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MIXED FINITE ELEMENT METHODS ON NON-MATCHING MULTIBLOCK GRIDS*

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Abstract. We consider mixed finite element methods for second order elliptic equations on non-matching multiblock grids. A mortar finite element space is introduced on the non-matching interfaces. We approximate in this mortar space the trace of the solution, and we impose weakly a continuity of flux condition. A standard mixed finite element method is used within the blocks. Optimal order convergence is shown for both the solution and its flux. Moreover, at certain discrete points, superconvergence is obtained for the solution, and also for the flux in special cases. Computational results using an efficient parallel domain decomposition algorithm are presented in confirmation of the theory.

Key words. Mixed finite element, mortar finite element, error estimates, superconvergence, multiblock, non-conforming grids

AMS subject classifications. 65N06, 65N12, 65N15, 65N22, 65N30

1. Introduction. Mixed finite element methods have become popular due to their local (mass) conservation property and good approximation of the flux variable. In many applications the complexity of the geometry or the behavior of the solution may warrant using a multiblock domain structure, wherein the domain is decomposed into non-overlapping blocks or subdomains with grids defined independently on each block. Typical examples include modeling faults and wells in subsurface applications. Faults are natural discontinuities in material properties. Locally refined grids are needed for accurate approximation of high gradients around wells.

In this work we consider second order linear elliptic equations that in porous medium applications model single phase Darcy flow. We solve for the pressure p and the velocity $\mathbf{u}$ satisfying

$$\mathbf{u} = -K\nabla p \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = f \quad \text{in } \Omega, \tag{1.2}$$

$$p = g \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^d$, $d = 2$ or 3, is a multiblock domain and K is a symmetric, uniformly positive definite tensor with $L^\infty(\Omega)$ components representing the permeability divided by the viscosity. The Dirichlet boundary conditions are considered merely for simplicity. To more clearly present our ideas, we also suppose that the problem is at least

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$H^{3/2+\varepsilon}$-regular, for some $\varepsilon > 0$. We have H^2-regularity, for example, if $f \in L^2(\Omega)$, $g \in H^{3/2}(\Omega)$, the components of $K \in C^{0,1}(\bar{\Omega})$, and Ω is convex or $\partial\Omega$ is smooth enough (see [21, 22, 19]). (Strictly speaking, this simplification excludes point or line sources and discontinuous K.)

A number of papers deal with the analysis and the implementation of the mixed methods applied to the above problem on conforming grids (see, e.g., [28, 26, 25, 8, 6, 7, 10, 14, 24, 29, 15, 17, 2, 1] and [27, 9]). Mixed methods on nested locally refined grids are considered in [16, 18]. These works apply the notion of "slave" or "worker" nodes to force continuity of fluxes across the interfaces. The results rely heavily on the fact that the grids are nested and cannot be extended to non-matching grids.

In the present work we employ a partially hybridized form [4, 9] of the mixed method to obtain accurate approximations on non-matching grids. We assume that Ω is a union of non-overlapping polygonal blocks, each covered by a conforming, affine finite element partition. Lagrange multiplier pressures are introduced on the interblock boundaries [4, 9, 20]. Since the grids are different on the two sides of the interface, the Lagrange multiplier space can no longer be the normal trace of the velocity space. A different boundary space is needed, which we call a mortar finite element space, using terminology from previous works on Galerkin and spectral methods (see [5] and references therein). As we show later in the analysis, the method is optimally convergent if the boundary space has one order higher approximability than the normal trace of the velocity space. Moreover, superconvergence for the pressure and, in the case of rectangular grids, for the velocity is obtained at certain discrete points. (See also [3] for a similar technique that avoids the use of a mortar space, at the expense of losing strict mass conservation.)

We allow the mortar space to consist of either continuous or discontinuous piecewise polynomials and obtain the same order of convergence in both cases. The method using discontinuous mortars provides better local mass conservation across the interfaces, but numerical observations suggest that this may lead to slightly bigger numerical error.

The method presented here has also been considered in [30] in the case of the lowest order Raviart-Thomas spaces [26, 25]. Here we take a somewhat different approach in the analysis, which allows us to relax a condition on the mortar grids needed to obtain optimal convergence and superconvergence. The relaxed condition is easily satisfied in practice.

An attractive feature of the scheme is that it can be implemented efficiently in parallel using non-overlapping domain decomposition algorithms. In particular, we modify the Glowinski-Wheeler algorithm [20, 13] to handle non-matching grids. Since this algorithm uses Lagrange multipliers on the interface, the only additional cost is computing projections of the mortar space onto the normal trace of the local velocity spaces and vice-versa.

