MODELING OF NATURALLY FRACTURED RESERVOIRS BY FORMAL HOMOGENIZATION TECHNIQUES*,¶

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ABSTRACT. Double porosity models are derived for various types of flow in naturally fractured reservoirs. A single component in a single phase and two-component miscible and immiscible flows are treated. These models are derived by homogenizing the appropriate equations describing flow in a highly discontinuous single porosity reservoir. The mathematical theory of homogenization is used only in its formal sense.

Keywords. porous medium, double porosity, fractured reservoir, homogenization

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§1. Introduction.

Petroleum reservoirs are often found in nature with many interconnected fracture planes throughout their extent in a fairly regular geometric pattern. These fractures, though small, have a profound effect on oil recovery processes since fluids can flow much more readily in the fractures than in the porous rock (the *matrix*). A concept of double porosity has been used to model such reservoirs (see, e.g., [3], [4], [7], [21], [30]). The reservoir is assumed to possess *two* porous structures. The matrix consists of essentially disjoint blocks that are ordinary porous media. Connecting these blocks is a fracture system that is itself a porous medium, though not an ordinary one.

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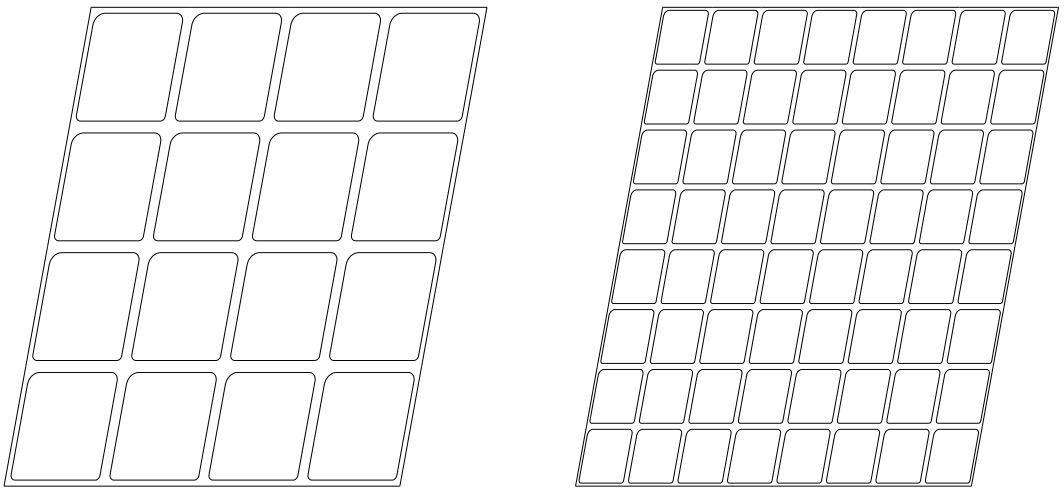
To describe the flow of fluid in such a reservoir, sets of equations are needed for the fracture system and for the matrix. Normally both sets of equations are assumed to be based on Darcy's law. The interactions between the two systems are crucial, and various proposals have been presented for describing these interactions (see, e.g., [2], [3], [4], [6], [7], [11], [12], [13], [17], [19], [18], [24], [25], [26], [28], [30]). The purpose of this paper is to derive unique forms of double porosity models for various flows in naturally fractured reservoirs by means of the mathematical theory of homogenization [8], [22]. This theory has been successful in deriving Darcy's law from general principles [27], but it has not been applied to fractured porous media to date (except in [5] and [16]).

Our reservoir $\Omega \subset \mathbb{R}^3$ is a two-connected domain with a periodic structure. We shall scale this periodic structure by a parameter $\epsilon > 0$ which will represent the size of the matrix blocks. The standard ($\epsilon = 1$) period is a cell $\mathcal{Q}$ consisting of a matrix block domain $\mathcal{Q}_m$ completely surrounded by a connected fracture domain $\mathcal{Q}_f$. The ϵ-reservoir consists of copies of $\epsilon\mathcal{Q}$ covering Ω. Each ϵ-cell is adjacent to its neighbors but not overlapping any of them (see Figure 1.1). There are four regions that are of special interest to us, namely, the external boundary of the reservoir $\partial\Omega$, the fractures Ω_f^ϵ, the matrix Ω_m^ϵ, and the matrix-fracture interface Γ^ϵ. These last three can be described as

$$\Omega_f^\epsilon = \Omega \cap \bigcup_{\xi\in\mathcal{A}} \epsilon(\mathcal{Q}_f + \partial\mathcal{Q} + \xi), \quad \Omega_m^\epsilon = \Omega \cap \bigcup_{\xi\in\mathcal{A}} \epsilon(\mathcal{Q}_m + \xi),$$
$$\text{and} \quad \Gamma^\epsilon = \Omega \cap \bigcup_{\xi\in\mathcal{A}} \epsilon(\partial\mathcal{Q}_m + \xi), \tag{1.1}$$

where $\mathcal{A}$ is an appropriate infinite lattice. We shall let the time interval of interest be denoted by $J = (0, T]$.



$\epsilon = 1$ $\qquad$ $\epsilon = \frac{1}{2}$

Figure 1.1

In each type of flow to be considered, our starting point is a microscopic model which consists of the proper equations describing Darcy flow in a porous medium

(see, e.g., [10], [20], [23]) posed over all of Ω. The porosity, permeability, etc., will be discontinuous across Γ^ϵ. The equations on Ω_m^ϵ will be scaled by appropriate powers of ϵ to conserve flow in some sense; consequently, the porous medium equations do not degenerate as ϵ tends to zero and the form of the matrix equations of the microscopic model and of the homogenized macroscopic model coincide. This technique has been used in several other papers [5], [14], [15], [29]. It allows us to derive a *double* porosity model rather than simply a modified single porosity model [1], [9] which would be inadequate to describe the complexity of flow in a naturally fractured reservoir.

We shall derive our models only in a formal sense [22]. We assume that our solutions behave in space as if they were functions of a macroscopic variable $x \in \Omega$ and, at each such x, a microscopic variable $y \in \mathcal{Q}_r$, $r = m$ or f. The macroscopic and microscopic scales are related by ϵ; that is, up to a translation,

$$y \sim \epsilon^{-1} x, \tag{1.2}$$

so

$$\nabla \sim \epsilon^{-1} \nabla_y + \nabla_x, \tag{1.3}$$

where ∇_z is the gradient with respect to the z-variables. The solutions are then assumed to have the asymptotic form

$$\psi^\epsilon(x,t) \sim \psi^0(x,y,t) + \epsilon\psi^1(x,y,t) + \cdots = \sum_{k=0}^{\infty} \epsilon^k \psi^k(x,y,t), \tag{1.4}$$

where the ψ^k are periodic in y for y in $\mathcal{Q}_f$. Next, functions of ψ^ϵ can be expanded by Taylor's Theorem as

$$\varphi(\psi^\epsilon) = \varphi(\psi^0) + \varphi'(\psi^0)(\psi^\epsilon - \psi^0) + \cdots \equiv \varphi(\psi^0) + \epsilon\varphi^1 + \epsilon^2\varphi^2 + \cdots. \tag{1.5}$$

Finally, if $\varphi^\epsilon(x) = \varphi(\epsilon^{-1}x)$ is periodic of period $\epsilon\mathcal{Q}$, then clearly

$$\varphi^\epsilon(x) \sim \varphi(y). \tag{1.6}$$

The assumption of the asymptotic relations (1.2)–(1.6) in our microscopic models will give separate equations in the coefficients of (1.4) for each power of ϵ. These will lead to a set of equations for the leading term alone. This, of course, is our limit or macroscopic model.

