CSEM Area A-CAM Preliminary Exam (CSE 386C-D)
May 31, 2016, 9:00 am – 12:00 pm

Work any 5 of the following 6 problems.

1. Let $X$ and $Y$ be normed linear spaces and $T : X \to Y$ a linear operator. We say that $T$ is bounded if it takes bounded sets to bounded sets.
   (a) Prove that $T$ is bounded if and only if there is a constant $C > 0$ such that
   $$\|Tx\|_Y \leq C\|x\|_X \quad \forall x \in X.$$  
   (b) Prove that $T$ is continuous if and only if $T$ is bounded.

2. Let $f_n$ be a sequence bounded both in $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$. Assume that $f_n$ converges pointwise almost everywhere to $f \in L^2(\mathbb{R}^d)$.
   (a) Prove that the entire sequence $f_n$ converges weakly to $f$ in $L^2(\mathbb{R}^d)$. [Hint: consider compactly-supported test functions.]
   (b) If additionally $\|f_n\|_{L^2} \to \|f\|_{L^2}$, prove that the entire sequence $f_n$ converges strongly to $f$ in $L^2(\mathbb{R}^d)$.

3. Define the linear operator $T : L^2([0, 1]) \to L^2([0, 1])$ by
   $$Tf(x) = \int_0^x \int_y^1 f(z) \, dz \, dy.$$  
   (a) Show that $T$ is self-adjoint.
   (b) Show that $T$ is compact.
   (c) Find an orthogonal basis for $L^2([0, 1])$ based on the eigenvalues of this operator. [Hint: differentiate twice and consider carefully the boundary conditions that must be satisfied.]

4. Let $\Omega \subset \mathbb{R}^d$ be a domain and let $w \in L^\infty(\Omega)$. Define
   $$H_w(\Omega) = \{ f \in L^2(\Omega) : \nabla(wf) \in (L^2(\Omega))^d \}.$$  
   (a) Give reasonable conditions on $w$ so that $H_w(\Omega) = H^1(\Omega)$.
   (b) Prove that $H_w(\Omega)$ is a Hilbert space. What is the inner product?
   (c) Suppose that $\Omega$ is bounded. Prove that there is a constant $C > 0$ such that for all $f \in H_w(\Omega)$ satisfying $\int_\Omega w(x) f(x) \, dx = 0$,
   $$\|f\|_{L^2(\Omega)} \leq C \left\{ \|\nabla(wf)\|_{L^2(\Omega)} + \|(1 - w)f\|_{L^2(\Omega)} \right\}.$$
5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \Omega$ and unit outward normal $\nu$. Given smooth functions $f(x), g(x)$ and $a(x)$, consider the following boundary-value problem (BVP) in non-divergence form:

$$\begin{align*}
-a \Delta u + u &= f \quad \text{in } \Omega \\
a \nabla u \cdot \nu &= g \quad \text{on } \partial \Omega.
\end{align*}$$

(a) Reformulate the BVP as a variational problem for $u \in H^1(\Omega)$. Indicate precisely the spaces for $f$ and $g$. Is the variational problem equivalent to the BVP?

(b) State reasonable (but not necessarily optimal) conditions on $a$ for which the Lax-Milgram Theorem would be applicable to the variational problem.

(c) Prove the existence of a solution to the variational problem.

6. Let $\phi(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $K(x) \in L^1(\mathbb{R})$. Use the contraction mapping principle to prove that the initial-value problem

$$\begin{align*}
\partial_t u &= K * (u + u^3), \quad x \in \mathbb{R}, \; t > 0, \\
u(x, 0) &= \phi(x)
\end{align*}$$

has a continuous and bounded solution $u = u(x, t)$, at least up to some time $T < \infty$. 
CSEM Area A-CAM Preliminary Exam (CSE 386C–D)
May 30, 2017, 9:00 a.m. – 12:00 noon

Work any 5 of the following 6 problems.

1. Let \( X \) be a NLS. Suppose \( x \in X, \{x_n\}_{n=0}^{\infty} \) is a sequence in \( X \), and \( M \subset X' \) is such that its span is dense in \( X' \). Prove that \( x_n \to x \) in \( X \) if and only if
   
   (i) the sequence \( \{\|x_n\|\}_{n=0}^{\infty} \) is bounded, and 
   (ii) for every \( f \in M \subset X' \), \( f(x_n) \to f(x) \).

2. Up to a constant multiple, the Legendre polynomial of degree \( n \) is

\[
P_n(x) = \frac{d^n}{dx^n} (x^2 - 1)^n.
\]

The Weierstrass approximation theorem says that for any function \( g \in C^0([-1,1]) \) and \( \epsilon > 0 \), there is a polynomial \( p \) such that \( |g(x) - p(x)| \leq \epsilon \) for any \( x \in [-1,1] \).

(a) Show that \( P_n \) has exact degree \( n \).

(b) Show that the Legendre polynomials form an orthogonal base for \( L^2((-1,1)) \). [Hint: For orthogonality, show that \( P_n \) is orthogonal to \( x^m \) for \( m < n \) using integration by parts.]

3. Let \( X \) be a Banach space and consider \( GL(X, X) \), the set of all isomorphisms from \( X \) to \( X \). Show that \( GL(X, X) \) is an open set of \( B(X, X) \). [Hint: Recall that \( (1 + x)^{-1} = \sum_{n=0}^{\infty} (-x)^n \).]

4. Consider the boundary value problem:

\[
\begin{align*}
-u_{xx} + (1 + y)u &= f, & \text{for } (x, y) \in (0, 1)^2, \\
u(0, y) &= 0, & u(1, y) = \cos(y), & \text{for } y \in (0, 1).
\end{align*}
\]

(a) Find the associated variational problem. In which space should \( f \) lie?

(b) Show that there exists a unique solution to this problem.
5. Let $(X, d)$ be a complete metric space and $T : X \to X$ be a contraction with contraction constant $\theta \in [0, 1)$ and fixed point $x \in X$. Suppose that $S : X \to X$ is an approximation to $T$ in the sense that for some $\epsilon > 0$,

$$d(T(z), S(z)) \leq \epsilon \quad \text{for all } z \in X.$$  

For fixed $x_0 = y_0 \in X$ and integer $m \geq 1$, let $x_m = T(x_{m-1})$ and $y_m = S(y_{m-1})$.

(a) Use induction to show that

$$d(x_m, y_m) \leq \frac{1 - \theta^m}{1 - \theta}.$$  

(b) We know that $d(x_m, x) \leq \frac{\theta^m}{1 - \theta}d(x_0, x_1)$. Use this fact to prove that

$$d(y_m, x) \leq \frac{1}{1 - \theta} (\epsilon + \theta^m d(y_0, y_1)).$$

6. Fix $g \in L^2(\mathbb{R}^d)$. For any $u \in H^1(\mathbb{R}^d)$, we define

$$J(u) = \int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2 - gu) \, dx.$$  

(a) Find the Euler-Lagrange equation associated to $J$.

(b) Find all the critical points of $J$ [Hint: You may use the Fourier transform.]

(c) Are those critical points maxima or minima of $J$?
1. \( X \text{ NLS}, x \in X, \exists x_n \leq x, M \leq x, \overline{\text{span}(M)} = X \)

\[ x_n \to x \iff (i) \text{ all bounded} \\
(ii) f(x_n) \to f(x) \forall f \in M. \]

(\Rightarrow) If \( x_n \to x \), we know that
\[ f(x_n) \to f(x) \quad \forall f \in X', \]
so (ii) holds.

Now for a fixed \( f \in X' \),
\[ |f(x_n)| \text{ is bounded} \quad \text{(since } f(x_n) \text{ converges)} \]
so
\[ |f(x_n)| = |E_{x_n}^f| \leq C_f \forall f \in X'. \]

By UBP:
\[ |E_{x_n}^f| \leq C \]
That is, \( \|E_{x_n}\| \leq \|x_n\| \text{ bounded}. \)

(\Leftarrow) Let \( g \in X' \), \( \epsilon > 0 \) and choose \( n, a; \epsilon \in \mathbb{F} \),
\( f_i \in M \) for \( i = 1, 2, \ldots, n \) s.t.
\[ \|g_i - \sum a_i f_i\| \leq \epsilon. \]

Then
\[ g(x_n) - g(x) = g(x_n - x) = (g - h)(x_n - x) + h(x_n - x) \]

\[ \Rightarrow \]
\[ |g(x_n - x)| \leq \|g - h\| (\|x_n\| + \|x\|) + |h(x_n - x)| \leq \epsilon \left( M + \|x\| + \|h(x_n - x)\| \right) \to 0 \text{ as } \epsilon \to 0, \ n \to \infty. \]
2. \( P_n = \frac{d^n}{dx^n} (x^2 - 1)^n \)

\( x \in \mathbb{C}, \epsilon > 0 \Rightarrow \exists \delta > 0 \) for \( |x - \rho(x)| \leq \epsilon \) for all \( x \in [\rho - \delta, \rho + \delta] \)

(a) \((x^2 - 1)^n \in P_{2n} \Rightarrow P_n \in P_n^* \)

Leading term of \((x^2 - 1)^n\) is \( x^{2n} \)

\Rightarrow leading term of \( P_n \) is \( \frac{(2n)!}{n!} x^n \)

(b) The set is clearly linearly independent.

For \( 1 \), ETS \( \perp \) on \( P_n \) to \( x^m \), \( m < n \).

\[ \sum_{i=1}^{n} P_n x^m = \sum_{i=1}^{n} D^n (x^2 - 1)^n x^m \]

\[ = D^{n-1} (x^2 - 1)^n x^m l^m - m \sum_{i=1}^{n} D^{n-1} (x^2 - 1)^n x^{m-1} \]

all terms have \((x^2 - 1)\) \Rightarrow term vanishes

\[ = \cdots \]

\[ = \pm \sum_{i=1}^{n} D(x^2 - 1)^n \cdot 0 = 0. \]

For density, note for \( f \in L^2, \exists g \in L^2 \) s.t. \( \|f-g\| \leq \epsilon \). Wannier gave \( \rho \times g \).

Now \( P \in \text{span } \{P_1, \ldots, P_n\} \)

for some \( n < \infty \), so \( \|f-P\| \leq \|f-g\| + \|g-P\| \leq \epsilon + 2 \epsilon = 3 \epsilon \to 0 \).

Thus we have \( \epsilon \) is bounded.
3. Let \( A \in \text{GL}(X, X) \).

