Contemporary Mathematics

AN OVERVIEW OF SUBGRID UPSCALING FOR ELLIPTIC PROBLEMS IN MIXED FORM

Todd Arbogast

Abstract. We present an overview of recent work dealing with upscaling second order elliptic problems in mixed form. We use a direct sum decomposition of the solution space into coarse and localized "subgrid" spaces. We use these to construct a two-scale variational form. A *numerical Greens function* approach allows for its efficient approximation. A three dimensional computational result representing flow in a porous medium illustrates the performance of the technique in approximating fine scales on coarse grids.

1. Introduction

Complex physical phenomena almost always occur on widely varying scales. A continuing challenge in mathematical, scientific, and computational modeling is to handle all the relevant scales properly. Fine scale effects often have a profound influence on coarser scales. It is imperative to express each modeled phenomenon appropriately on the scale of interest, and to properly account for their interactions.

The recent emergence of terascale computing has given promise to significant advances in computational science and engineering, making possible simulations of unprecedented sophistication and detail, and allowing the resolution of events that occur on many different spatial and temporal scales. However, in spite of these vastly expanded limits on computational power, we will continue into the foreseeable future to be thwarted in our efforts to understand the most complex multi-scale phenomena. This is due to the vast range of scales and the stochastic nature of certain problems, which preclude either (1) full resolution of all pertinent space and time scales for a sufficient number of stochastic realizations to obtain meaningful statistical predictions; or (2) the heterogeneity gives rise to severely ill-conditioned linear systems that cannot be easily solved. That is, many important problems remain and will remain computationally intractable.

Since merely relying on sheer technological improvements in computing power cannot solve the problem of simulating multi-scale phenomena, mathematical and

2000 *Mathematics Subject Classification.* 65N15, 65N30, 76S05.

Key words and phrases. second order elliptic, two-scale expansion, upscaling, subgrid, mixed method, variational multiscale method, numerical Greens function.

This work was supported by the U.S. National Science Foundation under grants DMS-9707015 and SBR-9873326.

computational modeling remains a key tool. Many multi-scale techniques have been developed over the last few hundred years. To mention just a few, these include statistical mechanics, asymptotic analysis, homogenization, and cellular automata, and various techniques based on expansion methods such as Taylor, Fourier, wavelet, and eigenfunction expansions and transforms.

We consider in this paper a relatively recently exploited type of expansion technique and its computational implementation. The expansion technique is based analytically on very general Hilbert space direct sum decompositions, and implemented on the discrete or computational level as a *variational multi-scale method* [**25, 26, 11**]. However, because of the way it handles the scales of the underlying equations, it is also referred to as a type of subgrid upscaling method [**6, 2**].

Many numerical methods involve computations on coarse and fine grids. The *subgrid methods* use a coarse grid and a subgrid within each coarse element. Although the subgrids form a fine grid, they are effectively decoupled from each other. Many numerical techniques may be classified as subgrid methods. An incomplete list includes methods involving (residual free) bubble functions and subgrid stabilization [**27, 25, 16, 17, 22, 23**], the variational multiscale method [**25, 26, 28**], and subgrid upscaling [**6, 2, 3, 1**]. Very closely related but somewhat distinct methods include those involving nonoverlapping domain decomposition [**30, 34, 33**], the generalized finite element method (if the elements are defined on a subgrid) [**8, 32**] and the multiscale finite element method [**24**].

Subgrid methods approximate some of the finer-scale aspects of the problem by incorporating information within each coarse grid element. Subgrid methods can provide improvements in stability, mathematical modeling (e.g., large eddy simulation), and numerical approximation.

In this paper we review some of the basic theory of subgrid methods and summarize work of the author in implementing these ideas for second order elliptic problems in mixed form. The method was developed and applied to simulate flow in heterogeneous porous media. We take the point of view that the porous medium flow problem can be expressed at some scale that can be resolved on a very fine grid. However, this fine grid gives rise to a discrete problem that is too large to be solved efficiently. Coarse grids are the rule in subsurface simulation, due to the large size of the physical domain, its extreme heterogeneity, the large number of equations that must be solved simultaneously, and the large number of stochastic realizations that must be investigated.

Our subgrid approach was developed to scale up fine grid information to coarse scales in an approximation to a nonlinear parabolic system governing two-phase flow in porous media. The technique allows upscaling of the usual parameters porosity and relative and absolute permeabilities, and also the location of wells and capillary pressure effects. Some of these are critical nonlinear terms that need to be resolved on the fine scale, or serious errors will result.

