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A FINITE DIFFERENCE METHOD FOR A TWO-SEX MODEL OF POPULATION DYNAMICS*

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This paper is dedicated to Jim Douglas, Jr., on the occasion of his 60th birthday.

Abstract. An explicit finite difference scheme is developed to approximate the solution of a nonlinear and nonlocal system of integro-differential equations that models the dynamics of a two-sex population. The scheme is unconditionally stable. The optimal rate of convergence of the scheme is demonstrated for the maximum norm. Results from a numerical simulation of U.S. population growth from 1970 to 1980 are presented; these compare favorably with the actual data.

Key words. two-sex population, finite difference, method of characteristics, simulation

AMS(MOS) subject classifications. 65M10, 65M20, 92A15, 65C20

1. Introduction.

In the year 1202 L. Fibonacci [5] tried to model the growth of a population (of rabbits in this case) by the well-known sequence of integers that carries his name. The first serious attempt to describe human population dynamics took place almost 600 years later when Malthus [12] introduced his mathematical model for population growth. Malthus' model assumed constant birth and death rates and is based on the assumption that the rate of change of the population is proportional to its size, $p(t)$; that is,

$$\frac{dp}{dt} = (\beta - \delta)\, p,$$

where β is the birth rate and δ the death rate. This leads to the well-known exponential model

$$p(t) = p^0 e^{(\beta-\delta)t},$$

where p^0 is the size of the initial population. This model is not valid for large time t since it leads to a population growing without bound.

Forty years later Verhulst [16] introduced a model that imposes a maximum size on the population based on the assumption that the rate of change of the population is proportional both to its size and to the difference between some maximal level M and the current size; that is,

$$\frac{dp}{dt} = kp(M-p) = (kM - kp)\, p.$$

So here it is assumed that the birth rate $\beta = kM$ is constant and the death rate $\delta = kp$ is proportional to the size of the population. This leads to the well-known logistic model

$$p(t) = \frac{M}{1 + \left(\dfrac{M-p^0}{p^0}\right) e^{-\beta t}}.$$

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It was only in 1922 that Lotka [11] finally introduced the idea of an age-dependent model. This was taken over by McKendrick [13] who first derived the model presented below and later by von Foerster [17] who made a mathematical analysis of that model. Let $u(x,t)$ be the age distribution function, where x denotes the age and t the time. The number of individuals in the age bracket $[x_1, x_2]$ is given by

$$p_0 \int_{x_1}^{x_2} u(x,t)\, dx,$$

where p_0 is a scaling factor such as the initial population or a power of ten. Let $\beta = \beta(x,t)$ and $\delta = \delta(x,t)$ be, respectively, the (prescribed) age-specific fertility and age-specific death rates of the population. The McKendrick-von Foerster model is given by

$$\left\{ \begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -\delta u, \qquad x > 0,\ t > 0, \\ &u(0,t) = B(t) = \int_0^\infty \beta(x,t) u(x,t)\, dx, \qquad t \geq 0, \\ &u(x,0) = u^0(x), \qquad x \geq 0, \end{aligned} \right. \tag{1.1}$$

where $B(t)$ is the birth rate and $u^0(x)$ is the initial age distribution (a probability density function if p_0 is the size of the initial population). If the function $B(t)$ were known, one would have the explicit solution of (1.1) given by

$$u(x,t) = \left\{ \begin{aligned} &u^0(x-t) \exp\left[-\int_0^t \delta(x-t+\tau, \tau)\, d\tau \right], \qquad x \geq t, \\ &B(t-x) \exp\left[-\int_0^x \delta(\tau, t-x+\tau)\, d\tau \right], \qquad t > x. \end{aligned} \right.$$

Substituting this expression into the integral that defines $B(t)$, one arrives at the following integral equation for B(t):

$$B(t) = f(t) + \int_0^t K(x,t) B(t-x)\, dx,$$

where the functions f and K involve only the data β, δ, and u^0. When β and δ are time-independent (then so is K) one has the *renewal equation*

$$B(t) = f(t) + \int_0^t K(t-x) B(x)\, dx$$

which can be solved using Laplace transforms.

A vast bibliography on a variety of models and their properties exists (see, for example, [2], [6], [14], [15]), but very little work on numerical methods for the approximation of their solutions has been done. Among the few papers in the literature, [1] and [4] discuss various methods for the approximation of $B(t)$, [3] analyzes a finite difference scheme along the characteristic direction for (1.1) with population-dependent death rate, and [10] a numerical scheme for a simplified version of (1.1) based on the explicit solution (with *prescribed* birth rate, but population dependent death rate).