For \( \varepsilon \) to be determined, consider

\[ B_\varepsilon(A) = \{ T \in B(X, X) : \|T - A\| < \varepsilon \} \]

Now

\[ T = T - A + A = A(I + A^{-1}(T - A)) \]

This is the composition of 2 invertible maps

if (claim) \( \|A^{-1}(T - A)\| < 1 \)

which is true if \( \|T - A\| < \frac{1}{\|A^{-1}\|} \equiv \varepsilon \).

To prove the claim (i.e., \( I + R \) inv. if \( \|R\| < 1 \))

\[ S_N = \sum_{n=0}^{N} (-R)^n = I - R + R^2 - \cdots + (-R)^N R^N \]

\[ S_N (I + R) = I + (-R)^N R^{N+1} = (I + R) S_N \]

Now \( \|R^{N+1}\| \leq \|R\|^{N+1} \to 0 \) as \( N \to \infty \)

Thus \( S_N \) is Cauchy \( \Rightarrow S_N \to S \in B(X, X) \)

So \( S_N \to (I + R)^{-1} \).
4. \begin{equation*}
\begin{aligned}
\Delta u + (1+\gamma) u &= f & (x, y) \in (0,1)^2 \\
u(0,y) &= 0, & u(1,y) &= \text{cay y}
\end{aligned}
\end{equation*}

(a) Let

\[ H = \{ v \in L^2((0,1)^2) : v_x \in L^2((0,1)^2) \} \]

\[ H_0 = \{ v \in H : v(0,y) = v(1,y) = 0 \ \forall y \} \]

The wave at \( x=0,1 \) exists because for a.e. \( y \), \( v(\cdot,y) \in H^1((0,1)) \).

Find \( u \in H_0 + x \text{cay y} \) st.

\[ (u_x, v_x) + (1+\gamma) u_x v_x = (f, v) \quad \forall v \in H_0 \]

We want \( f \in (H_0)' \).

(b) Let \( a(u,v) = (u_x, v_x) + (1+\gamma) u_x v_x \)

**Note:** \( H \) is Hilbert with IP

\[ \langle u, v \rangle = (u_x, v_x) + (u, v) \]

Completeness follows from the completeness of \( L^2 \):

\[ u_n \text{ Cauchy} \Rightarrow u_n \xrightarrow{L^2} u, \quad u_n \xrightarrow{H^1} \]

But

\[ \langle u_n, \varphi \rangle = \langle u_n, \varphi_x \rangle \Rightarrow \langle u, \varphi \rangle \]

\[ \Rightarrow \quad \varphi = u_x. \quad \text{Thus} \quad u_n \xrightarrow{H} u. \]

Now

\[ |a(u,v)| \leq ||u_x|| ||v_x|| + 2 ||u|| ||v|| \leq 3 ||u|| ||v|| \]

and

\[ a(u,v) = ||u_x||^2 + (1+\gamma) u_x v_x \geq ||u_x||^2 + ||v||^2 \]

\[ \text{Poincaré} \Rightarrow ||u_x|| \geq \gamma ||u_x|| \quad \forall y \Rightarrow ||u_x|| \geq \gamma ||u|| \]

\[ \text{Thus} \quad a(u,v) \geq \frac{1}{2} \min (1, \gamma) ||u||^2 \]

\[ \text{Lax - Milgram} \Rightarrow \exists! \text{ soln.} \]
(5. \ (x, d) \ \ T: X \to X \text{ contraction, } \Theta, \ \ T x = x.
S: X \to X, \ d(T(x), S(x)) \leq \varepsilon \ \ \forall x \in X.

x_0 = x_0, \ \ x_m = T(x_{m-1}), \ \ y_m = S(y_{m-1})

(a) We have that
\[ d(T(x), T(y)) \leq \Theta d(x, y) \]

Now,
(1) \[ d(x_0, y_0) = 0 \leq \varepsilon \leq \frac{1-\Theta}{1-\theta} = 0. \]
(2) Suppose \[ d(x_m, y_m) \leq \varepsilon \frac{1-\Theta^m}{1-\theta} \]

Consider \[ d(x_{m+1}, y_{m+1}) = d(Tx_m, Sy_m) \]
\[ \leq d(Tx_m, Ty_m) + d(Ty_m, Sy_m) \]
\[ \leq \Theta d(x_m, y_m) + \varepsilon \]
\[ \leq \varepsilon (\frac{1-\Theta^m}{1-\theta} + 1) = \varepsilon \frac{1-\Theta^m}{1-\theta} \]

(b) \[ d(x_m, x) \leq \frac{\Theta^m}{1-\theta} d(x_0, x_1) \]
\[ d(y_m, x) \leq d(y_m, x_m) + d(x_m, x) \]
\[ \leq \varepsilon \frac{1-\Theta^m}{1-\theta} + \frac{\Theta^m}{1-\theta} d(x_0, x_1) \]
\[ = \frac{1}{1-\theta} \left[ \varepsilon (1-\Theta^m) + \Theta^m (d(y_0, x_1) + d(y_1, x_1)) \right] \]
\[ \leq \varepsilon \]
\[ \leq \frac{1}{1-\theta} (\varepsilon + \Theta^m d(y_0, y_1)) \]
(a) $F(u) = \int_{\mathbb{R}^d} (|D^2 u|^2 + |u|^2 - g u) \, dx$

\[ \frac{\partial}{\partial u} F - \frac{2}{\partial (\partial F)} \left( \frac{\partial F}{\partial u_x} \right) = 0 \]

\[ \Rightarrow \quad 2u - g - 2 \sum_j u_{x_j x_j} = 0 \]

\[ \Rightarrow \quad -\Delta u + u = \frac{1}{2} g \]

(b) \( (1 + 13^2) \hat{u} = \frac{1}{2} \hat{g} \)

\[ \Rightarrow \quad \hat{u} = \frac{1}{2} \frac{\hat{g}}{1 + 13^2} \Rightarrow u = \frac{1}{2} \left( \frac{\hat{g}}{1 + 13^2} \right)^\nu \]

\[ \Rightarrow \quad u = \frac{1}{2} \left( \frac{1}{2\pi} \left( \frac{1}{1 + 13^2} \right) \right)^\nu \]

(c) Sine

\[ \int \left( (\nabla (u + v))^2 + |u + v|^2 - g(u + v) \right) \, dx \]

\[ = \epsilon \int (\nabla u + \nabla v)^2 + 2\nabla u \cdot \nabla v - g u + \epsilon^2 (|\nabla u|^2 + |v|^2) \, dx \]

\[ \Rightarrow 0 \quad \Rightarrow \text{minima.} \]
CSEM Area A-CAM Preliminary Exam (CSE 386C–D)
May 31, 2018, 9:00 a.m. – 12:00 noon

Work any 5 of the following 6 problems.

1. The set $\mathcal{X}$ of all sequences $\{x_n\}_{n=1}^{\infty}$ of complex numbers is a vector space. Let $0 < p < 1$ and let $X \subset \mathcal{X}$ be the set of all sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

(a) Show that $X$ is a vector space. [Hint: Show that $|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p).$]

(b) Show that the map taking $\{x_n\}_{n=1}^{\infty} \in X$ to $\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ is not a norm on $X$.

(c) Show that the map $d : X \times X \to \mathbb{R}$ defined by $d(\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ is a metric on $X$.

2. Open Mapping Theorem.

(a) State the Open Mapping Theorem.

(b) Suppose that $\| \cdot \|$ and $\| \cdot \|'$ are two norms on a vector space $X$. Suppose that both $(X, \| \cdot \|)$ and $(X, \| \cdot \|')$ are complete and there is a constant $C > 0$ such that

$$\|x\| \leq C\|x\|' \quad \text{for all } x \in X.$$

From the Open Mapping Theorem, show that the two norms are equivalent.

3. Let $\Omega = [a, b]$, $p, q \in (1, \infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$. Let $v \in L^q(\Omega)$. For every $u \in L^p(\Omega)$ define a function $Au$ by setting

$$(Au)(t) = \int_{a}^{t} v(s) u(s) \, ds \quad \text{for all } t \in \Omega.$$

(a) Show that $A$ maps $L^p(\Omega)$ into $L^q(\Omega)$ and is continuous.

(b) Explain why $A : L^p(\Omega) \to L^q(\Omega)$ is compact.

4. Suppose $(X, d_X)$ and $(Y, d_Y)$ are metric spaces. $Y$ is complete, $A \subset X$ is dense, and $T : A \to Y$ is uniformly continuous. Prove that there is a unique extension $\tilde{T} : X \to Y$ which is uniformly continuous.
5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $f \in L^2(\Omega)$, and $\epsilon > 0$. Suppose $u_\epsilon$ satisfies

\[-\epsilon \Delta u_\epsilon + u_\epsilon = f \quad \text{in } \Omega,
\]

\[u_\epsilon = 0 \quad \text{on } \partial \Omega.
\]

Show $u_\epsilon \to f$ in $L^2(\Omega)$ as $\epsilon \to 0$. [Hint: Bound appropriate norms of $u_\epsilon$ and $\sqrt{\epsilon} u_\epsilon$.]

6. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary and outer unit normal $\nu$. Let $b$ a constant vector and $f \in L^2(\Omega)$. Consider the fourth order problem

\[u + \Delta^2 u + b \cdot \nabla u = f \quad \text{in } \Omega,
\]

\[u = 0 \quad \text{and} \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial \Omega.
\]

(a) State the Lax-Milgram Theorem for a real Hilbert space.

(b) Develop a suitable variational form for the problem. [Be careful to handle the boundary values and define the Hilbert spaces you use.]

(c) Give a hypothesis on $|b|$ so that the Lax-Milgram theorem provides a unique solution to your variational problem. [Hint: Gårding's inequality gives a $C_G > 0$ such that $\|v\|^2_{H^2} \leq C_G \{\|u\|^2 + \|\Delta u\|^2\}$ for all $v \in H^2_0$.]
(a) (i) $\langle x, y \rangle = x \cdot y$

$\frac{1}{2} \| x + \frac{p}{q} y \|_p \leq \frac{1}{2} \left( \| x \|_p + \| y \|_p \right)$

$\Rightarrow \frac{\| x + y \|_p}{2} \leq \frac{1}{2} \left( \| x \|_p + \| y \|_p \right)$

$\Rightarrow \sum |x_n + y_n|^p \leq 2^{p-1} \left( \sum |x_n|^p + \sum |y_n|^p \right) < \infty$

(ii) $d \langle x, y \rangle = \| x - y \|_p$

$\sum |x_n - y_n|^p < \infty$.  