Very briefly, upscaling is achieved by decomposing the solution space into a direct sum of Hilbert spaces with desirable properties. In particular, one of the spaces is localized in space. The differential system is written as a variational problem and decomposed using this direct sum into a coarse-grid-scale operator coupled to a subgrid-scale operator. The subgrid-scale operator is approximated locally in space on each coarse-grid element. An influence function or numerical Greens function technique enables us to solve these subgrid-scale problems independently of

the coarse-grid approximation. The coarse-grid problem is modified to take into account the subgrid-scale solution and solved as a large linear system of equations. Finally, the coarse scale solution is corrected on the subgrid-scale, providing a fine-grid scale representation of the solution. In this approach, no explicit macroscopic coefficients nor pseudo-functions result.

In the next section we review the abstract theory of the multi-scale subgrid variational methods, developed originally by Hughes et al. [**26**] and Brezzi [**11**]. Once this context has been set forth, it is relatively easy to describe the application of the ideas to problems written in mixed form. This is done in Section 3. In the last section we present a numerical example to illustrate the results.

2. Background: Abstract Framework for Subgrid Methods

In outline form, subgrid methods involve the following set of ideas for a variational problem:

(1) A decomposition of the solution space into a direct sum of coarse and subgrid spaces, for which the latter in some sense can be localized;
(2) A separation of the problem scales via the test space;
(3) A closing of the system by defining an affine subgrid solution operator taking coarse information to the subgrid;
(4) An upscaling of the coarse problem using this solution operator to remove explicit reference to the subgrid;
(5) The solving of the coarse problem;
(6) The construction of a full two-scale solution by adding back in the subgrid scales using the subgrid solution operator.

We detail this approach below (following heavily the ideas in [**26, 11**]).

2.1. Construction of the two-scale variational form. Let X be a Hilbert space and $a : X \times X \to \mathbb{R}$ a continuous, coercive, bilinear form. Let $f \in X'$. Consider the problem: Find $u \in X$ such that

$$a(u, v) = f(v) \qquad \forall v \in X. \tag{2.1}$$

We impose what we call a two-scale direct sum decomposition of X,

$$X = X_c \oplus \delta X, \tag{2.2}$$

wherein we consider X_c the *coarse* space and δX the *subgrid* space. Then $u = u_c + \delta u$ is uniquely decomposed or expanded into coarse and subgrid parts. The nature of these parts depends on the type of direct sum decomposition we choose, and different decompositions result in different properties.

We then recast our variational problem into the form: Find $u_c \in X_c$ and $\delta u \in \delta X$ such that

$$a(u_c + \delta u, v_c) = f(v_c) \qquad \forall v_c \in X_c, \tag{2.3}$$

$$a(u_c + \delta u, \delta v) = f(\delta v) \qquad \forall \delta v \in \delta X. \tag{2.4}$$

We remark that if $a(v_1, v_2) = (Av_1, v_2)$ and $f(v) = (F, v)$, then

$$a(\delta u, \delta v) = f(\delta v) - a(u_c, \delta v) = (F - Au_c, \delta v),$$

so δu can be viewed as a residual correction.

To apply these ideas to upscaling through the use of subgrid modeling, we choose a coarse mesh $\mathcal{T}_H$. This coarse mesh is artificial at this point, but it will

become the coarse mesh of the discrete scheme later. We suppose that we have chosen the direct sum decomposition so that we can localize the bilinear and linear forms on the subgrid scale:

$$a(u,\delta v)=\sum_{E_c\in\mathcal{T}_H} a_{E_c}(u|_{E_c},\delta v|_{E_c}) \quad \text{and} \quad f(\delta v)=\sum_{E_c\in\mathcal{T}_H} f_{E_c}(\delta v|_{E_c}).$$

For a second order elliptic problem, we would probably take $X\subset H^1$ and $\delta X|_{E_c}=H_0^1(E_c)$ (i.e., the *bubble functions*), so that these localization conditions would hold.

The subgrid problem (2.4) can be used to define the subgrid or δ-solution operator $\delta u: X_c\to\delta X$ by

$$a(v_c+\delta u(v_c),\delta v)=f(\delta v) \qquad \forall \delta v\in\delta X. \tag{2.5}$$

This is an affine operator, with linear part $\delta\hat{u}: X_c\to\delta X$ defined by

$$a(v_c+\delta\hat{u}(v_c),\delta v)=0 \qquad \forall \delta v\in\delta X, \tag{2.6}$$

and constant part $\delta\bar{u}\in\delta X$ defined by

$$a(\delta\bar{u},\delta v)=f(\delta v) \qquad \forall \delta v\in\delta X. \tag{2.7}$$

If u_c were known, then $\delta u=\delta u(u_c)=\delta\hat{u}(u_c)+\delta\bar{u}$.

