$$

The coefficient in front of s_1 is the same as before, but the coefficient θ in front of s_2 is now of order β^2.

Estimation of τ_j. We now discuss the four terms contributing to the right-hand side above.

Term 1: $i\omega(e_z \times f_t, F_{t2,j})$ contributing to s_1. The estimate is identical with that for α_i. Skipping constant terms, we have:

$$\begin{aligned}
\sum_j \int_0^L |\tau_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(e_z \times f_t, F_{t2,j} + \delta F_{t2,j})|^2 && \text{(Lemma (ii)}_2\text{)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} [|(e_z \times f_t, F_{t2,j})|^2 + |(e_z \times f_t, \delta F_{t2,j})|^2] && \text{(Young's inequality)} \\
&\lesssim 2 \sum_j \int_0^L |(e_z \times f_t, F_{t2,j})|^2 && (\text{linearization}, \beta_j^{-2} \lesssim 1) \\
&\leq 2 \int_0^L \|e_z \times f_t\|^2\, dz = 2 \int_0^L \|f_t\|^2\, dz\,.
\end{aligned}$$

Term 2: $(\frac{1}{\epsilon} g_3, \operatorname{div} F_{t2,j})$ contributing to s_1. The situation is now different as $\operatorname{div} F_{t2,j} = \operatorname{div} \operatorname{grad} \phi_j = -\lambda_j \phi_j$.

$$\begin{aligned}
\sum_j \int_0^L |\tau_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t2,j} + \delta F_{t2,j}))|^2 && \text{(Lemma (ii)}_2\text{)} \\
&\leq 2 \sum_j \int_0^L \beta_j^{-2} [|(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t2,j}))|^2 + |(\frac{1}{\epsilon} g_3, \operatorname{div}(\delta F_{t2,j}))|^2] && \text{(Young's lemma)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} |(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t2,j}))|^2 && \text{(linearization)} \\
&\lesssim 2 \sum_j \int_0^L |(\frac{1}{\epsilon} g_3, \lambda_j^{\frac{1}{2}} \phi_j))|^2 && (\beta_j^{-2} \lambda_j \lesssim 1) \\
&\lesssim \|\frac{1}{\epsilon} g_3\|^2 \lesssim \|g_3\|^2\,.
\end{aligned}$$

Term 3: $i\omega(e_z \times g_t, G_{t2,j})$ contributing to s_2. We follow exactly the same reasoning as for Term 1. Lemma (ii)$_1$ gives us a better factor β^{-4} but there is a factor of order β^2 in front of s_2.

Term 4: $(f_3, \operatorname{div} G_{t2,j})$ contributing to s_2. Compared with the estimate for α_j, we lose again factor β_j^{-2} due to the term in front of s_2.

$$\begin{aligned}
\sum_j \int_0^L |\tau_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(f_3, \operatorname{div}(G_{t2,j} + \delta G_{t2,j}))|^2 && \text{(Lemma (ii)}_1\text{)} \\
&\leq 2 \sum_j \int_0^L \beta_j^{-2} [|(f_3, \operatorname{div}(G_{t2,j}))|^2 + |(f_3, \operatorname{div}(\delta G_{t2,j}))|^2] && \text{(Young's lemma)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} |(f_3, \operatorname{div}(G_{t2,j}))|^2 && \text{(linearization)} \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} \lambda_j |(f_3, \lambda_j^{1/2} \phi_j)|^2 && (\operatorname{div} G_{t2,j} = -\lambda \phi_j) \\
&\lesssim 2 \sum_j \int_0^L |(f_3, \lambda_j^{1/2} \phi_j)|^2 && (\beta_j^{-2} \lambda_j \approx O(1)) \\
&= 2\|f_3\|^2\,.
\end{aligned}$$

Estimation of $\boldsymbol{\eta_j}$. From the first equation in system (5.5), we get:

$$\eta = \frac{1}{\theta}(i\omega\beta' + s_1) \quad \Rightarrow \quad \|\eta\|^2_{L^2(I)} \lesssim \frac{1}{\lambda^2}\|\tau'\|^2_{L^2(I)} + \frac{1}{\lambda^2}\|s_1\|^2_{L^2(I)}$$

as $|s_1| \approx \lambda$. In order to estimate $\operatorname{curl} H_t$, we need a more demanding estimate for

$$\sum_j \lambda_j \|\eta_j\|^2_{L^2(I)} \lesssim \sum_j \lambda_j^{-1}\|\tau_j'\|^2_{L^2(I)} + \lambda_j^{-1}\|s_{1,j}\|^2_{L^2(I)} \,.$$

In the estimation of derivatives τ_j', we lose a factor of β_j^2 but we gain it back with the factor λ_j^{-1} above. We also need the additional factor when estimating the second term. The details are as follows.