Since we are primarily interested in the internal flow characteristics, and especially in the matrix-fracture interaction, we shall generally ignore the outer boundary $\partial\Omega$ below. We shall also assume that all of our quantities are smooth enough for the formal manipulations that we perform.

In the rest of the paper we consider three kinds of flows. In §2, the flow of a single component in a single phase is considered. This is done very briefly, as the formal results of this section have been shown rigorously in a related paper by the

current authors [5]. However, this simple case is highly illustrative of the kinds of results that we obtain for the other, two-component flows. In §3 we treat the flow of two completely miscible fluids in a naturally fractured reservoir. In §4 we treat the flow of two immiscible fluids (each component in a separate phase). Finally, in the last section we modify the geometry of our reservoir somewhat to derive a second model for immiscible flow. This second model is more sensitive to the possible gravitational segregation of the fluids.

We close this section with a comment on the notation. Generally speaking, fracture quantities are denoted by upper case letters and are defined on all of Ω (and used on Ω_f^ϵ), while the corresponding matrix quantities are denoted by the corresponding lower case letters and are defined on Ω_m^ϵ.

§2. A single component in a single phase. The simplest type of flow in a naturally fractured reservoir is that of a single component in a single phase. This simple situation is ideal for describing the kind of results that we obtain since the features of this model are in some sense common to all of the models contained herein.

For convenience of exposition, the definitions of our symbols are collected in Table 2.1. The matrix blocks have been assumed to possess identical properties, so ϕ and k are periodic of period $\epsilon\mathcal{Q}_m$ as noted. (More generally these may vary slowly over the reservoir: $\phi^\epsilon(x) = \phi(x, \epsilon^{-1}x)$, $k^\epsilon(x) = k(x, \epsilon^{-1}x)$.) In this section only, we shall ignore gravity for the sake of simplicity. The microscopic model appears below in (2.1)–(2.6). Here and throughout the subscript t denotes partial differentiation in time and ν denotes the outer unit normal vector with respect to the matrix domain.

Table 2.1 — Symbols for single component, single phase flow.

Symbol		Meaning
Fractures	Matrix	
$\rho^\epsilon(x,t)$	$\sigma^\epsilon(x,t)$	mass density
$\Phi^*(x)$	$\phi^\epsilon(x) = \phi(\frac{x}{\epsilon})$ [periodic]	porosity, bounded below
$K^*(x)$	$k^\epsilon(x) = k(\frac{x}{\epsilon})$	permeability, bounded below
	[periodic tensor]	(positive definite)
	μ	viscosity
	c	constant of compressibility
	$f(x,t)$	external source/sink
	$\rho_{init}(x)$	initial density

The flow in the fracture domain is controlled by

$$\Phi^* \rho_t^\epsilon - \nabla \cdot \left(\frac{K^*}{\mu c} \nabla \rho^\epsilon \right) = f \quad \text{in} \quad \Omega_f^\epsilon \times J, \tag{2.1}$$

$$\frac{K^*}{\mu c} \nabla \rho^\epsilon \cdot \nu = \epsilon^2 \frac{k^\epsilon}{\mu c} \nabla \sigma^\epsilon \cdot \nu \quad \text{on} \quad \Gamma^\epsilon \times J, \tag{2.2}$$

$$\rho^\epsilon = \rho_{init} \quad \text{on} \quad \Omega_f^\epsilon \times \{0\}. \tag{2.3}$$

The flow in the matrix domain is controlled by

$$\phi^\epsilon \sigma_t^\epsilon - \epsilon^2 \nabla \cdot \left(\frac{k^\epsilon}{\mu c} \nabla \sigma^\epsilon \right) = f \quad \text{in} \quad \Omega_m^\epsilon \times J, \tag{2.4}$$

$$\sigma^\epsilon = \rho^\epsilon \quad \text{on} \quad \Gamma^\epsilon \times J, \tag{2.5}$$

$$\sigma^\epsilon = \rho_{init} \quad \text{on} \quad \Omega_m^\epsilon \times \{0\}. \tag{2.6}$$

The equations (2.1) and (2.4) represent conservation of mass combined with Darcy's law and the equations of state

$$\rho \, d\rho = c \, dp, \qquad \sigma \, d\sigma = c \, dp,$$

where p is the pressure. The appropriate conditions on the matrix-fracture interface Γ^ϵ are (2.2), conservation of mass flux between the two regions, and (2.5), continuity of pressure (actually density).

The ϵ^2 in (2.2) and in (2.4) properly scales the matrix flux term. This can be argued as follows. The critical process in any naturally fractured porous medium is the transfer of fluid between the matrix and fractures. We cannot expect to have a reasonable homogenized model unless we preserve this transfer in some sense as ϵ tends to zero. For $\epsilon = 1$, the net fluid transferred is

$$\int_{\partial \Omega_m} \frac{k}{\mu c} \nabla \sigma \cdot \nu \, da(x);$$

this quantity should not degenerate or blow up as ϵ tends to zero. This implies that one of the coefficients, say permeability, must be scaled. Let us denote the unscaled permeability by $k^{*,\epsilon}$. Then a simple change of variables $y = \epsilon^{-1}(x - \xi)$ shows the following, where the sum is taken over the ϵ-cells contained in Ω at the lattice points $\xi \in \epsilon \mathcal{A}$ and where $|\cdot|$ denotes the volume of the set:

$$\begin{aligned}
\int_{\partial \Omega_m^\epsilon} \frac{k^{*,\epsilon}}{\mu c} \nabla \sigma^\epsilon \cdot \nu \, da(x) &= \sum_{\epsilon\text{-cells}} \int_{\partial(\epsilon \mathcal{Q}_m + \xi)} \frac{k^{*,\epsilon}}{\mu c} \nabla \sigma^\epsilon \cdot \nu \, da(x) \\
&= \sum_{\epsilon\text{-cells}} \int_{\partial \mathcal{Q}_m} \frac{k^*}{\mu c} \epsilon^{-1} \nabla \sigma^\epsilon \cdot \nu \, \epsilon^2 da(y) \\
&= \sum_{\epsilon\text{-cells}} \int_{\partial \mathcal{Q}_m} \epsilon^{-2} \frac{k^*}{\mu c} \nabla \sigma^\epsilon \cdot \nu \, da(y) \, |\epsilon \mathcal{Q}_m|.
\end{aligned}$$

As ϵ tends to zero, the sum becomes a Riemann integral, and so indeed $k^* = \epsilon^2 k$ as asserted.

Note that Φ^* and K^* represent the porosity and permeability of the fractures themselves; consequently, $\Phi^* \approx 1$ and K^* is very large. These change during the homogenization process. This feature is common to all of the models to be derived in this paper. The macroscopic fracture *system* porosity $\Phi(x)$ is always defined to be

$$\Phi = \frac{|\mathcal{Q}_f|}{|\mathcal{Q}|}\Phi^*. \tag{2.7}$$

We need here and later the auxiliary functions $\omega_j(y)$ which are $\mathcal{Q}$-periodic solutions to the problems

$$\nabla_y^2 \omega_j = 0 \quad \text{in} \quad \mathcal{Q}_f, \tag{2.8i}$$

$$\nabla_y \omega_j \cdot \nu = -e_j \cdot \nu \quad \text{on} \quad \partial\mathcal{Q}_m, \tag{2.8ii}$$

where e_j is the unit vector in the jth direction. These functions arise naturally as in (3.15) and (4.15) below. The macroscopic fracture system permeability $K(x)$ is the tensor

$$K = K^* \left[\frac{|\mathcal{Q}_f|}{|\mathcal{Q}|} I + \overline{(\partial_i \omega_j)} \right], \tag{2.9}$$

where I is the identity tensor, $\partial_j = \dfrac{\partial}{\partial x_j}$, the double subscripted symbol $\overline{(\partial_i \omega_j)}$ is the tensor whose (i,j)-component is as written, and the overbar denotes the local average

$$\overline{\varphi} = \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}_f} \varphi(y)\, dy. \tag{2.10}$$