The δ-solution operator is usually called a *closure operator,* since it is used to remove direct reference to the subgrid scale. Normally, the equations defining this operator are not well-posed, and some *closure assumption,* such as additional boundary conditions or dropping certain "small" terms, is imposed to define the operator. In this new framework, the closure operator is defined without adding additional assumptions to the problem. This is very important, as it is often difficult to understand the modeling errors committed by imposition of closure assumptions.

The upscaled problem is then given by substituting the δ-solution operator in the coarse scale equations (2.3). The result is the problem: Find $u_c\in X_c$ such that

$$a(u_c+\delta\hat{u}(u_c),v_c)=f(v_c)-a(\delta\bar{u},v_c) \qquad \forall v_c\in X_c, \tag{2.8}$$

or, in symmetric form (assuming a is symmetric),

$$a(u_c+\delta\hat{u}(u_c),v_c+\delta\hat{u}(v_c))=f(v_c)-a(\delta\bar{u},v_c) \qquad \forall v_c\in X_c. \tag{2.9}$$

This problem is defined entirely in terms of the coarse space X_c. We have modified the bilinear form and linear functional, and hence we have the term *multiscale variational method.* Once this is solved for u_c, the full two-scale solution is then

$$u=u_c+\delta u(u_c)=u_c+\delta\hat{u}(u_c)+\delta\bar{u}. \tag{2.10}$$

The above construction has been given on the analytical level, and so it is exact. A similar construction can be given on the discrete level. The advantage we are trying to exploit is that the subgrid problems are localized and expressed in terms of the δ-solution operator.

2.2. Discretization. We obtain a finite element approximation by defining within each coarse element $E_c\in\mathcal{T}_H$, a fine grid $\mathcal{T}_h(E_c)$ (see Fig. 1). Approximate X_c by X_H and $\delta X|_{E_c}$ by $\delta X_h(E_c)$, where H is the maximal diameter of the coarse elements, and h is the maximal diameter of the subgrid elements. By restricting the trial and test spaces in (2.8) or (2.9) and (2.6)–(2.7) to these finite dimensional subspaces, we obtain our finite element method.

The δ-operator must be evaluated, but the key is that it needs to be evaluated only on the space X_H. We use a technique involving influence functions or numerical

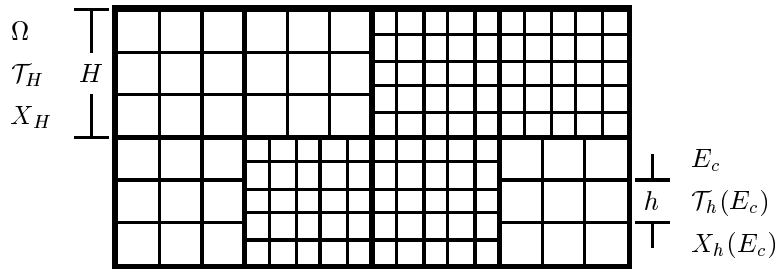


FIGURE 1. The two-scale discretization grid.

Greens functions, related to static condensation or Shur complements. For each $E_c \in \mathcal{T}_H$, let $\{v_{H,j}\}_j$ be a basis for $X_H|_{E_c}$. This space has low dimension. Since

$$u_H|_{E_c} = \sum_j \alpha_j \, v_{H,j},$$

we have that

$$\delta\hat{u}(u_H)|_{E_c} = \delta\hat{u}(u_H|_{E_c}) = \sum_j \alpha_j \delta\hat{u}(v_{H,j}).$$

Define the *numerical Greens function* $\delta\hat{u}_{h,j} \in \delta X_h$ by

$$a(v_{H,j} + \delta\hat{u}_{h,j}, \delta v_h) = 0 \qquad \forall \delta v_h \in \delta X_h. \tag{2.11}$$

Then

$$\delta\hat{u}_h(u_H)|_{E_c} = \sum_j \alpha_j \delta\hat{u}_{h,j}$$

and $\delta u_h(u_H)|_{E_c} = \delta\hat{u}_h(u_H)|_{E_c} + \delta\bar{u}_h|_{E_c}$, where $\delta\bar{u}_h \in \delta X_h$ satisfies

$$a(\delta\bar{u}_h, \delta v_h) = f(\delta v_h) \qquad \forall \delta v_h \in \delta X_h. \tag{2.12}$$

There are just a small number of these problems for each coarse element, and they are small and have multiple right-hand-sides, so a direct solver is appropriate. Moreover, the problems on each $E_c \in \mathcal{T}_H$ are independent, and so naturally parallelize. Thus this part of the computation is very fast and efficient.

