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Multiscale mortar mixed methods for heterogeneous elliptic problems

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Abstract. Consider solving a second order elliptic problem when the elliptic coefficient is highly heterogeneous. Generally, a numerical method either uses a very fine computational mesh to resolve the heterogeneities and therefore becomes computationally inefficient, or it performs efficiently on a coarse mesh but gives inaccurate results. Standard nonoverlapping domain decomposition using mortar spaces to couple together the subdomains efficiently handles these equations in parallel, but the issue of heterogeneity is not directly addressed. We define new mortar spaces that incorporate fine scale information obtained from local cell problems, using the theory of homogenization as a heuristic guide to limit the number of degrees of freedom in the mortar space. This gives computational efficiency in parallel, even when the subdomain problems are fully resolved on a fine mesh. In the case of an elliptic coefficient satisfying the two-scale separation assumption, the method is provably accurate with respect to the heterogeneity. Formally first and second order mortar space approximations are constructed explicitly in two dimensions. Numerical tests are presented for one medium with the two-scale separation assumption and two without it. The results show that these new homogenization based mortar spaces work much better than simple polynomial based mortar spaces, and that generally the second order spaces work better than the first order ones.

1. Introduction

We consider a second order elliptic problem with a heterogeneous coefficient (i.e., one that is highly variable or oscillatory in space) that models, for example, the single phase flow of fluid in the Earth's subsurface according to Darcy's Law. In mixed form [**BF91**, **RT91**, **BS94**], the problem is

$$\mathbf{u} = -a_\varepsilon \nabla p \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

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where $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is the problem domain, ν is the outer unit normal, $a_\varepsilon(x)$ is a symmetric, uniformly positive definite tensor coefficient with $L^\infty(\Omega)$ components representing the *permeability*, $f \in L^2(\Omega)$ is the source or sink term, and the unknowns are *pressure* $p(x)$ and *velocity* $\mathbf{u}(x)$. The homogeneous Neumann boundary condition is considered for simplicity.

Domain decomposition for mixed methods [**GW88**] has been developed as a divide and conquer strategy to increase parallelism in computations and to handle interdomain multiphysics. In a nonoverlapping domain decomposition approach, a relatively small mortar finite element space [**BMP94**, **ACWY00**, **APWY07**] can be introduced to reduce coupling between subdomains. Let Ω_i, $i = 1, 2, \ldots, n$, be nonoverlapping subdomains of Ω, let p_i and $\mathbf{u}_i$ be the pressure p and velocity $\mathbf{u}$ restricted to Ω_i, and let ν_i be the outer unit normal to $\partial\Omega_i$. We rewrite (1.1)–(1.3) in a domain decomposition setting as

$$\mathbf{u} = -a_\varepsilon \nabla p \quad \text{in } \Omega_i, \tag{1.4}$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega_i, \tag{1.5}$$

$$p_i = p_j \quad \text{on } \partial\Omega_i \cap \partial\Omega_j \equiv \Gamma_{ij}, \tag{1.6}$$

$$\mathbf{u}_i \cdot \nu_i + \mathbf{u}_j \cdot \nu_j = 0 \quad \text{on } \Gamma_{ij}, \tag{1.7}$$

$$\mathbf{u} \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{1.8}$$

Accurate approximation of (1.1)–(1.3) or (1.4)–(1.8) is difficult when the permeability coefficient a_ε is a highly varying function, where ε is some measure of the correlation length of the medium. When $\varepsilon \ll 1$, the medium is highly heterogeneous, and its resolution requires a fine computational mesh (see, e.g., the error estimate Theorem 2.1 and (2.7)–(2.8)). This is reasonable for the subdomain problems, since they can be computed independently in parallel without the need for communication. The mortar interface problem is not so easily solved in parallel, but it can be made computationally efficient if it is small in size.

Recently, one of the current authors in [**Arb11a**, **Arb11b**] suggested a new multiscale finite element space based on the homogenization microstructure theorem (see Theorem 3.1) to handle the heterogeneity. More recently, two of the current authors in [**AX12**] adapted the idea to define a new multiscale mortar space and numerically tested a formally first order mortar space approximation with only three degrees of freedom on each subdomain interface. In this paper, we extend the idea to give a formally second order multiscale mortar space approximation with five degrees of freedom on each subdomain interface.

Briefly, the idea is to efficiently sample the microstructure by solving local cell problems. Heuristically, homogenization theory tells us that these local solutions can be used implicitly to reconstruct the pressure p in terms of a fixed operator and a smooth homogenized function p_0 (see (3.6)). Rather than approximating p directly, we approximate p_0 by a polynomial (see (4.7)), which gives an efficient multiscale mortar space with only a few degrees of freedom per subdomain interface.

We close this introduction by outlining the paper. We first give a brief review of the domain decomposition mortar method and homogenization theory in Sections 2 and 3, respectively. Then in Section 4, we define the first and second order mortar space approximations based on solutions to a localized cell problem as in homogenization theory. We also note that in the case of an elliptic coefficient satisfying the two-scale separation assumption of periodic homogenization [**BLP78**, **JKO94**],

the method is provably accurate with respect to the heterogeneity, and we show that our new mortar method can be viewed as an implicitly defined multiscale finite element method. In Section 5, three numerical examples are given. Although our new mortar space was designed based on homogenization theory, which requires a locally periodic coefficient, our numerical tests on nonperiodic permeability fields show that the new method performs well for problems with general heterogeneities.