The microscopic model can be homogenized in the formal sense described in the introduction using some of the techniques of the following sections. We omit the details here and refer to [5] for a rigorous proof of the convergence of the homogenization process. The macroscopic model that results is described below in (2.11)–(2.15). The fracture system density ρ^0 is independent of y and satisfies the equations, posed over all of Ω,

$$\Phi \rho_t^0 + \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}_m} \phi(y) \sigma_t^0 \, dy - \nabla \cdot \left(\frac{K}{\mu c} \nabla \rho^0 \right) = f \quad \text{in} \quad \Omega \times J, \tag{2.11}$$

$$\rho^0 = \rho_{init} \quad \text{on} \quad \Omega \times \{0\}. \tag{2.12}$$

As the size ϵ of the matrix blocks tends to zero, a block $\mathcal{Q}_m$ is associated with each point $x \in \Omega$ of the reservoir. The equations on this block are given in scale

invariant form in terms of the variable $y \in \mathcal{Q}_m$ as

$$\phi(y)\sigma_t^0(x,y,t) - \nabla_y \cdot \left(\frac{k(y)}{\mu c} \nabla_y \sigma^0(x,y,t) \right) = f(x,t) \quad \text{in } \Omega \times \mathcal{Q}_m \times J, \tag{2.13}$$

$$\sigma^0(x,y,t) = \rho^0(x,t) \quad \text{on } \Omega \times \partial\mathcal{Q}_m \times J, \tag{2.14}$$

$$\sigma^0(x,y,0) = \rho_{init}(x) \quad \text{on } \Omega \times \mathcal{Q}_m. \tag{2.15}$$

The key features of this model are the form of the matrix source term

$$\frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}_m} \phi(y)\sigma_t^0(x,y,t)\, dy$$

of (2.11), which represents a source/sink of fluid from the fracture system (i.e. it takes into account the fluid that flows from the fractures into the matrix) and the condition (2.14) on the boundary of each matrix block. This boundary condition is a constant for $y \in \partial\mathcal{Q}_m$ for the block $\mathcal{Q}_m$ associated with x. These two features are reflected in the models to follow. This model is essentially identical to that derived from physical arguments and appearing in [4]; it is also the model employed in [13].

We have presented the results of this section only in a formal framework; that is, the convergence of the solutions $(\rho^\epsilon, \sigma^\epsilon)$ of (2.1)–(2.6) to the solution (ρ^0, σ^0) of (2.11)–(2.15) is not proved by the techniques used herein. As mentioned above, however, this convergence has been shown to hold in a rigorous sense [5] (with gravitational effects included in the equations). If we add an outer no-flow boundary condition

$$\frac{K^*}{\mu c} \nabla \rho^\epsilon \cdot \nu_\Omega = 0 \quad \text{on} \quad \partial\Omega \times J$$

to (2.1)–(2.3), then in the sense of weak L^2-convergence, $(\rho^\epsilon, \sigma^\epsilon)$ converges to (ρ^0, σ^0) as ϵ tends to zero, where to (2.11)–(2.12) we must add the outer boundary condition

$$\frac{K}{\mu c} \nabla \rho^0 \cdot \nu_\Omega = 0 \quad \text{on} \quad \partial\Omega \times J.$$

§**3. Incompressible miscible displacement.** In this section we consider the displacement of one fluid by another that is completely miscible with it. This model applies also to a solute being transported with a fluid. We assume that both fluids are incompressible and that they mix without changing volume.

Our list of symbols is given in Table 3.1. Note especially the parameters that depend on the solution itself. Again, the quantities that describe the properties of the matrix are periodic of period $\epsilon\mathcal{Q}_m$ as indicated. We have tacitly assumed that only molecular diffusion occurs in the fractures on the microscopic scale; that is, D^* is independent of $U^{*,\epsilon}$.

The microscopic model describing such miscible displacement in a naturally fractured reservoir consists of the set of equations (3.1)–(3.12) below. Pressure equations can be written as follows, with Darcy's law being expressed by (3.1) and (3.4)

Table 3.1 — Symbols for miscible flow.

Symbol		Meaning
Fractures	Matrix	
$U^{*,\epsilon}(x,t)$	$u^\epsilon(x,t)$	Darcy velocity
$P^\epsilon(x,t)$	$p^\epsilon(x,t)$	pressure
$C^\epsilon(x,t)$	$c^\epsilon(x,t)$	concentration (of one of the two components)
$\Phi^*(x)$	$\phi^\epsilon(x) = \phi(\frac{x}{\epsilon})$ [periodic]	porosity, bounded below
$K^*(x)$	$k^\epsilon(x) = k(\frac{x}{\epsilon})$ [periodic tensor]	permeability, bounded below (positive definite)
$D^*(x)$	$d^\epsilon(x,u^\epsilon) = d(\frac{x}{\epsilon},u^\epsilon)$ [periodic (in x) tensor]	diffusion coefficient, bounded below (positive definite)
$\rho(C)$	$\rho(c)$	density of the mixture
$\mu(C)$	$\mu(c)$	viscosity of the mixture
g		gravitational constant vector
$f(x,t)$		external source/sink
$f_+(x,t) = \max(f,0)$		external source (injection)
$f_-(x,t) = \min(f,0)$		external sink (production)
$C_{inj}(x,t)$		concentration of injected fluid
$C_{init}(x)$		initial concentration
$G(C) = \rho(C)g$	$G(c) = \rho(c)g$	
$\Lambda^*(x,C^\epsilon) = \dfrac{K^*(x)}{\mu(C^\epsilon)}$	$\lambda^\epsilon(x,c^\epsilon) = \lambda(\frac{x}{\epsilon},c^\epsilon) = \dfrac{k(\frac{x}{\epsilon})}{\mu(c^\epsilon)}$	

and conservation of mass by (3.2) and (3.5):

$$U^{*,\epsilon} = -\Lambda^*(C^\epsilon)[\nabla P^\epsilon + G(C^\epsilon)] \quad \text{in} \quad \Omega_f^\epsilon \times J, \tag{3.1}$$

$$\nabla \cdot U^{*,\epsilon} = f \quad \text{in} \quad \Omega_f^\epsilon \times J, \tag{3.2}$$

$$U^{*,\epsilon} \cdot \nu = \epsilon u^\epsilon \cdot \nu \quad \text{on} \quad \Gamma^\epsilon \times J, \tag{3.3}$$

$$u^\epsilon = -\lambda^\epsilon(c^\epsilon)[\epsilon\nabla p^\epsilon + G(c^\epsilon)] \quad \text{in} \quad \Omega_m^\epsilon \times J, \tag{3.4}$$

$$\epsilon\nabla \cdot u^\epsilon = f \quad \text{in} \quad \Omega_m^\epsilon \times J, \tag{3.5}$$

$$p^\epsilon = P^\epsilon \quad \text{on} \quad \Gamma^\epsilon \times J. \tag{3.6}$$

