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A LINEAR DEGENERATE ELLIPTIC EQUATION ARISING FROM TWO-PHASE MIXTURES*

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Abstract. We consider the linear degenerate elliptic system of two first order equations $\mathbf{u} = -a(\phi)(\nabla p - \mathbf{g})$ and $\nabla \cdot (b(\phi)\mathbf{u}) + \phi p = \phi^{1/2} f$, where a and b satisfy $a(0) = b(0) = 0$ and are otherwise positive, and the *porosity* $\phi \geq 0$ may be zero on a set of positive measure. This model equation has a similar degeneracy to that arising in the equations describing the mechanical system modeling the dynamics of partially melted materials, e.g., in the earth's mantle and in polar ice sheets and glaciers. In the context of mixture theory, ϕ represents the phase variable separating the solid one-phase ($\phi = 0$) and fluid-solid two-phase ($\phi > 0$) regions. The equations should remain well-posed as ϕ vanishes so that the free boundary between the one- and two-phase regions need not be found explicitly. Two main problems arise. First, as ϕ vanishes, one equation is lost. Second, it is shown by stability or energy bounds for the solution that the *pressure* p is not controlled outside the support of ϕ. After an appropriate scaling of the pressure and velocity, we obtain a mixed system for which we can show existence and uniqueness of a solution over the entire domain, regardless of where ϕ vanishes. The key is to define the appropriate Hilbert space containing the velocity, which must have a well defined scaled divergence and normal trace. We then develop for the scaled problem a mixed finite element method based on lowest order Raviart–Thomas elements which is stable and has an optimal convergence rate for sufficiently smooth solutions. We show some numerical results that verify the optimal rates of convergence for sufficiently regular solutions.

Key words. degenerate elliptic, mixed method, energy bounds, mixture theory, ice sheets, mantle dynamics

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1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ ($n = 1$, 2, or 3) be a domain, let $\phi : \Omega \to [0, \phi^*]$, $0 < \phi^* < \infty$, be a given differentiable function that we will call *porosity*, and let a and b lie in $C^1([0, \phi^*])$ with $a(0) = b(0) = 0$ and both positive on $(0, \phi^*]$. For the *velocity* $\mathbf{u}$ and the *pressure* p, we consider the linear degenerate elliptic boundary value problem

$$\mathbf{u} = -a(\phi)(\nabla p - \mathbf{g}) \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \big(b(\phi)\mathbf{u}\big) + \phi p = \phi^{1/2} f \quad \text{in } \Omega, \tag{1.2}$$

$$b(\phi)\mathbf{u} \cdot \nu = \phi^{1/2} g_{\mathrm{N}} \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\mathbf{g}$ and f drive the system and a Neumann boundary condition has been applied for some g_{N} (we will also treat a Robin boundary condition; see (5.1)). The choice of scaling in (1.2)–(1.3) in terms of ϕ will become clear as we develop the ideas. The critical factor here is that ϕ may vanish on a set of positive measure. This leads to a loss of control on p (see (2.4)), and a number of issues arise for numerical approximation. In fact, p may be unbounded outside the support of ϕ, which is

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difficult to approximate numerically. Moreover, if ϕ vanishes, say everywhere for simplicity, it appears that the first equation implies that $\mathbf{u} = 0$, but the second equation trivializes to $0 = 0$, leading to more numerical difficulties.

Degenerate elliptic equations have been approximated in many works, e.g., [16, 23, 5, 22, 11], using weighted Sobolev spaces and least squares techniques. But in these works, Ω is the support of ϕ, so the degeneracies are isolated to $\partial\Omega$ and interior sets of measure zero. It is important in some applications that ϕ is allowed to vanish on a set of positive measure.

The system (1.1)–(1.3) arises, for example, as a simplified mathematical model of mantle dynamics. Models of flow in the earth's mantle [1, 20, 19, 18, 3] are based on a mixture of fluid melt and matrix solid. Both fluid and matrix phases are assumed to exist at each point of the domain. The porosity $\phi \geq 0$ represents the relative volume of fluid melt to the bulk volume, and this quantity is very small (a few percent) within the mantle. Fluid melt is believed to form between rock crystal boundaries [27], forming a porous medium, and so the interstitial fluid velocity $\mathbf{v}_\mathrm{f}$ is governed by a Darcy law in terms of the fluid pressure p_f, such as

$$\phi(\mathbf{v}_\mathrm{f} - \mathbf{v}_\mathrm{s}) = -\frac{K(\phi)}{\mu_\mathrm{f}}(\nabla p_\mathrm{f} - \rho\mathbf{g}), \tag{1.4}$$

for some (relative) permeability $K(\phi)$, viscosity μ_f, and density ρ. The matrix solid is deformable, and it is modeled as a highly viscous fluid (velocity $\mathbf{v}_\mathrm{s}$, pressure p_s, viscosity μ_s) governed by a Stokes equation. Conservation of mass, assuming constant and equal phase densities (or a Boussinesq approximation), gives the mixture equation

$$\nabla \cdot \big(\phi\mathbf{v}_\mathrm{f} + (1-\phi)\mathbf{v}_\mathrm{s}\big) = 0, \tag{1.5}$$

and a compaction relation is given as

$$\mu_\mathrm{s}\nabla \cdot \mathbf{v}_\mathrm{s} = \phi(p_\mathrm{f} - p_\mathrm{s}). \tag{1.6}$$

Because these systems have been combined using mixture theory, one obtains a single, two-phase model that is assumed to hold even when one of the phases disappears. Such models have advantages in numerical approximation, since the free boundary between the one- and two-phase regions need not be determined, and the equations remain unaltered in a time-dependent problem when a phase disappears or forms in some region of the domain. A similar model arises in modeling two-phase flow within a nondeformable porous medium [9, 21, 10, 14] and in the modeling of partially melted ice, e.g., in glacier dynamics [17, 6, 26].

Nevertheless, the model (1.4)–(1.6) gives rise to a degenerate system when the fluid melt disappears. The Stokes part is well-posed, since there is always matrix rock present at each point of space (i.e., $\phi \leq \phi^* < 1$). Thus, we ignore the matrix part of the problem. Assuming a relative permeability of the form $K(\phi) = \mu_\mathrm{f}\,\phi\,a(\phi)$ and setting $b(\phi) = \phi$, we then extract the simplified mathematical model (1.1)–(1.3) with $\mathbf{u} = \mathbf{v}_\mathrm{f} - \mathbf{v}_\mathrm{s}$ and $p = p_\mathrm{f}$. In the mantle dynamics problem, $\phi^{-1/2}f$ would represent the matrix pressure p_s, and perhaps one would set $a(\phi) = \phi^{1+2\theta}$ for some $\theta \geq 0$.

2. A priori estimates and a change of dependent variables. Let $L^p(S)$ be the standard Lebesgue space of index p, $1 \leq p \leq \infty$. Later we will need the space $W^{k,p}(S)$, the standard Sobolev space of k weakly differentiable functions for which each derivative is in $L^p(S)$ (so $L^p(S) = W^{0,p}(S)$). We may omit S if $S = \Omega$. Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner-product or possibly duality pairing, and let $\langle\cdot,\cdot\rangle$ denote the $L^2(\partial\Omega)$ inner-product or duality pairing.

To develop intuition about the degenerate system (1.1)–(1.3), we proceed formally by assuming that there is a sufficiently smooth solution and that ϕ is reasonable. Once we understand the system on a formal level, we will return to a rigorous analysis of the problem in the next section. It will be convenient to define

$$c(\phi) = \sqrt{b(\phi)/a(\phi)} \quad \text{and} \quad d(\phi) = \sqrt{a(\phi)\, b(\phi)}.$$

The possibility that $c(\phi)$ is ill-defined will be discussed in the next section.

After multiplying (1.1) by $c(\phi)^2\psi$ and (1.2) by w, integrating, and integrating by parts, we obtain the weak form

$$(c(\phi)^2\mathbf{u}, \boldsymbol{\psi}) - \big(p, \nabla\cdot(b(\phi)\boldsymbol{\psi})\big) = (b(\phi)\mathbf{g}, \boldsymbol{\psi}), \tag{2.1}$$

$$\big(\nabla\cdot(b(\phi)\mathbf{u}), w\big) + (\phi p, w) = (\phi^{1/2} f, w), \tag{2.2}$$

where the test function $\boldsymbol{\psi}$ satisfies the homogeneous boundary condition (1.3) with $g_{\mathrm{N}} = 0$ on $\partial\Omega$.

Suppose that we can extend g_{N} to $\mathbf{u}_{\mathrm{N}}$ in Ω such that

$$b(\phi)\mathbf{u}_{\mathrm{N}}\cdot\nu = \phi^{1/2} g_{\mathrm{N}} \quad \text{on } \partial\Omega. \tag{2.3}$$

Taking $\boldsymbol{\psi} = \mathbf{u} - \mathbf{u}_{\mathrm{N}}$ and $w = p$, and also taking $w = \phi^{-1}\nabla\cdot(b(\phi)\mathbf{u})$, in (2.1)–(2.2) results in the a priori energy estimates

$$\begin{aligned} &\|c(\phi)\mathbf{u}\| + \|\phi^{1/2}p\| + \|\phi^{-1/2}\nabla\cdot(b(\phi)\mathbf{u})\| \\ &\quad \le C\big\{\|f\| + \|d(\phi)\mathbf{g}\| + \|c(\phi)\mathbf{u}_{\mathrm{N}}\| + \|\phi^{-1/2}\nabla\cdot(b(\phi)\mathbf{u}_{\mathrm{N}})\|\big\} \end{aligned} \tag{2.4}$$

in terms of the norm $\|\cdot\|_{k,S}$ of the Hilbert space $H^k(S) = W^{k,2}(S)$, where $\|\cdot\|_S = \|\cdot\|_{0,S}$ and $\|\cdot\| = \|\cdot\|_\Omega$. Assuming the data are given so that the right-hand side is bounded, as ϕ vanishes, we potentially lose control of p (and possibly $\mathbf{u}$). This makes sense, since p is the fluid pressure and there is no fluid phase. Nevertheless, we wish to have a well-posed two-phase mixture even as one phase disappears. We do this by making a change of dependent variables.