(b) Consider the $\Delta$ norm, for

$x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, 0, \ldots)$

$\Rightarrow \left( \sum |x_n - y_n|^p \right)^{1/p} = 2^{1/p} > 2$

$\Rightarrow \left( \sum |x_n|^p \right)^{1/p} + \left( \sum |y_n|^p \right)^{1/p} = 1 + 1 = 2$

But $2^{1/p} > 2$, so not a norm.

(c) $d \langle x, y \rangle = \sum |x_n - y_n|^p$

(i) $d \langle x, y \rangle = 0 \iff d \langle x, y \rangle = 0 \iff x = y$ for $\forall n$

(ii) $d \langle x, y \rangle = d \langle y, x \rangle$

(iii) $d \langle x, y \rangle \leq d \langle x, z \rangle + d \langle z, y \rangle$

$|x_n - y_n|^p = |x_n - z_n|^p + |z_n - y_n|^p$

$\leq 2^{p-1} \left( |x_n - z_n|^p + |z_n - y_n|^p \right) \left( \text{see (a)} \right)$

$\Rightarrow \sum |x_n - y_n|^p \leq 2 \sum |x_n - z_n|^p + \sum |z_n - y_n|^p$.  \(\blacksquare\)
2. Open Mapping
(a) Let $X, Y$ be Banach.
If $T : X \to Y$ is bounded, linear, surjective,
then $T$ is open (maps open sets to open sets).

(b) If $\| \cdot \|, \| \cdot \|'$, $(X, \| \cdot \|)$ & $(X, \| \cdot \|')$ complete,
$\|x\| \leq C \|x\|'$ \forall x \in X.$
Consider
$i : (X, \| \cdot \|') \to (X, \| \cdot \|)$ identity.
Note $i$ is bounded, surjective (and linear).
so $i$ is open $\Rightarrow$
$i : B_{\varepsilon}' \to Q \subseteq X$ open.
Now $\exists \varepsilon > 0$ s.t. $B_{\varepsilon} \subseteq Q$.
Given $x \in X,$
\[ \frac{\varepsilon}{2} \frac{x}{\|x\|} \in B_{\varepsilon} \cap B_{1} \]
$\Rightarrow \quad \frac{\varepsilon}{2} \frac{\|x\|'}{\|x\|} \leq 1 \Rightarrow \|x\|' \leq \frac{2}{\varepsilon} \|x\|.$
Thus the norms are equivalent.
3. \( Q = [a, b], \quad \forall p, q \in (1, \infty), \quad \frac{1}{p} + \frac{1}{q} = 1, \quad v \in L^q(\Omega). \quad A : L^p(\Omega) \rightarrow \text{Rem}. \)

\[(Au)(t) = \int_a^b v(s)u(s)ds \quad \forall t \in \Omega.\]

(a) \( A : \rightarrow L^p(\Omega) \)

\[\int_a^b |(Au)(t)|^p = \int_a^b \left( \int_a^b |v(s)|u(s)ds \right)^p dt \]

\[= \int_a^b \left( \|v\|_{L_b} \|u\|_{L_p} \right)^p dt \]

\[\leq (b-a)^p \|v\|_{L_b} \|u\|_{L_p}^p \]

\[\Rightarrow \|Au\|_{L_p} \leq (b-a)^p \|v\|_{L_b} \|u\|_{L_p}^p \]

So, \( A \) maps into \( L^p \) and is continuous.

(b) \( Au(t) = \int_a^b v(s)K_{a,s}u(s)ds \)

\[v \text{ density \& } C(\Omega) \text{ in } L^p, L^q \]

\[v_j \to v, \quad u_k \rightharpoonup u \]

\[A_j u = \lim_k A_j u_k, \quad A_j : C(\Omega) \to C(\Omega) \]

is compact by Ascoli-Arzela

\[A_j : L^p \to L^p \text{ compact by density} \]

\[A_j \to A \text{ is also compact.} \]
4. \( X, Y \) metric, \( Y \) complete, \( A \subseteq X \) dense
\[ T: A \rightarrow Y \text{ unif. cont. (on A)}. \]

\[ \forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \ \text{s.t.} \ A \ni x, y \in A, \]
\[ d_y(Tx, Ty) < \varepsilon \text{ whenever } d_x(x, y) < \delta_\varepsilon. \]

Let \( x \in X \) and \( x_n \rightarrow x, x_n \in A \)

Claim: \( T x_n \) Cauchy.

\( T x_n \) Cauchy \( \Rightarrow \exists N > 0 \) st.
\[ d(x_n, x_m) < \delta_\varepsilon \quad \forall \ n, m > N. \]

\( Y \) complete \( \Rightarrow \overline{T}(x_n) \rightarrow \xi = \overline{T}(x). \)

Claim: \( T \) unif. cont. If so, then \( \overline{T}x = Tx \ \forall x \in A \)
and \( \overline{T} \) is unique (since \( A \) dense).

Now \[ d_T(Tx, Ty) \leq d_T(Tx, T x_n) + d_T(T x_n, T y_m) + d_T(T y_m, Ty) \]
where \( x_n \rightarrow x, y_m \rightarrow y, x_n, y_m \in A \).

If \( N > 0 \) chosen so
\[ d_x(x_n, x_m) < \frac{\delta_\varepsilon}{3} + \frac{\delta_\varepsilon}{3} + \frac{\delta_\varepsilon}{3} \]
for \( n, m > N \) and \( d_x(x_n, y_m) < \frac{\delta_\varepsilon}{3}.
\]
\[ \Rightarrow d_y(Tx, Ty) \leq 3 \varepsilon. \text{ For } n, m \text{ large}. \]
5. \( \Omega \subset \mathbb{R}^d, \, f \in L^2, \, \varepsilon > 0 \)

\[ -\varepsilon \Delta u_\varepsilon + u_\varepsilon = f, \, \Omega \text{ j } u_\varepsilon = 0, \, \partial \Omega \]

Equi. to variational form is.

\[ \varepsilon (\nabla u_\varepsilon, \nabla v) + (u_\varepsilon, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \]

\( \varepsilon \| \nabla u_\varepsilon \|^2 + \| u_\varepsilon \|^2 = (f, u_\varepsilon) \leq \frac{1}{2} \| f \|^2 + \frac{1}{2} \| u_\varepsilon \|^2 \]

\( \Rightarrow \quad \varepsilon \| \nabla u_\varepsilon \|^2 + \frac{1}{2} \| u_\varepsilon \|^2 \leq \frac{1}{2} \| f \|^2 \]

\( \Rightarrow \quad \sqrt{\varepsilon} u_\varepsilon \text{ bounded in } H_0^1 \]

\[ \Rightarrow \quad u_\varepsilon \rightharpoonup u \text{ in } L^2 \]

\[ \sqrt{\varepsilon} u_\varepsilon \rightharpoonup \varrho \text{ in } H_0^1 \Rightarrow \sqrt{\varepsilon} u_\varepsilon \rightarrow \varrho \text{ in } L^2 \]

But \( \sqrt{\varepsilon} u_\varepsilon \rightarrow 0 \Rightarrow \varrho = 0 \)

Thus,

\[ 0 + (u, v) = (f, v) \quad \forall v \in H_0^1(\Omega) \]

\[ \Rightarrow \quad (u - f, v) = 0 \Rightarrow u = f \]
6. \( \Omega \subset \mathbb{R}^d \), \( b \in L^2 \)

\[
\begin{cases}
    \Delta^2 u + b \cdot \nabla u = f, & \Omega \\
    u = 0, & \nabla u \cdot n = 0, & \partial \Omega
\end{cases}
\]

(a) Let \( \mathcal{H} \) be a real Hilbert space with closed subspace \( \mathcal{H} \). Let \( B : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \)
be bilinear s.t.

(c) \( |B(x, y)| \leq M \|x\| \|y\| \quad \forall x, y \in \mathcal{H} \), \( \text{and} \)

(c) \( B(x, x) \geq \varepsilon \|x\|^2 \quad \forall x \in \mathcal{H} \), \( \text{coercive} \)

If \( x_0 \in \mathcal{H} \), \( F \in \mathcal{H}^* \), then \( \exists! u \in \mathcal{H} + x_0 \) s.t.

\[
B(u, v) = F(v) \quad \forall v \in \mathcal{H}.
\]

\[
\|u\| \leq \frac{1}{\varepsilon} \|F\| + (\frac{M}{\varepsilon} + 1) \|x_0\|
\]

(b) Find \( u \in \mathcal{H} = \{w \in L^2 : w = \omega, \Delta w = 0 \text{ on } \partial \Omega \} \) s.t.

\[
\begin{align*}
    (u, v) + (u, \Delta v) + (b \cdot \nabla u, v) &= (f, v) \quad \forall v \in \mathcal{H}.
\end{align*}
\]

(c) \( \text{LHS} = B(u, v) \), which is cont. For coercivity:

\[
\begin{align*}
    \|u\|^2 + \|\Delta u\|^2 + (b \cdot \nabla u, u) \\
    &\geq \|u\|^2 + \|\Delta u\|^2 - 16 \|\Delta u\| \|u\| \\
    &\geq \|u\|^2 + \|\Delta u\|^2 - \varepsilon \|u\|^2 \\
    &= (1 - \varepsilon) \|u\|^2 + \|\Delta u\|^2 - \varepsilon \|u\|^2 \\
    \text{Now } \epsilon (\|u\|^2 + \|\Delta u\|^2) &\geq \|\Delta u\|^2 \Rightarrow \text{Need}
\end{align*}
\]

\[
(1 - \varepsilon) \frac{\varepsilon}{\varepsilon} |b| < c \quad \Rightarrow \quad |b| < \frac{c}{\varepsilon}
\]
1. Let $X$ be a Banach space with dual space $X^*$ and duality pairing $\langle \cdot, \cdot \rangle$, and let $A, B : X \to X^*$ be linear maps.

(a) State the Closed Graph Theorem and what it means for an operator to be closed.

(b) Assuming $\langle Ax, y \rangle = \langle Ay, x \rangle$ for all $x, y \in X$, show that $A$ is bounded.

(c) Assuming $\langle Bx, x \rangle \geq 0$ for all $x \in X$, show that $B$ is bounded. [Hint: Suppose $B$ is not continuous at 0, so $x_n \to 0$ but $Bx_n \to y \neq 0$. For $w \in X$ such that $\langle y, w \rangle > 0$, consider $x_n + \epsilon w$.]