2. Mortar domain decomposition mixed method

Throughout, let $\Gamma = \bigcup_{i,j} \Gamma_{ij}$ and $\Gamma_i = \partial\Omega_i \bigcap \Gamma$ denote interior subdomain interfaces. For any $\omega \subset \Omega$ and $\gamma \subset \Gamma$, let $(\cdot,\cdot)_\omega$ and $\langle\cdot,\cdot\rangle_\gamma$ denote the $L^2(\omega)$ and $L^2(\gamma)$ inner products, respectively.

2.1. The variational form. Define the function spaces

$$\mathbf{V}_i = \{\mathbf{v} \in H(\mathrm{div};\Omega_i) : \mathbf{v}\cdot\nu|_{\partial\Omega\cap\partial\Omega_i} = 0\}, \qquad \mathbf{V} = \bigoplus_{i=1}^n \mathbf{V}_i,$$

$$W_i = L^2(\Omega_i), \qquad W = \Big\{w \in L^2(\Omega) : \int_\Omega w\,dx = 0\Big\},$$

$$M = H^{1/2}(\Gamma).$$

The variational form of (1.4)–(1.8) is: Find $\mathbf{u} \in \mathbf{V}$, $p \in W$, and $\lambda = p \in M$ such that for $1 \le i \le n$,

$$(a_\varepsilon^{-1}\mathbf{u},\mathbf{v})_{\Omega_i} - (p,\nabla\cdot\mathbf{v})_{\Omega_i} + \langle\lambda,\mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} = 0 \qquad \forall\,\mathbf{v}\in\mathbf{V}_i, \tag{2.1}$$

$$(\nabla\cdot\mathbf{u},w)_{\Omega_i} = (f,w)_{\Omega_i} \qquad \forall\,w\in W_i, \tag{2.2}$$

$$\sum_{i=1}^n \langle\mathbf{u}\cdot\nu_i,\mu\rangle_{\Gamma_i} = 0 \qquad \forall\,\mu\in M. \tag{2.3}$$

2.2. The Finite element approximation. Let $\mathcal{T}_{h,i}$ be a conforming, quasi-uniform, finite element partition of Ω_i with maximum element diameter h_i. Let $h = \max_i h_i$ and $\mathcal{T}_h = \bigcup_{i=1}^n \mathcal{T}_{h,i}$ be the finite element partition over the entire domain Ω. Let $\mathbf{V}_{h,i}\times W_{h,i} \subset \mathbf{V}_i \times W_i$ be any of the usual inf-sup stable mixed finite element spaces [**BF91**, **RT91**, **BS94**] defined over $\mathcal{T}_h$, and set $\mathbf{V}_h = \bigoplus_{i=1}^n \mathbf{V}_{h,i}$ and $W_h = \bigoplus_{i=1}^n W_{h,i}/\mathbb{R}$. Denote by $\mathcal{T}_{H,ij}$ a quasi-uniform finite element partition of Γ_{ij}, with maximal diameter of H_{ij} and $H = \max_{1\le i,j\le n} H_{ij}$. Let $M_{H,ij} \subset L^2(\Gamma_{ij})$ be the local mortar finite element space we will define later, and let $M_H = \bigoplus_{i\ne j} M_{H,ij}$.

In mixed finite element approximation of (2.1)–(2.3), we find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, and $\lambda_H \in M_H$ such that for $1 \le i \le n$,

$$(a_\varepsilon^{-1}\mathbf{u}_h,\mathbf{v})_{\Omega_i} - (p_h,\nabla\cdot\mathbf{v})_{\Omega_i} + \langle\lambda_H,\mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} = 0 \qquad \forall\,\mathbf{v}\in\mathbf{V}_{h,i}, \tag{2.4}$$

$$(\nabla\cdot\mathbf{u}_h,w)_{\Omega_i} = (f,w)_{\Omega_i} \qquad \forall\,w\in W_{h,i}, \tag{2.5}$$

$$\sum_{i=1}^n \langle\mathbf{u}_h\cdot\nu_i,\mu\rangle_{\Gamma_i} = 0 \qquad \forall\mu\in M_H. \tag{2.6}$$

If our mixed finite element spaces give approximation of order $\mathcal{O}(h^k)$ for $\mathbf{u}$ and $\mathcal{O}(h^\ell)$ for p, and if we use a mortar space M_H of piecewise continuous or discontinuous polynomials of degree $m-1$ over each $\mathcal{T}_{H,ij}$, then from [**APWY07**] we have the following a-priori estimates.