Here, (3.3) represents conservation of total mass flux between the fractures and the matrix, while (3.6) simply represents continuity of pressure. Secondly, we have concentration equations, (3.7) and (3.10), which conserve the mass of each component,

as well as the component mass flux equation (3.8) between the two domains:

$$\Phi^* C^\epsilon_t - \nabla \cdot [D^* \nabla C^\epsilon - U^{*,\epsilon} C^\epsilon] = C_{inj}\ f_+ + C^\epsilon f_- \qquad \text{in} \quad \Omega^\epsilon_f \times J, \tag{3.7}$$

$$D^* \nabla C^\epsilon \cdot \nu = \epsilon^2 d^\epsilon(u^\epsilon) \nabla c^\epsilon \cdot \nu \qquad \text{on} \quad \Gamma^\epsilon \times J, \tag{3.8}$$

$$C^\epsilon = C_{init} \qquad \text{on} \quad \Omega^\epsilon_f \times \{0\}, \tag{3.9}$$

$$\phi^\epsilon c^\epsilon_t - \epsilon \nabla \cdot [\epsilon d^\epsilon(u^\epsilon) \nabla c^\epsilon - u^\epsilon c^\epsilon] = C_{inj}\ f_+ + c^\epsilon f_- \qquad \text{in} \quad \Omega^\epsilon_m \times J, \tag{3.10}$$

$$c^\epsilon = C^\epsilon \qquad \text{on} \quad \Gamma^\epsilon \times J, \tag{3.11}$$

$$c^\epsilon = C_{init} \qquad \text{on} \quad \Omega^\epsilon_m \times \{0\}. \tag{3.12}$$

Equation (3.8) has been simplified by making use of (3.3) and (3.11); (3.11) just represents continuity of component concentration across Γ^ϵ. Note that the injected fluid is specified, while the production fluid splits between the two components according to the concentration. The flux terms have been scaled as described in the previous section to keep the matrix-fracture fluid transfer terms from blowing up as ϵ tends to zero. In this case, u^ϵ is defined by (3.4), so in fact the matrix permeability and the matrix diffusion coefficient have been scaled by ϵ^2, while gravity in the matrix has been compensated by an ϵ^{-1}. This preserves the original form of the total mass flux. If we do not compensate gravity, it will drop out of the matrix mass flux terms in the macroscopic model; there is little difference in practice.

In a heuristic sense, we now adopt the assumptions (1.2)–(1.6). Specifically, (1.4) gives us the expansions

$$U^{*,\epsilon} \sim \sum_{k=0}^{\infty} \epsilon^k U^{*,k}(x,y,t), \quad P^\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k P^k(x,y,t), \quad C^\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k C^k(x,y,t)$$

for $(x,y,t) \in \Omega \times \mathcal{Q}_f \times J$ (periodic in y) and

$$u^\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k u^k(x,y,t), \quad p^\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k p^k(x,y,t), \quad c^\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k c^k(x,y,t)$$

for $(x,y,t) \in \Omega \times \mathcal{Q}_m \times J$; moreover, it follows from (1.5) and (1.6) that

$$\phi^\epsilon(x) \sim \phi(y), \qquad k^\epsilon(x) \sim k(y),$$

$$\begin{aligned} \Lambda^*(C^\epsilon) &\sim \Lambda^*(C^0) + \epsilon \Lambda^1 + O(\epsilon^2), \\ \lambda^\epsilon(x, c^\epsilon) &\sim \lambda(y, c^0) + O(\epsilon), \\ G(\xi^\epsilon) &\sim G(\xi^0) + O(\epsilon), \quad \xi = C \text{ or } c, \\ d^\epsilon(x, u^\epsilon) &\sim d(y, u^0) + O(\epsilon). \end{aligned}$$

Upon substitution of these formal expansions into (3.1)–(3.12) and upon equating coefficients of like powers of ϵ, we obtain the following relations. From (3.1), in $\Omega \times \mathcal{Q}_f \times J$, the ϵ^{-1} and ϵ^0 terms give

$$(3.1/\text{-}1) \qquad 0 = -\Lambda^*(C^0)\nabla_y P^0,$$

$$(3.1/0) \qquad U^{*,0} = -\Lambda^*(C^0)[\nabla_y P^1 + \nabla_x P^0 + G(C^0)] + \Lambda^1 \nabla_y P^0.$$

From (3.2), in $\Omega \times \mathcal{Q}_f \times J$, the ϵ^{-1} and ϵ^0 terms give

$$(3.2/\text{-}1) \qquad \nabla_y \cdot U^{*,0} = 0,$$

$$(3.2/0) \qquad \nabla_y \cdot U^{*,1} + \nabla_x \cdot U^{*,0} = f.$$

From (3.3), on $\Omega \times \partial\mathcal{Q}_m \times J$, the ϵ^0 and ϵ^1 terms give

$$(3.3/0) \qquad U^{*,0} \cdot \nu = 0,$$

$$(3.3/1) \qquad U^{*,1} \cdot \nu = u^0 \cdot \nu.$$

From (3.4)–(3.6), the ϵ^0 terms give

$$(3.4/0) \qquad u^0 = -\lambda(y, c^0)[\nabla_y p^0 + G(c^0)] \quad \text{in} \quad \Omega \times \mathcal{Q}_m \times J,$$

$$(3.5/0) \qquad \nabla_y \cdot u^0 = f \quad \text{in} \quad \Omega \times \mathcal{Q}_m \times J,$$

$$(3.6/0) \qquad p^0 = P^0 \quad \text{on} \quad \Omega \times \partial\mathcal{Q}_m \times J.$$

From (3.7), in $\Omega \times \mathcal{Q}_f \times J$, the ϵ^{-2}, ϵ^{-1}, and ϵ^0 terms give

$$(3.7/\text{-}2) \qquad -\nabla_y \cdot (D^* \nabla_y C^0) = 0,$$

$$(3.7/\text{-}1) \qquad -\nabla_y \cdot [D^*(\nabla_y C^1 + \nabla_x C^0) - U^{*,0} C^0] - \nabla_x \cdot [D^* \nabla_y C^0] = 0,$$

$$(3.7/0) \qquad \Phi^* C^0_t - \nabla_y \cdot [D^*(\nabla_y C^2 + \nabla_x C^1) - U^{*,1} C^0 - U^{*,0} C^1]$$
$$- \nabla_x \cdot [D^*(\nabla_y C^1 + \nabla_x C^0) - U^{*,0} C^0] = C_{inj}\ f_+ + C^0\ f_-.$$

From (3.8), on $\Omega \times \partial\mathcal{Q}_m \times J$, the ϵ^{-1}, ϵ^0, and ϵ^1 terms give

$$(3.8/\text{-}1) \qquad D^* \nabla_y C^0 \cdot \nu = 0,$$

$$(3.8/0) \qquad D^*(\nabla_y C^1 + \nabla_x C^0) \cdot \nu = 0,$$

$$(3.8/1) \qquad D^*(\nabla_y C^2 + \nabla_x C^1) \cdot \nu = d(y, u^0) \nabla_y c^0 \cdot \nu.$$

From (3.9), on $\Omega \times \mathcal{Q}_f \times \{0\}$, the ϵ^0 term gives

$$(3.9/0) \qquad C^0 = C_{init}.$$

Finally, from (3.10)–(3.12), the ϵ^0 terms give

$$(3.10/0)\quad \phi(y)c^0_t - \nabla_y \cdot [d(y,u^0)\nabla_y c^0 - u^0 c^0] = C_{inj} f_+ + c^0 f_- \text{ in } \Omega \times \mathcal{Q}_m \times J,$$
$$(3.11/0)\quad c^0 = C^0 \qquad \text{on } \Omega \times \partial\mathcal{Q}_m \times J,$$
$$(3.12/0)\quad c^0 = C_{init} \qquad \text{on } \Omega \times \mathcal{Q}_m \times \{0\}.$$