Term 1: $(e_z \times f_t, F_{t2,j})$ contributing to s_1. This terms in painless. We do not need the additional factor $\lambda_j^{-1} \approx \beta_j^{-2}$.

$$\begin{aligned}
&\sum_j \int_0^L |(e_z \times f_t, F_{t2,j} + \delta F_{t2,j})|^2 \\
&\quad \lesssim 2\sum_j \int_0^L [|(e_z \times f_t, F_{t2,j})|^2 + |(e_z \times f_t, \delta F_{t2,j})|^2] \quad \text{(Young's inequality)} \\
&\quad \lesssim 2\sum_j \int_0^L |(e_z \times f_t, F_{t2,j})|^2 \quad \text{(linearization)} \\
&\quad \leq 2\int_0^L \|e_z \times f_t\|^2 \, dz = 2\int_0^L \|f_t\|^2 \, dz \,.
\end{aligned}$$

Term 2: $(\frac{1}{\epsilon}g_3, \operatorname{div} G_{t2,j})$ contributing to s_1. The presence of the additional factor is now essential.

$$\begin{aligned}
&\sum_j \beta_j^{-2} \int_0^L |(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t2,j} + \delta F_{t2,j}))|^2 \\
&\quad \leq 2\sum_j \int_0^L \beta_j^{-2}[|(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t2,j}))|^2 + |(\frac{1}{\epsilon}g_3, \operatorname{div}(\delta F_{t2,j}))|^2] \quad \text{(Young's lemma)} \\
&\quad \lesssim 2\sum_j \beta_j^{-2} \int_0^L |(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t2,j}))|^2 \quad \text{(linearization)} \\
&\quad \lesssim 2\sum_j \beta_j^{-2}\lambda_j \int_0^L |(\frac{1}{\epsilon}g_3, \lambda_j^{\frac{1}{2}}\phi_j))|^2 \quad (\operatorname{div} F_{t2,j} = -\lambda_j\phi_j) \\
&\quad \lesssim 2\sum_j \int_0^L |(\frac{1}{\epsilon}g_3, \lambda_j^{\frac{1}{2}}\phi_j))|^2 \quad (\beta_j^{-2}\lambda_j \lesssim 1) \\
&\quad \leq 2\|\frac{1}{\epsilon}g_3\|^2 \lesssim \|g_3\|^2 \,.
\end{aligned}$$

5.4.1 Final Result

We arrive at our final result.

THEOREM 2
Assume that $\epsilon = 1 + \delta\epsilon$ where the perturbation $\delta\epsilon$ is sufficiently[5] *small in L^∞-norm, and that it vanishes near the boundary*[6]*. There exists then a constant $C > 0$ such that*

$$\|E\|^2 + \|H\|^2 \le CL^2 \left(\|\boldsymbol{\nabla} \times E - i\omega H\|^2 + \|\boldsymbol{\nabla} \times H + i\omega\epsilon E\|^2 \right)$$

for all (E, H) from the domain of the operator. ∎

Proof: We have proved the theorem under the simplifying assumption of distinct (simple) eigenvalues β_i^2 of the homogeneous waveguide problem, see Assumption A in Appendix A. Extending the proof to the case of multiple eigenvalues requires techniques discussed in Appendix A. ∎

6 Conclusions

We have extended the stability analysis for homogeneous electromagnetic waveguides from [5] to the case of a non-homogeneous waveguide with a perturbed dielectric constant $\epsilon = 1 + \delta\epsilon$. The analysis was done using the classical (formal) perturbation theory for eigenproblems involving a self-adjoint operator under the assumption of 'smallness' of perturbation $\delta\epsilon$ but with no assumptions on its derivatives. In particular, the results hold for discontinuous perturbations $\delta\epsilon$ and are therefore applicable to the case of step-index fibers.