The plan of this paper is as follows. In §2 we describe a model for the dynamics of a two-sex population such as that of humans. In §3 we propose a numerical method for the approximation of its solution, and we prove that this method converges optimally to the exact solution. Finally, in §4 we present results from a numerical simulation run using this method and compare them with actual demographic statistics.

2. A two-sex model. The major defect common to all of the preceding models is that they do not consider the partition of the population into sexes. When this is done, a coupled system of partial differential equations of McKendrick type results (see, for example, [7]). Let $u_m(x,t)$ and $u_f(y,t)$ denote, respectively, the age distributions of males and females in the population, where x and y refer to the ages of males and females, respectively. These age distributions must satisfy the following integro-differential systems:

$$\begin{cases} \dfrac{\partial u_m}{\partial t} + \dfrac{\partial u_m}{\partial x} = -\delta_m(x,t)u_m, \qquad x > 0, \quad t > 0, \\ \qquad u_m(0,t) = B_m(t) = \displaystyle\int_0^\infty \int_0^\infty \beta_m(x,y,t)c(x,y,t)\,dx\,dy, \qquad t > 0, \\ \qquad u_m(x,0) = u_m^0(x), \qquad x \geq 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \dfrac{\partial u_f}{\partial t} + \dfrac{\partial u_f}{\partial y} = -\delta_f(y,t)u_f, \qquad y > 0, \quad t > 0, \\ \qquad u_f(0,t) = B_f(t) = \displaystyle\int_0^\infty \int_0^\infty \beta_f(x,y,t)c(x,y,t)\,dx\,dy, \qquad t > 0, \\ \qquad u_f(y,0) = u_f^0(y), \qquad y \geq 0, \end{cases} \tag{2.2}$$

where $c(x,y,t)$ is the distribution of couples with male of age x and female of age y, $\delta_m(x,t)$ and $\delta_f(y,t)$ are, respectively, the age specific death rates, and β_m and β_f give, respectively, the productivity of such couples for male and female progeny (that is, $\beta_m(x,y,t)$ is the average number of "sons" and $\beta_f(x,y,t)$ the average number of "daughters" born at time t to a couple with male of age x and female of age y). The function $c = c(x,y,t)$ is the solution of the following nonlinear initial-boundary value problem:

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{\partial c}{\partial x} + \dfrac{\partial c}{\partial y} = -\sigma(x,y,t)c + \mu(x,y,t;s_m,s_f), \qquad x > 0,\ y > 0,\ t > 0, \\ \qquad c(x,0,t) = c(0,y,t) = 0, \qquad x \geq 0,\ y \geq 0,\ t > 0, \\ \qquad c(x,y,0) = c^0(x,y), \qquad x \geq 0,\ y \geq 0, \end{cases} \tag{2.3}$$

where the first term on the right hand side of the differential equation describes the change in c due to the separation of couples (by death, annulment, or divorce) while the second term describes the source of couples, which depends on the distribution of "single" males and females,

$$\begin{cases} s_m(x,t) = u_m(x,t) - \displaystyle\int_0^\infty c(x,y,t)\,dy, \\ s_f(y,t) = u_f(y,t) - \displaystyle\int_0^\infty c(x,y,t)\,dx. \end{cases} \tag{2.4}$$

The function μ is called the *marriage function* and is empirically chosen in several papers (for example see [6], [8], and [9]) as homogeneous of degree one in s_m and s_f.

In this model the total population $p(t)$ is given by

$$p(t) = p_0 \left[\int_0^\infty u_m(x,t)\,dx + \int_0^\infty u_f(y,t)\,dy \right]. \tag{2.5}$$

The initial and the boundary conditions must be compatible:

$$\begin{cases} u_m^0 = B_m(0) \qquad \text{and} \qquad u_f^0 = B_f(0), \\ c^0(0,y) = c^0(x,0) = 0, \qquad x, y \geq 0, \\ c^0, u_m^0, u_f^0 \geq 0, \\ \displaystyle\int_0^\infty c^0(x,y)\,dy \leq u_m^0(x) \qquad (\text{i.e., } s_m^0 \geq 0), \\ \displaystyle\int_0^\infty c^0(x,y)\,dx \leq u_f^0(y) \qquad (\text{i.e., } s_f^0 \geq 0). \end{cases} \tag{2.6}$$

The initial age-distributions of individuals and couples, u_m^0, u_f^0, c^0, must be compactly supported (for biological reasons). These conditions clearly imply that $c \geq 0$, which in turn implies that $u_m, u_f \geq 0$ and that u_m, u_f have compact support. If we set $\mu(s_m, s_f) \equiv 0$ for s_m or $s_f \leq 0$, then in fact $s_m, s_f \geq 0$. To see this for s_m, for example, note that

$$\frac{\partial s_m}{\partial t} + \frac{\partial s_m}{\partial x} = -\delta_m s_m + (1 - \delta_m) \int_0^\infty \delta_f(y,t) c(x,y,t)\,dy + \int_0^\infty \alpha(x,y,t) c(x,y,t)\,dy - \int_0^\infty \mu(x,y,t,s_m,s_f)\,dy,$$

where $\alpha(x,y,t) = \sigma(x,y,t) - \delta_m(x,t) - \delta_f(y,t) + \delta_m(x,t)\delta_f(y,t)$ is the annulment and divorce rate; since $s_m^0 \geq 0$ and the characteristic slope is positive when s_m is negative, s_m must stay nonnegative. Finally, then, c has compact support for any t.