The discrete upscaled problem is then: Find $u_H = \sum_j \alpha_j \, v_{H,j} \in X_H$ such that

$$\sum_j \alpha_j \, a(v_{H,j} + \delta\hat{u}_{h,j}, v_{H,i} + \delta\hat{u}_{h,i}) = f(v_{H,i}) - a(\delta\bar{u}_h, v_{H,i}) \quad \forall v_{H,i} \in X_H. \tag{2.13}$$

We have modified the matrix and right-hand-side. After solving this problem, the full two-scale solution is given by

$$u_h = u_H + \delta u_h = \sum_j \alpha_j \, (v_{H,j} + \delta\hat{u}_{h,j}) + \delta\bar{u}_h. \tag{2.14}$$

We use modified test functions as in the generalized and multiscale finite element methods; however, we have also modified the right-hand-side term.

2.3. Error analysis. The error is easily analyzed if the approximation is viewed as a whole. Let

$$X_{H,h} = X_H + \sum_{E_c \in \mathcal{T}_h} \delta X_h(E_c) \tag{2.15}$$

denote our two-scale finite element space. Then the combination of the equations is the problem: Find $u_h \in X_{H,h}$ such that

$$a(u_h, v_h) = f(v_h) \qquad \forall v_h \in X_{H,h}. \tag{2.16}$$

It is then easy to analyze the error [**20, 9**]. Cia's Lemma gives the estimate that

$$\|u - u_h\| \le C \inf_{v_h \in X_{H,h}} \|u - v_h\| \le C \inf_{v_H \in X_H} \|u - v_H\|, \tag{2.17}$$

for some constant C independent of the coarse and subgrid meshes. Thus accuracy is reduced simply to a question of approximation theory. Generally, we can only expect approximation accuracy to some order in the coarse scale parameter H, the diameter of the coarse elements, and not in the fine scale parameter h, the diameter of the subgrid elements. The somewhat surprising result is that we obtain approximation accuracy on the fine scale when applying the method to elliptic equations in mixed form (for the scalar variable).

3. Subgrid Upscaling for Problems in Mixed Form

The above ideas were applied to and developed for second order elliptic equations in mixed form in the papers [**6, 4, 2, 5, 3, 1**]. Consider the model problem

$$\nabla \cdot \mathbf{u} = f \qquad \text{in } \Omega, \tag{3.1}$$

$$\mathbf{u} = -K\nabla p \qquad \text{in } \Omega, \tag{3.2}$$

$$\mathbf{u} \cdot \nu = 0 \qquad \text{on } \partial\Omega, \tag{3.3}$$

where $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain, $f \in L^2(\Omega)/\mathbb{R}$, and K is a uniformly bounded and positive definite rank two tensor. With the Hilbert spaces

$$W = L^2(\Omega)/\mathbb{R},$$
$$\mathbf{V} = H(\mathrm{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \nabla \cdot \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \nu = 0 \text{ on } \partial\Omega\},$$

and the $L^2(\Omega)$ inner product $(\,\cdot\,,\,\cdot\,)$, a mixed variational formulation is: Find $p \in W$ and $\mathbf{u} \in \mathbf{V}$ such that

$$a(\mathbf{u}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) \qquad \forall\ \mathbf{v} \in \mathbf{V}, \tag{3.4}$$

$$(\nabla \cdot \mathbf{u}, w) = (f, w) \qquad \forall\ w \in W, \tag{3.5}$$

where $a(\mathbf{u}, \mathbf{v}) = (K^{-1}\mathbf{u}, \mathbf{v})$.

3.1. Upscaling the mixed problem. We base our two-scale decomposition or expansion on local mass conservation. Both W and $\mathbf{V}$ must be decomposed. Again, we begin by defining a coarse computational grid $\mathcal{T}_H$ on Ω, and, for simplicity, let

$$W_c \supset \{w_c \in W : w_c \text{ is constant } \forall \text{ coarse elements } E_c \in \mathcal{T}_H\},$$
$$\delta W = W_c^{\perp},$$
$$\mathbf{V}_c^1 = \{\mathbf{v} \in \mathbf{V} : \nabla \cdot \mathbf{v} \in W_c\},$$
$$\delta\mathbf{V}^1 = \{\delta\mathbf{v} \in \mathbf{V} : \nabla \cdot \delta\mathbf{v} \in \delta W,\ \delta\mathbf{v} \cdot \nu = 0 \text{ on } \partial E_c\ \forall\ E_c \in \mathcal{T}_H\},$$