THEOREM 2.1. *There exists C, independent of h and H, such that for $1 \le r \le k$, $0 \le s \le \ell$, and $0 < t \le m$,*

$$\begin{aligned}
\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_0 &\le C\|f\|_s h^s, \\
\|\mathbf{u} - \mathbf{u}_h\|_0 &\le C\big\{\|\mathbf{u}\|_r h^r + \|p\|_{t+1/2} H^{t-1/2} + \|\mathbf{u}\|_{r+1/2} h^r H^{1/2}\big\}, \\
\|p - p_h\|_0 &\le C\big\{\|p\|_s h^s + \|p\|_{t+1/2} H^{t+1/2} \\
&\qquad + \|f\|_s h^s H + \|\mathbf{u}\|_r h^r H + \|\mathbf{u}\|_{r+1/2} h^r H^{3/2}\big\}.
\end{aligned}$$

To be computationally feasible, we usually assume that $h < \varepsilon < H$. However, recall that the gradients of the solution $(\mathbf{u}, p)$ also depend on ε, i.e.,

$$\|\nabla p\|_0 = \mathcal{O}(\varepsilon^{-1}) \text{ and } \|D^k p\|_0 = \mathcal{O}(\varepsilon^{-k}),$$

and similar for $\mathbf{u}$. Thus Theorem 2.1 implies

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C\big\{(h/\varepsilon)^r + (H/\varepsilon)^{t-1/2}/\varepsilon + (h/\varepsilon)^r (H/\varepsilon)^{1/2}\big\}, \tag{2.7}$$

$$\|p - p_h\|_0 \le C\big\{(h/\varepsilon)^s[1+H] + (H/\varepsilon)^{t+1/2} + (h/\varepsilon)^r[1 + (H/\varepsilon)^{1/2}]H\big\}. \tag{2.8}$$

The approximation is poor when $h < \varepsilon < H$, so multiscale techniques are required.

3. Resolving heterogeneities using homogenization theory

Homogenization is a classic mathematical theory to resolve heterogeneities in porous media [**BLP78**, **JKO94**]. The key assumption in periodic homogenization theory is the two-scale separation of $a_\varepsilon(x)$, that is,

$$a_\varepsilon(x) = a(x, x/\varepsilon), \tag{3.1}$$

where $a(x, y)$ is periodic in y in the unit cell $\mathbf{Y} = [0, 1]^d$. Now $a(x, y)$ is assumed to vary slowly in $x \in \Omega$, and these variations can be resolved by H, but as $\varepsilon \to 0$, $y = x/\varepsilon$ varies more and more rapidly (i.e., a_ε becomes more heterogeneous).

The homogenized problem is formulated as

$$\mathbf{u}_0 = -a_0 \nabla p_0 \quad \text{in } \Omega, \tag{3.2}$$

$$\nabla \cdot \mathbf{u}_0 = f \quad \text{in } \Omega, \tag{3.3}$$

$$\mathbf{u}_0 \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

The true solution $(\mathbf{u}, p)$ of (1.1)–(1.3) converges to the homogenized solution $(\mathbf{u}_0, p_0)$ as $\varepsilon \to 0$. Here, the homogenized coefficient tensor $a_0(x)$ is given by

$$a_{0,ij}(x) = \int_{\mathbf{Y}} a(x, y)\Big(\delta_{ij} + \frac{\partial \omega_j(x, y)}{\partial y_i}\Big)\, dy, \quad i, j = 1, \dots, d,$$

where $\omega_j(x, y)$, for each fixed $x \in \Omega$, is the y-periodic solution of the cell problem

$$-\nabla_y \cdot \big[a(x, y)\big(\nabla_y \omega_j(x, y) + \mathbf{e}_j\big)\big] = 0 \quad \text{in } \Omega \times \mathbf{Y},\ j = 1, \dots, d, \tag{3.5}$$

with $\mathbf{e}_j \in \mathbb{R}^d$ being the jth Cartesian unit vector. We can further correct the homogenization solution $(\mathbf{u}_0, p_0)$ to first order expansion [**JKO94**, **MV97**, **AB06**].

THEOREM 3.1. *Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d)^T$ and define the* first order corrector *by*

$$p_\varepsilon^1(x) = p_0(x) + \varepsilon \boldsymbol{\omega}(x, x/\varepsilon) \cdot \nabla p_0(x). \tag{3.6}$$

If $p_0 \in H^2(\Omega)$, then there is some constant C, depending on the solutions to the cell problems but not on ε, such that

$$\|p - p_\varepsilon^1\|_0 \le C\varepsilon \|p_0\|_2. \tag{3.7}$$

Moreover, if $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, *then*

$$\|\nabla(p - p_\varepsilon^1)\|_0 \le C\big\{\varepsilon\|\nabla p_0\|_1 + \sqrt{\varepsilon}\,\|\nabla p_0\|_{0,\infty}\big\}. \tag{3.8}$$

4. A multiscale mortar space based on homogenization

We remark that the two-scale assumption (3.1) is used above for theoretical analysis and error estimation. It is not used in this section to define our multiscale mortar space (which follows the construction in [**AX12**]).