The rest of the paper is organized as follows. The mixed finite element method with mortar elements is presented in the next section. In Section 3 we construct a projection operator onto the space of weakly continuous (with respect to the mortars) velocities and analyze its approximation properties. Sections 4 and 5 are devoted to the error analysis of the velocity and the pressure, respectively. In Section 6 the method is reformulated as an interface problem. A substructuring domain decomposition algorithm for the solution of the interface problem is discussed in Section 7. Numerical results confirming the theory are presented in Section 8.

2. Formulation of the method. A weak solution of (1.1)–(1.3) is a pair $\mathbf{u} \in H(\mathrm{div};\Omega)$, $p \in L^2(\Omega)$ such that

$$(K^{-1}\mathbf{u},\mathbf{v}) = (p, \nabla\cdot\mathbf{v}) - \langle g, \mathbf{v}\cdot\nu\rangle_{\partial\Omega}, \quad \mathbf{v}\in H(\mathrm{div};\Omega), \tag{2.1}$$

$$(\nabla\cdot\mathbf{u}, w) = (f, w), \qquad w \in L^2(\Omega), \tag{2.2}$$

It is well known (see, e.g., [9, 27]) that (2.1)–(2.2) has an unique solution.

Let $\Omega = \cup_{i=1}^n \Omega_i$ be decomposed into non overlapping subdomain blocks Ω_i, and let $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \cup_{i,j=1}^n \Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma = \partial\Omega_i \backslash \partial\Omega$ denote interior block interfaces. Let

$$\mathbf{V}_i = H(\mathrm{div};\Omega_i), \qquad \mathbf{V} = \bigoplus_{i=1}^n \mathbf{V}_i,$$

$$W_i = L^2(\Omega_i), \qquad W = \bigoplus_{i=1}^n W_i = L^2(\Omega).$$

If the solution $(\mathbf{u},p)$ of (2.1)–(2.2) belongs to $H(\mathrm{div};\Omega)\times H^1(\Omega)$, it is easy to see that it satisfies, for $1 \le i \le n$,

$$(K^{-1}\mathbf{u},\mathbf{v})_{\Omega_i} = (p,\nabla\cdot\mathbf{v})_{\Omega_i} - \langle p, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} - \langle g, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\backslash\Gamma}, \quad \mathbf{v}\in\mathbf{V}_i, \tag{2.3}$$

$$(\nabla\cdot\mathbf{u}, w)_{\Omega_i} = (f,w)_{\Omega_i}, \qquad w\in W_i, \tag{2.4}$$

where ν_i is the outer unit normal to $\partial\Omega_i$ (see also [9, pp. 91–92]). We will further assume that these problems posed over each Ω_i are at least $H^{3/2+\varepsilon}$-regular.

Let $\mathcal{T}_{h,i}$ be a conforming, quasi-uniform finite element partition of Ω_i, $1 \le i \le n$, allowing for the possibility that $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ need not align on $\Gamma_{i,j}$. Let $\mathcal{T}_h = \cup_{i=1}^n \mathcal{T}_{h,i}$. Let

$$\mathbf{V}_{h,i} \times W_{h,i} \subset \mathbf{V}_i \times W_i$$

be any of the usual mixed finite element spaces, (i.e., the RTN spaces [28, 26, 25]; BDM spaces [8]; BDFM spaces [7]; BDDF spaces [6], or CD spaces [10]). We assume that the order of the spaces is the same on every subdomain. Let

$$\mathbf{V}_h = \bigoplus_{i=1}^n \mathbf{V}_{h,i}, \qquad W_h = \bigoplus_{i=1}^n W_{h,i}.$$

Although the normal components of vectors in $\mathbf{V}_h$ are continuous between elements within each block Ω_i, there is no such restriction across Γ. Recall that

$$\nabla\cdot\mathbf{V}_{h,i} = W_{h,i},$$

and that there exists a projection Π_i of $(H^{1/2+\varepsilon}(\Omega_i))^d \cap \mathbf{V}_i$ onto $\mathbf{V}_{h,i}$ (for any $\varepsilon > 0$), satisfying amongst other properties that for any $\mathbf{q} \in (H^{1/2+\varepsilon}(\Omega_i))^d \cap \mathbf{V}_i$,

$$(\nabla\cdot(\Pi_i\mathbf{q} - \mathbf{q}), w)_{\Omega_i} = 0, \qquad w \in W_{h,i} \tag{2.5}$$

$$\langle(\mathbf{q} - \Pi_i\mathbf{q})\cdot\nu_i, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i} = 0, \qquad \mathbf{v}\in\mathbf{V}_{h,i}. \tag{2.6}$$

Note that, since $\mathbf{q} \in (H^{1/2+\varepsilon}(\Omega_i))$, $\mathbf{q}\cdot\nu|_e \in H^\varepsilon(e)$ for any element face (or edge) e; therefore $\Pi_i\mathbf{q}$ is well defined.