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[5] To justify the perturbation analysis.
[6] See Assumption D in Appendix A.

A Perturbation Analysis Continued

In this section we provide additional results obtained from the perturbation analysis. The results below require the perturbation analysis for the adjoint F and G problems that follows the same lines as for the E and H problems. We skip the details and present only the final results that were used in Section 4.

A.1 Computation of Scaling Coefficients c, d, θ, ν

A.1.1 First Family of Eigenvectors

We first investigate perturbations of E, F eigenvectors $\nabla \times \psi_i$ and the corresponding H, G eigenvectors $\nabla \psi_i$.

Perturbations of coefficients c, d. Let $\nabla \times \psi + \delta E$ and $\nabla \psi + \delta H$ be the perturbed E and H eigenvectors corresponding to a perturbed eigenvalue $\beta + \delta\beta$. Here ψ is a Neumann eigenvector of the Laplacian, $-\Delta\psi = \mu\psi$, and $\beta^2 = \omega^2 - \mu$. The perturbed coefficient $c + \delta c$ is defined by the relation:

$$(c + \delta c)(\nabla\psi + \delta H) = \frac{\beta - \delta\beta}{\omega\beta^2} \left[-\nabla \operatorname{curl}(\nabla \times \psi + \delta E) + \omega^2 e_z \times (1 + \delta\epsilon)(\nabla \times \psi + \delta E) \right] .$$

We first compute the value of c. Testing with $\nabla\psi$, we get,

$$c = \frac{1}{\omega\beta}([-\nabla \underbrace{(-\Delta\psi)}_{=\mu\psi} + \omega^2 \underbrace{(e_z \times (\nabla \times \psi)}_{=\nabla\psi}], \nabla\psi) = \frac{1}{\omega\beta} \underbrace{[-\mu + \omega^2]}_{=\beta^2} = \frac{\beta}{\omega} .$$

Linearizing both sides, and testing with $\nabla\psi$, we get,

$$\begin{aligned}
\delta c \underbrace{(\nabla\psi, \nabla\psi)}_{=1} + c \underbrace{(\delta H, \nabla\psi)}_{=0} &= -\tfrac{\delta\beta}{\omega\beta^2} \underbrace{(-\nabla \operatorname{curl} \nabla \times \psi + \omega^2 \nabla\psi, \nabla\psi)}_{=\beta^2} \\
&\quad + \tfrac{1}{\omega\beta} \underbrace{(-\nabla \operatorname{curl} \delta E + \omega^2 e_z \times \delta E, \nabla\psi)}_{=0} \\
&\quad + \tfrac{1}{\omega\beta}\omega^2 \underbrace{(\delta\epsilon\, e_z \times (\nabla \times \psi), \nabla\psi)}_{=(\delta\epsilon\nabla\psi, \nabla\psi)} .
\end{aligned}$$

In the end,

$$\delta c = -\frac{\delta\beta}{\omega} + \frac{\omega}{\beta}(\delta\epsilon\nabla\psi, \nabla\psi) .$$

Similarly, perturbation δd is defined by the relation:

$$(d + \delta d)(\nabla\psi + \delta G) \approx \frac{\beta - \delta\beta}{\omega\beta^2} \left[-\nabla \times ((1 - \delta\epsilon) \operatorname{div}(\nabla \times \psi + \delta F)) + \omega^2 (e_z \times (\nabla \times \psi + \delta F)) \right] .$$

This yields:

$$d = \frac{\omega}{\beta} \quad \text{and} \quad \delta d = -\delta\beta \frac{\omega}{\beta^2} .$$

Perturbations of coefficients entering the decoupled system of equations. We are now ready to compute the perturbation of coefficient

$$\theta = -\frac{\omega\beta}{2c}[1+cd] = -\frac{\omega\beta}{2}[\frac{1}{c}+d] = -\omega^2 .$$