although we can be even more general [**1**]. It turns out that $W = W_c \oplus \delta W$ and $\mathbf{V} = \mathbf{V}_c^1 + \delta\mathbf{V}^1$; moreover, there exist subspaces $\mathbf{V}_c \subset \mathbf{V}_c^1$ and $\delta\mathbf{V} \subset \delta\mathbf{V}^1$ such that $\nabla \cdot \mathbf{V}_c = W_c$ and $\nabla \cdot \delta\mathbf{V} = \delta W$ [**1**]. This two-scale expansion is used to define and separate scales in the variational form (3.4)–(3.5). The inf-sup condition holds over both $W_c \times \mathbf{V}_c$ and $\delta W \times \delta\mathbf{V}$, with constants independent of the coarse mesh [**1**]. Thus the saddle-point theory of Babuška [**7**] and Brezzi [**10**] can be used to prove the existence of the affine δ-solution operator. In this simple case, we have

$$\delta p = \delta p(\mathbf{u}_c) = \delta\hat{p}(\mathbf{u}_c) + \delta\bar{p},$$
$$\delta\mathbf{u} = \delta\mathbf{u}(\mathbf{u}_c) = \delta\hat{\mathbf{u}}(\mathbf{u}_c) + \delta\bar{\mathbf{u}}$$

(in general, the operators also have a linear term involving p_c), where, for $\mathbf{v}_c \in \mathbf{V}_c$, these solve the variational problems: Find $\delta\hat{p}(\mathbf{v}_c) \in \delta W$ and $\delta\hat{\mathbf{u}}(\mathbf{v}_c) \in \delta\mathbf{V}$ such that

$$a(\mathbf{v}_c + \delta\hat{p}, \mathbf{v}) = (\delta\hat{p}, \nabla \cdot \mathbf{v}) \qquad \forall\ \delta\mathbf{v} \in \delta\mathbf{V}, \tag{3.6}$$
$$(\nabla \cdot \delta\hat{\mathbf{u}}, \delta w) = 0 \qquad \forall\ \delta w \in \delta W; \tag{3.7}$$

and: Find $\delta\bar{p} \in \delta W$ and $\delta\bar{\mathbf{u}} \in \delta\mathbf{V}$ such that

$$a(\delta\bar{p}, \mathbf{v}) = (\delta\bar{p}, \nabla \cdot \mathbf{v}) \qquad \forall\ \delta\mathbf{v} \in \delta\mathbf{V}, \tag{3.8}$$
$$(\nabla \cdot \delta\bar{\mathbf{u}}, \delta w) = (f, \delta w) \qquad \forall\ \delta w \in \delta W. \tag{3.9}$$

The upscaled problem, in symmetric form, is: Find $(p_c, \mathbf{u}_c) \in W_c \times \mathbf{V}_c$ such that

$$a(\mathbf{u}_c + \delta\hat{\mathbf{u}}(\mathbf{u}_c), \mathbf{v}_c + \delta\hat{\mathbf{u}}(\mathbf{v}_c)) = (p_c, \nabla \cdot \mathbf{v}_c) - a(\delta\bar{\mathbf{u}}, \mathbf{v}_c) \qquad \forall\ \mathbf{v}_c \in \mathbf{V}_c, \tag{3.10}$$
$$(\nabla \cdot \mathbf{u}_c, w_c) = (f, w_c) \qquad \forall\ w_c \in W_c, \tag{3.11}$$

and the full solution is

$$p = p_c + \delta p(\mathbf{u}_c) = p_c + \delta\hat{p}(\mathbf{u}_c) + \delta\bar{p}, \tag{3.12}$$
$$\mathbf{u} = \mathbf{u}_c + \delta\mathbf{u}(\mathbf{u}_c) = \mathbf{u}_c + \delta\hat{\mathbf{u}}(\mathbf{u}_c) + \delta\bar{\mathbf{u}}. \tag{3.13}$$

These equations are exact, since no closure assumption is made. Moreover, the equations maintain strict local conservation on both coarse and subgrid scales. Since $\delta\mathbf{V}$ has a homogeneous Neumann boundary condition, $\delta\hat{\mathbf{u}}$ is a local operator.

3.2. Mixed method discretization. The numerical approximation spaces on the coarse and subgrid scales can be any of the usual mixed spaces [**31, 29, 14, 12, 13, 18, 15**]. We select $\mathbf{V}_H^* \times W_H^* \subset \mathbf{V} \times W$ on $\mathcal{T}_H$, and, for each coarse element $E_c \in \mathcal{T}_H$, we select $\delta W_h(E_c) \times \delta\mathbf{V}_h(E_c)$ on $\mathcal{T}_h(E_c)$, with $\delta W_h(E_c) \perp 1$ and $\delta\mathbf{V}_h(E_c)$ satisfying the homogeneous Neumann boundary condition on ∂E_c. It may be that the coarse and δ-spaces are not linearly independent, so we may need to reduce $\mathbf{V}_H^* \times W_H^*$ to $\mathbf{V}_H \times W_H$ so that

$$W_{H,h} = W_H^* + \delta W_h = W_H \oplus \delta W_h \subset W,$$
$$\mathbf{V}_{H,h} = \mathbf{V}_H^* + \delta\mathbf{V}_h = \mathbf{V}_H \oplus \delta\mathbf{V}_h \subset \mathbf{V}.$$

Since $\nabla \cdot \delta\mathbf{V}_h = \delta W_h$ and $\nabla \cdot \mathbf{V}_{H,h} = W_{H,h}$, we maintain discrete local mass conservation.