2. Let $\Omega = [0, 1]$ and $1 \leq p < \infty$ be given and consider the sequence of functions $g_n \in L^p(\Omega)$ defined by $g_n(x) = n^{1/p}e^{-nx}$. Show that as $n \to \infty$:

(a) $g_n(x)$ converges pointwise to zero for each fixed $x \in (0, 1]$ and for any $p \geq 1$;

(b) $g_n$ does not converge strongly to zero in $L^p(\Omega)$ for any $p \geq 1$;

(c) $g_n$ converges weakly to zero in $L^p(\Omega)$ if $p > 1$, but not if $p = 1$.

3. Prove the Mazur Separation Lemma, which says that if $X$ is a normed linear space, $Y$ a linear subspace of $X$, $w \in X$ but $w \not\in Y$, and $d = \text{dist}(w, Y) = \inf_{y \in Y} \|w - y\|_X > 0$,

then there exists $f \in X^*$ such that $\|f\|_X \leq 1$, $f(w) = d$, and $f(z) = 0$ for all $z \in Y$. [Hint: Begin by working in $Z = Y + Fw$.]

4. Let $\Omega = (0, 1)^2$ and consider the boundary value problem (BVP)

$$\begin{align*}
-u_{xx} + u_{xy} - u_{yy} &= f & \text{in } \Omega, \\
-u_x + u_y - u &= g & \text{on } \Gamma_L = \{(0, y) : y \in (0, 1)\}, \\
u &= 0 & \text{on } \Gamma_* = \partial \Omega \setminus \Gamma_L.
\end{align*}$$

Let $H = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_*\}$, which is a Hilbert space.

(a) Find the corresponding variational problem for $u \in H$ and test functions $v \in H$. Also give the function spaces containing $f$ and $g$.

(b) Show the general Poincaré type inequality: There exists $\gamma > 0$ such that

$$\|\nabla v\|_{L^2(\Omega)}^2 + \int_{\Gamma_L} v^2 \geq \gamma \|v\|_{L^2(\Omega)}^2 \quad \forall v \in H.$$

(c) Show that there is a unique solution to the variational problem.
5. For fixed $T > 0$, let $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and Lipschitz continuous in the second argument, i.e., there is some $L > 0$ such that

$$\|g(t, v) - g(t, w)\| \leq L \|v - w\| \quad \forall v, w \in \mathbb{R}^d, t \in [0, T],$$

where $\| \cdot \|$ is the norm on $\mathbb{R}^d$. For any $u_0 \in \mathbb{R}^d$, consider the initial value problem (IVP) $u'(t) = g(t, u(t))$ and $u(0) = u_0$.

(a) Write this IVP as the fixed point of a functional $G : C^0([0, T]; \mathbb{R}^d) \to C^0([0, T]; \mathbb{R}^d)$.

(b) Normally, we use the $L^\infty([0, T])$-norm for $C^0([0, T]; \mathbb{R}^d)$. Show that the function $||| \cdot ||| : C^0([0, T]; \mathbb{R}^d) \to [0, \infty)$, defined by

$$|||v||| = \sup_{0 \leq t \leq T} (e^{-Lt} \|v(t)\|),$$

is a norm equivalent to the $L^\infty([0, T])$-norm.

(c) In terms of this new norm, show that $G$ is a contraction.

(d) Explain how we conclude that there is a unique solution $u \in C^1([0, \infty); \mathbb{R}^d)$ to the IVP for all time.

6. Consider finding extremals to the problem: Find $u, v \in C^1_0([0, 1])$ minimizing

$$F(u, v, u', v') = \int_0^1 \left( (u')^2 + (v')^2 + 2uv \right) dx.$$

(a) Find the Euler-Lagrange (EL) equations for this problem.

(b) Reduce the EL equations to a single equation and find its solution. [Hint: The fourth roots of unity are $\pm 1$ and $\pm i$.]

(c) Find the extremal to the problem, up to solving a $4 \times 4$ system of linear equations.

(d) If we add the constraint that $\int_0^1 u^2 v' \, dx = 0$, what EL equations do we get?
1. X Banach, \( A, B : X \to X^* \) linear.

(a) Closed Graph Theorem:
Let \( X \) and \( Y \) be Banach spaces and \( T : X \to Y \) linear. Then: \( T \) is continuous (bounded)
\[ \iff T \text{ is closed.} \]
\( T \) is closed if whenever \( x_n \to x, T x_n \to y \), then \( y = T x \).

(b) \( \langle Ax, y \rangle = \langle A x, y \rangle \) \( \forall x, y \in X \).
Suppose \( x_n \to x \), \( A x_n \to y \).
Then \( \langle A x_n, z \rangle = \langle A z, x_n \rangle \) \( \forall z \in X \)
\[ \rightarrow \langle A x, z \rangle = \langle A z, x \rangle = \langle A x, z \rangle \]
\[ \Rightarrow A x = y, \text{ and } A \text{ continuous (bounded).} \]

(c) \( \langle B x, x \rangle \geq 0 \) \( \forall x \in X \).
ETS for \( x_n \to 0 \), \( B x_n \to y \).
Suppose not: \( y \neq 0 \) so \( \exists w \in X, \langle y, w \rangle \neq 0 \).
Consider
\[ 0 \leq \langle B(x_n + \varepsilon w), x_n + \varepsilon w \rangle \]
\[ \rightarrow \langle y + \varepsilon B w, \varepsilon w \rangle \]
\[ = \varepsilon \langle y, w \rangle + \varepsilon^2 \langle B w, w \rangle \]
\[ \geq 0. \]
Let \( \varepsilon \to 0 \) so last term negligible.
Contradiction, since \( \varepsilon \) can be + or -.
So \( y = 0 \) and \( B \) cont.
2. \( D = [0, 1] \), \( 1 < p < \infty \), \( g_n(x) = \frac{n^p}{e^{nx}} - nx^{\frac{p}{n}} \).

(a) \( g_n(x) = \frac{n^p}{e^{nx}} \to \frac{1}{x^{\frac{p}{n}}} \quad \text{Höpital} \quad \frac{1}{x e^{nx}} \to 0. \)

(b) \[ \|g_n\|_p = \int_0^1 n e^{-nx} \, dx = -\frac{1}{n} e^{-nx} \bigg|_0^1 = \frac{1}{n} \left(1 - e^{-n} \right) \to \frac{1}{p} = 0. \]

So \( g_n \to 0. \)

(c) Let \( g(x) = \frac{1}{x^{\frac{p}{n}}} \), \( \frac{1}{p} + \frac{1}{n} = 1. \)

If \( p > 1 \), then by density suppose \( h \in L^p \).

Then \( \exists x_\ast > 0 \) s.t. \( h(x) = 0 \) for \( x < x_\ast \).

Now \[ \left| \int_0^1 g_n h \right| \leq \int_0^1 g_n \, |h|, \]

so suppose \( h \geq 0. \)

Note \( \frac{d}{dn} \left( \frac{n^p}{e^{nx}} h \right) = \frac{n^p}{e^{nx}} \left( \frac{1}{p} - x \right) h \)

\( \leq 0 \) for \( n \) large enough, and \( x \geq x_\ast \) (so \( \forall x \)).

Thus \( g_n h \) is monotone, so \( \text{MCT} \Rightarrow \)

\[ \lim \int g_n h = \int \lim g_n h = 0. \]

That is, \( g_n \to 0. \)

But for \( p = 1 \), \( L^1 = L^\infty \). Consider \( h \equiv 1. \)

Then \[ \int_0^1 n e^{-nx} = -e^{-nx} \bigg|_0^1 = 1 - e \to 1 \neq 0. \]
3. \( \mathbf{X} \) is NLS, \( Y \) is subspace, \( w \in X \setminus Y \).

\[
d = \text{dist}(w, Y) = \inf_{y \in Y} \|w - y\| > 0.
\]

Work in \( Z = Y + \text{lin} w \)

\( z \in Z \Rightarrow \exists! y \in Y, \lambda \in \mathbb{F} \) s.t.

\[
z = y + \lambda w
\]

(for otherwise \( \exists y \neq y' \), \( \lambda \neq \lambda' \), \( w \not\in Y \)).

Let \( \varphi : Z \to \mathbb{F} \)

\( \varphi(z) = \lambda d \) (well defined).

Now \( \varphi \) is linear and

\[
\| \varphi(y + \lambda w) \|_2 = \| \varphi(y) + \lambda \varphi(w) \|_2 = \| \lambda \varphi(w) \|_2 = \lambda \| \varphi(w) \|_2
\]

\[
= \inf_{z \in \mathbb{F}} \frac{\|w - z\|}{\|y + \lambda w\|} \leq 1.
\]

\[
\Rightarrow \| \varphi \| \leq 1.
\]

Extend (using Hahn–Banach) to \( X \).
4. \( Q = (0,1)^2 \)

\[ \begin{cases} -u_{xx} + u_{yy} - u_yg = f, & x, \ y \\
-\Delta u = u = 0, & \Gamma \\
u = \partial \mathcal{E}/\sqrt{L} \end{cases} \]

\[ H = \{ v \in H^1 : v = 0 \text{ on } \Gamma \} \]

(a) \( (u_x, v_x) - \langle u_x, v \rangle_{L^2} - (u_y, v_x) + \langle u_y, v \rangle_{L^2} + (u_y, v_y) \) 

\[ \Rightarrow \]

\[ B(y, v) = (u_x, v_x) - (u_y, v_x) + (u_y, v_y) + \langle u, v \rangle_{L^2} = (\mathcal{F}, v) - \langle g, v \rangle_{L^2} \]

So \( f \in H^\ast, \ g \in (H^2(\Omega))^\ast \)

(b) Suppose not, so \( \exists v, \mathbf{n} \) s.t. \( \| v \|_{L^2} = 1 \)

but \( \| \nabla v \|_{L^2}^2 + S_x v_x^2 \leq n \).

\[ \Rightarrow \text{ (sub.)) } \]

\[ \nabla v \rightarrow 0, \quad S_x v_x^2 \rightarrow 0 \]

\( \| v \|_{H^1} \leq 2 \Rightarrow v \rightarrow v, \quad v \rightarrow v \)

But \( \nabla v = 0 \Rightarrow v = 0 \) contradicting \( \| v \|_{L^2} = 1 \).

(c) Lax-Milgram, Linear form good by \( \mathcal{F} \).