Let the *scaled velocity* and *scaled pressure* be defined as

$$\mathbf{v} = c(\phi)\mathbf{u} = \sqrt{b(\phi)/a(\phi)}\,\mathbf{u}, \tag{2.5}$$

$$q = \phi^{1/2}p, \tag{2.6}$$

respectively, since we have control of these quantities. Since $b(\phi)\mathbf{u} = d(\phi)\mathbf{v}$, the system (1.1)–(1.3) becomes

$$\mathbf{v} = -d(\phi)\big(\nabla(\phi^{-1/2}q) - \mathbf{g}\big) \quad \text{in } \Omega, \tag{2.7}$$

$$\nabla\cdot(d(\phi)\mathbf{v}) + \phi^{1/2}q = \phi^{1/2}f \quad \text{in } \Omega, \tag{2.8}$$

$$d(\phi)\mathbf{v}\cdot\nu = \phi^{1/2}g_{\mathrm{N}} \quad \text{on } \partial\Omega. \tag{2.9}$$

If we divide the second equation by $\phi^{1/2}$, this system is antisymmetric, since the formal adjoint of $-d(\phi)\nabla(\phi^{-1/2}(\cdot))$ is $\phi^{-1/2}\nabla\cdot(d(\phi)(\cdot))$. Now the energy estimates (2.4) are transformed using (2.5)–(2.6) to read more simply as

$$\begin{aligned} &\|\mathbf{v}\| + \|q\| + \|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v})\| \\ &\quad \le C\big\{\|f\| + \|d(\phi)\mathbf{g}\| + \|\mathbf{v}_{\mathrm{N}}\| + \|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_{\mathrm{N}})\|\big\}, \end{aligned} \tag{2.10}$$

where $\mathbf{v}_{\mathrm{N}} = c(\phi)\mathbf{u}_{\mathrm{N}}$.

3. The space $H_{\phi,d}$(div). We return to mathematical rigor. Based on the previous section, we are led to define the space

$$H_{\phi,d}(\mathrm{div};\Omega) = \big\{\mathbf{v} \in (L^2(\Omega))^n \,:\, \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}) \in L^2(\Omega)\big\}, \tag{3.1}$$

wherein we must assume that ϕ behaves well enough to support the definition. The natural condition to impose is

$$\phi^{-1/2}d(\phi) \in L^\infty(\Omega) \quad \text{and} \quad \phi^{-1/2}\nabla d(\phi) \in (L^\infty(\Omega))^n. \tag{3.2}$$

The meaning is then clear: To define $H_{\phi,d}(\mathrm{div})$, we interpret

$$\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}) = \phi^{-1/2}\nabla d(\phi)\cdot\mathbf{v} + \phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}, \tag{3.3}$$

and so we simply require that $\phi^{-1/2}d(\phi)$ times the weak divergence of $\mathbf{v}$ lies in L^2. To ensure that the formal adjoint operator $-d(\phi)\nabla(\phi^{-1/2}(\cdot))$ is well defined, we also ask that

$$\phi^{-3/2}d(\phi)\nabla\phi \in (L^\infty(\Omega))^n. \tag{3.4}$$

We remark that our conditions (3.2) and (3.4) are equivalent to the requirements that $\phi^{-1/2}d(\phi) \in W^{1,\infty}(\Omega)$ and $\phi^{-1/2}\nabla d(\phi) \in (L^\infty(\Omega))^n$.

LEMMA 3.1. *If* (3.2) *and* (3.4) *hold, then* $H_{\phi,d}(\mathrm{div};\Omega)$ *is a Hilbert space with the inner-product*

$$(\mathbf{u},\mathbf{v})_{H_{\phi,d}(\mathrm{div})} = (\mathbf{u},\mathbf{v}) + \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}), \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v})\big). \tag{3.5}$$

Moreover, $H(\mathrm{div};\Omega) = \big\{\mathbf{v} \in (L^2(\Omega))^n \,:\, \nabla\cdot\mathbf{v} \in L^2(\Omega)\big\} \subset H_{\phi,d}(\mathrm{div};\Omega)$.

Proof. It is clear that $H_{\phi,d}(\mathrm{div})$ is a linear space and that $(\cdot,\cdot)_{H_{\phi,d}(\mathrm{div})}$ is an inner-product. We must show that the space is complete. Let $\{\mathbf{u}_k\}_{k=1}^\infty \subset H_{\phi,d}(\mathrm{div})$ be a Cauchy sequence, which is to say that

$$\|\mathbf{u}_\ell - \mathbf{u}_k\|^2_{H_{\phi,d}(\mathrm{div})} = \|\mathbf{u}_\ell - \mathbf{u}_k\|^2 + \|\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{u}_\ell - \mathbf{u}_k))\|^2 \longrightarrow 0$$

as $\ell, k \to \infty$. As a consequence, there is $\mathbf{u} \in (L^2)^n$ such that $\mathbf{u}_k \to \mathbf{u}$ as $k \to \infty$, and there is $\xi \in L^2$ such that $\xi_k = \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}_k) \to \xi$ as $k \to \infty$.

To obtain strong convergence of the full divergence term, multiply by a test function $\psi \in C_0^\infty(\Omega)$, integrate, integrate by parts, and use the product rule for the gradient operator to compute

$$\begin{aligned}
\big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}_k), \psi\big) &= -\big(\mathbf{u}_k, d(\phi)\nabla(\phi^{-1/2}\psi)\big)\\
&= \frac{1}{2}(\mathbf{u}_k, \phi^{-3/2}d(\phi)\nabla\phi\,\psi) - (\mathbf{u}_k, \phi^{-1/2}d(\phi)\nabla\psi)\\
&\longrightarrow \frac{1}{2}(\mathbf{u}, \phi^{-3/2}d(\phi)\nabla\phi\,\psi) - (\mathbf{u}, \phi^{-1/2}d(\phi)\nabla\psi)\\
&= -\big(\mathbf{u}, d(\phi)\nabla(\phi^{-1/2}\psi)\big)\\
&= \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}), \psi\big).
\end{aligned}$$

We conclude that $\xi_k = \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u}_k)$ converges weakly to $\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u})$ in L^2. Therefore, $\xi = \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{u})$, and the proof of completeness is finished.

For any $\mathbf{v} \in H(\mathrm{div})$, both $\mathbf{v} \in (L^2)^n$ and $\nabla\cdot\mathbf{v} \in L^2$, and so (3.3) implies that $\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}) \in L^2$, and the final assertion of the lemma holds. ∎

We wish to apply the Neumann boundary condition. Define the normal trace operator $\gamma_{\phi,d} : H_{\phi,d}(\mathrm{div};\Omega) \to H^{-1/2}(\partial\Omega)$ using the integration by parts formula

$$\langle \gamma_{\phi,d}(\mathbf{v}), w\rangle = \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}), w\big) + \big(\mathbf{v}, d(\phi)\nabla(\phi^{-1/2}w)\big), \tag{3.6}$$

wherein $w \in H^{1/2}(\partial\Omega)$ has been extended to $w \in H^1(\Omega)$. Note that (3.2) and (3.4) imply that the operator is well defined by the right-hand side, and that we can interpret $\gamma_{\phi,d}(\mathbf{v}) = \phi^{-1/2}d(\phi)\mathbf{v}\cdot\nu$.

LEMMA 3.2. *If* (3.2) *and* (3.4) *hold, then* $\gamma_{\phi,d} : H_{\phi,d}(\mathrm{div};\Omega) \to H^{-1/2}(\partial\Omega)$ *is well defined by* (3.6)*, and there is a constant* $C > 0$ *such that*

$$\|\gamma_{\phi,d}(\mathbf{v})\|_{-1/2,\partial\Omega} = \|\phi^{-1/2}d(\phi)\mathbf{v}\cdot\nu\|_{-1/2,\partial\Omega} \le C\|\mathbf{v}\|_{H_{\phi,d}(\mathrm{div})} \tag{3.7}$$

for any $\mathbf{v} \in H_{\phi,d}(\mathrm{div};\Omega)$.