4.1. Interface error in the mortar method. Let the weakly continuous velocities [**ACWY00**] be

$$\mathbf{V}_{h,0} = \Big\{\mathbf{v} \in \mathbf{V}_h : \sum_{i=1}^n \langle \mathbf{v}|_{\Omega_i} \cdot \nu_i, \mu\rangle_{\Gamma_i} = 0 \;\forall\, \mu \in M_H\Big\}, \tag{4.1}$$

and reformulate (2.4)–(2.6) as: Find $\mathbf{u}_h \in \mathbf{V}_{h,0}$ and $p \in W_h$ such that

$$(a_\varepsilon^{-1}\mathbf{u}_h, \mathbf{v}) - \sum_{i=1}^n (p_h, \nabla\cdot\mathbf{v})_{\Omega_i} = 0 \qquad \forall\, \mathbf{v} \in \mathbf{V}_{h,0}, \tag{4.2}$$

$$\sum_{i=1}^n (\nabla\cdot\mathbf{u}_h, w)_{\Omega_i} = (f, w) \quad \forall\, w \in W_h. \tag{4.3}$$

Subtracting (4.2)–(4.3) from (2.1)–(2.2), we obtain equations for the error (recalling $p = \lambda$ on Γ)

$$(a_\varepsilon^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{v})_\Omega - \sum_{i=1}^n \big[(p - p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle p, \mathbf{v}\cdot\nu\rangle_{\Gamma_i}\big] = 0 \quad \forall\, \mathbf{v} \in \mathbf{V}_{h,0}, \tag{4.4}$$

$$\sum_{i=1}^n (\nabla\cdot(\mathbf{u} - \mathbf{u}_h), w)_{\Omega_i} = 0 \quad \forall\, w \in W_h. \tag{4.5}$$

The non-conforming error term $\langle p, \mathbf{v}\cdot\nu\rangle_{\Gamma_i}$ arises because although p is continuous, it is not *weakly* continuous. However, $\mathbf{v}$ is in the weakly continuous space, so

$$\langle p, \mathbf{v}\cdot\nu\rangle_{\Gamma_i} = \langle p - \mu, \mathbf{v}\cdot\nu\rangle_{\Gamma_i} \quad \forall\, \mu \in M_H, \tag{4.6}$$

leads to coarse H-level approximation error. We next use results from the homogenization theory heuristically as a guide to improve the approximation of p in M_H.

4.2. Formal first and second order approximations. From Theorem 3.1, we should expect that although the solution p of (2.1)–(2.3) is not smooth, it is a fixed operator of a smooth function p_0. Thus we should approximate

$$\begin{aligned}\lambda(x) = p(x) \approx p_\varepsilon^1(x) &= \big(1 + \varepsilon\boldsymbol{\omega}(x, x/\varepsilon)\cdot\nabla\big)p_0(x)\\ &\approx \big(1 + \varepsilon\boldsymbol{\omega}(x, x/\varepsilon)\cdot\nabla\big)q(x),\end{aligned} \tag{4.7}$$

where $q(x)$ is a piecewise polynomial.

Since we may not in general have a local period Y for $a_\varepsilon(x)$, we may also have no cell problem (3.5) defining $\varepsilon\boldsymbol{\omega}(x, x/\varepsilon)$. We approximate the local microstructure near each Γ_{ij} on $\overline{\Omega_i \cup \Omega_j}$ by finding the periodic solution to

$$-\nabla\cdot\big[a_\varepsilon(x)\big(\nabla\omega_k^{\Gamma_{ij}}(x, y) + \mathbf{e}_k\big)\big] = 0 \quad \text{in } \overline{\Omega_i \cup \Omega_j},\ k = 1, \dots, d. \tag{4.8}$$

Let Γ_{ij}^* to be an extension of Γ_{ij} in the normal direction into $\overline{\Omega_i \cup \Omega_j}$, and let $\mathbb{P}_{m-1}(\mathcal{T}_{H,ij}^*)$ to be the piecewise (continuous or discontinuous) polynomials of degree

$m-1$ defined over the interface mesh $\mathcal{T}_{H,ij}$ and extended in the normal direction of the same degree. Then we define [**AX12**]

$$M_H = \Big\{\lambda \in L^2(\Gamma) : \lambda\big|_e = \big(1+\boldsymbol{\omega}^{\Gamma_{ij}}\cdot\nabla\big)q\big|_e,\ q\in\mathbb{P}_{m-1}(\mathcal{T}^*_{H,ij}),\ e\in\mathcal{T}_{H,ij}\Big\},$$

wherein the extended polynomials were restricted back to Γ.

In a two dimensional example, suppose we use only a single finite element over each interface Γ_{ij}. We linearly map an interface Γ_{ij} and its neighboring strip in both normal directions to a master rectangle $[-\eta,\eta]\times[0,H]$, where $x=0$ gives the restriction to Γ_{ij}. If we choose q to be a linear polynomial as in [**AX12**],

$$q(x,y) = a+bx+cy,$$

then

$$\begin{aligned} p(x) &\approx (1+\boldsymbol{\omega}^{\Gamma_{ij}}(x,y)\cdot\nabla)(a+bx+cy)\\ &= a+b\big[x+\omega_1^{\Gamma_{ij}}(x,y)\big]+c\big[y+\omega_2^{\Gamma_{ij}}(x,y)\big], \end{aligned}$$

and the formally first order mortar approximation on $\{0\}\times[0,H]$ is

$$\lambda_H(y) = a+b\,\omega_1^{\Gamma_{ij}}(0,y)+c\big[y+\omega_2^{\Gamma_{ij}}(0,y)\big].$$

Similarly, we could choose q to be a quadratic polynomial,

$$q(x,y) = a+bx+cy+dx^2+exy+fy^2,$$

and then we have the formally second order mortar approximation

$$\begin{aligned} \lambda_H(y) = a+b\,\omega_1^{\Gamma_{ij}}(0,y)+c\big[y+\omega_2^{\Gamma_{ij}}(0,y)\big]\\ +e\,y\,\omega_1^{\Gamma_{ij}}(0,y)+f\big[y^2+2y\,\omega_2^{\Gamma_{ij}}(0,y)\big]. \end{aligned}$$

Notice that we have three, not two, degrees of freedom for first order approximation on the one dimensional interface. Similarly, we have five, not three, degrees of freedom for second order approximation.