**Continuity:** \( |B(v, w)| \leq (\| u_x \|_{L^2} + \| u_y \|_{L^2}) (\| v_x \|_{L^2} + \| v_y \|_{L^2}) + \| u \|_{H^1} \)

\[ \leq \| u \|_{H^1} \| v \|_{H^1} + \| v \|_{H^1} \]

**Coercivity:** \( B(v, v) \geq \| v \|_{H^1}^2 - (\| v_x \|_{L^2}^2 + \| v_y \|_{L^2}^2 + S_x v_x^2) \)

\[ \geq \frac{1}{2} \| v \|_{H^1}^2 \quad \text{by (b), } 2 \| v_x \|_{L^2}^2 + 2 \| v_y \|_{L^2}^2 \]
5. \( u' = g(t, u(t)) \quad u(0) = u_0 \cdot \]

\[
\Rightarrow u(t) - u(0) = \int_0^t g(s, u(s)) \, ds
\]

\[
G(u) = u_0 + \int_0^t g(s, u(s)) \, ds
\]

\[
\|v\|_1 = \sup_{\alpha \leq t \leq T} \left( e^{-\lambda t} \|v(t)\| \right)
\]

Note: \( \|v\|_1 \leq \|v\|_\infty, \|v\|_1 \geq e^{-\lambda T} \|v\|_\infty \)

so \( \|v\|_1 \) equiv. to \( \|v\|_\infty \) \rightarrow

\( \|v\|_1 \) satisfies the zero property

Scaling clearly okay

\[
\|v + w\|_1 = \sup_{\alpha \leq t \leq T} \left( e^{-\lambda t} \|v + w\| \right) \leq \sup_{\alpha \leq t \leq T} \left( e^{-\lambda t} \|v\| + \|w\| \right) \\
\leq \sup_{\alpha \leq t \leq T} e^{-\lambda t} \|v\| + \sup_{\alpha \leq t \leq T} e^{-\lambda t} \|w\| \\
= \|v\|_1 + \|w\|_1
\]

so \( \|v\|_1 \) is a norm.

\[
e^{-\lambda t} \|G(u) - G(w)\| = e^{-\lambda t} \left| \int_0^t (g(s, v) - g(s, w)) \, ds \right| \\
\leq Le^{-\lambda t} \int_0^t \|v - w\| \leq Le^{-\lambda t} \int_0^t e^{-\lambda s} \|v - w\| \leq Le^{-\lambda t} \int_0^t e^{\lambda s} \|v - w\| \\
= e^{-\lambda t} \int_0^t e^{\lambda s} \|v - w\| \leq e^{-\lambda t} \int_0^t (1 - e^{\lambda s}) \|v - w\| \Rightarrow \|G(u) - G(w)\| \leq e^{-\lambda t} \|v - w\| \cdot \theta = e^{-\lambda t} \|v - w\| \cdot \Theta = e^{-\lambda t} < 1.
\]

(d) Banach contraction mapping Thm \( \Rightarrow \exists ! \)

\( u \in C^0([0, T]) \) s.t. \( G(u) = u \) (IVP)

But \( \Rightarrow u \in C^0 \), then \( G(u) \in C^0 \)

Finally, let \( T \to \infty \).
6. \( u, v \in C^1_{0,1}([0,1]) \), \( F(u, v, u', v') = \int_0^1 \left[ (u')^2 + (v')^2 + 2uv \right] dx 

\( \hat{p}_{y_i} = (p_{y_i})' \), \( i = 1, 2 \)

\[
\begin{align*}
2v &= 2u'' \\
2u &= 2v'' \Rightarrow \begin{cases} v = u'' \\
u = v'' \end{cases}
\end{align*}
\]

(6) \( u = v'' = u''' \)

\( \Rightarrow u = e^{rt}, \quad r^4 = 1 \quad (r = \pm 1, \pm i) \)

\( u(x) = Ae^x + Be^{-x} + Ce^{ix} + De^{-ix} \)

(6) \( v(x) = u'' 

= Ae^x + Be^{-x} - Ce^{ix} - De^{-ix} \)

\[
\begin{cases}
\begin{align*}
u(0) &= A + B + C + D = 0 \\
u(1) &= Ae^x + Be^{-x} + Ce^{ix} + De^{-ix} = 1 \\
v(0) &= A + B - C - D = 0 \\
v(1) &= Ae^x + Be^{-x} - Ce^{ix} - De^{-ix} = 1.
\end{align*}
\end{cases}
\]

(d) \( H = \int_0^1 \left[ (u')^2 + (v')^2 + 2uv + \lambda u^2 v' \right] dx \)

\[
\begin{align*}
2v + 2\lambda uv' &= 2u'' \Rightarrow \begin{cases} v + \lambda uv' = u'' \\
u - \frac{1}{2} \lambda (u^2)' = v'' \end{cases}
\end{align*}
\]
1. A problem on continuous operators.

(a) Define the topological dual of a Banach space.

(b) Define the weak topology on a Banach space.

(c) Let $X, Y$ be Banach spaces and $A : X \to Y$ be a linear operator. Prove that $A$ is continuous if and only if it is weakly continuous (i.e., it is continuous when $X$ and $Y$ are equipped with their weak topologies).

Solution.

(a) The topological dual $X'$ of a normed space $X$ consists of all linear and continuous functionals defined on $X$. For a complex space $X$, we may define the topological dual as the space of all anti-linear and continuous functionals on $X$. Either space is equipped with the norm

$$l \in X', \quad \|l\|_{X'} := \sup_{x \in X, x \neq 0} \frac{|l(x)|}{\|x\|_X} = \sup_{\|x\|_X \leq 1} |l(x)| = \sup_{\|x\|_X = 1} |l(x)|.$$

For a reflexive Banach space, the supremum is actually attained and can be replaced with maximum. The dual space is always complete, no matter whether $X$ is complete or not.

(b) The weak topology on a Banach space $X$ is a locally convex topology defined by a family of seminorms

$$X \ni x \mapsto |\langle x', x \rangle| = |x'(x)|, \quad x' \in X'.$$

Due to the definiteness of the duality pairing (proved using Hahn-Banach Theorem), the family of seminorms satisfies the axiom of separation which implies that the weak topology is well-defined.

(c) We first prove that weak continuity of $A$ implies strong continuity of $A$. Assume, to the contrary, that there exists a sequence $x_n$ such that $\|x_n\|_X \to 0$ but $\|Ax_n\|_Y \not\to 0$. At the cost of replacing $x_n$ with a subsequence, we can assume that there exists $\epsilon > 0$ such that $\|Ax_n\|_Y \geq \epsilon$. Define

$$\bar{x}_n = \frac{x_n}{\|x_n\|_{X'}^{1/2}}.$$

Then,

$$\|\bar{x}_n\|_X = \frac{\|x_n\|_X^{1/2}}{\|x_n\|_{X'}^{1/2}} \to 0 \quad \text{and} \quad \|A\bar{x}_n\|_Y \to \infty.$$
As the strong convergence implies weak convergence, \( \bar{x}_n \to 0 \) and, by weak continuity of \( A \), \( A\bar{x}_n \to 0 \) in \( Y \). But every weakly convergent sequence must be bounded, a contradiction.

Assume now that \( A \) is strongly continuous.

**Lemma:** Let \( X \) be an arbitrary topological vector space, and \( Y \) be a normed space. Let \( A \in \mathcal{L}(X,Y) \). The following conditions are equivalent to each other.

(i) \( A : X \to Y \) (with weak topology) is continuous.
(ii) \( f \circ A : X \to \mathbb{R}(\mathbb{C}) \) is continuous \( \forall f \in Y' \).

(i) \( \Rightarrow \) (ii). Any linear functional \( f \in Y' \) is also continuous on \( Y \) with weak topology. Composition of two continuous functions is continuous.

(ii) \( \Rightarrow \) (i). Take an arbitrary \( B(I_0, \epsilon) \), where \( I_0 \) is a finite subset of \( Y' \). By (ii),

\[
\forall g \in I_0 \exists B_g, \text{ a neighborhood of } 0 \text{ in } X : x \in B_g \Rightarrow |g(A(x))| < \epsilon.
\]

It follows from the definition of filter of neighborhoods that

\[
B = \bigcap_{g \in I_0} B_g
\]

is also a neighborhood of \( 0 \). Consequently,

\[
x \in B \Rightarrow |g(A(x))| < \epsilon \Rightarrow Ax \in B(I_0, \epsilon).
\]

To conclude the final result, it is sufficient now to show that, for any \( g \in Y' \),

\[
g \circ T : X \text{ (with weak topology)} \to \mathbb{R}
\]

is continuous. But \( g \circ T \), as a composition of continuous functions, is a strongly continuous linear functional and, consequently, it is continuous in the weak topology as well (compare the discussion in the book).

2. Projections on a Hilbert space. Let \( X \) and \( Y \) be Hilbert spaces, \( P : X \to Y \) and \( Q : Y \to X \) be bounded linear operators, and suppose that \( QP : X \to X \) is an orthogonal projection operator. Let \( U_1 = R(QP) \) and \( U_2 = N(QP) \), i.e., the image (or range) and null space (or kernel) of the operator, respectively. Moreover, let \( V_1 = R(P) \).

(a) What does it mean to say \( X = U_1 \oplus U_2 \)? Show that \( U_1 \) and \( U_2 \) are orthogonal to each other.

(b) Prove that \( U_1 \) and \( V_1 \) are isomorphic.

(c) Show directly that \( P^*Q^* : X \to X \) is an orthogonal projection.

(d) If \( N(Q) \cap R(PQ) = \{0\} \), show that \( PQ : Y \to Y \) is a projection operator (not necessarily orthogonal).
Solution.

(a) The symbols \( X = U_1 \oplus U_2 \) mean that \( X = \{ u_1 + u_2 : u_i \in U_i, i = 1, 2 \} \) and \( U_1 \cap U_2 = \{0\} \). For \( u_i \in U_i \), we know that \( u_1 = QP u_1 \) and \( QP u_2 = 0 \), so
\[
\langle u_1, u_2 \rangle_X = \langle QP u_1, u_2 - QP u_2 \rangle_X = 0
\]
by the definition of orthogonal projection.

(b) Consider the map \( T = P|_{U_1} : U_1 \to V_1 \), that is bounded and linear. Every \( v \in V_1 \) has some \( u \in X \) such that \( Pu = v \). However, there are (unique) \( u_i \in U_i \) such that \( u = u_1 + u_2 \), and so \( Tu_1 = Pu_1 = Pu = v \) shows that \( T \) maps onto \( V_1 \). To finish, we need to show that \( T \) maps one-to-one, i.e., that \( Tu_1 = 0 \) implies that \( u_1 = 0 \). But \( 0 = Tu_1 = Pu_1 \), so also \( QP u_1 = 0 \). Thus \( u_1 \in U_1 \cap U_2 \), and so \( u_1 = 0 \).