Finally, we apply the homogeneous boundary condition to define

$$H_{\phi,d,0}(\mathrm{div};\Omega) = \big\{\mathbf{v} \in H_{\phi,d}(\mathrm{div};\Omega) \ : \ \gamma_{\phi,d}(\mathbf{v}) = \phi^{-1/2}d(\phi)\mathbf{v}\cdot\nu = 0 \text{ on } \partial\Omega\big\}, \tag{3.8}$$

and we let the image of the normal trace operator be denoted by

$$H^{-1/2}_{\phi,d}(\partial\Omega) = \gamma_{\phi,d}\big(H_{\phi,d}(\mathrm{div};\Omega)\big) \subset H^{-1/2}(\partial\Omega). \tag{3.9}$$

4. A scaled weak formulation and unique existence of the solution. In this section, we set up and analyze a weak form of (2.7)–(2.9). To apply the essential Neumann boundary condition (2.9), we define a lifting of the Neumann data g_{N} to a function $\mathbf{v}_{\mathrm{N}} \in H_{\phi,d}(\mathrm{div};\Omega)$ such that

$$\gamma_{\phi,d}(\mathbf{v}_{\mathrm{N}}) = \phi^{-1/2}d(\phi)\mathbf{v}_{\mathrm{N}}\cdot\nu = g_{\mathrm{N}}, \tag{4.1}$$

which can be found provided that $g_{\mathrm{N}} \in H^{-1/2}_{\phi,d}(\partial\Omega)$. We test (2.7) and $\phi^{-1/2}$ times (2.8) to obtain our scaled weak formulation: Find $\mathbf{v} \in H_{\phi,d,0}(\mathrm{div};\Omega) + \mathbf{v}_{\mathrm{N}}$ and $q \in L^2(\Omega)$ such that

$$(\mathbf{v},\boldsymbol{\psi}) - \big(q, \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) = (d(\phi)\mathbf{g},\boldsymbol{\psi}) \quad \forall\boldsymbol{\psi} \in H_{\phi,d,0}(\mathrm{div};\Omega), \tag{4.2}$$

$$\big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}), w\big) + (q,w) = (f,w) \qquad \forall w \in L^2(\Omega). \tag{4.3}$$

We require that $f \in L^2(\Omega)$ and $d(\phi)\mathbf{g} \in (L^2(\Omega))^n$.

We saw the a priori energy estimates (2.10) for (4.2)–(4.3), which imply that if there is a solution to the problem, then it is unique. To prove existence of a solution, we use a stabilized variational formulation [13]. Let $\delta \ge 0$. The stabilized formulation is constructed by taking (4.2) for $\boldsymbol{\psi} \in H_{\phi,d,0}(\mathrm{div};\Omega)$ and adding (4.3) with w replaced by $\breve{w} + \delta\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi}) \in L^2(\Omega)$ for $\breve{w} \in L^2(\Omega)$. That is,

$$\begin{aligned}(\mathbf{v},\boldsymbol{\psi}) &- \big(q,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}),\breve{w}\big) + (q,\breve{w})\\ &+ \delta\big\{\big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}),\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \big(q,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big)\big\}\\ &= (d(\phi)\mathbf{g},\boldsymbol{\psi}) + (f,\breve{w}) + \delta\big(f,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big)\\ &\qquad\qquad \forall(\boldsymbol{\psi},\breve{w}) \in H_{\phi,d,0}(\mathrm{div};\Omega)\times L^2(\Omega).\end{aligned} \tag{4.4}$$

All of these problems are equivalent to the problem with $\delta = 0$, which is the original problem (4.2)–(4.3), because for any given δ and $\boldsymbol{\psi} \in H_{\phi,d,0}(\mathrm{div};\Omega)$, we can replace the scalar test function $\breve{w}$ by $w - \delta\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi}) \in L^2(\Omega)$ for any $w \in L^2(\Omega)$.

The stabilized bilinear form $a_\delta : \big(H_{\phi,d,0}(\mathrm{div}) \times L^2\big) \times \big(H_{\phi,d,0}(\mathrm{div}) \times L^2\big) \to \mathbb{R}$ is defined by

$$
\begin{aligned}
(4.5)\quad & a_\delta\big((\mathbf{v}_0, q), (\boldsymbol{\psi}, w)\big) \\
&= (\mathbf{v}_0, \boldsymbol{\psi}) - \big(q, \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0), w\big) + (q, w) \\
&\quad + \delta\big\{\big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0), \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \big(q, \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big)\big\},
\end{aligned}
$$

and the linear form $b_\delta : H_{\phi,d,0}(\mathrm{div}) \times L^2 \to \mathbb{R}$ is defined by

$$
\begin{aligned}
(4.6)\quad b_\delta(\boldsymbol{\psi}, w) &= (d(\phi)\mathbf{g}, \boldsymbol{\psi}) - (\mathbf{v}_\mathrm{N}, \boldsymbol{\psi}) + \big(f - \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_\mathrm{N}), w\big) \\
&\quad + \delta\big\{\big(f - \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_\mathrm{N}), \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big)\big\}.
\end{aligned}
$$

Now (4.4) is the problem: Find $(\mathbf{v}_0, q) \in H_{\phi,d,0}(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$
(4.7)\quad a_\delta\big((\mathbf{v}_0, q), (\boldsymbol{\psi}, w)\big) = b_\delta(\boldsymbol{\psi}, w) \quad \forall(\boldsymbol{\psi}, w) \in H_{\phi,d,0}(\mathrm{div};\Omega) \times L^2(\Omega).
$$

With $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_\mathrm{N}$ and $\delta = 0$, this problem is (4.2)–(4.3). By construction, however, all of the problems (4.7) are equivalent for any $\delta \ge 0$.

The two forms (4.5)–(4.6) are clearly continuous, i.e., bounded. We claim that for any $\delta \in (0, 2)$, the bilinear form a_δ is coercive. To see this, simply compute

$$
\begin{aligned}
a_\delta\big((\mathbf{v}_0, q), (\mathbf{v}_0, q)\big) &= \|\mathbf{v}_0\|^2 + \|q\|^2 + \delta\|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0)\|^2 + \delta\big(q, \phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0)\big) \\
&\ge \|\mathbf{v}_0\|^2 + \left(1 - \frac{1}{2}\delta\right)\|q\|^2 + \frac{1}{2}\delta\|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_0)\|^2 \\
&\ge \frac{1}{2}\delta\|\mathbf{v}_0\|_{H_{\phi,d}(\mathrm{div})} + \left(1 - \frac{1}{2}\delta\right)\|q\|^2.
\end{aligned}
$$

Therefore, we can apply the Lax–Milgram theorem to conclude that (4.7) has a unique solution for $\delta \in (0, 2)$. By the equivalence of the weak problems, we then have a solution for any $\delta \ge 0$, and in particular one for our problem, which is (4.7) with $\delta = 0$, i.e., (4.2)–(4.3) (and for which we already showed the solution is unique). We have proven the following theorem.

THEOREM 4.1. *Let* (3.2) *and* (3.4) *hold,* $f \in L^2(\Omega)$, $d(\phi)\mathbf{g} \in (L^2(\Omega))^n$, *and* $g_\mathrm{N} \in H^{-1/2}_{\phi,d}(\partial\Omega)$. *If* $\mathbf{v}_\mathrm{N}$ *is defined by* (4.1), *then there is a unique solution to the problem* (4.2)–(4.3), *and the energy estimates* (2.10) *and* (2.4) *hold.*

5. Some extensions of the results.

We can handle Dirichlet and Robin boundary conditions, and in some cases, we can show that $p \in L^2(\Omega)$.

5.1. Dirichlet and Robin boundary conditions.

Instead of Neumann boundary conditions (1.3), we could impose Dirichlet or Robin boundary conditions of the form

$$
(5.1)\quad \phi p - \kappa^2 b(\phi)\mathbf{u}\cdot\nu = \phi^{1/2} g_\mathrm{R} \quad \text{on } \partial\Omega,
$$

where $\kappa \ge 0$ is a bounded function and g_R is given. Recalling (2.5)–(2.6), the scaled version is

$$
(5.2)\quad q - \kappa^2\phi^{-1/2} d(\phi)\mathbf{v}\cdot\nu = g_\mathrm{R} \quad \text{on } \partial\Omega,
$$

and the scaled weak form is as follows: Find $\mathbf{v} \in H_{\phi,d}(\mathrm{div};\Omega)$ and $q \in L^2(\Omega)$ such that

$$(5.3)\qquad (\mathbf{v},\boldsymbol{\psi}) - \big(q, \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) + \langle \kappa^2\phi^{-1}(d(\phi))^2\mathbf{v}\cdot\nu, \boldsymbol{\psi}\cdot\nu\rangle$$
$$= (d(\phi)\mathbf{g},\boldsymbol{\psi}) - \langle g_{\mathrm{R}}, \phi^{-1/2}d(\phi)\boldsymbol{\psi}\cdot\nu\rangle \qquad \forall\boldsymbol{\psi}\in H_{\phi,d}(\mathrm{div};\Omega),$$

$$(5.4)\qquad \big(\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}), w\big) + (q,w) = (f,w) \qquad \forall w \in L^2(\Omega).$$

We require that $g_{\mathrm{R}} \in H^{1/2}(\partial\Omega)$ (actually, we require merely that g_{R} be in the dual space $(H^{-1/2}_{\phi,d}(\partial\Omega))^*$) and, as before, $f \in L^2(\Omega)$ and $d(\phi)\mathbf{g} \in (L^2(\Omega))^n$. Using the trace Lemma 3.2, the a priori energy estimates are

$$(5.5)\qquad \|\mathbf{v}\| + \|q\| + \|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v})\| + \|\kappa\phi^{-1/2}d(\phi)\mathbf{v}\cdot\nu\|_{\partial\Omega}$$
$$\le C\big\{\|f\| + \|d(\phi)\mathbf{g}\| + \|g_{\mathrm{R}}\|_{1/2,\partial\Omega}\big\},$$

and the analogue to Theorem 4.1 can be proved in a similar way. We need to modify a_δ by extending it to $\big(H_{\phi,d}(\mathrm{div})\times L^2\big)\times\big(H_{\phi,d}(\mathrm{div})\times L^2\big)$ and adding a term, obtaining

$$\tilde a_\delta\big((\mathbf{v},q),(\boldsymbol{\psi},w)\big) = a_\delta\big((\mathbf{v},q),(\boldsymbol{\psi},w)\big) + \langle\kappa^2\phi^{-1}(d(\phi))^2\mathbf{v}\cdot\nu, \boldsymbol{\psi}\cdot\nu\rangle.$$