We have:

$$\begin{aligned} \delta\theta &= -\frac{\omega}{2}[\frac{1}{c}+d]\,\delta\beta + \frac{\omega\beta}{2}\frac{1}{c^2}\,\delta c - \frac{\omega\beta}{2}\,\delta d \\ &= -\frac{\omega^2}{\beta}\delta\beta + \frac{\omega^3}{2\beta}[-\frac{\delta\beta}{\omega} + \frac{\omega}{\beta}(\delta\epsilon\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi)] + \frac{\omega^2}{2\beta}\delta\beta \\ &= -\frac{\omega^2}{\beta}\delta\beta + \frac{\omega^4}{2\beta^2}(\delta\epsilon\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi) . \end{aligned}$$

Finally, using formula (3.9) for $\delta\beta^2$ and utilizing $\delta\beta^2 = 2\beta\,\delta\beta$, we obtain,

$$\delta\theta = -\frac{\omega^4}{2\beta^2}(\delta\epsilon\boldsymbol{\nabla}\times\psi, \boldsymbol{\nabla}\times\psi) + \frac{\omega^4}{2\beta^2}(\delta\epsilon\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi) = 0 .$$

We proceed with the coefficient

$$\nu = -\frac{\omega\beta}{2d}[1+cd] = -\frac{\omega\beta}{2}[\frac{1}{d}+c] = -\beta^2 .$$

We have:

$$\begin{aligned} \delta\nu &= -\frac{\omega}{2}[\frac{1}{d}+c]\,\delta\beta + \frac{\omega\beta}{2}\frac{1}{d^2}\,\delta d - \frac{\omega\beta}{2}\,\delta c \\ &= -\beta\,\delta\beta - \frac{\beta}{2}\delta\beta - \frac{\omega\beta}{2}[-\frac{\delta\beta}{\omega} + \frac{\omega}{\beta}(\delta\epsilon\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi)] \\ &= -\beta\,\delta\beta - \frac{\omega^2}{2}(\delta\epsilon\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi) . \end{aligned}$$

Using again formula (3.9) for $\delta\beta^2$, we obtain,

$$\delta\nu = -\frac{\omega^2}{2}(\delta\epsilon\boldsymbol{\nabla}\times\psi, \boldsymbol{\nabla}\times\psi) - \frac{\omega^2}{2}(\delta\epsilon\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi) = -\omega^2(\delta\epsilon\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi) .$$

A.1.2 Second Family of Eigenvectors

Next, we record the results for similar computations for perturbations of the second family of E, F eigenvectors: $\boldsymbol{\nabla}\phi_j$ and the corresponding H, G eigenvectors $\boldsymbol{\nabla}\times\phi_j$.

$$\begin{aligned} c &= -\frac{\omega}{\beta} \\ d &= -\frac{\beta}{\omega} \\ \theta = -\frac{\omega\beta}{2c}[1+cd] &= \beta^2 \\ \nu = -\frac{\omega\beta}{2d}[1+cd] &= \omega^2 . \end{aligned}$$

The perturbations are as follows:

$$\begin{aligned} \delta c &= \frac{\omega}{\beta^2}\delta\beta - \frac{\omega}{\beta}(\delta\epsilon\,\boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\phi) \\ \delta d &= \frac{\delta\beta}{\omega} - \frac{\lambda^2}{\omega\beta}(\delta\epsilon\,\phi, \phi) \end{aligned}$$

and,

$$\begin{aligned}\delta\theta &= \beta\,\delta\beta + \frac{\beta^3}{2\omega}\delta c - \frac{\omega\beta}{2}\delta d \\ &= \beta\,\delta\beta - \frac{\beta^2}{2}(\delta\epsilon\,\boldsymbol{\nabla}\times\phi, \boldsymbol{\nabla}\times\phi) + \frac{\lambda^2}{2}(\delta\epsilon\,\phi,\phi) \\ &= \lambda^2(\delta\epsilon\,\phi,\phi) \\ \delta\nu &= \frac{\omega^2}{\beta}\delta\beta + \frac{\omega^3}{2\beta}\delta d - \frac{\omega\beta}{2}\delta c \\ &= \frac{\omega^2}{\beta}\delta\beta - \frac{\omega^2\lambda^2}{2\beta^2}(\delta\epsilon\,\phi,\phi) + \frac{\omega^2}{2}(\delta\epsilon\,\boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\phi) \\ &= \omega^2(\delta\epsilon\,\boldsymbol{\nabla}\phi,\boldsymbol{\nabla}\phi)\,.\end{aligned}$$

The results for both families are summarised in Table 4.