We shall now analyze these asymptotic relations. First note that (3.7/-2) and (3.8/-1) imply that

$$(3.13)\qquad C^0 = C^0(x,t) \quad \text{only},$$

and (3.1/-1) says that

$$(3.14)\qquad P^0 = P^0(x,t) \quad \text{only}.$$

Next we relate P^1 to P^0 and C^1 to C^0. To this end, note that P^1 satisfies (3.1/0), (3.2/-1), and (3.3/0), which can be rewritten as

$$\begin{aligned} &\nabla_y \cdot [\Lambda^*(C^0)\nabla_y P^1] = 0 && \text{in} \quad \Omega \times \mathcal{Q}_f \times J, \\ &\Lambda^*(C^0)\nabla_y P^1 \cdot \nu = -\Lambda^*(C^0)[\nabla_x P^0 + G(C^0)] \cdot \nu && \text{on} \quad \Omega \times \partial\mathcal{Q}_m \times J, \end{aligned}$$

where we have used (3.13) and (3.14) several times. Since $\Lambda^*(C^0)$ is independent of y, this can be solved easily with the aid of the functions $\omega_j(y)$ introduced above in (2.8):

$$(3.15)\qquad P^1(x,y,t) = \sum_{j=1}^{3} \omega_j(y)\big[\partial_j P^0(x) + G_j\big(C^0(x)\big)\big] + \alpha(x,t)$$

for some function α since $\mathcal{Q}_f$ is connected and P^1 is (assumed) periodic in y across $\partial\mathcal{Q}$. Similarly, from (3.7/-1), (3.2/-1), and (3.8/-1),

$$(3.16)\qquad C^1(x,y,t) = \sum_{j=1}^{3} \omega_j(y)\,\partial_j C^0(x) + \beta(x,t)$$

for some β.

We shall now derive a macroscopic Darcy law for the fracture flow. Using (3.15), (3.1/0) becomes

$$\begin{aligned} (3.17)\qquad U^{*,0} &= -\Lambda^*(C^0)\Big\{\sum_{j=1}^{3} \nabla_y \omega_j [\partial_j P^0 + G_j(C^0)] + \nabla_x P^0 + G(C^0)\Big\} \\ &= -\Lambda^*(C^0)\,[I + (\partial_i \omega_j)]\,[\nabla_x P^0 + G(C^0)]. \end{aligned}$$

This relation involves the microscopic variable y. We locally average this equation (recall (2.10)) to obtain the macroscopic equation. Hence let

$$U^0 = \overline{U^{*,0}} \tag{3.18}$$

and define $K(x)$ by (2.9) above; we obtain

$$U^0 = -\frac{K}{\mu(C^0)}\left[\nabla P^0 - G(C^0)\right] \quad \text{in} \quad \Omega \times J. \tag{3.19}$$

Note that this is Darcy's law, since U^0 is a macroscopic volumetric flow rate.

We shall now verify that the total mass is conserved. Using (3.2/0), we have that

$$\nabla_x \cdot U^0 = \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} \nabla_x \cdot U^{*,0}\, dy = \frac{|\mathcal{Q}_f|}{|\mathcal{Q}|} f - \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} \nabla_y \cdot U^{*,1}\, dy.$$

The last term above can be related to a matrix quantity by using the divergence theorem and the flux condition (3.3/1). With also periodicity and (3.5/0), we see that

$$\begin{aligned} -\int_{\mathcal{Q}_f} \nabla_y \cdot U^{*,1}\, dy &= \int_{\partial \mathcal{Q}_f} U^{*,1} \cdot \nu_{\mathcal{Q}_f}\, da(y) = -\int_{\partial \mathcal{Q}_m} U^{*,1} \cdot \nu\, da(y) \\ &= -\int_{\partial \mathcal{Q}_m} u^0 \cdot \nu\, da(y) = -\int_{\mathcal{Q}_m} \nabla_y \cdot u^0\, dy = -|\mathcal{Q}_m| f. \end{aligned} \tag{3.20}$$

Hence, indeed,

$$\nabla \cdot U^0 = f \quad \text{in} \quad \Omega \times J. \tag{3.21}$$

It remains to show that the homogenized fracture concentration C^0 satisfies an equation expressing conservation of the mass of the component it represents. The local average of (3.7/0) is

$$\begin{aligned} &\Phi C^0_t - \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} \nabla_y \cdot [D^*(\nabla_y C^2 + \nabla_x C^1) - U^{*,1} C^0 - U^{*,0} C^1]\, dy \\ &\quad - \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} \nabla_x \cdot [D^*(\nabla_y C^1 + \nabla_x C^0) - U^{*,0} C^0]\, dy = \frac{|\mathcal{Q}_f|}{|\mathcal{Q}|}(C_{inj} f_+ + C^0 f_-), \end{aligned} \tag{3.22}$$

where Φ was defined previously in (2.7). The first integral can be rewritten by an argument similar to that given in (3.20) above. With (3.8/1), (3.3/0), (3.3/1), (3.11/0), and then (3.10/0), we derive the matrix source term as follows:

$$\begin{aligned} &\int_{\mathcal{Q}_f} \nabla_y \cdot [D^*(\nabla_y C^2 + \nabla_x C^1) - U^{*,1} C^0 - U^{*,0} C^1]\, dy \\ &\quad = \int_{\partial \mathcal{Q}_f} [D^*(\nabla_y C^2 + \nabla_x C^1) - U^{*,1} C^0 - U^{*,0} C^1] \cdot \nu_{\mathcal{Q}_f}\, da(y) \\ &\quad = -\int_{\partial \mathcal{Q}_m} [d(y, u^0)\nabla_y c^0 - u^0 c^0] \cdot \nu\, da(y) \\ &\quad = -\int_{\mathcal{Q}_m} \nabla_y \cdot [d(y, u^0)\nabla_y c^0 - u^0 c^0]\, dy \\ &\quad = \int_{\mathcal{Q}_m} (-\phi(y) c^0_t + c^0 f_-)\, dy + |\mathcal{Q}_m| C_{inj} f_+. \end{aligned} \tag{3.23}$$

The first part of the second integral of (3.22) can be rewritten with (3.16) as

$$\int_{\mathcal{Q}_f} \nabla_x \cdot (D^* \nabla_y C^1)\, dy = \int_{\mathcal{Q}_f} \nabla_x \cdot \left[D^* \nabla_y \left(\sum_{j=1}^{3} \omega_j \partial_j C^0 \right) \right] dy$$
$$= |\mathcal{Q}| \nabla_x \cdot \left[\sum_{j=1}^{3} D^* \overline{\nabla_y \omega_j}\, \partial_j C^0 \right].$$

Hence we obtain our desired equation:

$$\Phi C_t^0 + \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}_m} \phi(y) c_t^0\, dy - \nabla \cdot (D \nabla C^0 - U^0 C^0)$$
$$= C_{inj} f_+ + \frac{1}{|\mathcal{Q}|} \left\{ |\mathcal{Q}_f| C^0 + \int_{\mathcal{Q}_m} c^0\, dy \right\} f_- \quad \text{in} \quad \Omega \times J, \tag{3.24}$$

where we have set

$$D(x) = D^*(x) \left[\frac{|\mathcal{Q}_f|}{|\mathcal{Q}|} I + \overline{(\partial_i \omega_j)} \right]. \tag{3.25}$$

The formal derivation is now complete. The macroscopic model consists of several equations. The fracture system follows a set of equations that are independent of y and consist of a pressure equation (3.19) and (3.21) as well as a concentration equation (3.24), plus the initial condition (3.9/0). To these, external boundary conditions must of course be added. At each point $x \in \Omega$ we have a matrix block, the flow in which is described by the pressure equation (3.4/0)–(3.6/0) and the concentration equation (3.10/0)–(3.12/0).

This is a "zeroth order" model because the size of the matrix blocks strictly tended to zero. It possesses the features described in the last section, the matrix source term (the second term of (3.24)) and the constant in y-space boundary conditions (3.6/0) and (3.11/0). As noted in [2], such a zeroth order model is inadequate because no *viscous* displacement may occur between the matrix and fracture systems. There are no pressure or concentration gradients over the surfaces of the blocks, so fluid only transfers between the two systems by diffusion. We have, however, discovered the leading terms of what should be the correct model. The model of [2] incorporates these leading terms and certain higher order ones to allow for viscous displacement.