We also redefine b_δ as $\tilde b_\delta : H_{\phi,d}(\mathrm{div})\times L^2 \to \mathbb{R}$ such that

$$\tilde b_\delta(\boldsymbol{\psi},w) = (d(\phi)\mathbf{g},\boldsymbol{\psi}) + (f,w) - \langle g_{\mathrm{R}}, \phi^{-1/2}d(\phi)\boldsymbol{\psi}\cdot\nu\rangle + \delta\big(f, \phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big).$$

THEOREM 5.1. *Let* (3.2) *and* (3.4) *hold,* $f \in L^2(\Omega)$, $d(\phi)\mathbf{g} \in (L^2(\Omega))^n$, *and* $g_{\mathrm{R}} \in H^{1/2}(\partial\Omega)$. *Then there is a unique solution to the problem* (5.3)–(5.4)*, and the energy estimates* (5.5) *hold.*

5.2. A condition for the pressure to be in L^2. In some cases the solution is more regular than implied by (2.4). Proceeding formally from (1.1)–(1.2), we have the single equation

$$(5.6)\qquad -\nabla\cdot\big((d(\phi))^2\nabla p\big) + \phi p = \phi^{1/2}f - \nabla\cdot\big((d(\phi))^2\mathbf{g}\big).$$

We multiply by $\phi^{-1}p$ and integrate by parts using the homogeneous Neumann boundary condition (1.3) to see that

$$\big((d(\phi))^2\nabla p, \nabla(\phi^{-1}p)\big) + \|p\|^2 = (\phi^{-1/2}f,p) + \big((d(\phi))^2\mathbf{g}, \nabla(\phi^{-1}p)\big).$$

After formally expanding the derivative terms, we obtain

$$\begin{aligned}&\big(\phi^{-1}(d(\phi))^2\nabla p,\nabla p\big) + \|p\|^2\\ &\quad= (\phi^{-1/2}f,p) + \big(\phi^{-1}(d(\phi))^2\mathbf{g},\nabla p\big) - \big(\phi^{-2}(d(\phi))^2\nabla\phi\cdot\mathbf{g},p\big)\\ &\qquad+ \big(\phi^{-2}(d(\phi))^2\nabla\phi\cdot\nabla p, p\big)\\ &\quad\le \epsilon\big\{\|p\|^2 + \|\phi^{-1/2}d(\phi)\nabla p\|^2\big\} + \|\phi^{-3/2}d(\phi)\nabla\phi\|_{L^\infty(\Omega)}\|\phi^{-1/2}d(\phi)\nabla p\|\,\|p\|\\ &\qquad+ C_\epsilon\big\{\|\phi^{-1/2}f\|^2 + \|\phi^{-1/2}d(\phi)\mathbf{g}\|^2 + \|\phi^{-2}(d(\phi))^2\nabla\phi\cdot\mathbf{g}\|^2\big\}\end{aligned}$$

for any $\epsilon > 0$, and so

$$(5.7)\quad \|\phi^{-1/2}d(\phi)\nabla p\| + \|p\| \le C\big\{\|\phi^{-1/2}f\| + \|\phi^{-1/2}d(\phi)\mathbf{g}\| + \|\phi^{-2}(d(\phi))^2\nabla\phi\cdot\mathbf{g}\|\big\},$$

provided that $\|\phi^{-3/2}d(\phi)\nabla\phi\|_{L^\infty(\Omega)} < 2$. For greater generality, we have chosen to work with the scaled pressure. However, it is interesting to note that p may in fact be stable in some cases, and then it is only the loss of an equation that is problematic for numerical approximation.

6. Mixed finite element methods. We now discuss discrete versions of our scaled systems (4.2)–(4.3) for Neumann and (5.3)–(5.4) for Dirichlet and Robin boundary conditions. We assume that $\Omega \subset \mathbb{R}^n$ is a polygonal domain and impose over it a quasi-uniform, conforming finite element mesh $\mathcal{T}_h$ of simplices or rectangular parallelepipeds having maximal diameter h. We will approximate $(\mathbf{v}, q)$ in the lowest order Raviart–Thomas (RT_0) finite element space $\mathbf{V}_h \times W_h$ [24, 12, 25], although other mixed elements could be used. If $n = 2$, the mesh may also contain convex quadrilaterals, which then use the lowest order Arbogast–Correa (AC_0) mixed finite elements [2]. We also define the space $\mathbf{V}_{h,0} = \{\mathbf{v}_h \in \mathbf{V}_h : \mathbf{v}_h \cdot \nu = 0 \text{ on } \partial\Omega\}$.

Later we will make use of the usual projection operators associated with RT_0 or AC_0. Let $\mathcal{P}_{W_h} : L^2(\Omega) \to W_h$ denote the $L^2(\Omega)$ projection operator, which projects a function onto the space of piecewise constant functions. We sometimes abbreviate this operator when applied to a function w as $\hat{w} = \mathcal{P}_{W_h} w$. Moreover, let $\pi : H(\mathrm{div}; \Omega) \cap L^{2+\epsilon}(\Omega) \to \mathbf{V}_h$ (any $\epsilon > 0$) denote the standard Raviart–Thomas or Fortin operator that preserves element average divergence and edge normal fluxes [24, 12, 25, 2].

To simplify the treatment of boundary conditions, when using Neumann conditions, let $\beta_{\mathrm{N}} = 1$ and $\beta_{\mathrm{R}} = 1 - \beta_{\mathrm{N}} = 0$, and when using Robin conditions, let $\beta_{\mathrm{N}} = 0$ and $\beta_{\mathrm{R}} = 1$. Also let $\tilde{\mathbf{V}}_h = \beta_{\mathrm{N}} \mathbf{V}_{h,0} + \beta_{\mathrm{R}} \mathbf{V}_h$. The mixed finite element method for Neumann (4.2)–(4.3) or Robin (5.3)–(5.4) boundary conditions is as follows: Find $\mathbf{v}_h \in \tilde{\mathbf{V}}_h + \beta_{\mathrm{N}} \mathbf{v}_{\mathrm{N}}$ and $q_h \in W_h$ such that

$$(6.1)\quad \begin{aligned} &(\mathbf{v}_h, \boldsymbol{\psi}) - \big(q_h, \phi^{-1/2} \nabla \cdot (d(\phi) \boldsymbol{\psi})\big) + \beta_{\mathrm{R}} \langle \kappa^2 \phi^{-1} (d(\phi))^2 \mathbf{v}_h \cdot \nu, \boldsymbol{\psi} \cdot \nu \rangle \\ &\quad = (d(\phi) \mathbf{g}, \boldsymbol{\psi}) - \beta_{\mathrm{R}} \langle g_{\mathrm{R}}, \phi^{-1/2} d(\phi) \boldsymbol{\psi} \cdot \nu \rangle \qquad \forall \boldsymbol{\psi} \in \tilde{\mathbf{V}}_h, \end{aligned}$$

$$(6.2)\quad \big(\phi^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_h), w\big) + (q_h, w) = (f, w) \qquad \forall w \in W_h.$$

When using Neumann boundary conditions, we could modify the scheme to find $\mathbf{v}_h \in \mathbf{V}_{h,0} + \pi \mathbf{v}_{\mathrm{N}}$ instead. The divergence and boundary terms involve division by ϕ, so some care is needed in the implementation. Everything is well defined by the assumptions (3.2) and (3.4), so at a quadrature point, simply set the term to zero when ϕ vanishes.

To show unique solvability and stability of the Robin case, we require a discrete version of the trace Lemma 3.2.

LEMMA 6.1. *If (3.2) and (3.4) hold and the finite element mesh $\mathcal{T}_h$ is quasi-uniform, then there is a constant $C > 0$ such that*

$$(6.3)\qquad \|\phi^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_h)\| \le C \big\{ \|\mathbf{v}_h\| + \|\mathcal{P}_{W_h} [\phi^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_h)]\| \big\}$$

for any $\mathbf{v}_h \in \mathbf{V}_h$. Moreover, for some possibly different constant $C > 0$,

$$(6.4)\qquad \begin{aligned} \|\gamma_{\phi,d}(\mathbf{v}_h)\|_{-1/2,\partial\Omega} &= \|\phi^{-1/2} d(\phi) \mathbf{v}_h \cdot \nu\|_{-1/2,\partial\Omega} \\ &\le C \big\{ \|\mathbf{v}_h\| + \|\mathcal{P}_{W_h} [\phi^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_h)]\| \big\} \end{aligned}$$

for any $\mathbf{v}_h \in \mathbf{V}_h$.