3. The numerical method. For $0 \leq t \leq T$ we shall consider the approximation of the age distribution functions $u_m(x,t)$ and $u_f(y,t)$ and of the couple distribution function $c(x,y,t)$ in a two-sex population with population dependent birth and death rates. We shall, therefore, replace β_m, β_f, δ_m, δ_f, and σ in (2.1) by

$$\begin{cases} \beta_g = \beta_g(x,y,t;p), \qquad g \text{ (gender)} = m, f, \\ \delta_g = \delta_g(z,t;p), \qquad (g,z) = (m,x) \text{ or } (f,y), \\ \sigma = \sigma(x,y,t;p). \end{cases} \tag{3.1}$$

For simplicity we shall take p_0 in (2.5) to be one.

We shall assume that the initial-boundary value problem (2.1)–(2.5), (3.1) has a unique solution which has all the regularity necessary for the approximation and analysis of the scheme we define below.

We shall discretize the differential equations (2.1)–(2.3) by using finite difference schemes in the characteristic direction τ. Note that $\tau = \frac{1}{\sqrt{2}}(1,1)$ in (2.1) and (2.2) and that $\tau = \frac{1}{\sqrt{3}}(1,1,1)$ in (2.3). Let N be a positive integer and let $\Delta t = T/N$ and $t^n = n\Delta t$. The approximation will be made using a uniform age-time grid

$$\mathcal{G} = \{x_i = i\Delta t \,:\, i \geq 0\} = \{y_j = j\Delta t \,:\, j \geq 0\};$$

that is, we use the same time increment for the ages as for the time. For a function $\psi = \psi(x, y, t; r, s)$ we let

$$\psi^n_{i,j}(r,s) = \psi(x_i, y_j, t^n; r, s), \qquad r, s \in \mathbb{R},$$

where i, j, r, or s will be suppressed as appropriate.

Our scheme will compute simultaneously approximations $U^n_{m,i}$ of $u^n_{m,i}$, $U^n_{f,j}$ of $u^n_{f,j}$, $C^n_{i,j}$ of $c^n_{i,j}$, $S^n_{g,i,j}$ of $s^n_{g,i,j}$ $(g = m, f)$, and P^n of p^n as follows. Initialize the scheme by

$$\text{(3.2)} \qquad \begin{cases} U^0_{m,i} = u^0_{m,i}, & i \geq 0, \\ U^0_{f,j} = u^0_{f,j}, & j \geq 0, \\ C^n_{i,j} = c^n_{i,j}, & i, j \geq 0, \\ S^0_m, S^0_f \text{ given by (3.6) below with } n = 0, \\ P^0 \text{ given by (3.7) below with } n = 0. \end{cases}$$

For $n > 0$, use the following (explicit) scheme:

$$(3.3) \qquad \begin{cases} \dfrac{U_{m,i}^n - U_{m,i-1}^{n-1}}{\Delta t} = -\delta_{m,i}^n(P^{n-1})U_{m,i}^n, & i \geq 1, \\ U_{m,0}^n = \displaystyle\sum_{i,j=1}^{\infty} \beta_{m,i,j}^n(P^{n-1})\, C_{i,j}^n\, (\Delta t)^2, \end{cases}$$

$$(3.4) \qquad \begin{cases} \dfrac{U_{f,j}^n - U_{f,j-1}^{n-1}}{\Delta t} = -\delta_{f,j}^n(P^{n-1})U_{f,j}^n, & j \geq 1, \\ U_{f,0}^n = \displaystyle\sum_{i,j=1}^{\infty} \beta_{f,i,j}^n(P^{n-1})\, C_{i,j}^n\, (\Delta t)^2, \end{cases}$$

$$(3.5) \qquad \begin{cases} \dfrac{C_{i,j}^n - C_{i-1,j-1}^{n-1}}{\Delta t} = -\sigma_{i,j}^n(P^{n-1})\, C_{i,j}^n + \mu_{i,j}^n(S_{m,i-1}^{n-1}, S_{f,j-1}^{n-1}), & i,j \geq 1, \\ C_{0,j}^n = C_{i,0}^n = 0, \qquad i,j \geq 0, \end{cases}$$

$$(3.6) \qquad \begin{cases} S_{m,i}^n = U_{m,i}^n - \displaystyle\sum_{j=1}^{\infty} C_{i,j}^n\, \Delta t, & i \geq 0, \\ S_{f,j}^n = U_{f,j}^n - \displaystyle\sum_{i=1}^{\infty} C_{i,j}^n\, \Delta t, & j \geq 0, \end{cases}$$

$$(3.7) \qquad P^n = \sum_{i=0}^{\infty} U_{m,i}^n\, \Delta t + \sum_{j=0}^{\infty} U_{f,j}^n\, \Delta t.$$