§4. Incompressible, immiscible displacement in a reservoir with moderate-sized matrix blocks. We now consider the displacement of one fluid by another (for convenience, oil by water) when the two fluids are incompressible and do not mix. Hence, two phases exist in the reservoir. The displacing fluid is the wetting phase.

Our symbols are given in Table 4.1. Generally one assumes that $f_{o,inj} \equiv \max(f_o, 0) = 0$ and that production splits according to the mobility: $f_{\theta,prod} \equiv$

Table 4.1 — Symbols for immiscible flow.

Symbol		Meaning
Fractures	Matrix	
θ		phase: o (oil) or w (water)
$P_\theta^\epsilon(x,t)$	$p_\theta^\epsilon(x,t)$	pressure of θ phase
$S_\theta^\epsilon(x,t)$	$s_\theta^\epsilon(x,t)$	saturation of θ phase
$S^\epsilon(x,t)$	$s^\epsilon(x,t)$	saturation of o phase, say
$\Phi^*(x)$	$\phi^\epsilon(x) = \phi(\frac{x}{\epsilon})$ [periodic]	porosity, bounded below
$K^*(x)$	$k^\epsilon(x) = k(\frac{x}{\epsilon})$	permeability, bounded below
	[periodic tensor]	(positive definite)
$K_{r\theta}(S^\epsilon)$	$k_{r\theta}(s^\epsilon)$	relative θ permeability
$P_c(S^\epsilon)$	$p_c(s^\epsilon)$	capillary pressure
$S_{r\theta}$	$s_{r\theta}$	residual θ saturation
	ρ_θ	density of θ phase
	μ_θ	viscosity of θ phase
	g	gravitational constant vector
	$f_\theta(x,t)$	external θ source/sink
	$S_{\theta,init}(x)$	initial θ saturation
	$G_\theta = \rho_\theta g$	
$\Lambda_\theta^*(x,S^\epsilon)$ $= \frac{K^*(x)K_{r\theta}(S^\epsilon)}{\mu_\theta}$	$\lambda_\theta^\epsilon(x,s^\epsilon) = \lambda_\theta(\frac{x}{\epsilon},s^\epsilon)$ $= \frac{k(\frac{x}{\epsilon})k_{r\theta}(s^\epsilon)}{\mu_\theta}$	

$\min(f_\theta, 0) = \frac{\Lambda_\theta}{\Lambda_o+\Lambda_w} f_{prod}$, where f_{prod} is the total production rate. We do not require this, however. It is important to note that the relative permeability of the θ phase tends to zero as the θ saturation tends to its residual value. Also, the capillary function tends to zero as the water saturation becomes residual, and it blows up at the residual oil saturation.

The microscopic model is presented below. Conservation of mass of each component (phase) combined with Darcy's law gives (4.1) and (4.4), while (4.2) represents conservation of mass flux between the two regions and (4.5) represents continuity

of the phase pressures across Γ^ϵ. For $\theta = o$ or w,

$$\Phi^* S^\epsilon_{\theta,t} - \nabla \cdot [\Lambda^*_\theta(S^\epsilon)(\nabla P^\epsilon_\theta + G_\theta)] = f_\theta(S^\epsilon) \quad \text{in} \quad \Omega^\epsilon_f \times J, \tag{4.1}$$

$$\Lambda^*_\theta(S^\epsilon)(\nabla P^\epsilon_\theta + G_\theta) \cdot \nu = \epsilon \lambda^\epsilon_\theta(s^\epsilon)(\epsilon \nabla p^\epsilon_\theta + G_\theta) \cdot \nu \quad \text{on} \quad \Gamma^\epsilon \times J, \tag{4.2}$$

$$S^\epsilon_\theta = S_{\theta,init} \quad \text{on} \quad \Omega^\epsilon_f \times \{0\}, \tag{4.3}$$

$$\phi^\epsilon s^\epsilon_{\theta,t} - \epsilon \nabla \cdot [\lambda^\epsilon_\theta(s^\epsilon)(\epsilon \nabla p^\epsilon_\theta + G_\theta)] = f_\theta(s^\epsilon) \quad \text{in} \quad \Omega^\epsilon_m \times J, \tag{4.4}$$

$$p^\epsilon_\theta = P^\epsilon_\theta \quad \text{on} \quad \Gamma^\epsilon \times J, \tag{4.5}$$

$$s^\epsilon_\theta = S_{\theta,init} \quad \text{on} \quad \Omega^\epsilon_m \times \{0\}. \tag{4.6}$$

To these equations we have the usual properties that the fluid fills the volume,

$$S_o + S_w = 1 \quad \text{in} \quad \Omega^\epsilon_f \times J, \tag{4.7}$$

$$s_o + s_w = 1 \quad \text{in} \quad \Omega^\epsilon_m \times J, \tag{4.8}$$

and that the phase pressures are related by a strictly monotone function of the saturation,

$$P^\epsilon_o - P^\epsilon_w = P_c(S^\epsilon) \quad \text{in} \quad \Omega^\epsilon_f \times J, \tag{4.9}$$

$$p^\epsilon_o - p^\epsilon_w = p_c(s^\epsilon) \quad \text{in} \quad \Omega^\epsilon_m \times J. \tag{4.10}$$

The matrix permeabilities have been scaled by ϵ^2 and gravity has been compensated. Again, if gravity is not compensated, it drops out of the matrix equations of the macroscopic model.

In the heuristic sense of (1.2)–(1.6), we expand the solution S^ϵ_θ, s^ϵ_θ, P^ϵ_θ, and p^ϵ_θ and we expand the functions Λ^*_θ, λ^ϵ_θ, P_c, p_c, and f_θ. We are led to the following formal relations. From (4.1), in $\Omega \times \mathcal{Q}_f \times J$, the ϵ^{-2}, ϵ^{-1}, and ϵ^0 terms yield

$$-\nabla_y \cdot [\Lambda^*_\theta(S^0)\nabla_y P^0_\theta] = 0, \tag{4.1/-2}$$

$$\begin{aligned} &-\nabla_y \cdot [\Lambda^*_\theta(S^0)(\nabla_y P^1_\theta + \nabla_x P^0_\theta + G_\theta) + \Lambda^1_\theta \nabla_y P^0_\theta] \\ &\quad -\nabla_x \cdot [\Lambda^*_\theta(S^0)\nabla_y P^0_\theta] = 0, \end{aligned} \tag{4.1/-1}$$

$$\begin{aligned} &\Phi^* S^0_{\theta,t} - \nabla_y \cdot [\Lambda^*_\theta(S^0)(\nabla_y P^2_\theta + \nabla_x P^1_\theta) \\ &\qquad + \Lambda^1_\theta(\nabla_y P^1_\theta + \nabla_x P^0_\theta + G_\theta) + \Lambda^2_\theta \nabla_y P^0_\theta] \\ &\quad -\nabla_x \cdot [\Lambda^*_\theta(S^0)(\nabla_y P^1_\theta + \nabla_x P^0_\theta + G_\theta) + \Lambda^1_\theta(S^0)\nabla_y P^0_\theta] = f_\theta(S^0). \end{aligned} \tag{4.1/0}$$