As usual, the finite element method is given by replacing the full spaces by the corresponding finite dimensional spaces. We solve the system using the numerical Greens function approach described above (see also [**2, 3**] for more details).

3.3. Convergence results. If $\mathcal{P}_{W_{H,h}} : W \to W_{H,h}$ is L^2-projection, we have the following approximation result [**1**].

LEMMA 3.1. *If W_H approximates as $O(H^L)$, δW_h approximates to order $O(h^\ell)$, and $L \geq \ell$, then*

$$\|w - \mathcal{P}_{W_{H,h}} w\|_0 \leq C|w|_L H^{L-\ell} h^\ell.$$

Note that we have fine-scale approximation. For the vector variable, we have the existence of a π-projection operator ([**1**]; cf., [**21**]).

LEMMA 3.2. *There exists an operator $\pi : \mathbf{V} \cap H^1(\Omega) \to \mathbf{V}_{H,h}$ such that*

$$\nabla \cdot \pi \mathbf{v} = \mathcal{P}_{W_{H,h}} \nabla \cdot \mathbf{v},$$
$$\|\mathbf{v} - \pi \mathbf{v}\|_0 \leq C|\mathbf{v}|_K H^K,$$

where $\|\cdot\|_j$ is the usual $H^j(\Omega)$-norm, C is independent of the coarse and subgrid meshes, and K is the approximation order of the coarse velocity space.

These approximation results lead to the following error estimate [**1**].

THEOREM 3.3. *If $L \geq \ell$, $\Omega \subset \mathbb{R}^d$, and there is some fixed $\gamma > 0$ such that as $H \to 0$, the coarse meshes satisfy the property that*

$$msr(E_c) \geq \gamma (diam(E_c))^d \qquad \forall E_c \in \mathcal{T}_H,$$

where $msr(E_c)$ is the measure and $diam(E_c)$ is the diameter of E_c, then

$$\|K^{-1/2}(\mathbf{u} - \mathbf{u}_h)\|_0 \leq \inf_{\substack{\mathbf{v}_h \in \mathbf{V}_{H,h} \\ \nabla \cdot \mathbf{v}_h = \mathcal{P}_{W_{H,h}} \nabla \cdot \mathbf{u}}} \|K^{-1/2}(\mathbf{u} - \mathbf{v}_h)\|_0 \leq C|\mathbf{u}|_K H^K = O(H^K),$$

$$\nabla \cdot \mathbf{u}_h = \mathcal{P}_{W_{H,h}} \nabla \cdot \mathbf{u},$$

$$\|\mathcal{P}_{W_{H,h}} p - p_h\|_0 \leq C\{|f|_L H^{L-\ell} h^{\ell+1} + |\mathbf{u}|_K H^{K+1}\} = O(H^{L-\ell} h^{\ell+1} + H^{K+1}),$$

$$\|p - p_h\|_0 \leq C\{[|p|_L + |f|_L h] H^{L-\ell} h^\ell + |\mathbf{u}|_K H^{K+1}\} = O(H^{L-\ell} h^\ell + H^{K+1}).$$

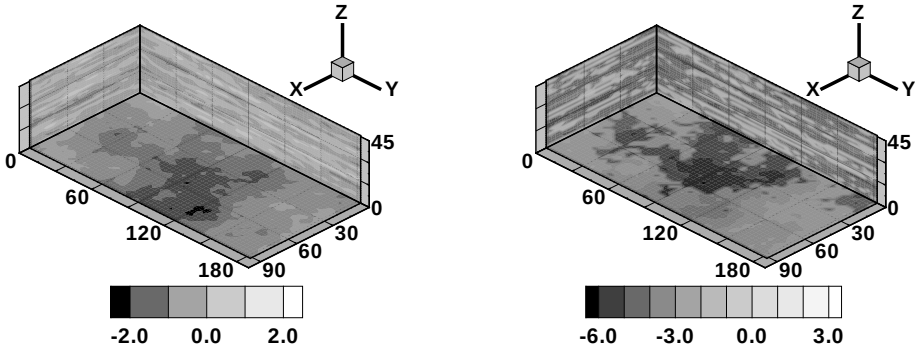
In all mixed finite element spaces, $L = K$ or $L = K - 1$. If the lowest order Raviart-Thomas spaces (RT0) are used on both the coarse and fine scales, we have the RT0-RT0 two-scale space for which $K = L = \ell = 1$ and