Proof. The triangle inequality gives that

$$\begin{aligned} &\|\phi^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_h)\| \\ &\quad \le \|\mathcal{P}_{W_h} [\phi^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_h)]\| + \|\phi^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_h) - \mathcal{P}_{W_h} [\phi^{-1/2} \nabla \cdot (d(\phi) \mathbf{v}_h)]\|. \end{aligned}$$

We compute

$$
\begin{aligned}
&\|\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h) - \mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\| \\
&\quad = \|\phi^{-1/2}\nabla d(\phi)\cdot\mathbf{v}_h - \mathcal{P}_{W_h}[\phi^{-1/2}\nabla d(\phi)\cdot\mathbf{v}_h] \\
&\qquad\qquad + \phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}_h - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}_h]\| \\
&\quad \le 2\|\phi^{-1/2}\nabla d(\phi)\cdot\mathbf{v}_h\| + \|\phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}_h - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)\nabla\cdot\mathbf{v}_h]\| \\
&\quad \le C\|\mathbf{v}_h\| + \big\|\big(\phi^{-1/2}d(\phi) - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)]\big)\nabla\cdot\mathbf{v}_h\big\|,
\end{aligned}
$$

since $\phi^{-1/2}\nabla d(\phi) \in (L^\infty(\Omega))^n$ by assumption (3.2). Furthermore, for the last term,

$$
\begin{aligned}
&\big\|\big(\phi^{-1/2}d(\phi) - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)]\big)\nabla\cdot\mathbf{v}_h\big\| \\
&\quad \le \|\phi^{-1/2}d(\phi) - \mathcal{P}_{W_h}[\phi^{-1/2}d(\phi)]\|_{L^\infty(\Omega)}\|\nabla\cdot\mathbf{v}_h\| \\
&\quad \le Ch\|\phi^{-1/2}d(\phi)\|_{W^{1,\infty}(\Omega)}\|\nabla\cdot\mathbf{v}_h\| \\
&\quad \le C\|\phi^{-1/2}d(\phi)\|_{W^{1,\infty}(\Omega)}\|\mathbf{v}_h\|,
\end{aligned}
$$

using [15] for the approximation of the L^2-projection in L^∞ and an inverse estimate, since the finite element mesh $\mathcal{T}_h$ is assumed to be quasi-uniform. Because $\phi^{-1/2}d(\phi) \in W^{1,\infty}(\Omega)$ by assumptions (3.2) and (3.4), the first result (6.3) is established. The discrete trace bound (6.4) then follows directly from the trace Lemma 3.2. ∎

Substituting into (6.1)–(6.2) the discrete solution $\boldsymbol{\psi} = \mathbf{v}_h - \beta_{\mathrm{N}}\mathbf{v}_{\mathrm{N}} \in \tilde{\mathbf{V}}_h$ and $w = q_h + \mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)] \in W_h$ shows the stability result

$$
\begin{aligned}
\text{(6.5)}\qquad &\|\mathbf{v}_h\| + \|q_h\| + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_h)]\| + \beta_{\mathrm{R}}\|\kappa\phi^{-1/2}d(\phi)\mathbf{v}_h\cdot\nu\|_{\partial\Omega} \\
&\quad \le C\big\{\|f\| + \|d(\phi)\mathbf{g}\| \\
&\qquad + \beta_{\mathrm{R}}\|g_{\mathrm{R}}\|_{1/2,\partial\Omega} + \beta_{\mathrm{N}}\big(\|\mathbf{v}_{\mathrm{N}}\| + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\mathbf{v}_{\mathrm{N}})]\|\big)\big\}.
\end{aligned}
$$

Uniqueness of the solution is therefore established, and existence follows because the discrete system over a basis is a square linear system.

With the notation $|E|$ for the measure (area or volume) of $E \in \mathcal{T}_h$, the local average of ϕ restricted to the element $E \in \mathcal{T}_h$ is

$$
\hat{\phi}|_E = \frac{1}{|E|}\int_E \phi\,dx.
$$

We define the piecewise constant discrete pressure $p_h \in W_h$ by setting for all $E \in \mathcal{T}_h$

$$
\text{(6.6)}\qquad p_h|_E = \begin{cases} 0 & \text{if } \hat{\phi}|_E = 0,\\ (\hat{\phi}^{-1/2}q_h)|_E & \text{if } \hat{\phi}|_E \ne 0,\end{cases}
$$

and the discrete velocity $\mathbf{u}_h \in \tilde{\mathbf{V}}_h + \beta_{\mathrm{N}}\mathbf{v}_{\mathrm{N}}$ is defined by setting, for all element edges ($n = 2$) or faces ($n = 3$) e,

$$
\text{(6.7)}\qquad \mathbf{u}_h\cdot\nu|_e = \begin{cases} 0 & \text{if } b_e \equiv \displaystyle\int_e b(\phi)\,ds = 0,\\ b_e^{-1}\displaystyle\int_e d(\phi)\,ds\ \mathbf{v}_h\cdot\nu|_e & \text{if } b_e \ne 0,\end{cases}
$$

so that $\pi(b(\phi)\mathbf{u}_h) = \pi(d(\phi)\mathbf{v}_h)$.

7. An analysis of the error of the mixed method. For simplicity of exposition, we continue the discussion only for the Robin system ($\beta_N = 0$). Since $\mathbf{V}_h \subset H(\mathrm{div}) \subset H_{\phi,d}(\mathrm{div})$, we can take the difference of the true weak formulation (5.3)–(5.4) with discrete test functions and (6.1)–(6.2), which leads to the system

$$
\begin{aligned}
(7.1)\qquad & (\mathbf{v}-\mathbf{v}_h,\boldsymbol{\psi}) - \big(q-q_h,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big)\\
&\quad + \langle \kappa^2\phi^{-1}(d(\phi))^2(\mathbf{v}-\mathbf{v}_h)\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle = 0 \quad \forall\boldsymbol{\psi}\in\mathbf{V}_h,\\
(7.2)\qquad & \big(\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\mathbf{v}_h)),w\big) + (q-q_h,w) = 0 \quad \forall w\in W_h.
\end{aligned}
$$

We modify this system by introducing our two projection operators to see that

$$
\begin{aligned}
&(\pi\mathbf{v}-\mathbf{v}_h,\boldsymbol{\psi}) - \big(\hat q-q_h,\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})]\big) + \langle\kappa^2\phi^{-1}(d(\phi))^2(\pi\mathbf{v}-\mathbf{v}_h)\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle\\
&\quad = -(\mathbf{v}-\pi\mathbf{v},\boldsymbol{\psi}) + \big(q-\hat q,\phi^{-1/2}\nabla\cdot(d(\phi)\boldsymbol{\psi})\big) - \langle\kappa^2\phi^{-1}(d(\phi))^2(\mathbf{v}-\pi\mathbf{v})\cdot\nu,\boldsymbol{\psi}\cdot\nu\rangle,\\
&\big(\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))],w\big) + (\hat q-q_h,w)\\
&\quad = -\big(\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\pi\mathbf{v}))],w\big).
\end{aligned}
$$

Assuming that $\mathbf{v}$ is sufficiently regular to compute $\pi\mathbf{v}$, the test functions $\boldsymbol{\psi} = \pi\mathbf{v} - \mathbf{v}_h \in \mathbf{V}_h$ and $w = \hat q - q_h + \mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))] \in W_h$ lead us to

$$
\begin{aligned}
(7.3)\qquad & \|\pi\mathbf{v}-\mathbf{v}_h\|^2 + \|\hat q-q_h\|^2\\
&\quad + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))]\|^2 + \|\kappa\phi^{-1/2}d(\phi)(\pi\mathbf{v}-\mathbf{v}_h)\cdot\nu\|^2_{\partial\Omega}\\
&\le C\big\{\|\mathbf{v}-\pi\mathbf{v}\|^2 + \|q-\hat q\|^2\\
&\quad + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\pi\mathbf{v}))]\|^2 + \|\kappa\phi^{-1/2}d(\phi)(\mathbf{v}-\pi\mathbf{v})\cdot\nu\|^2_{\partial\Omega}\\
&\quad + \big|\big(q-\hat q,\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))\big)\big|\big\}.
\end{aligned}
$$

The last term on the right-hand side is troublesome. It arises because we could not substitute the test function $\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h)) \notin W_h$ for w.