From (4.2), on $\Omega \times \partial\mathcal{Q}_m \times J$, the ϵ^{-1}, ϵ^0, and ϵ^1 terms give

$$\Lambda^*_\theta(S^0)\nabla_y P^0_\theta \cdot \nu = 0, \tag{4.2/-1}$$

$$\Lambda^*_\theta(S^0)(\nabla_y P^1_\theta + \nabla_x P^0_\theta + G_\theta) \cdot \nu + \Lambda^1_\theta \nabla_y P^0_\theta \cdot \nu = 0, \tag{4.2/0}$$

$$\begin{aligned} &\Lambda^*_\theta(S^0)(\nabla_y P^2_\theta + \nabla_x P^1_\theta) \cdot \nu + \Lambda^1_\theta(\nabla_y P^1_\theta + \nabla_x P^0_\theta + G_\theta) \cdot \nu + \Lambda^2_\theta \nabla_y P^0_\theta \cdot \nu \\ &\qquad = \lambda_\theta(y, s^0)(\nabla_y p^0_\theta + G_\theta) \cdot \nu. \end{aligned} \tag{4.2/1}$$

Finally, we only need the ϵ^0 terms from the other equations (4.3)–(4.10):

$$(4.3/0) \qquad S^0_\theta = S_{\theta,init} \qquad \text{on} \quad \Omega \times \mathcal{Q}_f \times \{0\},$$

$$(4.4/0) \qquad \phi(y) s^0_{\theta,t} - \nabla_y \cdot [\lambda_\theta(y, s^0)(\nabla_y p^0_\theta + G_\theta)] = f_\theta(s^0) \quad \text{in} \quad \Omega \times \mathcal{Q}_m \times J,$$

$$(4.5/0) \qquad p^0_\theta = P^0_\theta \qquad \text{on} \quad \Omega \times \partial\mathcal{Q}_m \times J,$$

$$(4.6/0) \qquad s^0_\theta = S_{\theta,init} \qquad \text{on} \quad \Omega \times \mathcal{Q}_m \times \{0\},$$

$$(4.7/0) \qquad S^0_o + S^0_w = 1 \qquad \text{in} \quad \Omega \times \mathcal{Q}_f \times J,$$

$$(4.8/0) \qquad s^0_o + s^0_w = 1 \qquad \text{in} \quad \Omega \times \mathcal{Q}_m \times J,$$

$$(4.9/0) \qquad P^0_o - P^0_w = P_c(S^0) \qquad \text{in} \quad \Omega \times \mathcal{Q}_f \times J,$$

$$(4.10/0) \qquad p^0_o - p^0_w = p_c(s^0) \qquad \text{in} \quad \Omega \times \mathcal{Q}_m \times J.$$

The analysis of these asymptotic relations proceeds in a manner very similar to that given in §3 for the miscible case even though the equations are quite different. First, we would like to conclude that the S^0_θ and P^0_θ are independent of y. Since the Λ^*_θ are degenerate, however, we must be careful. We can easily conclude from an energy estimate of (4.1/-2) and (4.2/-1) that

$$(4.11) \qquad \Lambda^*_\theta (S^0)^{\frac{1}{2}} \nabla_y P^0_\theta = 0.$$

Now (4.9/0) implies that

$$(4.12) \qquad \nabla_y P^0_o - \nabla_y P^0_w = P_c'(S^0) \nabla_y S^0,$$

so $\Lambda^*_o(S^0)^{\frac{1}{2}} \Lambda^*_w(S^0)^{\frac{1}{2}} P_c'(S^0) \nabla_y S^0 = 0$. This shows that S^0 is piecewise constant in y (assuming at most the values $S_{r,o}$, $1 - S_{r,w}$, and possibly one value between these two). Assuming that S^0 is continuous, or at least that $\nabla_y S^0$ is, say, in $L^2(\Omega)$, we are led to conclude with (4.7/0) that

$$(4.13) \qquad S^0_\theta = S^0_\theta(x,t) \quad \text{only.}$$

Now (4.11) shows that at least one of the P^0_θ is independent of y, and then (4.12) shows this of them both:

$$(4.14) \qquad P^0_\theta = P^0_\theta(x,t) \quad \text{only.}$$

Next, (4.1/-1) and (4.2/0) allow us to write P^1_θ in terms of P^0_θ as

$$(4.15) \qquad P^1_\theta = \sum_{j=1}^{3} \omega_j(y) \left[\partial_j P^0_\theta(x,t) + G_{\theta,j}\right] + \alpha_\theta(x,t),$$

at least if neither S^0_θ is residual, where the ω_j have been defined above in (2.8) and the α_θ are independent of y.

Finally, the local average of (4.1/0) leads us to a macroscopic equation for each θ that represents mass conservation and Darcy's law. This average is

(4.16)
$$\Phi S^0_{\theta,t} - \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} \nabla_y \cdot [\Lambda^*_\theta(S^0)(\nabla_y P^2_\theta + \nabla_x P^1_\theta) + \Lambda^1_\theta(\nabla_y P^1_\theta + \nabla_x P^0_\theta + G_\theta)]\, dy$$
$$- \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_f} \nabla_x \cdot [\Lambda^*_\theta(S^0)(\nabla_y P^1_\theta + \nabla_x P^0_\theta + G_\theta)]\, dy = \frac{|\mathcal{Q}_f|}{|\mathcal{Q}|} f_\theta(S^0),$$

where (2.7) defines $\Phi(x)$. As in (3.23), we rewrite the first integral above as

$$\int_{\mathcal{Q}_m} [-\phi(y) s^0_{\theta,t} + f_\theta(s^0)]\, dy,$$

using (4.2/1) and (4.4/0). Now with (4.15) being used to rewrite the P^1_θ term in the second integral, (4.16) becomes our macroscopic equation

(4.17)
$$\Phi S^0_{\theta,t} + \frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}_m} \phi(y) s^0_{\theta,t}\, dy - \nabla \cdot [\Lambda_\theta(S^0)(\nabla P^0_\theta + G_\theta)]$$
$$= \frac{1}{|\mathcal{Q}|}\left\{ |\mathcal{Q}_f| f_\theta(S^0) + \int_{\mathcal{Q}_m} f_\theta(s^0)\, dy \right\} \quad \text{in} \quad \Omega \times J.$$

Here we define K by (2.9) above and then set

(4.18)
$$\Lambda_\theta(S^0) = \frac{K K_{r,\theta}(S^0)}{\mu_\theta}.$$

In summary, the fracture system, defined on $\Omega \times J$, consists of (4.17), (4.3/0), (4.7/0), and (4.9/0). For each $x \in \Omega$, we have a matrix block, the flow in which is described by (4.4/0)–(4.6/0), (4.8/0), and (4.10/0). Again our two features are reflected in the model by (4.13) and (4.14). As a consequence, there is no viscous displacement in this model as well. However, when capillary imbibition is the dominant force, our "zeroth order" model is reasonable. It is essentially the model described and numerically and computationally analyzed in [6], [11], [12], and [13].

The constancy in y-space of the boundary conditions (4.5/0) does not allow the two phases to segregate along $\partial\mathcal{Q}_m$ due to gravitational effects. Such segregation is believed to have an important effect on the matrix-fracture interaction, at least when the height of the matrix blocks is relatively large. Hence, the present model is appropriate only for reservoirs with moderate-sized matrix blocks. In the next section we consider the case of large matrix blocks.

