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## Abstracts

### Mixed variational multiscale methods and multiscale finite elements

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(joint work with Kirsten J. Boyd)

A longstanding open problem in applied mathematics is to accurately approximate a function that possesses scales smaller than the level of practical discretization. In this work, we consider the approximation on a coarse grid of spacing  $H$  of the solution  $(\mathbf{u}, p)$  to a second order elliptic problem in mixed form:

$$\begin{aligned} \mathbf{u} &= -K\nabla p && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= f && \text{in } \Omega, \\ \mathbf{u} \cdot \nu &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega \subset \mathbf{R}^d$ ,  $d = 2$  or  $3$ , is a bounded domain,  $K$  is uniformly elliptic and bounded, and  $\nu$  is the outward unit normal vector. We assume that  $K$  and possibly  $f$  exhibit microstructure (i.e., variability or heterogeneity) on a small scale  $\epsilon < H$ , which induces similar  $\epsilon$ -scale variation into the solution.

In 1983, Babuška and Osborn [7] gave a practical strategy for problems in one-dimension by defining what they called the *generalized* finite element method, which uses nonpolynomial basis functions. The idea was to solve the differential system locally over the elements (on a mesh that adequately resolves the scale  $\epsilon$ ), and to piece these local solutions together to form the finite element basis functions.

In 1997, Hou and Wu [12, 13] took up this idea and defined *multiscale* finite elements with at least two advances. First, they suggested the use of *oversampling*, i.e., solving the local problems over a larger domain to capture more of the local microstructure. In multiple dimensions, this results in a nonconforming method, because the basis functions do not piece together continuously. A second advance was to give a multiscale error analysis of the method, illuminating the dependence of the error on both  $H$  and  $\epsilon$ . Later, in 2002, Chen and Hou [11] extended the ideas to mixed finite elements.

Beginning in 1995, Hughes [14, 15] and, independently, Brezzi [8] developed an alternate variational approach, which is called the *variational multiscale* method. The Hilbert space of trial solutions and test functions is divided into two pieces through a direct sum decomposition. The two pieces in some sense represent coarse and fine scales. This decomposition splits both the solution and the variational problem (i.e., the test functions) into coarse and fine scales. If one omits the fine-scale equation and the fine-scale component of the solution, traditional finite element analysis results. However, approximation of the fine-scale components can lead to a better overall approximation, such as greater numerical stability.

Beginning in 1998, Arbogast et al. [6, 1, 2, 3] developed a mixed variant of the variational multiscale method, with the goal of improving the quality of the

approximation itself. A coarse grid is used to decompose the solution space  $H(\text{Div}) = \bar{\mathbf{V}} \oplus \mathbf{V}'$  and  $L^2/\mathbf{R} = \bar{W} \oplus W'$  so as to fulfill two main objectives. First, the decomposition preserves an important physical principle. It conserves mass on both the coarse grid scale and on the fine, or *subgrid*, scales. Secondly, it achieves an important localization property needed for efficient numerical approximation. The subgrid scales from different coarse elements do not interact.

The two-scale variational form can be *upscaled*, meaning that the subgrid parts of the solution can be removed from the equations. If we denote  $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \in \bar{\mathbf{V}} \oplus \mathbf{V}'$  and  $p = \bar{p} + p' \in \bar{W} \oplus W'$ , then the upscaling operator takes the coarse space  $\bar{\mathbf{V}}$  to the fine space  $\mathbf{V}' \times W'$ , so that  $\bar{\mathbf{u}}$  is mapped to  $(\mathbf{u}', p')$ . This operator is *not* a linear operator; it is *affine*. The linear part is *anti-diffusive* [4]. The constant component of the upscaling operator takes into account the source function  $f$  (and external boundary conditions). This is important, since in some problems, such as flow in porous media,  $f$  represents wells, which is a small scale feature that must be resolved in the variational framework.

By approximating  $(\bar{\mathbf{u}}, \bar{p})$  in the upscaled equation by functions in a standard mixed finite element space, one is lead to a coarse grid mixed finite element method. The standard mixed basis functions are modified by the linear part of the upscaling operator, as in the generalized finite element method. In fact, compared to the usual mixed method on a coarse grid, we obtain a linear system of the same size and sparsity but with a modified matrix and right-hand side vector (since the upscaling operator is affine). Because of these affine terms, this method is not simply a generalized finite element method. However, when the source function has no fine scale component, we obtain exactly the method of Chen and Hou provided that oversampling is not used and we restrict our elements to RT0 [4], the lowest order Raviart-Thomas elements [16]. It is the variational multiscale framework that properly picks up the fine-scale components of  $f$ .

Since the construction of the multiscale basis on a fine subgrid parallelizes naturally, the method is very efficient [2]. Numerical examples show that the method can capture significant fine-scale detail even on very coarse grids. Moreover, it can pick up small-scale effects from wells of diameter much less than  $H$  [1, 2, 5].

Multiscale and variational aspects of the method have recently been put on a sound theoretical foundation. The pressure and velocity errors are well approximated. For example, using BDM1, the first order Brezzi-Douglas-Marini spaces [10, 9], on the coarse scale  $H$  and RT0 on a fine grid of spacing  $h$  to approximate the upscaling operator, the error in  $\mathbf{u}$  is  $O(H^2)$  and the error in  $p$  is  $O(h + H^3)$  [3]. The multiscale analysis of Chen and Hou extends to the full variational multiscale method. The main assumptions are that the microstructure of  $K$  varies periodically on a scale  $\epsilon$ , and that the upscaling operator is exact (i.e., not approximated). The error bound for both  $\mathbf{u}$  and  $p$  is  $O(\epsilon + H^m + \sqrt{\epsilon/H})$ , where  $m = 1$  if RT0 is used and  $m = 2$  if BDM1 is used [4].

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