Table 4: Coefficients δ and ν for the two fames of eigenvectors

	First family	Second family
$\theta+\delta\theta$	$-\omega^2$	$-\beta^2-\omega^2(\delta\epsilon\,\boldsymbol{\nabla}\psi,\boldsymbol{\nabla}\psi)$
$\nu+\delta\nu$	$\beta^2+\lambda^2(\delta\epsilon\,\phi,\phi)$	$\omega^2(1+(\delta\epsilon\,\boldsymbol{\nabla}\phi,\boldsymbol{\nabla}\phi))$

A.2 Are the Perturbed Eigenvalues $\boldsymbol{\beta^2+\delta\beta^2}$ Real?

We recall the main results concerning the E eigenvalues for the homogeneous waveguide obtained in [5].

Lemma 5
Let (λ_i,ϕ_i) and $(\mu_j\psi_j)$ denote the Dirichlet and Neumann eigenpairs of the Laplacian in domain D. The eigenvalues β_i^2 are classified into the following three families.

(a) *$\beta^2=\omega^2-\mu_j$ with μ_j distinct from all λ_i. The corresponding eigenvectors are curls:*

$$E=\boldsymbol{\nabla}\times\psi_j\,,$$

with multiplicity of β^2 equal to the multiplicity of μ_j.

(a) *$\beta^2=\omega^2-\lambda_i$ with λ_i distinct from all μ_j. The corresponding eigenvectors are gradients:*

$$E=\boldsymbol{\nabla}\phi_i\,,$$

with multiplicity of β^2 equal to the multiplicity of λ_i.

(c) $\beta^2 = \omega^2 - \mu_j = \omega^2 - \lambda_i$ *for* $\mu_j = \lambda_i$*. The corresponding eigenvectors are linear combinations of curls and gradients:*

$$E = a\boldsymbol{\nabla} \times \psi_j + b\boldsymbol{\nabla}\phi_i\,, \quad a, b \in \mathbb{C}\,,$$

with multiplicity of β^2 *equal to the sum of multiplicities of* μ_j *and* λ_i*.*

∎

Analogous results hold for the H eigenproblem (2.11).

In Section 3, under the assumption of distinct (simple) eigenvalues, we derived the following formula for the perturbation of E eigenvalues and eigenvectors:

$$(\beta_i^2 - \beta_k^2)(\delta E_i, E_k) + \delta\beta_i^2 \delta_{ik} = \omega^2(\delta\epsilon E_i, E_k) - \big(\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} E_i), E_k\big) + \big(\boldsymbol{\nabla} \operatorname{div}(\delta\epsilon E_i), E_k\big)\,. \tag{A.1}$$

For $k = i$, we obtain the formula for the perturbation $\delta\beta_i^2$:

$$\delta\beta_i^2 = \omega^2(\delta\epsilon E_i, E_i) - \big(\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} E_i), E_i\big) + \big(\boldsymbol{\nabla} \operatorname{div}(\delta\epsilon E_i), E_i\big)\,. \tag{A.2}$$

The case of multiple eigenvalues will be discussed momentarily. We investigate now whether the perturbed eigenvalues remain real. The first term on the right-hand side of (A.2) is always real. The second term is real as well as,

$$-\big(\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} E_i), E_i\big) = \big(\delta\epsilon \operatorname{div} E_i, \operatorname{div} E_i\big)\,.$$

Assumption D: Perturbation $\delta\epsilon$ is zero near the boundary ∂D.

We use the assumption to rewrite the third term as:

$$\big(\boldsymbol{\nabla} \operatorname{div}(\delta\epsilon E_i), E_i\big) = -\big(\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_i\big)\,.$$

Consider now the three cases discussed in Lemma 5. In the first case, $E_i = \boldsymbol{\nabla} \times \psi_i$, the term is zero. For the second case, $E_i = \boldsymbol{\nabla}\phi_i$, the term is:

$$-\big(\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_i\big) = -\big(\operatorname{div}(\delta\epsilon\boldsymbol{\nabla}\phi_i), -\lambda_i\phi_i\big) = \lambda_i\big(\operatorname{div}(\delta\epsilon\boldsymbol{\nabla}\phi_i), \phi_i\big) = -\lambda_i(\delta\epsilon\boldsymbol{\nabla}\phi_i, \boldsymbol{\nabla}\phi_i)$$

which is real as well. This concludes the analysis for the case when Dirichlet and Neumann eigenvalues of the Laplace operator are distinct.