4.3. Implicitly defined multiscale finite elements. Define the bi-linear form $d_H : M_H\times M_H\to\mathbb{R}$ and linear functional $g_H : M_H\to\mathbb{R}$ by

$$d_H(\lambda,\mu) = -\sum_{i=1}^n \langle \mathbf{u}_h^*(\lambda)\cdot\nu_i,\mu\rangle_{\Gamma_i},$$

$$g_H(\mu) = \sum_{i=1}^n \langle \bar{\mathbf{u}}_h\cdot\nu_i,\mu\rangle_{\Gamma_i},$$

where $(\mathbf{u}_h^*(\lambda),p_h^*(\lambda))\in\mathbf{V}_h\times W_h$ solves (wherein λ is given, $f=0$)

$$\begin{aligned} (a_\varepsilon^{-1}\mathbf{u}_h^*(\lambda),\mathbf{v})_{\Omega_i}-(p_h^*(\lambda),\nabla\cdot\mathbf{v})_{\Omega_i} &= -\langle\lambda,\mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} && \forall\,\mathbf{v}\in\mathbf{V}_{h,i},\\ (\nabla\cdot\mathbf{u}_h^*(\lambda),w)_{\Omega_i} &= 0 && \forall\,w\in W_{h,i}, \end{aligned}$$

and $(\bar{\mathbf{u}}_h,\bar{p}_h)\in\mathbf{V}_h\times W_h$ solves (wherein $\lambda=0$, f is given)

$$\begin{aligned} (a_\varepsilon^{-1}\bar{\mathbf{u}}_h,\mathbf{v})_{\Omega_i}-(\bar{p}_h,\nabla\cdot\mathbf{v})_{\Omega_i} &= 0 && \forall\,\mathbf{v}\in\mathbf{V}_{h,i},\\ (\nabla\cdot\bar{\mathbf{u}}_h,w)_{\Omega_i} &= (f,w)_{\Omega_i} && \forall\,w\in W_{h,i}. \end{aligned}$$

The equivalent coarse variational problem is [**GW88**]: Find $\lambda_H\in M_H$ such that

$$d_H(\lambda_H,\mu) = g_h(\mu)\quad\forall\,\mu\in M_H. \tag{4.9}$$

Let $\{\mu_\ell\}$ be a basis for $M_H = \text{span}\{\mu_\ell\}$. Define $\mathbf{v}_\ell = \mathbf{u}_h^*(\mu_\ell)$, $w_\ell = p_h^*(\mu_\ell)$, and

$$N_{h,H} = \text{span}\big\{(\mathbf{v}_\ell, w_\ell)\big\} = \text{span}\big\{\big(\mathbf{u}_h^*(\mu_\ell), p_h^*(\mu_\ell)\big)\big\} \subset \mathbf{V}_h \times W_h.$$

It is easy to show that $d_H(\lambda,\mu) = (a_\varepsilon^{-1}\mathbf{u}_h^*(\lambda), \mathbf{u}_h^*(\mu))$. Then we can reformulate the coarse variational problem (4.9) as: Find $(\mathbf{u}_h, p_h) \in N_{h,H} + (\bar{\mathbf{u}}_h, \bar{p}_h)$ such that

$$(a_\varepsilon^{-1}\mathbf{u}_h, \mathbf{v}) = (f, w) \qquad \forall (\mathbf{v}, w) \in N_{h,H}.$$

The discrete space $N_{h,H}$ incorporates fine-scale information, and is thus a multiscale finite element space [**EH09**]. In this sense, the multiscale mortar method can be viewed as a multiscale finite element method, with the subdomains being coarse elements [**Arb11b**, **AX12**]. This is a very unusual multiscale mixed finite element, in that each basis function has weakly zero flow, but not zero flow, on all of its element edges, and pressures and velocities are intrinsically coupled together.

4.4. A-priori error estimates. Under certain technical conditions [**AX12**], we have the following bounds on the velocity and pressure errors.