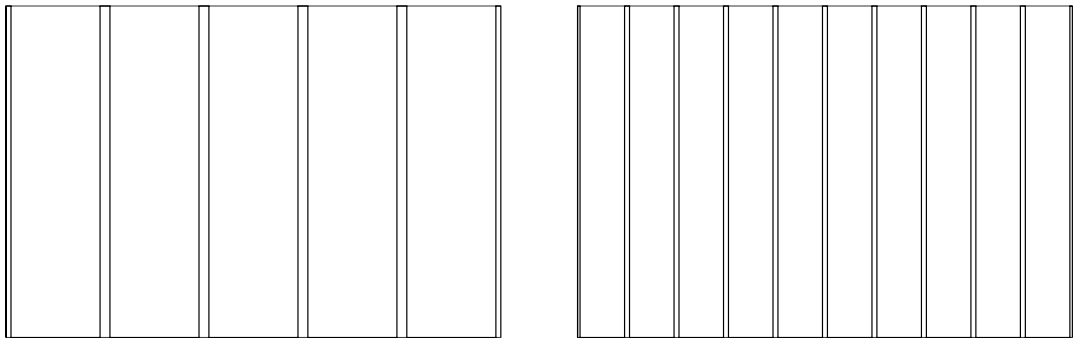
§5. Incompressible, immiscible displacement in a reservoir with large matrix blocks. To this point we have homogenized the reservoir by letting the diameter of the matrix blocks tend to zero. As mentioned at the end of the last

section, in that case gravity has no vertical distance on $\partial\mathcal{Q}_m$ to act over as far as the matrix-fracture interaction is concerned. To allow for a gravitational segregation of the fluids on the surfaces of the blocks, we shall now homogenize the reservoir only in the horizontal directions; that is, the horizontal cross-section of the blocks will tend to zero area but the height will remain fixed.

We restrict the geometry (see Figure 5.1) by requiring that $\Omega = \tilde{\Omega} \times \mathcal{I}$ and $\mathcal{Q} = \tilde{\mathcal{Q}} \times \mathcal{I}$, where $\tilde{\Omega}$, $\tilde{\mathcal{Q}} \subset \mathbb{R}^2$ are the horizontal cross sections and $\mathcal{I} \subset \mathbb{R}^1$ is the height. Also, for $\tilde{\mathcal{Q}}_m$ strictly contained in $\tilde{\mathcal{Q}}$ and surrounded by $\tilde{\mathcal{Q}}_f$ we have $\mathcal{Q}_m = \tilde{\mathcal{Q}}_m \times \mathcal{I}$ and $\mathcal{Q}_f = \tilde{\mathcal{Q}}_f \times \mathcal{I}$. Note that we have allowed no horizontal fractures. The scaled reservoir is then as described in (1.1) provided only that we interpret each nonsuperscripted ϵ as a tensor:

$$\epsilon \quad \longmapsto \quad \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.1}$$

and that we drop the top and bottom faces of Γ^ϵ.



$\epsilon = 1$ $\qquad\qquad$ $\epsilon = \frac{1}{2}$

Figure 5.1

Our symbols are defined in Table 4.1, where the periodic matrix quantities have the tensor ϵ of (5.1) and are periodic in the horizontal direction only; i.e., of period $\epsilon\tilde{\mathcal{Q}}$. Note that $G_\theta = (0, 0, G_{\theta,3})$ in our coordinate system. The microscopic model consists of the equations of the last section, namely (4.1)–(4.10), with the interpretation (5.1) in (4.2) and (4.4). (To these we should add an external no-flow condition on $\tilde{\Omega} \times \partial\mathcal{I}$.)

To homogenize this reservoir, we assume that $y \in \tilde{\mathcal{Q}}$ is such that, up to a translation,

$$y \sim \epsilon^{-1}\tilde{x}, \qquad \tilde{x} = (x_1, x_2). \tag{5.2}$$

Also, we set

$$\nabla_y = \left(\frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial y_2}, \quad 0 \right). \tag{5.3}$$

Then our heuristic relations (1.3)–(1.6) hold, where the periodicity of (1.4), when $(y, x_3) \in \mathcal{Q}_f$, and (1.6) are assumed only in the horizontal directions.

Fracture quantities are defined on $\Omega \times \tilde{\mathcal{Q}}_f = \{(x, y) : x \in \Omega \text{ and } (y, x_3) \in \mathcal{Q}_f\}$, which is conveniently written as $\Omega \times \mathcal{Q}_f$ by abusing the notation somewhat. Similar statements hold for matrix quantities and for quantities on the matrix-fracture interface.

Now, the formal relations that we obtain include those of the previous section (4.1/$\cdot$)–(4.3/0) and (4.5/0)–(4.10/0). The relations (4.2/$\cdot$) hold since $\nu_3 = 0$ on Γ^ϵ. From (4.4), in $\Omega \times \mathcal{Q}_m \times J$, the ϵ^0 term gives

$$\phi(\hat{y}) s^0_{\theta,t} - \nabla_{\hat{y}} \cdot [\lambda_\theta(\hat{y}, s^0) \nabla_{\hat{y}}\, p^0_\theta + G_\theta] = f_\theta(s^0), \tag{5.4}$$

where $\hat{y} = (y, x_3) \in \mathcal{Q}_m \subset \mathbb{R}^3$.

As far as the analysis of these relations goes, we first obtain (4.13) and (4.14) from (4.1/-2) and (4.2/-1) as in the previous section. Define $\tilde{\omega}_j,\ j = 1, 2$, periodic across $\partial\tilde{\mathcal{Q}}$ by

$$\nabla^2_y \tilde{\omega}_j = 0 \qquad \text{in} \quad \tilde{\mathcal{Q}}_f, \tag{5.5i}$$

$$\nabla_y \tilde{\omega}_j \cdot \nu = -e_j \cdot \nu \quad \text{on} \quad \partial\tilde{\mathcal{Q}}_m. \tag{5.5ii}$$

If we now set $\omega = (\tilde{\omega}_1, \tilde{\omega}_2, 0)$, we obtain (4.15) as before. Finally, we take our local average of (4.1/0) in two dimensions only; that is, if we replace $\mathcal{Q}$ by $\tilde{\mathcal{Q}}$ and $\mathcal{Q}_f$ by $\tilde{\mathcal{Q}}_f$, we obtain (4.16), where (2.7) continues to define Φ. Manipulation from this leads to our final equation:

$$\begin{aligned} &\Phi\, S^0_{\theta,t} + \frac{1}{|\tilde{\mathcal{Q}}|} \int_{\tilde{\mathcal{Q}}_m} \{\phi(\hat{y})\, s^0_{\theta,t} - \partial_3[\lambda_\theta(\hat{y}, s^0)(\partial_3 p^0_\theta + G_{\theta,3})]\}\, dy \\ &- \nabla \cdot [\Lambda_\theta(S^0)(\nabla P^0_\theta - G_\theta)] = \frac{1}{|\tilde{\mathcal{Q}}|} \left\{ |\tilde{\mathcal{Q}}_f|\, f_\theta(S^0) + \int_{\tilde{\mathcal{Q}}_m} f_\theta(s^0)\, dy \right\} \quad \text{in} \quad \Omega \times J, \end{aligned} \tag{5.6}$$

where Λ_θ is defined in (4.18) above and

$$K(x) = \frac{1}{|\tilde{\mathcal{Q}}|}\, K^*(x) \left[|\tilde{\mathcal{Q}}_f| I + \int_{\tilde{\mathcal{Q}}_m} (\partial_i\, \tilde{\omega}_j)\, dy \right], \tag{5.7}$$

where the final row and column of the tensor above defined by integrals of $\partial_i\, \tilde{\omega}_j$ are zero.

The fracture system of our final model is described by (5.6), (4.3/0), (4.7/0), and (4.9/0) on $\Omega \times J$. At each point $\tilde{x} \in \tilde{\Omega}$, we have an infinitely thin matrix block, the flow in which is governed by (5.4), (4.5/0)–(4.6/0), (4.8/0), and (4.10/0) on $(\hat{y}, t) \in \mathcal{Q}_m \times J$. The boundary condition (4.5/0) is constant in y-space, that is, in horizontal planes only; it varies in the vertical direction. The matrix source term produces fluid in horizontal planes. It has the added component

$$-\frac{1}{|\tilde{\mathcal{Q}}|} \int_{\tilde{\mathcal{Q}}_m} \partial_3[\lambda_\theta(\hat{y}, s^0)(\partial_3 p^0_\theta + G_{\theta,3})]\, dy$$

which accounts for vertical flow contained entirely within the matrix block. This model was described originally in [11].

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