To advance a time step, equation (3.5) must be solved first, (3.3)–(3.4) next, and (3.6)–(3.7) last.

It is not hard to see from (3.2)–(3.5) that

$$C_{i,j}^n,\ U_{m,i}^n,\ U_{f,j}^n \geq 0, \qquad i,j,n \geq 0,$$

and that there are only finitely many i, j for which they are nonzero. However, it is the case that

$$S_{m,i}^0, S_{f,j}^0 \geq -O(\Delta t), \qquad i,j \geq 1,$$

which may be negative, so that the extension of the marriage function μ by zero mentioned at the end of §2 is necessary.

This scheme converges at the optimal rate Δt to the solutions of the two-sex model.

THEOREM. *Let the solutions* u_m, u_f, *and* $c(x,y,t)$ *of* (2.1)–(2.5), (3.1) *be twice continuously differentiable with bounded derivatives through the second order. Denote*

$$\xi = c - C, \qquad v_g = u_g - U_g, \qquad \text{and} \qquad \eta_g = s_g - S_g, \qquad g = m, f.$$

Then there exists a constant Q independent of Δt such that the error bound

$$\|\zeta\|_{l^\infty(l^\infty)} = \max_{0\le i,j\le\infty} \max_{0\le n\le N} |\zeta_{i,j}^n| \le Q\Delta t \tag{3.8}$$

holds for $\zeta = \xi,\ v_g,\ \eta_g\ (g = m, f)$.

Proof. Note that the characteristic derivatives can be approximated as follows:

$$\begin{cases} \left(\dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial x} + \dfrac{\partial\psi}{\partial y}\right)(x_i, y_j, t^n) = \dfrac{\psi_{i,j}^n - \psi_{i-1,j-1}^{n-1}}{\Delta t} + O\left(\left\|\dfrac{\partial^2\psi}{\partial\tau^2}\right\|_{L^\infty}\Delta t\right), \\ \left(\dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial z}\right)(z_k, t^n) = \dfrac{\psi_k^n - \psi_{k-1}^{n-1}}{\Delta t} + O\left(\left\|\dfrac{\partial^2\psi}{\partial\tau^2}\right\|_{L^\infty}\Delta t\right), \qquad z_k = x_i \text{ or } y_j. \end{cases} \tag{3.9}$$

Also, the following quadrature is obvious:

$$\int_0^\infty\int_0^\infty \psi(x,y)\,dx\,dy = \sum_{i,j=1}^\infty \psi_{i,j}\,(\Delta t)^2 + O\left(\left(\left\|\frac{\partial\psi}{\partial x}\right\|_{L^\infty} + \left\|\frac{\partial\psi}{\partial y}\right\|_{L^\infty}\right)\Delta t\right), \tag{3.10}$$

with a similar result holding for functions of one variable.

We can derive from (2.1)–(2.5), and (3.1)–(3.10) the following error equations:

$$v_{m,i}^0 = v_{f,j}^0 = \xi_{i,j}^0 = 0, \qquad i, j \ge 0, \tag{3.11}$$

and, for $n \geq 1$,

$$\tag{3.12} \begin{cases} \dfrac{v_{m,i}^n - v_{m,i-1}^{n-1}}{\Delta t} = -\delta_{m,i}^n(P^{n-1})v_{m,i}^n \\ \qquad\qquad + \left[\delta_{m,i}^n(P^{n-1}) - \delta_{m,i}^n(p^n)\right] u_{m,i}^n + O(\Delta t), \qquad i \geq 1, \\ v_{m,0}^n = \displaystyle\sum_{i,j=1}^{\infty} \beta_{m,i,j}^n(P^{n-1})\xi_{i,j}^n \, (\Delta t)^2 \\ \qquad\qquad - \displaystyle\sum_{i,j=1}^{\infty} \left[\beta_{m,i,j}^n(P^{n-1}) - \beta_{m,i,j}^n(p^n)\right] \, c_{i,j}^n \, (\Delta t)^2 + O(\Delta t), \end{cases}$$