THEOREM 4.1. *Suppose the two-scale separation assumption* (3.1) *holds. Then there exists a constant C, independent of h, H, L (the maximal diameter of the subdomains), and ε, such that for $1 \le r \le k$, $0 \le s \le \ell$, and $0 < t \le m$,*

$$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0 \le C\|f\|_s h^s, \tag{4.10}$$

$$\|p - p_h\|_0 \le \|\hat{p} - p_h\|_0 + C\|p\|_s h^s, \tag{4.11}$$

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_0 + \|\hat{p}-p_h\|_0 \le C\big\{&\big[\|\mathbf{u}\|_r + \|\mathbf{u}\|_{r+1/2}((H+\varepsilon)/L)^{1/2}\big]h^r \\ &+ H^{t-1}(H+\varepsilon)(Lh)^{-1/2}\|p_0\|_{t+1/2} + \varepsilon\|p_0\|_2 + \varepsilon^{1/2}\|\nabla p_0\|_{0,\infty}\big\},\end{aligned} \tag{4.12}$$

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_0 + \|\hat{p}-p_h\|_0 \le C\big\{&\|\mathbf{u}\|_r((H+\varepsilon)/L)^{1/2}h^{r-1/2} \\ &+ H^{t-1}(H+\varepsilon)(Lh)^{-1/2}\|p_0\|_{t+1/2} + \varepsilon\|p_0\|_2 + \varepsilon^{1/2}\|\nabla p_0\|_{0,\infty}\big\}.\end{aligned} \tag{4.13}$$

Here we can see that the error is small whenever $h < \varepsilon < H \le L$.

5. Numerical results

In the previous section, we noted theoretically that our new mortar method works well under the two-scale separation assumption (3.1). Here we first verify the theory with a test using a streaked permeability field with a locally periodic microstructure. We then present numerical results for permeability fields a_ε that do not possess an obvious two-scale structure.

For simplicity, all of our numerical tests are conducted on rectangular grids with rectangular subdomains. The subdomain problems are approximated in $\mathbf{V}_{h,i}\times W_{h,i}$, which we take to be the lowest order Raviart-Thomas space RT0 [**RT77**], for which $k = \ell = 1$. We take one element per interface $\Gamma_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$ (so $H = L$). Four mortar spaces M_H are tested for each example, they are:

(1) P1M, linear polynomial mortars with 2 degrees of freedom per edge;
(2) P2M, quadratic polynomial mortars with 3 degrees of freedom per edge;
(3) MS1, formally first order multiscale mortars based on homogenization with 3 degrees of freedom per edge;
(4) MS2, formally second order multiscale mortars based on homogenization with 5 degrees of freedom per edge.

All test examples use a rectangular domain with a quarter five-spot well pattern for f, that is, injection in the lower left corner and extraction (or production) in the upper right corner. For each interface Γ_{ij}, the cell problem is defined over the region $\mathbf{Y} = \Omega_i \cup \Gamma_{ij} \cup \Omega_j$, as described above and in [**Arb11a**, **AX12**]. We compare our numerical results with the reference fine-scale RT0 solution, since the true solution is not known analytically.

5.1. A streaked permeability. We first test a streaked permeability field as shown in Fig. 1, where a locally periodic structure can be observed. Also notice that this is a strongly anisotropic permeability field. From Table 1, we observe that by increasing the order of the polynomial space from P1M to P2M, the performance does not improve at all. On the other hand, we obtain a immediate improvement when turning to the homogenization side by using MS1, and we can further improve the performance by applying MS2. A similar performance can be found in the multiscale finite element method [**Arb11a**], where anisotropic problems are better handled with a homogenization-based element.

TABLE 1. Streaked permeability. Relative errors in the pressure and velocity for the mortar spaces relative to the 20×20 reference RT0 solution, using a 2×2 coarse grid and 10×10 subgrid.

	Pressure error		Velocity error	
Method	ℓ^2	ℓ^∞	ℓ^2	ℓ^∞
P1M	0.5964	0.1741	0.6357	0.7889
P2M	0.5615	0.1588	0.6656	1.0684
MS1	0.1755	0.0792	0.4095	0.3595
MS2	0.0305	0.0195	0.1491	0.2264

5.2. A moderately heterogeneous permeability. The permeability field of our second example is moderately heterogeneous, being locally isotropic and geostatistically mildly correlated. It is depicted in Fig. 2 on a logarithmic scale ranging over four orders of magnitude. The domain is 40 meters square and the fine grid is uniformly 40×40.

From Fig. 2, we can see that generally the MS1 and MS2 velocities are closer to the fine-scale RT0 velocity than P1M and P2M. Recall that P2M and MS1 use mortar spaces with the same number of degrees of freedom. Therefore, we can reduce the relative ℓ^2-velocity error from 25.6% to 10.7% without increasing the complexity of the interface problems by using our new mortar space. Moreover, although two more degrees of freedom per edge are introduced in MS2, we can get a 0.15% ℓ^2-pressure error and a 4.1% ℓ^2-velocity error in return, which is quite accurate.