(c) For \( u, w \in X \), we compute
\[
0 = \langle QP u - u, w \rangle_X = \langle u, P^* Q^* w - w \rangle_X,
\]
which shows that \( P^* Q^* \) is also an orthogonal projection operator.

(d) For \( y \in Y \), we know that \( QPQ PQ y = PQ Q y \), since \( QP \) is a projection. But then
\[
0 = QPQ PQ y - PQ Q y = Q( PQ PQ y - PQ y) = QP(Q PQ y - Q y).
\]
Thus \( PQQ PQ y - PQ y \in N(Q) \) and clearly \( PQQ PQ y - PQ y \in R(PQ) \), so \( PQQ PQ y = PQ y \).

3. Hilbert basis. Let \( H \) be a separable Hilbert space and let \( \{ e_n \}_{n=1}^\infty \) be a maximal orthonormal set (i.e., a Hilbert basis). Let \( \{ \lambda_n \}_{n=1}^\infty \) be a bounded sequence of real numbers, and define the linear operator \( A : H \to H \) by
\[
A x = \sum_{n=1}^\infty \lambda_n \langle x, e_n \rangle e_n.
\]

(a) Show that \( A \) is continuous and self-adjoint.

(b) Show that each \( \lambda_n \) is an eigenvalue with eigenvector \( e_n \).

(c) Show that if \( \lambda_n \to 0 \), then \( A \) is compact. [Hint: Consider the operator \( A_N \) defined by a truncated sum, and show that \( A_N \) converges to \( A \)]

Solution.

(a) If \( x_m \to 0 \), then \( \| x_m \|^2 = \sum_{n=1}^\infty | \langle x_m, e_n \rangle |^2 \to 0 \). Thus
\[
\| Ax_m \| = \sum_{n=1}^\infty | \lambda_n |^2 | \langle x, e_n \rangle |^2 \leq \max_n | \lambda_n |^2 \sum_{n=1}^\infty | \langle x_m, e_n \rangle |^2 \to 0.
\]
That is, \( A \) is continuous at 0, and so continuous everywhere.

Now
\[
\langle Ax, y \rangle = \sum_{n=1}^\infty \lambda_n \langle x, e_n \rangle \langle y, e_n \rangle = \sum_{n=1}^\infty \langle x, e_n \rangle \lambda_n \langle y, e_n \rangle = \langle x, Ay \rangle
\]
is clearly self adjoint (since \( \lambda_n \) is real).
(b) Compute

\[(A - \lambda I)x = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n - \lambda \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n = \sum_{n=1}^{\infty} (\lambda_n - \lambda) \langle x, e_n \rangle e_n , \]

and note that this cannot be invertible when \( \lambda = \lambda_n \) for some \( n \). Moreover, \( Ae_n = \lambda_n e_n \) is clear by orthonormality of the basis.

(c) Consider the operators

\[ A_N x = \sum_{n=1}^{N} \lambda_n \langle x, e_n \rangle e_n . \]

Each has finite dimensional range, and is hence compact. Moreover,

\[ \| A_N x - Ax \|^2 = \left\| \sum_{n=N+1}^{\infty} \lambda_n \langle x, e_n \rangle e_n \right\|^2 = \sum_{n=N+1}^{\infty} |\lambda_n|^2 |\langle x, e_n \rangle|^2 \to 0 , \]

so \( A_n \to A \) and \( A \) is compact.

4. Closed operators. All spaces are real. Consider the operator

\[ A : D(A) \to L^2(0,1), \quad Au = u' + u , \]

\[ D(A) := \{ u \in L^2(0,1) : Au \in L^2(0,1), \, u(0) = 0, \, u(1) = 0 \} , \]

where the derivative is understood in the sense of distributions.

(a) Interpret \( D(A) \) in terms of Sobolev spaces.

(b) Show that \( A \) is a closed operator.

(c) Prove that \( A \) is bounded below in \( L^2(0,1) \).

(d) Compute the \( L^2 \)-adjoint \( A^* , L^2(0,1) \supset D(A^*) \ni v \mapsto A^* v \in L^2(0,1) \).

(e) Compute the null space of the adjoint operator \( A^* \).

(f) For an appropriate right-hand side \( f \), discuss the well-posedness of the problem:

\[ \begin{cases} 
    u \in D(A), \\
    Au = f.
\end{cases} \]

Solution.

(a) We have

\[ u, u' + u \in L^2(0,1) \iff u, u' \in L^2(0,1) \iff u \in H^1(0,1). \]

Consequently, \( D(A) = H^1_0(0,1) \).
(b) We need to show that
\[ D(A) \ni u_n \rightarrow u, \quad Au_n \rightarrow w \quad \Rightarrow \quad u \in D(A), \quad Au = w. \]
All convergence is in the \( L^2 \)-sense. Let \( \phi \in D(0,1) \). We have
\[
\begin{align*}
(u_n, -\phi') + (u_n, \phi) &= (-u'_n + u_n, \phi) \rightarrow (w, \phi) \\
\downarrow & \quad \downarrow \\
(u, -\phi') & \quad (u, \phi)
\end{align*}
\]
This proves that \(-u' + u = w\) and, therefore, \( u \in H^1(0,1) \). Moreover, \( u_n \rightarrow u \) in \( H^1(0,1) \). Continuous embedding of \( H^1(0,1) \) into \( C([0,1]) \) implies that,
\[
u(x) = \lim_{n \to \infty} u_n(x) = 0 \quad \text{for} \quad x = 0, 1.
\]
Consequently, \( u \in D(A) \).
(c) We have
\[
||Au||^2 = ||u'||^2 + ||u||^2 + 2(u', u).
\]
But
\[
2(u', u) = \int_0^1 \frac{d}{dx}(u^2) = u^2|_0^1 = 0.
\]
Consequently,
\[
||Au||^2 = ||u'||^2 + ||u||^2 \geq ||u||^2.
\]
(d) Integration by parts and BC’s on \( u \) reveal that
\[
D(A^*) = H^1(0,1) \quad A^*v = -v' + v.
\]
(e) We get
\[
D(A^*) = \{ce^x : c \in \mathbb{R}\}.
\]
(f) According to the Closed Range Theorem for Closed Operators, the equation has a unique solution \( u \) for every \( f \in L^2(0,1) \) such that \( f \in \mathcal{N}(A^*)^\perp \), i.e.,
\[
\int_0^1 f(x)e^x = 0.
\]
5. Variational formulations. Consider the ultraweak variational formulation of the previous problem, i.e.,
\[
\begin{align*}
\begin{cases}
\forall u \in L^2(0,1) =: U, & \\
\int_0^1 uA^*v \, dx = \int_0^1 f v \, dx & \forall v \in D(A^*) = H^1(0,1) =: V,
\end{cases}
\end{align*}
\]
where $A^*$ denotes the $L^2$-adjoint of $A$, $A^* v = -v' + v$, and $f \in L^2(0,1)$. [Hint: For this problem, use results of the previous problem.]

(a) Define the operator $B : U \to V'$ and its conjugate corresponding to the bilinear form $b(u,v)$.

(b) State the Babuška-Nečas Theorem for Hilbert spaces.

(c) Use this theorem to investigate the well-posedness of the variational formulation.

Solution.

(a) If the bilinear form $b(u,v)$ is continuous (trivially in our case), then the operator

$$B : U \to V', \quad \langle Bu, v \rangle := b(u,v), \quad v \in V, u \in U,$$

is always well-defined, linear and continuous. The map setting $b$ into $B$ is an isometric isomorphism. The conjugate operator,

$$B' : V'' \to V', \quad \langle B'v, u \rangle = b(u,v) \quad u \in U, v \in V,$$

is also well-defined, linear and continuous with the norm equal to that of $B$.

(b) If the bilinear form satisfies the inf-sup condition,

$$\sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U \iff \|Bu\|_{V'} \geq \gamma \|u\|_U$$

and $l \in V'$ vanishes on the null space of the transpose operator,

$$l(v) = 0 \quad \forall v \in V_0 := \{ v \in V : b(w,u) = 0 \quad \forall w \in U \},$$

then there exists a unique solution $u$ to the variational problem and

$$\|u\|_U \leq \gamma^{-1} \|l\|_{V'}.$$

(c) We first prove the inf-sup condition. It is sufficient to find a $v \in H^1(0,1)$ such that $A^* v = u$ and

$$\|v\| \leq C \|A^* v\| = C \|u\|.$$

Once we control the $L^2$-norm of $v$, we control also the $L^2$-norm of its derivative,

$$\|v'\| \leq \| -v' + v \|_{A^* v} + \|v\| \leq (1 + C) \|A^* v\| = (1 + C) \|u\|,$$

and, consequently,

$$\|v\|_{H^1(0,1)} = \|v\|^2 + \|v'\|^2 \leq \left(1 + C^2 + C^2\right) \|u\|^2.$$

We have then

$$\sup_{v \in H^1(0,1)} \frac{|b(u,v)|}{\|v\|_{H^1}} \geq \frac{\|u\|_{L^2}^2}{\|v\|_{L^2}^2} \geq \frac{1}{C_1} \frac{\|u\|_{L^2}^2}{\|u\|_{L^2}^2} = \frac{1}{C_1} \|u\|_{L^2}^2.$$
Next, we determine the null space of the transpose operator. Clearly,

\[ 0 = \int_0^1 u A^* v \text{ } \forall u \in L^2(0, 1) \Rightarrow A^* v = 0. \]

This gives,

\[ N(B') = \{ ce^x : c \in \mathbb{R} \}. \]

Consequently, by the Babuška-Nečas Theorem, for every \( l \in (H^1(0, 1))' \) that satisfies the compatibility condition

\[ l(e^x) = 0, \]

the variational problem has a unique solution \( u \) that depends continuously upon \( l \). Note that the right-hand side may be more general than an \( L^2 \)-function. For the \( L^2 \)-function \( f \),

\[ l(v) = \int_0^1 f v, \]

so the function \( f \) must be \( L^2 \)-orthogonal to \( e^x \).