$$\tag{3.13} \begin{cases} \dfrac{v_{f,j}^n - v_{f,j-1}^{n-1}}{\Delta t} = -\delta_{f,j}^n(P^{n-1})v_{f,j}^n \\ \qquad\qquad + \left[\delta_{f,j}^n(P^{n-1}) - \delta_{f,j}^n(p^n)\right] u_{f,j}^n + O(\Delta t), \qquad j \geq 1, \\ v_{f,0}^n = \displaystyle\sum_{i,j=1}^{\infty} \beta_{f,i,j}^n(P^{n-1})\xi_{i,j}^n \, (\Delta t)^2 \\ \qquad\qquad - \displaystyle\sum_{i,j=1}^{\infty} \left[\beta_{f,i,j}^n(P^{n-1}) - \beta_{f,i,j}^n(p^n)\right] \, c_{i,j}^n \, (\Delta t)^2 + O(\Delta t), \end{cases}$$

$$\tag{3.14} \begin{cases} \dfrac{\xi_{i,j}^n - \xi_{i-1,j-1}^{n-1}}{\Delta t} = -\sigma_{i,j}^n(P^{n-1})\,\xi_{i,j}^n + \left[\mu_{i,j}^n(s_{m,i}^n, s_{f,j}^n) - \mu_{i,j}^n(S_{m,i-1}^{n-1}, S_{f,j-1}^{n-1})\right] \\ \qquad\qquad + \left[\sigma_{i,j}^n(P^{n-1}) - \sigma_{i,j}^n(p^n)\right] c_{i,j}^n + O(\Delta t), \qquad i,j \geq 1, \\ \xi_{0,j}^n = \xi_{i,0}^n = 0, \qquad i,j \geq 0, \end{cases}$$

$$\tag{3.15} \begin{cases} \eta_{m,i}^n = v_{m,i}^n - \displaystyle\sum_{j=1}^{\infty} \xi_{i,j}^n \, \Delta t + O(\Delta t), \qquad i \geq 0, \\ \eta_{f,j}^n = v_{f,j}^n - \displaystyle\sum_{i=1}^{\infty} \xi_{i,j}^n \, \Delta t + O(\Delta t), \qquad j \geq 0, \end{cases}$$

$$\tag{3.16} p^n - P^n = \sum_{i=0}^{\infty} v_{m,i}^n \, \Delta t + \sum_{j=0}^{\infty} v_{f,j}^n \, \Delta t + O(\Delta t).$$

Note that (3.16) implies that

$$\tag{3.17} |p^n - P^n| \leq \|v_m^n\|_{l^1} + \|v_f^n\|_{l^1} + O(\Delta t), \qquad n \geq 0,$$

where

$$\|\psi\|_{l^1} = \sum_i |\psi_i| \, \Delta t \quad \text{or} \quad \sum_{i,j} |\psi_{i,j}| \, (\Delta t)^2.$$

We shall also use the notation

$$\|\psi_{\cdot\,,j}\|_{l^1} = \sum_i |\psi_{i,j}| \, \Delta t \qquad \text{and} \qquad \|\psi_{i,\,\cdot}\|_{l^1} = \sum_j |\psi_{i,j}| \, \Delta t.$$

It is also clear that

$$\tag{3.18} p^n - p^{n-1} = O(\Delta t).$$

Easily (3.12)–(3.13) and (3.17)–(3.18) lead to, for $k, n \geq 1$ and $k = i$ or j,

$$\left[1+\delta_{g,k}^{n}(P^{n-1})\Delta t\right] v_{g,k}^{n} = v_{g,k-1}^{n-1} + O\left(\left\|\frac{\partial \delta_g}{\partial p}\right\|_{L^\infty} \|u_g\|_{L^\infty}\left[\|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1}\right]\Delta t\right) + O\left((\Delta t)^2\right), \qquad g = m, f,$$

which implies that

$$|v_{g,k}^{n}| \leq |v_{g,k-1}^{n-1}| + Q\left(\|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1}\right)\Delta t + O\left((\Delta t)^2\right), \qquad g = m, f, \tag{3.19}$$

where Q is a constant that is independent of i, j, n, and Δt but its value may vary from place to place below. It also follows from (3.12)–(3.13) and (3.17)–(3.18) that

$$|v_{g,0}^{n}| \leq Q\left(\|\xi^n\|_{l^1} + \|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1}\right) + O(\Delta t), \qquad g = m, f. \tag{3.20}$$

Multiplying (3.19) and (3.20) by Δt and summing over k we arrive at the estimate

$$\|v_g^n\|_{l^1} \leq \|v_g^{n-1}\|_{l^1} + Q\left(\|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1} + \|\xi^n\|_{l^1}\right)\Delta t + O\left((\Delta t)^2\right), \qquad g = m, f. \tag{3.21}$$