5.3. A channelized permeability from SPE10. Finally, we test the 80th layer of the Tenth Society of Petroleum Engineers Comparative Solution Project (SPE10) [**Chr01**], which is shown in Fig. 3. Obviously, the permeability does not fulfill the two-scale separation assumption (3.1). In Fig. 3, one can see that the velocities of P1M and P2M exhibit extreme inaccuracies that resemble points of singularity, making these methods perform poorly (see Table 3). On the other hand, MS1 and MS2 control the ℓ^2-velocity error within a reasonable range. Although

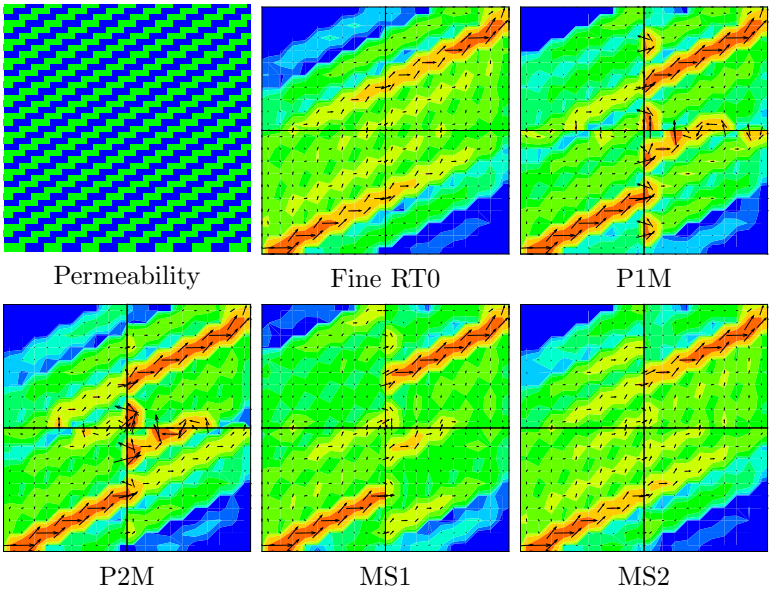

FIGURE 1. Streaked permeability. The permeability, on a 20×20 grid, has only two values 1 and 200. Velocities are computed by RT0 on the fine grid, and by the mortar methods on a 2×2 coarse grid of subdomains with a 10×10 subgrid. Color depicts speed on a log scale from 0.001 (blue) to 1 (red). Arrows show velocities.

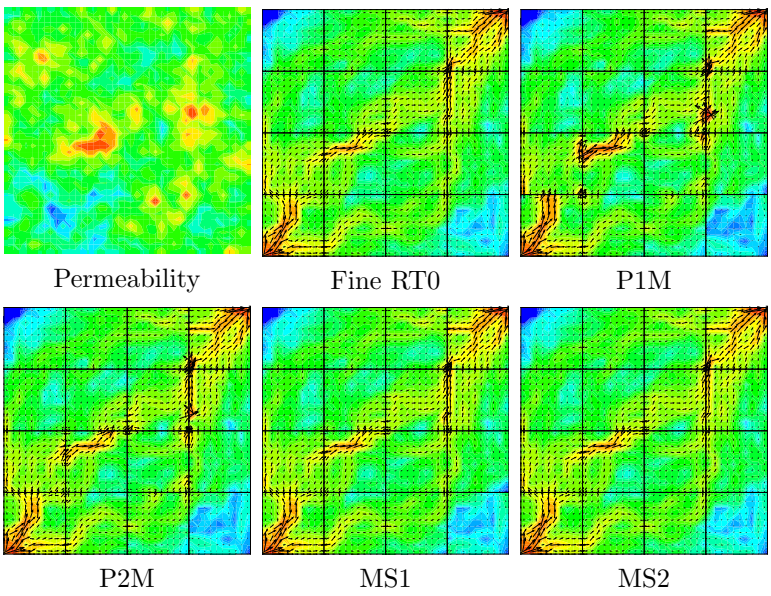

FIGURE 2. Moderate heterogeneity. The 40×40 permeability is shown on a log scale from about 0.32 to 3200 millidarcy. Velocities are computed by RT0 on the fine grid and by mortars on a 4×4 grid of subdomains with a 10×10 subgrid. Color depicts speed, on a log scale from 0.6 to 0.0006. Arrows show velocities.

TABLE 2. Moderate heterogeneity test. Relative errors in the pressure and velocity for the mortar spaces relative to the 40×40 reference RT0 solution, using a 4×4 coarse grid and 10×10 subgrid.

	Pressure error		Velocity error	
Method	ℓ^2	ℓ^∞	ℓ^2	ℓ^∞
P1M	0.1989	0.1452	0.4157	0.8042
P2M	0.0431	0.0353	0.2564	0.5267
MS1	0.0111	0.0137	0.1072	0.1432
MS2	0.0015	0.0020	0.0410	0.0688

TABLE 3. SPE10-80 test. Relative errors in the pressure and velocity for the mortar spaces relative to the 60×220 reference RT0 solution, using a 3×11 coarse grid and 20×20 subgrid.

	Pressure error		Velocity error	
Method	ℓ^2	ℓ^∞	ℓ^2	ℓ^∞
P1M	0.0846	0.0452	0.6584	2.0868
P2M	0.0437	0.0204	0.5287	2.0156
MS1	0.0127	0.0090	0.1459	0.4860
MS2	0.0093	0.0066	0.1143	0.5985

MS2 gives a better ℓ^2-error than MS1, it is a marginal improvement; moreover, MS2 shows somewhat greater ℓ^∞-velocity error.