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C|\mathbf{u}|_1 H = O(H),$$
$$\|p - p_h\|_0 \leq C\{[|p|_1 + |f|_1 h]h + |\mathbf{u}|_1 H^2\} = O(h + H^2).$$

This method does not give a particularly good approximation, since the velocity approximation is limited to first order on the coarse scale. A better choice is to use the Brezzi-Douglas-Duràn-Fortin spaces (BDDF1) on the coarse level, for which $K = 2$ and $L = \ell = 1$. (In 2-D, the BDDF1 spaces are due to Brezzi-Douglas-Marini, BDM1.) Then we have the BDDF1-RT0 two-scale space and

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C|\mathbf{u}|_2 H^2 = O(H^2),$$
$$\|p - p_h\|_0 \leq C\{[|p|_1 + |f|_1 h]h + |\mathbf{u}|_2 H^3\} = O(h + H^3).$$

This method gives a good estimate, since the velocity, while still only approximated in H, is now second order. The pressure is approximated on the fine scale (up to a higher order coarse perturbation). If, say, $H = \sqrt{h}$, we have good fine scale approximation of both variables, even though the work is concentrated in the upscaled problem defined over the coarse mesh. Thus the method is both very efficient and very accurate.



Horizontal permeability Vertical permeability

FIGURE 2. Log (base 10) of Rock permeability in milliDarcies, at the injection corner.

4. A Numerical Example

We close this paper with a numerical example representing a petroleum waterflood. The system has two fluid phases, water and oil. Each phase is governed by a conservation equation combined with Darcy's law. Their sum is an elliptic equation, called the *pressure equation*, that governs the total fluid velocity. The water conservation equation is then called the *saturation equation*; it is advection dominated. Only the elliptic pressure equation needs to be upscaled. For more details on the equations, see, e.g., [**5, 3**].

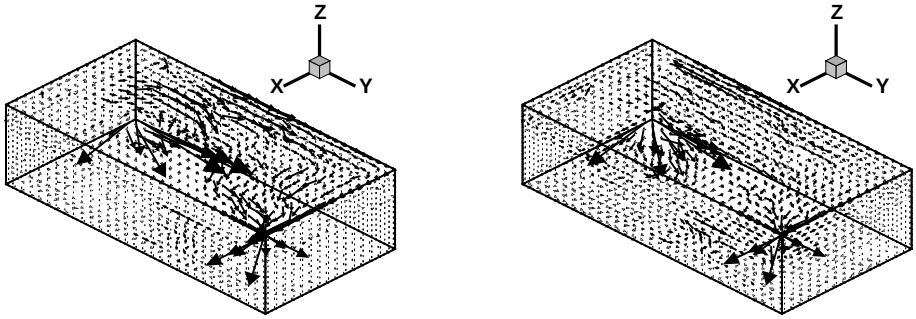
The domain is $300 \times 600 \times 150$ feet and initially filled with oil and about 30% water in gravitational equilibrium. There is a water injection well at one of the corners, and a production well at the opposite corner.

The domain is discretized by a uniform fine grid of size $30 \times 60 \times 30$ (so the fine grid elements are of size $10 \times 10 \times 5$ feet). We upscale the pressure equation to $6 \times 12 \times 6$ with a $5 \times 5 \times 5$ subgrid in each coarse element, for an upscaling factor of 125. The saturation equation is solved on the fine scale.

The problem is particularly difficult, since it has the highly stratified permeability field depicted in Fig. 2. The permeability and porosity are a corner portion of the data from the second test example of the Tenth Society of Petroleum Engineers Comparative Solution Project [**19**], which is designed to test upscaling techniques. From the figure it should be clear that the permeability field is not at all well resolved on the coarse grid. This is especially true in the vertical direction.