For a similar analysis of ξ, we first see that

$$\left[1+\sigma_{i,j}^{n}(P^{n-1})\Delta t\right]\xi_{i,j}^{n} = \xi_{i-1,j-1}^{n-1} + \left[\mu_{i,j}^{n}(s_{m,i}^{n}, s_{f,j}^{n}) - \mu_{i,j}^{n}(S_{m,i-1}^{n-1}, S_{f,j-1}^{n-1})\right]\Delta t + O\left(\|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1}\right)\Delta t + O\left((\Delta t)^2\right). \tag{3.22}$$

Next note that

$$\left|\mu_{i,j}^{n}(s_{m,i}^{n}, s_{f,j}^{n}) - \mu_{i,j}^{n}(S_{m,i-1}^{n-1}, S_{f,j-1}^{n-1})\right| = O\left(\left[\left\|\frac{\partial \mu}{\partial s_m}\right\|_{L^\infty} + \left\|\frac{\partial \mu}{\partial s_f}\right\|_{L^\infty}\right]\left[\,|s_{m,i}^{n} - S_{m,i-1}^{n-1}| + |s_{f,j}^{n} - S_{f,j-1}^{n-1}|\,\right]\right) \tag{3.23}$$

and

$$s_{g,k}^{n} - S_{g,k-1}^{n-1} = \eta_{g,k-1}^{n-1} + O\left(\left\|\frac{\partial s}{\partial \tau}\right\|_{L^\infty}\Delta t\right), \qquad (g,k) = (m,i) \text{ or } (f,j). \tag{3.24}$$

We see now from (3.15) and (3.22)–(3.24), that

$$\begin{aligned} |\xi_{i,j}^{n}| &\leq |\xi_{i-1,j-1}^{n-1}| + Q\Big(|\eta_{m,i-1}^{n-1}| + |\eta_{f,j-1}^{n-1}| \\ &\qquad + \|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1}\Big)\Delta t + O\left((\Delta t)^2\right) \\ &\leq |\xi_{i-1,j-1}^{n-1}| + Q\Big(|v_{m,i-1}^{n-1}| + |v_{f,j-1}^{n-1}| + \|\xi_{i-1,\cdot}^{n-1}\|_{l^1} \\ &\qquad + \|\xi_{\cdot,j-1}^{n-1}\|_{l^1} + \|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1}\Big)\Delta t + O\left((\Delta t)^2\right). \end{aligned} \tag{3.25}$$

We sum now on i and j to show that

$$\|\xi^n\|_{l^1} \leq \|\xi^{n-1}\|_{l^1} + Q\left(\|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1} + \|\xi^{n-1}\|_{l^1}\right)\Delta t + O\left((\Delta t)^2\right). \tag{3.26}$$

Combining (3.21) and (3.26) we see that, since $\Delta t \le T$,

$$(3.27)\quad \|v_g^n\|_{l^1} \le \|v_g^{n-1}\|_{l^1} + Q\Delta t(1+QT)\left(\|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1} + \|\xi^{n-1}\|_{l^1}\right) + O\left((\Delta t)^2\right).$$

Now a combination of (3.27) and (3.26) shows that

$$(3.28)\quad \|v_m^n\|_{l^1} + \|v_f^n\|_{l^1} + \|\xi^n\|_{l^1} \le (1+Q\Delta t)\left(\|v_m^{n-1}\|_{l^1} + \|v_f^{n-1}\|_{l^1} + \|\xi^{n-1}\|_{l^1}\right) + O\left((\Delta t)^2\right).$$

Since

$$Q\sum_{k=0}^{N-1}(1+Q\Delta t)^k(\Delta t)^2 = \left[(1+Q\Delta t)^N - 1\right]\Delta t \le Q'\Delta t,$$

recursive use of (3.28) and (3.11) leads to the preliminary estimate

$$(3.29)\quad \|v_m\|_{l^\infty(l^1)} + \|v_f\|_{l^\infty(l^1)} + \|\xi\|_{l^\infty(l^1)} \le Q\Delta t.$$

Using this estimate in (3.19)–(3.20), we arrive at

$$|v_{g,k}^n| \le |v_{g,k-1}^{n-1}| + O\left((\Delta t)^2\right), \qquad (g,k) = (m,i) \text{ or } (f,j),$$
$$|v_{g,0}^n| = O(\Delta t),$$

which used recursively yields the theorem for $\zeta = v_m$ and v_f.