We continue by noting that anything in W_h is orthogonal to $q-\hat q$. We also expand the divergence terms using the product rule for differentiation to estimate

$$
\begin{aligned}
&\big|\big(q-\hat q,\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))\big)\big|\\
&\quad = \big|\big(q-\hat q,(I-\mathcal{P}_{W_h})\big[\phi^{-1/2}\nabla d(\phi)\cdot(\pi\mathbf{v}-\mathbf{v}_h) + \phi^{-1/2}d(\phi)\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\big]\big)\big|\\
&\quad \le C\,\|q-\hat q\|\,\big\{\|\pi\mathbf{v}-\mathbf{v}_h\| + \|(I-\mathcal{P}_{W_h})[\phi^{-1/2}d(\phi)\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)]\|\big\},
\end{aligned}
$$

and, since $\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\in W_h$,

$$
\begin{aligned}
&\|(I-\mathcal{P}_{W_h})[\phi^{-1/2}d(\phi)\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)]\|\\
&\quad = \|[(I-\mathcal{P}_{W_h})\phi^{-1/2}d(\phi)]\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\|\\
&\quad \le \|(I-\mathcal{P}_{W_h})\phi^{-1/2}d(\phi)\|_{L^\infty(\Omega)}\|\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\|\\
&\quad \le Ch\,\|\phi^{-1/2}d(\phi)\|_{W^{1,\infty}(\Omega)}\|\nabla\cdot(\pi\mathbf{v}-\mathbf{v}_h)\|\\
&\quad \le C\|\pi\mathbf{v}-\mathbf{v}_h\|,
\end{aligned}
$$

using [15] again for the approximation of the L^2-projection in L^∞ and an inverse estimate, since the finite element mesh $\mathcal{T}_h$ is quasi-uniform. Combining the two previous results, we have that

$$
(7.4)\qquad \big|\big(q-\hat q,\phi^{-1/2}\nabla\cdot(d(\phi)(\pi\mathbf{v}-\mathbf{v}_h))\big)\big| \le C\,\|q-\hat q\|\,\|\pi\mathbf{v}-\mathbf{v}_h\|.
$$

Finally, (7.4), (7.3), and the triangle inequality lead us to the estimate

$$\begin{aligned}&\|\mathbf{v}-\mathbf{v}_h\| + \|q-q_h\| \\ &\qquad + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\mathbf{v}_h))]\| + \|\kappa\phi^{-1/2}d(\phi)(\mathbf{v}-\mathbf{v}_h)\cdot\nu\|_{\partial\Omega} \\ &\quad \le C\big\{\|\mathbf{v}-\pi\mathbf{v}\| + \|q-\hat{q}\| \\ &\qquad + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\pi\mathbf{v}))]\| + \|\kappa\phi^{-1/2}d(\phi)(\mathbf{v}-\pi\mathbf{v})\cdot\nu\|_{\partial\Omega}\big\}.\end{aligned}$$

Similar estimates can be shown to hold for the system using Neumann boundary conditions. In this case, the test function $\boldsymbol{\psi} = \pi\mathbf{v} - \mathbf{v}_h + \mathbf{v}_{\mathrm{N}} - \pi\mathbf{v}_{\mathrm{N}} \in \mathbf{V}_{h,0}$ is required. We have shown the following theorem.

THEOREM 7.1. *Let* (3.2) *and* (3.4) *hold,* $f \in L^2(\Omega)$, $d(\phi)\mathbf{g} \in (L^2(\Omega))^n$, *and assume that the finite element mesh* $\mathcal{T}_h$ *is quasi-uniform. Let* $(\mathbf{v}, q)$ *be either the solution to* (4.2)–(4.3) *with* $\mathbf{v}_N \in H_{\phi,d}(\mathrm{div};\Omega)$ (*and set* $\beta_{\mathrm{N}} = 1$, $\beta_{\mathrm{R}} = 0$) *or the solution to* (5.3)–(5.4) *with* $g_{\mathrm{R}} \in H^{1/2}(\partial\Omega)$ (*and set* $\beta_{\mathrm{N}} = 0$, $\beta_{\mathrm{R}} = 1$). *Let* $(\mathbf{v}_h, q_h)$ *be the solution to the mixed method* (6.1)–(6.2). *Assume that* $\mathbf{v}, \beta_{\mathrm{N}}\mathbf{v}_{\mathrm{N}} \in H(\mathrm{div};\Omega) \cap L^{2+\epsilon}(\Omega)$ *for some* $\epsilon > 0$. *Then*

$$\begin{aligned}&\|\mathbf{v}-\mathbf{v}_h\| + \|q-q_h\| + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\mathbf{v}_h))]\| \\ &\qquad + \beta_{\mathrm{R}}\|\kappa\phi^{-1/2}d(\phi)(\mathbf{v}-\mathbf{v}_h)\cdot\nu\|_{\partial\Omega} \\ &\quad \le C\big\{\|\mathbf{v}-\pi\mathbf{v}\| + \|q-\hat{q}\| + \|\mathcal{P}_{W_h}[\phi^{-1/2}\nabla\cdot(d(\phi)(\mathbf{v}-\pi\mathbf{v}))]\| \\ &\qquad + \beta_{\mathrm{R}}\|\kappa\phi^{-1/2}d(\phi)(\mathbf{v}-\pi\mathbf{v})\cdot\nu\|_{\partial\Omega} + \beta_{\mathrm{N}}\|\mathbf{v}_{\mathrm{N}}-\pi\mathbf{v}_{\mathrm{N}}\|_{H_{\phi,d}(\mathrm{div};\Omega)}\big\} \\ &\quad \le C\big\{\|\mathbf{v}-\pi\mathbf{v}\|_{H(\mathrm{div};\Omega)} + \|q-\hat{q}\| \\ &\qquad + \beta_{\mathrm{R}}\|\kappa(\mathbf{v}-\pi\mathbf{v})\cdot\nu\|_{\partial\Omega} + \beta_{\mathrm{N}}\|\mathbf{v}_{\mathrm{N}}-\pi\mathbf{v}_{\mathrm{N}}\|_{H(\mathrm{div};\Omega)}\big\}.\end{aligned} \tag{7.5}$$

The last estimate of the theorem follows from (3.2) and (3.4). If the solution is sufficiently regular, the approximation is of order $\mathcal{O}(h)$.

8. Some closed form solutions in one dimension. Before presenting numerical results, it is instructive to consider a few closed form solutions to the problem in one dimension. Let $\Omega = (-1,1)$, $a(\phi) = b(\phi) = \phi$ (so $d(\phi) = \phi$, $c(\phi) = 1$), $\mathbf{g} = 0$, and $\tilde{f} = \phi^{-1/2}f$, and reduce the mixed system (1.1)–(1.3) to

$$-(\phi^2 p')' + \phi p = \phi\tilde{f}, \quad -1 < x < 1, \tag{8.1}$$

$$\phi^{3/2}(-1)\,p'(-1) = \phi^{3/2}(1)\,p'(1) = 0. \tag{8.2}$$

This is a Sturm–Liouville problem. By our energy estimates (2.4), we require that $u = v = -\phi p' \in L^2(-1,1)$ when $\phi^{1/2}\tilde{f} \in L^2(-1,1)$.

For $\alpha > 0$, let us simplify to the porosity

$$\phi(x) = \begin{cases} 0, & x < 0, \\ x^\alpha, & x > 0. \end{cases} \tag{8.3}$$

The conditions (3.2) and (3.4) hold if and only if $\alpha \ge 2$. Now (8.1)–(8.2) becomes

$$-x^\alpha\, p'' - 2\alpha\, x^{\alpha-1}\, p' + p = \tilde{f}, \quad 0 < x < 1, \quad \text{and} \quad p'(1) = 0.$$

When $\alpha = 2$, we have the Euler equation

$$-\,x^2\, p'' - 4x\, p' + p = \tilde{f}, \quad 0 < x < 1. \tag{8.4}$$

In this case the Euler exponents satisfy $r(r-1)+4r-1=0$, and so

$$r_1 = \frac{-3+\sqrt{13}}{2} \approx 0.3 > 0 \quad \text{and} \quad r_2 = \frac{-3-\sqrt{13}}{2} \approx -3.3 < 0, \tag{8.5}$$

and the solution to the homogeneous equation is $p_{\text{hom}}(x) = c_1 x^{r_1} + c_2 x^{r_2}$. The boundary condition and the requirement that $u = -\phi p' \in L^2(0,1)$ show that the solution is unique. Variation of parameters gives the solution to the nonhomogeneous equation as

$$p(x) = \frac{-1}{r_1 - r_2}\left\{x^{r_1}\left(\int \frac{\tilde{f}(x)}{x^{r_1+1}}\,dx + c_1\right) - x^{r_2}\left(\int \frac{\tilde{f}(x)}{x^{r_2+1}}\,dx + c_2\right)\right\}.$$

If we restrict to

$$\tilde{f}(x) = x^{\beta}, \quad 0 < x < 1, \tag{8.6}$$

then, provided $\beta \neq r_1, r_2$ and $0 < x < 1$, we have the closed form solutions

$$p(x) = \frac{-x^{\beta}}{(\beta - r_1)(\beta - r_2)} + C_1\, x^{r_1} + C_2\, x^{r_2}.$$

To get $f = \phi^{1/2}\tilde{f} = x^{1+\beta} \in L^2(0,1)$, restrict to $\beta > -3/2$. Then $u = -\phi p' \in L^2(0,1)$ implies that $C_2 = 0$, and the boundary condition determines C_1. If we arbitrarily set $p = 0$ for $x < 0$, the solution can be expressed as

$$q(x) = x\,p(x) \quad \text{and} \quad p(x) = \begin{cases} 0, & -1 < x \leq 0, \\ \dfrac{\beta x^{r_1} - r_1 x^{\beta}}{r_1(\beta - r_1)(\beta - r_2)}, & 0 < x < 1, \end{cases} \tag{8.7}$$

$$v(x) = u(x) = \begin{cases} 0, & -1 < x \leq 0, \\ \dfrac{-\beta(x^{r_1+1} - x^{\beta+1})}{(\beta - r_1)(\beta - r_2)}, & 0 < x < 1. \end{cases} \tag{8.8}$$

9. Some numerical results.

In this section we test the convergence of our proposed numerical scheme (6.1)–(6.2), (6.6)–(6.7) using Dirichlet boundary conditions. We fix the domain $\Omega = (-1,1)^n$ and use a uniform rectangular mesh of $m = 2/h$ elements in each coordinate direction.