Finding a solution \( v \in H^1(0, 1) \), \( A^* v = u \in L^2(0, 1) \) is an undetermined problem. We may fix \( v \) by adding an extra BC: \( v(0) = 0 \). You can now find \( v \) explicitly (this is an elementary problem), or you can consider an auxiliary problem

\[
\begin{align*}
    v &\in H^1(0, 1), \ v(0) = 0, \\
    Lv &:= -v' + v = u.
\end{align*}
\]

By the same argument as in the previous problem, operator \( L \) is bounded below,

\[ \| -v' + v \|^2 = \|v'\|^2 + v(1)^2 + \|v\|^2 \geq \|v\|^2. \]

The adjoint,

\[ D(L^*) := \{ u \in H^1(0, 1) : u(1) = 0 \}, \quad L^* u = -u' + u, \]

has a trivial null space. The Closed Range Theorem for Closed Operators implies thus that there exists a unique solution \( v \in D(L), \ Lv = A^* v = u, \) and \( \|v\| \leq \|u\| \).

6. Nonlinear equations. Let \( X \) be a Banach space and \( T : X \to X \) a bounded linear operator. Let \( g : X \to X \) be a nonlinear mapping that is \( C^1 \) and has \( g(0) = 0 \) and \( Dg(0) = 0 \). For \( f \in X \), we want to solve

\[ F(u) = u + T g(u) = f \]

We consider the map \( G(u) = u + \alpha (F(u) - f) \) for some \( \alpha \in \mathbb{R} \).

(a) Show that \( G(u) \) is a contractive map for small enough \( u \) and properly chosen \( \alpha \).

(b) Use the Banach contraction mapping theorem to show that there is a solution to \( F(u) = f \), provided \( f \) is sufficiently small.

(c) Compute \( DF(u)(v) \) from the definition of the Fréchet derivative.

(d) Solve \( F(u) = f \) using the inverse function theorem, provided \( f \) is sufficiently small.
Solution.

(a) Let \( u, v \in X \) and compute

\[
G(u) - G(v) = u - v + \alpha(F(u) - F(v)) = (1 + \alpha)(u - v) + \alpha T(g(u) - g(v)),
\]

so that

\[
\|G(u) - G(v)\| \leq \|1 + \alpha\|\|u - v\| + \|\alpha\|\|T\|\|g(u) - g(v)\|.
\]

Since \( Dg(0) = 0 \) and \( g \) is \( C^1 \), given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for \( w \in B_\delta(0) \),

\[
\|Dg(w)\| \leq \epsilon.
\]

Therefore the mean value theorem shows that

\[
\|g(u) - g(v)\| \leq \epsilon\|u - v\| \quad \forall \, u, v \in B_\delta(0).
\]

Take, for example, \( \alpha = -\frac{1}{2} \) and \( \frac{1}{2}\epsilon\|T\| < \frac{1}{4} \) (which defines \( \delta \)). Then \( G \) is contractive (with constant \( \frac{3}{4} \)) on \( B_\delta(0) \).

(b) It remains to show that \( G : B_\delta(0) \to B_\delta(0) \). Compute

\[
\|G(u)\| \leq \|G(u) - G(0)\| + \|G(0)\| \leq \frac{3}{4}\|u\| + \|\alpha f\|.
\]

Requiring \( \|f\| < \frac{\delta}{4\|\alpha\|} \) completes the proof.

(c) We compute

\[
F(u + v) - F(u) = v + T(g(u + v) - g(u)) = v + T(Dg(u)(v) + R_g(u,v))
\]

\[
= v + T(Dg(u)(v)) + TR_g(u,v),
\]

where \( \|R_g(u,v)\| = o(\|v\|) \). But then \( \|TR_g\| \leq \|T\|\|R_g\| = o(\|v\|) \), so

\[
DF(u)(v) = v + TDg(u)(v).
\]

(d) We note that \( F \) is \( C^1 \) and \( DF(0) = I \) is invertible. Thus the inverse function theorem gives open sets \( U, V \subset X \) such that \( 0 \in U \) and \( F(0) = 0 \in V \) such that \( F \) is a diffeomorphism from \( U \) to \( V \). Thus we can solve the problem for \( f \in V \).
CSEM Area A-CAM Preliminary Exam (CSE 386C–D)
May 28, 2021, about any 3 hours from 9:00 a.m. to 3:00 p.m.

You may use the class textbooks and your own notes on this exam.

Work any 5 of the following 6 problems.

1. Let the field be real and \( \mathbb{P} \) denote the vector space of all polynomials in \( x \in \mathbb{R} \); that is, \( \mathbb{P} = \left\{ p(x) = \sum_{k=0}^{n} c_k x^k : n \text{ is a nonnegative integer and } c_k \in \mathbb{R} \right\} \). Let \( \| \cdot \| : \mathbb{P} \to [0, \infty) \) be defined for such \( p \) as \( \|p\| = \max_{0 \leq k \leq n} |c_k| \).
   
   (a) Show \( \| \cdot \| \) is a norm on \( \mathbb{P} \).
   
   (b) Show that the NLS \((\mathbb{P}, \| \cdot \|)\) is not complete.
   
   (c) Let \( m \geq 0 \) and \( T_m : \mathbb{P} \to \mathbb{R} \) be defined by \( T_m p = \sum_{k=0}^{\min(m,n)} c_k \), which is clearly linear. Show that each \( T_m \) is bounded.
   
   (d) Since \( \mathbb{P} \) is not Banach, the Uniform Boundedness Principle need not hold. In fact, show that \( \sup_m |T_m p| < \infty \) for each \( p \in \mathbb{P} \) but \( \sup_m \|T_m\| = \infty \).

2. Let \( \Omega \) be some set and \( (H, \langle \cdot, \cdot \rangle) \) be a Hilbert space of functions \( f : \Omega \to \mathbb{F} \) (\( \mathbb{F} \) is \( \mathbb{R} \) or \( \mathbb{C} \)). Suppose that there is a constant \( C(x) \) such that \( |f(x)| \leq C(x)\|f\| \) for all \( f \in H \).
   
   (a) Show that if \( f, g \in H \) and \( x \in \Omega \), then \( |f(x) - g(x)| \leq C(x)\|f - g\| \).
   
   (b) Show that there exists a function \( K : \Omega \times \Omega \to \mathbb{F} \) (called a reproducing kernel) such that for each fixed \( x \in \Omega \), \( K(\cdot, x) \in H \) and \( f(x) = \langle f, K(\cdot, x) \rangle \) for all \( f \in H \).
   
   [Hint: Use the Riesz representation theorem.]
   
   (c) Show that \( K(x, y) = \overline{K(y, x)} \) (i.e., \( K \) is conjugate symmetric). Be sure to justify that \( K(x, \cdot) \in H \) for each \( x \in \Omega \).

3. Let \( H \) be a complex Hilbert space and \( A \) a bounded linear operator on \( H \). Define \( |A| = (A^*A)^{1/2} \).

   (a) Show that \( |A| \) is a well defined, bounded linear, self-adjoint operator. [Hint: Use Theorem 4.26.]
   
   (b) Show that \( \| |A|x \| = \|Ax\| \) for all \( x \in H \).
   
   (c) Show that \( H = R(|A|) \oplus N(|A|) \) and that \( N(|A|) = N(A) \).
4. Half Laplacian in $\mathbb{R}$. Let $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. For $u \in H^1(\mathbb{R}^2_+)$, we denote by $\tilde{u}$ the Fourier transform in $x$ only, i.e.,
$$
\tilde{u}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x, y) e^{-ix\xi} \, dx.
$$
Take $f \in H^1(\mathbb{R})$, and consider $u$ the solution to
$$
\begin{cases}
\partial_x^2 u + \partial_y^2 u = 0, & (x, y) \in \mathbb{R}^2_+,
\quad u(x, 0) = f(x), & x \in \mathbb{R}.
\end{cases}
$$
(a) Find the equation verified by $\tilde{u}$.
(b) Show that there exists a unique solution of (1) such that $\nabla u \in L^2(\mathbb{R}^2_+)$, and give a formula for $\tilde{u}$. [Hint: Solutions to the ODE $y'' - \omega^2 y = 0$ are of the form $Ae^{-\omega t} + Be^{\omega t}$]
(c) For $f \in H^1(\mathbb{R})$, we define $\Delta^\alpha f$, for $0 < \alpha < 1$ a real number, through the Fourier transform as $\hat{\Delta}^\alpha f = |\xi|^{2\alpha} \hat{f}$. Show that for $u$ solving (1), we have
$$
-\partial_y u(x, 0) = \Delta^{1/2} f.
$$
(d) Show that
$$
\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx \, dy = \int_{\mathbb{R}} f \Delta^{1/2} f \, dx = \int_{\mathbb{R}} |\Delta^{1/4} f|^2 \, dx.
$$
5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, $f \in L^2(\Omega)$, and $\alpha > 0$. Consider the boundary value problem
$$
\begin{cases}
-\Delta u + u = f & \text{in } \Omega,
\quad \partial u / \partial \nu + \alpha u = 0 & \text{on } \partial \Omega.
\end{cases}
$$
(a) For this problem, formulate a variational principle
$$
B(u, v) = (f, v) \quad \forall v \in H^1(\Omega).
$$
(b) Show that this problem has a unique weak solution.
6. Given $I = [0, b]$, consider the problem of finding $u : I \to \mathbb{R}$ such that
$$
\begin{cases}
u'(s) = g(s) f(u(s)) & \text{for a.e. } s \in I,
\quad u(0) = \alpha,
\end{cases}
$$
where $\alpha \in \mathbb{R}$ is a given constant, $g \in L^p(I)$, $p \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ are given functions. We suppose that $f$ is Lipschitz continuous and satisfies $f(0) = 0$.
(a) Consider the functional
$$
F(u) = \alpha + \int_0^s g(\sigma) f(u(\sigma)) \, d\sigma.
$$
Show that $F$ maps $C^0(I)$ into $C^0(I) \cap W^{1,p}(I)$. Moreover, show that $u \in C^0(I) \cap W^{1,p}(I)$ is the solution to (2) if and only if it is a fixed point of $F$.
(b) Show that there exists $b$ small enough, not depending on $\alpha$, such that $F$ has a unique fixed point in $C^0(I)$. 