Combining the completed estimates in (3.8) with (3.25) and (3.29) we see that

$$(3.30)\quad |\xi_{i,j}^n| \le |\xi_{i-1,j-1}^{n-1}| + Q\left(\|\xi_{i-1,\cdot}^{n-1}\|_{l^1} + \|\xi_{\cdot,j-1}^{n-1}\|_{l^1}\right)\Delta t + O\left((\Delta t)^2\right).$$

Summing on j and using (3.29) again leads to the bound

$$\|\xi_{i,\cdot}^n\|_{l^1} \le (1+Q\Delta t)\|\xi_{i-1,\cdot}^{n-1}\|_{l^1} + O\left((\Delta t)^2\right),$$

which used recursively shows that $\|\xi_{i,\cdot}^n\|_{l^1} \le Q\Delta t$. Similarly, $\|\xi_{\cdot,j}^n\|_{l^1} \le Q\Delta t$. Hence (3.30) becomes

$$|\xi_{i,j}^n| \le |\xi_{i-1,j-1}^{n-1}| + O\left((\Delta t)^2\right),$$

and the rest of the theorem follows from a final recursion argument.

Remark. As a corollary to the theorem, if the differential solution is smooth, the approximate solution is uniformly bounded. This result can be easily obtained directly independently of the smoothness of the differential solution. Moreover, full unconditional stability under small perturbations of the approximate solution can be shown by an entirely similar argument to that given above.

4. Some numerical results. To check the validity of the model and to test our approximation scheme, we ran a simulation of the growth of the population of the United States from 1970 to 1980 using the initial distributions of males, females, and couples from the 1970 population census[1,2]. Vital statistics for births, deaths,

[1] U.S. Bureau of the Census, *Census of Population: 1970, Detailed Characteristics,* Final Report PC(1)-D1, U.S. Summary, U.S. Government Printing Office, Washington, D.C., 1973.

[2] U.S. Bureau of the Census, *Census of Population: 1970, Marital Status,* Final Report PC(2)-4C, U.S. Summary, U.S. Government Printing Office, Washington, D.C., 1972.

marriages, and marriage annulments/divorces were taken from the U. S. Department of Health and Human Services (and its precursor the Department of Health, Education, and Welfare)[3,4]. Based on these figures, we constructed the functions needed by the two-sex model.

A few brief remarks on the construction of these functions are in order since they are not always available in the form needed. Each of these functions was taken to be population independent. The time dependence was simply taken to be a linear interpolation of the figures from 1970 to those of 1980. (Of course, in practical situations one would need to extrapolate data from earlier times.) In some cases, data cross tabulated by ages of males and females is needed. Where such data was unavailable, it was constructed by forming the products of the data tabulated by the age of either sex alone. The marriage function given by Keyfitz [8] was used:

$$\mu(x,y,t;s_m,s_f) = \nu(x,y,t)\frac{s_m(x,t)s_f(y,t)}{s_m(x,t)+s_f(y,t)}\rho\left(\frac{s_m(x,t)}{s_f(y,t)}\right),$$

where ρ is a function used to bias the harmonic mean, with $\rho(0) = \rho(\infty) = 1$. We took ρ to be identically one.

Because the fertility of unmarried females is significant (and known), we modified the scheme to include a source of births from single females. In the model, this source for males consists of the addition of the term

$$\int_0^\infty \beta_m^s(y,t)s_f(y,t)\,dy$$

to the births in (2.1), where $\beta_m^s(y,t)$ is the fertility rate of unmarried women for the birth of males. A similar term is needed for the births of females in (2.2). The modification to the approximation scheme is straightforward, and the convergence theorem can be demonstrated with only minor changes in the proof.

We ran our simulation with a time step $\Delta t = 0.125$ years to obtain the predicted distribution of males, females, and couples in 1980. We compare these figures with the actual distributions (given by the 1980 census[5,6]) in Tables 1 and 2.

Note that the model, as stated, does not take into account migration into and out of the United States. Therefore, we must at least add to the figure obtained from the simulation the net number of immigrants during the decade simulated. This number, according to figures from the Immigration and Naturalization Service[7], is 4,336,001. We should note further that the population figures given by the 1980 census are quite

[3] Health Resources Administration, National Center for Health Statistics: *Vital Statistics of the United States, 1970,* Vols. I–III, DHEW publication nos. (HRA) 75-1100 to 75-1103, Public Health Service, U.S. Government Printing Office, Washington, D.C., 1974–75.

[4] National Center for Health Statistics: *Vital Statistics of the United States, 1980,* Vols. I–III, DHHS publication nos. (PHS) 85-1100 to 85-1103, Public Health Service, U.S. Government Printing Office, Washington, D.C., 1984–85.

[5] U.S. Bureau of the Census, *Census of Population: 1980, Vol. 1, Characteristics of Population,* PC 80-1-B1, U.S. Government Printing Office, Washington, D.C., 1981.

[6] U.S. Bureau of the Census, *Census of Population: 1980, Vol. 2, Subject Reports: Marital Characteristics,* PC 80-2-4C, U.S. Government Printing Office, Washington, D.C., 1984.