Multiple Eigenvalues

The third case is the most difficult to analyze as we are dealing with a multiple eigenvalue. The standard perturbation theory does not cover the case. Formula (A.2) is invalid for the case of a multiple eigenvalue because the right-hand side may depend upon the choice of an eigenvector from the eigenspace. Additionally, formula (A.1) does not allow to compute components of δE_i corresponding to other eigenvectors from the same eigenspace. Instead, we have to restrict the original perturbed operator to the eigenspace corresponding to the multiple eigenvalue, and consider the perturbed eigenvalue problem directly. Let us assume for simplicity that both $\lambda = \mu$ are simple eigenvalues, i.e., we are dealing with a double eigenvalue $\beta^2 = \omega^2 - \lambda = \omega^2 - \mu$. Recall the linearized operator:

$$\begin{aligned} A_\epsilon E \quad &:= \boldsymbol{\nabla} \times \operatorname{curl} E - \omega^2 \epsilon E - \boldsymbol{\nabla}(\tfrac{1}{\epsilon} \operatorname{div}(\epsilon E)) \\ &\approx \boldsymbol{\nabla} \times \operatorname{curl} E - \omega^2 (1 + \delta\epsilon) E - \boldsymbol{\nabla}((1 - \delta\epsilon) \operatorname{div}((1 + \delta\epsilon) E)) \\ &\approx \underbrace{\boldsymbol{\nabla} \times \operatorname{curl} E - \omega^2 E - \boldsymbol{\nabla}(\operatorname{div} E)}_{=AE} + \underbrace{\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} E) - \boldsymbol{\nabla}(\operatorname{div}(\delta\epsilon) E) - \omega^2 \delta\epsilon E}_{\delta A\, E} . \end{aligned}$$

Let ψ, ϕ be Neumann and Dirichlet Laplace eigenvectors corresponding to a common eigenvalue $\mu = \lambda$, and let $E_1 = \boldsymbol{\nabla} \times \psi$ and $E_2 = \boldsymbol{\nabla}\phi$ be the two eigenvectors spanning the two-dimensional eigenspace corresponding to eigenvalue $\beta^2 = \omega^2 - \lambda$ of operator A. We have:

$$\begin{aligned} \delta A\, E_1 \quad &= -\boldsymbol{\nabla}\big(\operatorname{div}(\delta\epsilon E_1)\big) - \omega^2 \delta\epsilon E_1 \\ \delta A\, E_2 \quad &= \underbrace{(\boldsymbol{\nabla}(\delta\epsilon \operatorname{div} \boldsymbol{\nabla}\phi)}_{=-\lambda \boldsymbol{\nabla}(\delta\epsilon\phi)} - \big(\boldsymbol{\nabla} \operatorname{div}(\delta\epsilon \boldsymbol{\nabla}\phi)\big) - \omega^2 \delta\epsilon \boldsymbol{\nabla}\phi . \end{aligned}$$

The perturbed operator restricted to the eigenspace in terms of its spectral components looks as follows.

$$\begin{pmatrix} (\delta A\, E_1, E_1) & (\delta A\, E_1, E_2) \\ (\delta A\, E_2, E_1) & (\delta A\, E_2, E_2) \end{pmatrix} = \begin{pmatrix} -\omega^2 \underbrace{(\delta\epsilon \boldsymbol{\nabla} \times \psi, \boldsymbol{\nabla} \times \psi)}_{=:a} & (\lambda - \omega^2) \underbrace{(\delta\epsilon \boldsymbol{\nabla} \times \psi, \boldsymbol{\nabla}\phi)}_{=:c} \\ -\omega^2 \underbrace{(\delta\epsilon \boldsymbol{\nabla} \times \psi, \boldsymbol{\nabla}\phi)}_{=:c} & (\lambda - \omega^2) \underbrace{(\delta\epsilon \boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\phi)}_{=:b} - \lambda^2 \underbrace{(\delta\epsilon\phi, \phi)}_{=:d} \end{pmatrix}$$

Let further simplify the matrix notation to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