Area A-CAM
May 2021
1. \[ P = \{ p = \sum_{k=0}^{n} c_k x^k \} \]
\[ ||p|| = \max |c_k| \]
(a) Norm (i) \[ ||p|| > 0, \quad ||p|| = 0 \iff c_k = 0 \forall k \]
\[ \iff p = 0 \]
(ii) \[ ||c p|| = ||\sum c_k x^k|| = \max |c c_k| = |c| \max |c_k| = |c||p|| \]
(iii) \[ ||p + \bar{q}|| = \max |c_k + d_k| \]
\[ \leq \max |c_k| + \max |d_k| = ||p|| + ||\bar{q}|| \]
(b) Let \[ p_n = 1 + \frac{1}{2}x + \ldots + \frac{1}{n}x^n \]
Then \( p_n \) is Cauchy. \( (m > n) \)
\[ ||p_n - p_m|| = \frac{1}{m+1} \rightarrow 0 \]
But \( p_n \nrightarrow p \) with finite degree
\[ (\text{if } \deg p = m, \quad \text{then } ||p_n - p|| \geq \frac{1}{m+1} ) \]
Thus \( p \) is not complete
(c) \[ \tilde{P} p = \sum_{k=0}^{\min(n,m)} c_k \]
\[ ||\tilde{P} p|| \leq \sum_{k=0}^{\min(n,m)} |c_k| \leq \min(n,m) ||p|| \]
\[ \leq m ||p|| \]
(d) \[ \sup ||\tilde{P} p|| \leq n ||p|| < \infty \]
\[ \sup ||\tilde{P}|| \geq \sup \frac{||\tilde{P} p||}{||p||}, \quad p = 1 + x + x^2 + \ldots + x^n \]
\[ \geq \sup ||\tilde{P} p|| = \min(n, m) = n \rightarrow \infty \]
2. \[ H = \{ f : \Omega \rightarrow \mathbb{F}^3 \} \]
\[ |f(x)| \leq c(x) \| f \| \quad \forall f \in H \]

(a) \[ f, g \in H, x \in \Omega \Rightarrow \]
\[ |f(x) - g(x)| = |(f-g)(x)| \leq c(x) \| f-g \| \]

(b) Let \( T_x : H \rightarrow \mathbb{F} \) be \( T_x f = f(x) \).

Then \( T_x \) is a linear functional (by definition of + sc. mult. of funcns).

Riesz \( \Rightarrow \) \( \exists \ \tilde{g} = k(\cdot, x) \in H \) st.
\[ f(x) = \langle f, k(\cdot, x) \rangle \quad \forall f \in H. \]

(c) \( k(\cdot, x) \in H \Rightarrow (by \ (\tilde{g})) \)

\[ k(\tilde{g}_x) = \langle k(\cdot, x), k(\cdot, \tilde{g}) \rangle \]
\[ = \langle k(\cdot, \tilde{g}), k(\cdot, x) \rangle = k(x, \tilde{g}). \]

Note: \( k(x, \tilde{g}) = k(\cdot, x) \in H. \)
3. \( H \) complex Hilbert. \( A \in \mathcal{B}(H, H) \). \( |A| = (A^*A)^{1/2} \)

(a) Let \( T = A^*A \in \mathcal{B}(H, H) \)
\[
\langle Tx, x \rangle = \langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \langle x, A^*Ax \rangle = \langle x, T^2x \rangle
\]
\[
\Rightarrow T = T^2
\]
\[
\langle Tx, x \rangle = \|Ax\|^2 \geq 0 \Rightarrow T \geq 0
\]

Thm 4.26 \( \Rightarrow T \) has a unique, pos. sq. root \( (A^*A)^{1/2} \in \mathcal{B}(H, H) \)

Since \( (A^*A)^{1/2} \geq 0 \), it is self-adjoint.

(b) \( \|Ax\|^2 = \langle |A|x, |A|x \rangle \)
\[
= \langle |A|^2x, x \rangle = \langle T^2x, x \rangle
\]
\[
= \langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2
\]

(c) Let \( R = \mathcal{R}(1|A|) \)

Then \( H = R \oplus R^\perp \)
\[
x \in R^\perp \iff \langle x, \gamma \rangle = 0 \quad \forall \gamma \in \mathcal{R}(1|A|)
\]
\[
\iff \langle x, |A|z \rangle = 0 \quad \forall z \in H
\]
\[
\iff \langle 1|A|x, z \rangle = 0 \quad \forall z \in H
\]
\[
\iff x \in N(1|A|)
\]
Thus \( R^\perp = N(1|A|) \) and \( H = \mathcal{R}(1|A|) \oplus N(1|A|) \)

But \( x \in N(1|A|) \iff \|1|A|x\| = 0 \iff \|Ax\| = 0 \iff x \in N(A) \)

So \( N(1|A|) = N(A) \).
\( 4. \quad R^2 = S(x, y) \quad y > 0 \quad u = \frac{1}{2\pi} \int_{\mathbb{R}} u(x, y) e^{-ix} \, dx \)

\[ \begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u(x, 0) = f(x) \in H^1 \end{cases} \]

(a) \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} - |y|^2 u = 0 \]

(b) \( u = A e^{-i|y|^2/4} + B e^{i|y|^2/4} \)

But \( u(3, 0) = \hat{f} \) & \( u \) blows up if \( B \neq 0 \)

so \( u(3, 0) = \hat{f}(3) e^{-i3|y|^2/4} \)

\[ u(x, y) = f^{-1}_{3\sqrt{2}} \left( \hat{f}(3) e^{-i3|y|^2/4} \right) \]

(2pi) \( f \times f^{-1}_{3\sqrt{2}} (e^{-i3|y|^2/4}) \)

(c) \( \Delta u = 13 |y|^2 \hat{f} \)

\[ \frac{\partial^2 u}{\partial y^2} - |y|^2 u(x, y) = (2\pi)^{-1/2} \hat{f}(x) \left( \frac{\partial^2}{\partial y^2} f^{-1}_{3\sqrt{2}} (e^{-i3|y|^2/4}) \right) \]

\[ = (2\pi)^{-1/2} \hat{f}(x) \left( f^{-1}_{3\sqrt{2}} (e^{-i3|y|^2/4}) \right) ^2 \]

\[ = -13/4 \hat{f}(3) \left( f^{-1}_{3\sqrt{2}} (e^{-i3|y|^2/4}) \right) = -13/4 \hat{f}(x) e^{-i3|y|^2/4} \]

\( \Rightarrow \quad \Delta u = 13 |y|^2 \hat{f} \)

(d) \[ \int_{\mathbb{R}^2} |\nabla u|^2 \, dx dy = (\nabla u, \nabla u)_{L^2} = - (\Delta u, u)_{L^2} + \int_{\mathbb{R}^2} \Delta u \cdot u \, dx dy \]

\[ = \int_{\mathbb{R}} (\Delta u)^2 \, dx = \int_{\mathbb{R}} 13 |y|^2 \hat{f} \cdot 13 |y|^2 \hat{f} \]

\[ = \int_{\mathbb{R}} |\Delta u|^2 \, dx \]
\( \Omega \) \( \alpha > 0 \) \( f \in L^2 \) \( \{ \Delta u + \alpha u = f, \Omega \} \)

\[(a) \quad (\Delta u, v) + (u, v) = (f, v) \]
\[= (\nabla u, \nabla v) - (\nabla u, v) \]
\[= (\nabla u, \nabla v) + (\alpha u, v) \]
\[\Rightarrow \quad B(u, v) = (\nabla u, \nabla v) + \alpha (u, v) \]
\[= (f, v) \quad \forall v \in H^1 \]

where we want \( u \in H^1 \) as well.

(b) Use Lax-Milgram.

\[ (f, v) \] given a \( \text{cont. lin. form} \)
\[ \forall f \in L^2 \subseteq (H^1)^* \]
\[ |B(u, v)| \leq ||\nabla u|| ||\nabla v|| + \alpha ||u|| ||v|| \]
\[ \leq \lambda ||u|| ||v||^2 + \alpha ||u||^2 \]
\[ \Rightarrow \quad (B(u, v))^2 \geq 2 \alpha ||u||^2 \]

We need a Poincaré type:
\[ ||u||^2 \leq C (||\nabla u|| + ||u||_{L^2(\Omega)}) \]

Suppose not. Then \( \exists u_n \) s.t.
\[ (u_n)^2 \geq n (||\nabla u|| + ||u||_{L^2(\Omega)}) \]
\[ \Rightarrow \quad u_n \rightharpoonup u \quad H^1 \quad (u_n \rightharpoonup u \quad L^2) \]
\[ \nabla u_n \rightarrow 0 \quad L^2, \quad u_n(\Omega) \rightarrow 0 \quad L^2(\Omega) \]
\[ \Rightarrow \quad u = 0 \]
But \( ||u|| = 1 \), contradiction.
6. \( I = [0,b] \)
   \[ u'(s) = g(s) f(u(s)) \]
   , a.e. \( s \in I \)

   \[ u(0) = \alpha \in \mathbb{R} \]

   \[ g \in L^p(I), p > 1; f: \mathbb{R} \to \mathbb{R}, f(0) = 0, \text{Lipschitz} \]

   \[ F(u) = \alpha + \int_0^s g(\sigma) f(u(\sigma)) d\sigma \]

   \[ u \in C^0 \Rightarrow F(u) \in C^0 \Rightarrow F(u) \in L^p \]

   \[ (F(u))' = g(s) f(u(s)) \in L^p \]

   \[ \in \ell^\infty \text{ since Lipschitz} \]

   If

   \[ u = \alpha + \int_0^s g(\sigma) f(u(\sigma)) d\sigma \]

   Then

   \[ \int u' = g(s) f(u(s)) \]

   \[ u(0) = \alpha \]

   \[ (b) \left\| F(u) - F(v) \right\| = \left\| \int_0^s g(\sigma) (f(u(\sigma)) - f(v(\sigma))) d\sigma \right\| \leq ||g||_{L^p(0,b)} \cdot ||f(u-v)||_{L^1(0,b)} \]

   \[ \leq ||g||_{L^p(0,b)} \cdot \left\| u - v \right\|_{L^1} \]

   \[ \Theta < 1 \text{ if } b \text{ small enough. (say } \Theta = \frac{1}{2} \) \]

   \[ \Rightarrow F \text{ contractive} \]

   \[ ||F(u)||_{L^\infty} = ||F(u) - F(0) + \alpha||_{L^\infty} \leq \Theta (||u||_{L^\infty} + \alpha) \leq \Theta R + \alpha \leq R \]

   \[ \Rightarrow \alpha \leq (1-\Theta) R \Rightarrow R = \frac{\alpha}{1-\Theta} = 2\alpha \]

   Thus \( F: B_R(0) \to B_R(0) \)

   and \( F \) has a fixed pt. in \( B_R(0) \)