[7] U.S. Department of Justice, Immigration and Naturalization Service, *1981 Statistical Yearbook of the Immigration and Naturalization Service.*

TABLE 1
Distribution of males and females

Age bracket (yrs.)	Males			Females		
	Calculated (1000's)	Actual (1000's)	Error (%)	Calculated (1000's)	Actual (1000's)	Error (%)
0–4	8510	8362	1.8	8088	7986	1.3
5–9	8964	8539	5.0	8537	8161	4.6
10–14	8661	9316	7.0	8346	8926	6.5
15–19	10122	10755	5.9	9787	10413	6.0
20–24	10467	10663	1.8	9997	10655	6.2
25–29	9538	9705	1.7	9155	9816	6.7
30–34	7596	8677	12.5	8181	8884	7.9
35–39	6417	6862	6.5	6748	7104	5.0
40–44	5425	5708	5.0	5756	5961	3.4
45–49	5194	5388	3.6	5564	5702	2.4
50–54	5419	5621	3.6	5915	6089	2.9
55–59	5214	5482	4.9	5886	6133	4.0
60–64	4493	4670	3.8	5254	5418	3.0
65–69	3667	3903	6.0	4576	4880	6.2
70–74	2748	2854	3.7	3773	3945	4.4
75–79	1766	1848	4.4	2268	2946	23.0
80–84	1030	1019	1.1	425	1916	77.8
85 and over	758	682	11.1	426	1559	72.7
Total	105989	110053	3.7	108682	116493	6.7

difficult to compare with those from the 1970 census due, among other factors, to the great number of illegal immigrants during that decade which were counted in the 1980 census. It is sufficient to subtract the 19,264,000 deaths from the 33,308,000 births recorded in the U.S. in those ten years[8] to obtain the intrinsic increase of the population, 14,044,000 (where we have rounded the figures to the nearest thousand). If we add this figure to the total population recorded by the 1970 census (203,210,000), the difference between this figure and the total population recorded by the 1980 census (226,546,000) should give the net migration into the U.S. in that decade: 9,292,000. Even without considering emigration, the immigration figure given by the INS falls short by about 5 million. Hence, a percentage error of at least 4.1% is to be expected.

As we can see from Table 1, the model describes quite accurately the age distribution of both sexes as they evolve in time, which is the object of this model. The distribution of couples by age of the partners is used as a tool to try to better model the births in the population. As is very well known [7], modeling the evolution of the couples distribution function in time is extremely difficult, mostly for lack of an accurate description of the marriage function μ. As can be seen in Table 2, there is a substantial error in the distribution of couples by cross tabulated ages; however, the distribution of couples by the age of one partner only is quite reasonable. This is related to the fact that we were forced by lack of data to construct the age specific marriage rate function $\nu(x, y, t)$ as a product of the data for each sex alone. In conclusion, then, these results show the overall validity of the two-sex model, as well as the usefulness of our approximation scheme.

[8] U.S. National Center for Health Statistics, *Vital Statistics of the United States*, annual, and unpublished data.

TABLE 2
Distribution of couples
(*numbers in thousands*)

	Age of wife (yrs.)						
	15–24	25–34	35–44	45–54	55–64	All ages	
Age of husband (yrs.)							
	1681.0	993.2	103.9	34.6	41.3	2865.7	Calculated
15–24	2638.8	332.5	14.4	5.0	3.1	2995.6	Actual
	36.3%	198.7%	621.5%	586.2%	1232.3%	4.3%	Error
	2680.8	7007.7	416.5	36.8	32.8	10182.4	·
25–34	2226.7	9010.4	479.9	28.5	5.9	11754.4	·
	20.4%	22.2%	13.2%	29.1%	455.9%	13.4%	·
	284.9	2602.3	5685.1	243.4	25.1	8845.6	·
35–44	119.9	3094.9	6346.1	374.3	26.7	9967.1	·
	137.6%	15.9%	10.4%	35.0%	6.0%	11.3%	·
	71.9	181.6	2397.4	5169.9	407.4	8258.3	·
45–54	18.5	263.8	2725.4	5423.4	495.8	8965.5	·
	288.6%	31.2%	12.0%	4.7%	17.8%	7.9%	·
	60.6	61.1	181.3	2442.4	4533.3	7648.2	·
55–64	7.2	44.5	263.7	2725.8	4900.6	8392.3	·
	741.7%	37.3%	31.2%	10.4%	7.5%	8.9%	·
	4797.0	10858.2	8811.6	8133.2	7045.1	43937.4	·
All ages	5014.5	12756.1	9868.7	8827.4	7720.8	49513.9	·
	4.3%	14.9%	10.7%	7.9%	8.8%	11.3%	·

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