The characteristic equation for perturbed eigenvalues $\delta\lambda$ reads as follows.

$$(\delta\beta^2)^2 - (\delta\beta)(A + D) + (AD - BC) = 0$$

with the discriminant

$$\Delta = (A + D)^2 - 4(AD - BC) = (A - D)^2 + 4BC .$$

This gives:

$$\begin{aligned} \Delta \quad &= (-\omega^2 a - (\lambda - \omega^2) b + \lambda^2 d)^2 - 4\omega^2 (\lambda - \omega^2) c^2 \\ &= (\omega^2 a + (\lambda - \omega^2) b - \lambda^2 d)^2 - 4\omega^2 (\lambda - \omega^2) c^2 . \end{aligned}$$

Can discriminant Δ be negative ? It is certainly positive for $\lambda \leq \omega^2$. In other words, the propagating modes remain propagating or become purely evanescent. However, for $\lambda > \omega^2$, it is difficult to exclude the possibility of the discriminant becoming negative[7].

[7] Among other things, we tried to compute the derivative $d\Delta/d\lambda$ at $\lambda = \omega^2$, but we could not show that it must be positive.

The analysis of multiple eigenvalues can now easily be extended to the case of multiple Neumann and Dirichlet eigenvalues for the Laplace operator. For simplicity, let us consider only the case of double eigenvalues, relevant for the cylindrical waveguide and the step-index fiber. Recall again the formula for the perturbed operator,

$$(\delta A\, E, F) = -(\delta\epsilon \operatorname{div} E, \operatorname{div} F) + \big(\operatorname{div}(\delta\epsilon E), \operatorname{div} F\big) - \omega^2(\delta\epsilon E, F)\,.$$

In the case of multiple Neumann eigenvalue μ, $E_1 = \boldsymbol{\nabla}\times\psi_1, E_2 = \boldsymbol{\nabla}\times\psi_2$, and we obtain

$$\begin{pmatrix} (\delta A\, E_1, E_1) & (\delta A\, E_1, E_2) \\ (\delta A\, E_2, E_1) & (\delta A\, E_2, E_2) \end{pmatrix} = -\omega^2 \begin{pmatrix} \underbrace{(\delta\epsilon\boldsymbol{\nabla}\times\psi_1, \boldsymbol{\nabla}\times\psi_1)}_{=:A} & \underbrace{(\delta\epsilon\boldsymbol{\nabla}\times\psi_1, \boldsymbol{\nabla}\times\psi_2)}_{=:B} \\ \underbrace{(\delta\epsilon\boldsymbol{\nabla}\times\psi_2, \boldsymbol{\nabla}\times\psi_1)}_{=:B} & \underbrace{(\delta\epsilon\boldsymbol{\nabla}\times\psi_2, \boldsymbol{\nabla}\times\psi_2)}_{=:D} \end{pmatrix}$$

Upon inspection, we see that the discriminant of the characteristic equation, is always positive.

$$\delta = (A+D)^2 - 4(AD - B^2) = (A-D)^2 + 4B^2\,.$$

For the step-index fiber, $B = 0$, and $A = D$. The perturbed eigenvalue remains a double eigenvalue.

In the case of multiple Dirichlet eigenvalue λ, $E_1 = \boldsymbol{\nabla}\phi_1, E_2 = \boldsymbol{\nabla}\phi_2$, and we obtain

$$(\delta A\, E_i, E_j) = (\lambda - \omega^2)(\delta\epsilon\boldsymbol{\nabla}\phi_i, \boldsymbol{\nabla}\phi_j) - \lambda^2(\delta\epsilon\phi_i, \phi_j) \quad i,j = 1,2\,.$$

For the step-index fiber, the off-diagonal terms are zero, and the diagonal terms are equal. The perturbed eigenvalue remains double. One can show in a similar way that, in the general case, perturbed eigenvalues $\delta\beta^2$ remain real.

Lemma 6

Assume that the perturbation $\delta\epsilon$ vanishes near the boundary ∂D. In the first two cases discussed in Lemma 5 and in the third case for $\lambda \leq \omega^2$, the perturbed eigenvalues $\beta_i^2 + \delta\beta_i^2$ are always real. In the third case, for $\lambda > \omega^2$, they may be complex. ▮