We implement the tests in terms of manufactured solutions in which closed form expressions for ϕ and p are given, and from these f and Dirichlet boundary conditions (i.e., $\kappa = 0$) are computed. In all tests, we take $\mathbf{g} = 0$ and $a(\phi) = b(\phi) = \phi$ (so $d(\phi) = \phi$, $c(\phi) = 1$, and $\mathbf{u} = \mathbf{v}$). In this case, (3.4) follows from (3.2), so we only check the latter condition.

The code was developed using the `deal.II` finite element library [8, 7].

9.1. A simple Euler equation in one dimension.

We begin with a test case corresponding to our closed form solution (8.7)–(8.8) of the Euler equation (8.4). In this case, it is easy to see that in terms of the potential singularity near $x = 0$, $|q| \sim |u| \sim |x|^{1.3} + |x|^{1+\beta}$ and $|p| \sim |x|^{0.3} + |x|^{\beta}$, and so for any $\epsilon > 0$,

$$q, u \in H^{\min(1.8, 3/2+\beta)-\epsilon} \quad \text{and} \quad p \in H^{\min(0.8, 1/2+\beta)-\epsilon}.$$

Since $\phi(x) = x^2$, $0 < x < 1$, we have that $\phi^{-1/2}\phi' = 2 \in L^\infty(0,1)$, and our conditions (3.2) and (3.4) on ϕ are satisfied.

Table 1

Euler equation. Shown are the relative discrete L^2-norm errors of q, p, and $\mathbf{u}$ for various numbers of elements $m \times m$ and for four values of β. The convergence rate corresponds to a superconvergent approximation, restricted by the regularity of the true solution.

		Scaled pressure q		Pressure p		Velocity u	
β	m	error	rate	error	rate	error	rate
0.5	32	0.019721	—	0.027441	—	0.007878	—
	64	0.009919	0.991	0.015205	0.852	0.003816	1.046
	128	0.004975	0.996	0.008590	0.824	0.001885	1.018
	256	0.002492	0.998	0.004932	0.801	0.000938	1.006
	512	0.001247	0.999	0.002863	0.785	0.000468	1.002
0	32	0.021504	—	0.031925	—	0.009957	—
	64	0.010815	0.992	0.019032	0.746	0.004722	1.076
	128	0.005424	0.996	0.012029	0.662	0.002289	1.045
	256	0.002716	0.998	0.007966	0.595	0.001125	1.024
	512	0.001359	0.999	0.005437	0.551	0.000558	1.013
-0.5	32	0.023505	—	0.124442	—	0.013274	—
	64	0.011860	0.987	0.120184	0.050	0.006533	1.023
	128	0.005968	0.991	0.117710	0.030	0.003231	1.016
	256	0.002999	0.993	0.115756	0.024	0.001604	1.010
	512	0.001506	0.994	0.113991	0.022	0.000799	1.006
-1	32	0.024534	—	0.467424	—	0.028559	—
	64	0.012366	0.988	0.502215	-0.104	0.019402	0.558
	128	0.006221	0.991	0.528012	-0.072	0.013445	0.529
	256	0.003132	0.990	0.546552	-0.050	0.009412	0.515
	512	0.001583	0.984	0.559693	-0.034	0.006622	0.507
-1.5	32	0.074891	—	0.706240	—	0.268648	—
	64	0.072461	0.048	0.717895	-0.024	0.259974	0.047
	128	0.071367	0.022	0.723448	-0.011	0.252191	0.044
	256	0.070585	0.016	0.726124	-0.005	0.245192	0.041
	512	0.069885	0.014	0.727429	-0.003	0.238866	0.038

We consider five values of β (which is the parameter in the source function $\tilde{f} = x^\beta$ or $f = x^{1+\beta}$, $0 < x < 1$), $\beta = 1/2$, 0, $-1/2$, -1, and $-3/2$. The numerical results are presented in Table 1. Based on the regularity of the solution, for q and u we expect convergence to be approximately $\mathcal{O}(h^1)$ for $\beta \geq -1/2$ and $\mathcal{O}(h^{0.5})$ for $\beta = -1$, and we expect no convergence for $\beta = -3/2$. Indeed, we see these rates in the table, although q exhibits better convergence in this test when $\beta = -1$. On the other hand, for p we expect convergence to be about $\mathcal{O}(h^{0.8})$ for $\beta = 1/2$ and $\mathcal{O}(h^{0.5})$ for $\beta = 0$, and we expect no convergence for $\beta \leq -1/2$. Again, we see these rates in the table.

9.2. A smooth solution test in two dimensions. For the next series of tests, we assume that $p = \cos(6xy^2)$ is smooth and that ϕ is given by

$$\phi = \begin{cases} 0, & x \leq -3/4 \text{ or } y \leq -3/4, \\ (x+3/4)^\alpha (y+3/4)^{2\alpha} & \text{otherwise.} \end{cases} \tag{9.1}$$

We note that $\phi^{-1/2}\nabla\phi$ is in $\big(L^\infty((-1,1)^2)\big)^2$ if and only if $\alpha \geq 2$. Nevertheless, we consider the five values $\alpha = 2$, 1, $1/2$, $3/8$, and $1/4$. Owing to the singularity in x along $x = -3/4$, we see that for any $\epsilon > 0$,

$$q \in H^{(\alpha+1)/2-\epsilon} \quad \text{and} \quad \mathbf{u} \in \big(H^{\alpha+1/2-\epsilon}\big)^2.$$

Results are given in Table 2. We see $\mathcal{O}(h)$ convergence for q and $\mathbf{u}$ when $\alpha = 2$ and 1. For these two largest values of α, q and $\mathbf{u}$ are (nearly) sufficiently regular

Table 2

Smooth p two-dimensional test. Shown are the relative discrete L^2-norm errors of q, p, and $\mathbf{u}$ for various numbers of elements $m \times m$ and for three values of α defining ϕ. The convergence rate is better than expected for low values of α.

		Scaled pressure q		Pressure p		Velocity $\mathbf{u}$	
α	m	error	rate	error	rate	error	rate
2.0	16	0.168371	—	0.091681	—	0.227056	—
	32	0.085350	0.980	0.050988	0.846	0.112325	1.015
	64	0.043004	0.989	0.029729	0.778	0.056147	1.000
	128	0.021588	0.994	0.018401	0.692	0.028124	0.997
	256	0.010816	0.997	0.011964	0.621	0.014091	0.997
1.0	16	0.109458	—	0.078700	—	0.191054	—
	32	0.055158	0.989	0.041996	0.906	0.094530	1.015
	64	0.027715	0.993	0.023506	0.837	0.047351	0.997
	128	0.013912	0.994	0.014135	0.734	0.023880	0.988
	256	0.006980	0.995	0.009043	0.644	0.012153	0.974
0.5	16	0.086993	—	0.075720	—	0.166463	—
	32	0.044064	0.981	0.039040	0.956	0.082994	1.004
	64	0.022311	0.982	0.020722	0.914	0.042063	0.980
	128	0.011329	0.978	0.011606	0.836	0.021808	0.948
	256	0.005780	0.971	0.006963	0.737	0.011962	0.866
0.375	16	0.082881	—	0.076193	—	0.162887	—
	32	0.042577	0.961	0.040198	0.923	0.085878	0.924
	64	0.022666	0.910	0.023732	0.760	0.052387	0.713
	128	0.013659	0.731	0.017894	0.407	0.040953	0.355
	256	0.010643	0.360	0.017134	0.063	0.039050	0.069
0.25	16	0.089039	—	0.092354	—	0.179598	—
	32	0.061751	0.528	0.074371	0.312	0.126078	0.510
	64	0.057174	0.111	0.076599	-0.043	0.115282	0.129
	128	0.060753	-0.088	0.084176	-0.136	0.118547	-0.040
	256	0.066000	-0.120	0.092146	-0.131	0.124752	-0.074

to support first order convergence, but $\alpha = 1$ is insufficient for the condition (3.2). Perhaps this condition is overly restrictive for convergence. In fact, it may be enough that $\phi^{-1/2}\nabla\phi \in \big(L^2((-1,1)^2)\big)^2$, which is true here if and only if $\alpha > 1$.

There is some degradation in the convergence rate for q and $\mathbf{u}$ when $\alpha = 0.5$, and we see poor convergence behavior for the two smallest values of α. Even though the pressure p is smooth (at least where $\phi > 0$), its approximation shows poor convergence for all values of α.

We depict the solutions p and q in Figure 1. Although p was chosen to be smooth, we have displayed $p = 0$ in the one-phase region, since it is ill-defined there. Therefore, p is not smooth on the boundary between the one- and two-phase regions $\mathcal{B} = \{x = -3/4, y \geq -3/4\} \cup \{x \geq -3/4, y = -3/4\}$. We also display the scaled pressure q, which is well behaved for $\alpha = 2$ and degenerates near $\mathcal{B}$ as α decreases (i.e., as $\phi^{1/2}\nabla\phi$ loses its regularity).

The reader should note that $\mathcal{B}$ lies on a grid line. If we take an odd number of elements, we will avoid this. Results are shown in Table 3. When $\alpha = 2$ or $\alpha = 1$, we see errors and rates of convergence similar to those for the case of $\mathcal{B}$ being resolved by the grid in Table 2. However, the errors and convergence rates are worse for the more challenging cases of $\alpha \leq 1/2$.