6. Conclusions

Nonoverlapping domain decomposition using mortar spaces to couple together the subdomains is an efficient way to numerically solve second order elliptic problems (1.1)–(1.3) in parallel. Heterogeneity in the elliptic coefficient a_ε can be handled within the mortar space by using ideas from homogenization theory. Local cell problem can be solved, which then allow implicit reconstruction of the pressure p in terms of a fixed operator and a smooth homogenized function p_0 through (3.6). Approximation of p_0 by a polynomial, as in (4.7), gives a multiscale mortar space with only a few degrees of freedom per subdomain interface, resulting in computational efficiency in parallel. In the two-scale separation case, we have good approximation properties (Theorem 4.1).

In two space dimensions, formally first order mortar spaces were constructed in [**AX12**] and reviewed here, and the formally second order mortar spaces were constructed explicitly here. We can generally expect more accurate numerical results when using homogenization based mortar spaces in domain decomposition, even without a two-scale microstructure. Usually the formally second order mortar approximation based on homogenization (MS2) give a better result than the first order mortars (MS1), and both generally perform much better the simple polynomial mortar space approximations (P1M and P2M).

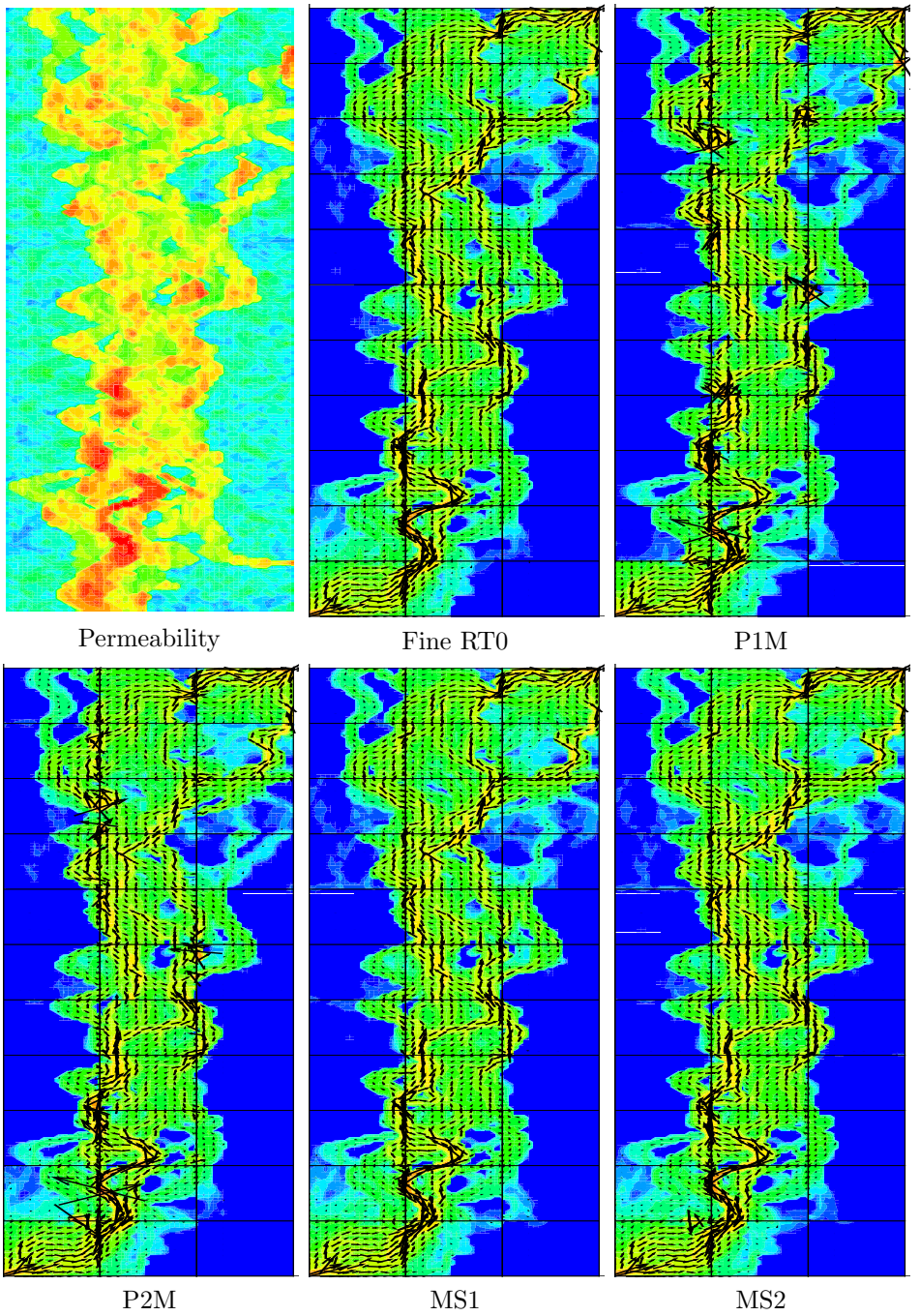

FIGURE 3. SPE10-80 test. The permeability is given on a 60×220 grid plotted using a log scale from 1.9e-11 (red) to 1.0e-18 (blue) m^2. The fine-scale RT0 speed and velocity are plotted on a log scale from 1.3 (red) to 1.0e-3 (blue). The mortar results use a 3×11 coarse grid with a 20×20 subgrid.

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