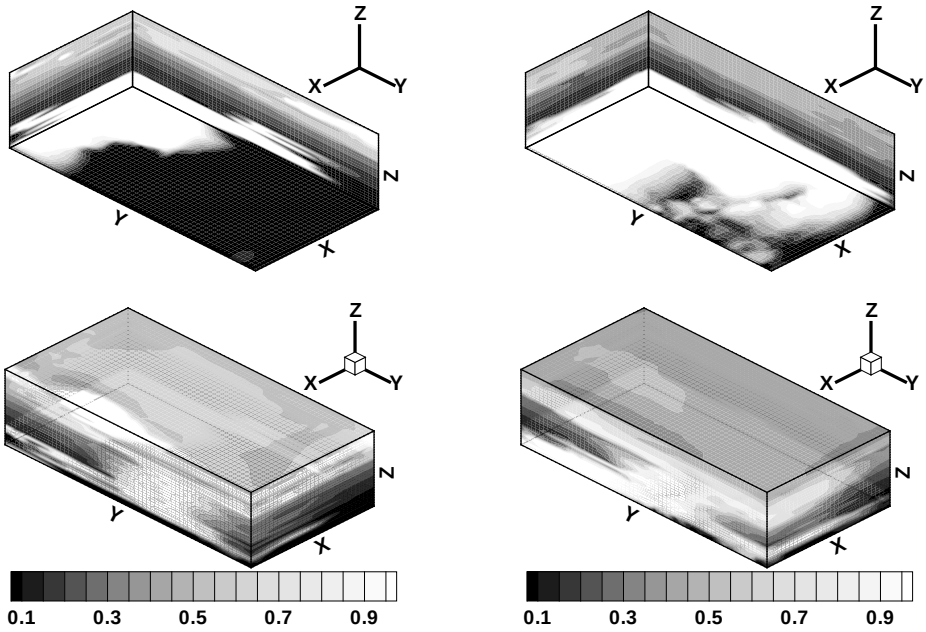
The velocity profile is depicted in Fig.3. It is difficult to assess the quality of the approximation, but it does appear to have captured the main features and the layering of the medium. Instead, we concentrate on the effects of the velocity field, namely, the flow of the fluid through time within this velocity field. The saturation distribution at 150 days of simulation time is shown in Figs. 4 and 5. The upscaled saturation contours are somewhat smeared, as is typical for upscaling and averaging techniques. However, the level of detail is remarkable, considering both the difficulty of the problem and the coarseness of the grid (a mere 432 coarse elements compared to 54,000).

The fine scale solution was solved using the first order approximation space RT0. The upscaling technique used the BDDF1-RT0 two-scale space. Both were



Fine $30 \times 60 \times 30$ Upscaled to $6 \times 12 \times 6$

FIGURE 3. Total fluid velocity at 150 days for fine and upscaled solutions.



Fine $30 \times 60 \times 30$ Upscaled to $6 \times 12 \times 6$

FIGURE 4. Water saturation at 150 days for fine and upscaled solutions at the injection corner (top figures) and production corner (bottom figures).

solved using Jacobi preconditioned conjugate gradients. On a 2.8 GHz Pentium 4 computer, to solve the pressure equation once, the code takes about 303 seconds on the fine scale and 33 seconds when upscaled, for a speed-up of about a factor of 9. The factor is not closer to 125 since the coarse problem in the upscaling technique uses the second order approximation space BDDF1, which is considerably bigger than RT0. The subgrid problems themselves actually add very little to the computation time. The saturation equation is solved 20 times for each pressure solve for a total of about 15–20 seconds, verifying that there is no need to upscale

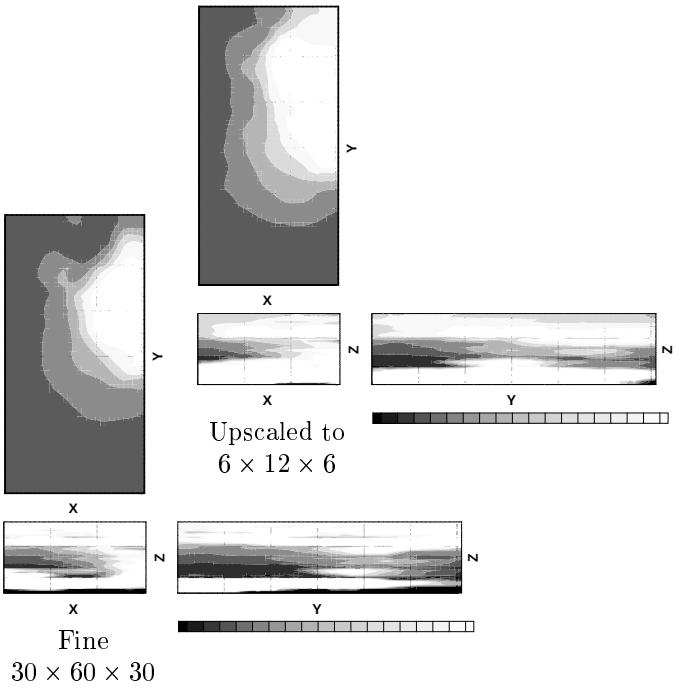


FIGURE 5. Central cross-sections of water saturation at 150 days for the fine and upscaled solution. Black is 0 and white is 1.

the saturation equation. Overall, this problem was solved about 7 times faster with the upscaling technique compared to the fine scale solution.

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Department of Mathematics, C1200; and Texas Institute for Computational and Applied Mathematics, C0200; The University of Texas at Austin; Austin, TX 78712
E-mail address: arbogast@ticam.utexas.edu