9.3. A nonsmooth solution test in two dimensions. For the final series of tests, we assume that ϕ is given by (9.1) with $\alpha = 2$, but we impose the nonsmooth pressure solution

$$p = y(y-3x)(x+3/4)^{\beta}, \quad \beta = -1/4, -1/2, \text{ or } -3/4. \tag{9.2}$$

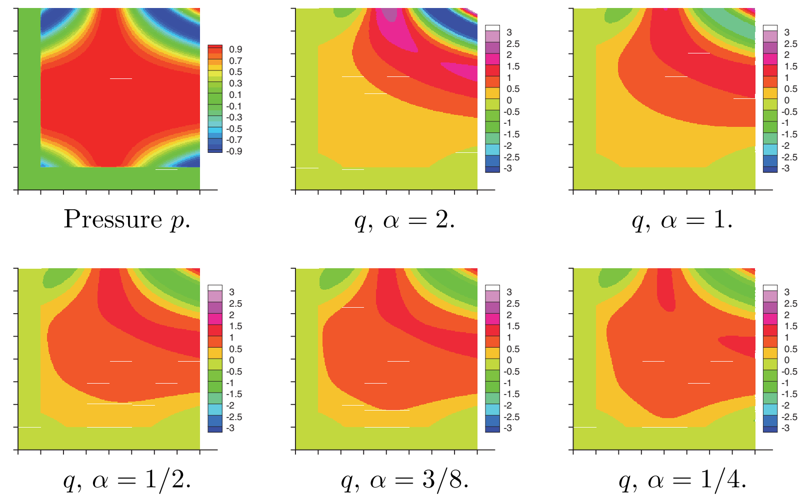
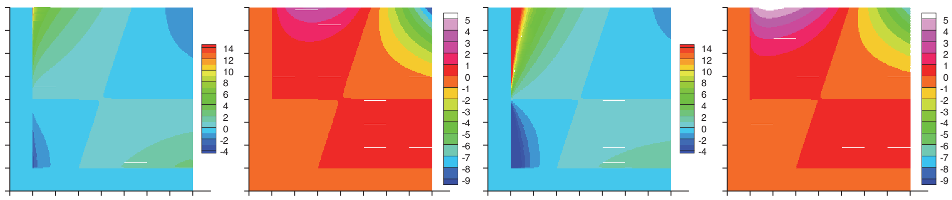


FIG. 1. *Smooth p two-dimensional test. Shown are the pressure p and scaled pressure q for various values of α defining ϕ. The pressure is smooth, except on the boundary of the support of ϕ (i.e., $x = -3/4$ or $y = -3/4$). The scaled pressure becomes less regular as α decreases near the boundary.*

TABLE 3

Smooth p two-dimensional test. Shown are the relative discrete L^2-norm errors of q, p, and $\mathbf{u}$ for various odd numbers of elements $m \times m$ and for $\alpha = 2$ and 0.25 defining ϕ. The convergence is similar to the case of grids that resolve the boundary between the one- and two-phase regions when $\alpha = 2$, but not for $\alpha = 1/4$.

		Scaled pressure q		Pressure p		Velocity $\mathbf{u}$	
α	m	error	rate	error	rate	error	rate
2.0	17	0.158736	—	0.093725	—	0.213317	—
	33	0.082806	0.939	0.053970	0.796	0.108907	0.970
	65	0.042348	0.967	0.032562	0.729	0.055285	0.978
	129	0.021422	0.983	0.020726	0.652	0.027907	0.986
	257	0.010774	0.991	0.013747	0.592	0.014036	0.991
1.0	17	0.103762	—	0.078254	—	0.178981	—
	33	0.053759	0.949	0.041893	0.901	0.091192	0.973
	65	0.027388	0.973	0.022864	0.874	0.046193	0.981
	129	0.013836	0.985	0.013206	0.792	0.023283	0.988
	257	0.006960	0.991	0.008070	0.710	0.011705	0.992
0.5	17	0.084208	—	0.076555	—	0.162552	—
	33	0.045726	0.881	0.047379	0.692	0.091236	0.833
	65	0.026708	0.776	0.036328	0.383	0.058930	0.631
	129	0.018298	0.546	0.032643	0.154	0.046176	0.352
	257	0.014852	0.301	0.030929	0.078	0.041385	0.158
0.375	17	0.084986	—	0.088812	—	0.170818	—
	33	0.054351	0.645	0.069672	0.350	0.112138	0.607
	65	0.043071	0.336	0.066478	0.068	0.090562	0.308
	129	0.040022	0.106	0.067148	-0.014	0.084275	0.104
	257	0.039486	0.019	0.068153	-0.021	0.082818	0.025
0.25	17	0.100156	—	0.116319	—	0.202030	—
	33	0.084350	0.248	0.110231	0.078	0.162366	0.315
	65	0.082428	0.033	0.113804	-0.046	0.151568	0.099
	129	0.083791	-0.024	0.117700	-0.049	0.150139	0.014
	257	0.085413	-0.028	0.120492	-0.034	0.151005	-0.008



p, $\beta = -1/4$. q, $\beta = -1/4$. p, $\beta = -3/4$. q, $\beta = -3/4$.

FIG. 2. *Nonsmooth p two-dimensional test. Shown are the pressure p and scaled pressure q for two values of β in* (9.2)*. The pressures are smooth, except on the boundary of the support of ϕ (i.e., $x = -3/4$ or $y = -3/4$). The two pressures become less regular near the boundary as β decreases.*

TABLE 4

Nonsmooth p two-dimensional test. Shown are the relative discrete L^2-norm errors of q, p, and $\mathbf{u}$ *for various odd numbers of elements $m \times m$ and for $\beta = -1/4$ and $-3/4$ defining p in* (9.2).

		Scaled pressure q		Pressure p		Velocity $\mathbf{u}$	
β	m	error	rate	error	rate	error	rate
$-1/4$	17	0.113422	—	0.219468	—	0.164061	—
	33	0.059622	0.928	0.170744	0.362	0.090493	0.858
	65	0.030586	0.963	0.138230	0.305	0.048777	0.892
	129	0.015493	0.981	0.114252	0.275	0.025908	0.913
	257	0.007798	0.991	0.095306	0.262	0.013628	0.927
$-1/2$	17	0.113601	—	0.464281	—	0.165351	—
	33	0.060562	0.907	0.436380	0.089	0.090903	0.863
	65	0.031451	0.945	0.417038	0.065	0.048833	0.896
	129	0.016115	0.965	0.401995	0.053	0.025859	0.917
	257	0.008201	0.975	0.389323	0.046	0.013566	0.931
$-3/4$	17	0.111486	—	0.684562	—	0.168008	—
	33	0.061565	0.857	0.676109	0.018	0.092050	0.868
	65	0.033496	0.878	0.673052	0.007	0.049354	0.899
	129	0.018222	0.878	0.672571	0.001	0.026147	0.917
	257	0.009990	0.867	0.673278	-0.002	0.013765	0.926

This pressure and the scaled pressure $q = \phi^{1/2}p$ are depicted in Figure 2, where one can see clearly the degeneracy in p near $x = -3/4$ and that q is better behaved. In the case $\beta = -1/4$, q and components of $\mathbf{u}$ lie in $H^{1.25-\epsilon}$ and are relatively smooth, whereas when $\beta = -3/4$, q and components of $\mathbf{u}$ lie only in $H^{0.75-\epsilon}$ (any $\epsilon > 0$). We use grids that do *not* resolve the interface between the one- and two-phase regions. The discrete errors and convergence rates are shown in Table 4. The scaled pressure converges as expected, and the velocity seems to be converging a bit better than expected. The pressure barely converges at all, or even may diverge.

10. Summary and conclusions. We considered a two-phase mixture of matrix solid and fluid melt, which can degenerate as the porosity ϕ vanishes. Energy estimates suggested that the pressure p is uncontrolled; moreover, an equation is lost when ϕ vanishes, making it difficult to handle the equations numerically.

We changed variables to a scaled set that remains bounded in the energy estimates. To formulate a well-posed mixed weak problem in the scaled variables, we defined precisely the Hilbert space $H_{\phi,d}(\text{div})$ within which the scaled velocity resides. The key hypotheses were that $\phi^{-1/2}d(\phi) \in W^{1,\infty}(\Omega)$ and $\phi^{-1/2}\nabla d(\phi) \in (L^\infty(\Omega))^n$. Moreover, a normal trace operator was defined to handle boundary conditions. Existence and uniqueness of a solution to the weak formulation was obtained from the Lax–Milgram theorem.

We defined a mixed finite element method based on lowest order Raviart–Thomas and Arbogast–Correa spaces. The method is stable, and an error analysis showed optimal rates of convergence for sufficiently smooth solutions.

In a simple case in one dimension, the equations reduce to an Euler equation for which a closed form solution was computed. Numerical tests of this problem showed that the mixed method achieved optimal rates of convergence with respect to the regularity of the solution. Convergence of the true pressure was relatively poor. A numerical test for a two-dimensional problem using $d(\phi) = \phi$ also exhibited optimal convergence rates when ϕ was reasonable. In this test, it was necessary that $\phi^{-1/2}\nabla\phi \in L^2$, which is weaker than being in L^∞. Moreover, meshes that did not match the boundary of the one-phase region showed no degradation of results from cases with meshes that match this boundary when $\phi^{-1/2}\nabla\phi \in L^2$.

We plan to present an easy to implement cell-centered finite difference (CCFD) approximation of this mixed method in [4]. The CCFD method is stable and locally mass conservative, and numerical tests show that it maintains optimal convergence rates and even achieves superconvergence. The techniques developed in this paper will be applied to the simulation of the mechanics of mantle dynamics in [3].

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