REMARK 2.1. Our $H^{3/2+\varepsilon}$-regularity assumption insures that we can apply Π_i to the flux arising from our elliptic problem. We note in passing that Mathew [23] shows that if $\mathbf{q} \in (H^\varepsilon(\Omega_i))^d$, $0 < \varepsilon < 1$, and $\nabla \cdot \mathbf{q} = 0$, then $\Pi_i \mathbf{q}$ is well defined and

$$\|\Pi_i \mathbf{q}\|_{0,\Omega_i} \le C \|\mathbf{q}\|_{\varepsilon,\Omega_i} \tag{2.7}$$

$$\|\Pi_i \mathbf{q} - \mathbf{q}\|_{0,\Omega_i} \le C h^\varepsilon \|\mathbf{q}\|_{\varepsilon,\Omega_i}, \tag{2.8}$$

where $\|\cdot\|_r$ is the H^r-norm. His argument, given for Raviart-Thomas spaces, can be trivially extended (see also [21], Section 1.5) to show that, for any of the mixed spaces under consideration, $\Pi_i \mathbf{q}$ is well defined for any $\mathbf{q} \in (H^\varepsilon(\Omega_i))^d \cap \mathbf{V}_i$ and

$$\|\Pi_i \mathbf{q}\|_{0,\Omega_i} \le C(\|\mathbf{q}\|_{\varepsilon,\Omega_i} + \|\nabla \cdot \mathbf{q}\|_{0,\Omega_i}). \tag{2.9}$$

Thus we could reduce our regularity requirements, at the expense of greatly increasing technical aspects of the analysis largely unrelated to those of the mortar element techniques.

Let the mortar interface mesh $\mathcal{T}_{h,i,j}$ be a quasi-uniform finite element partition of $\Gamma_{i,j}$. Denote by $\Lambda_{h,i,j} \subset L^2(\Gamma_{i,j})$ the mortar space on $\Gamma_{i,j}$, containing at least either the continuous or discontinuous piecewise polynomials of degree $k+1$ on $\mathcal{T}_{h,i,j}$, where k is associated with the degree of the polynomials in $\mathbf{V}_h \cdot \nu$. More precisely, if $d = 3$ and e is a triangle of the mesh, we take $\Lambda_{h,i,j}|_e = P_{k+1}(e)$, the set of polynomials of degree less than or equal to k on e. If e is a rectangle, we take $\Lambda_{h,i,j}|_e = Q_{k+1}(e)$, the set of polynomials on e for which the degree in each variable separately is less than or equal to k. Now let

$$\Lambda_h = \bigoplus_{1 \le i < j \le n} \Lambda_{h,i,j}$$

be the mortar finite element space on Γ. In the following we treat any function $\mu \in \Lambda_h$ as extended by zero on $\partial\Omega$. An additional assumption on the space Λ_h and hence $\mathcal{T}_{h,i,j}$ will be made below in (2.14) and (3.18). We remark that $\mathcal{T}_{h,i,j}$ need not be conforming if a discontinuous space is used.

In the mixed finite element approximation of (2.1)–(2.2), we seek $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, $\lambda_h \in \Lambda_h$ such that, for $1 \le i \le n$,

$$(K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} - \langle g, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \setminus \Gamma}, \mathbf{v} \in \mathbf{V}_{h,i}, \tag{2.10}$$

$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_{h,i}, \tag{2.11}$$

$$\sum_{i=1}^n \langle \mathbf{u}_h \cdot \nu_i, \mu \rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h. \tag{2.12}$$

Strictly within each block Ω_i, we have a standard mixed finite element method, and (2.11) enforces local conservation over each grid cell. Moreover, since $\mathbf{u}_h \cdot \nu$ is continuous on any element face (or edge) $e \not\subset \Gamma \cup \partial\Omega$, we have local mass conservation across interior element faces. From (2.3) we see that λ_h approximates the pressure p on the block interfaces Γ. Equation (2.12) enforces weak continuity of flux across these interfaces (weakly with respect to the mortar space Λ_h).

For each subdomain Ω_i, define a projection $\mathcal{Q}_{h,i} : L^2(\Gamma_i) \to \mathbf{V}_{h,i} \cdot \nu_i|_{\Gamma_i}$ such that, for any $\phi \in L^2(\Gamma_i)$,

$$\langle \phi - \mathcal{Q}_{h,i}\phi, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}. \tag{2.13}$$

Let, for $\phi \in L^2(\Gamma)$, $\mathcal{Q}_h\phi = \bigoplus_{i=1}^n \mathcal{Q}_{h,i}\phi$.

LEMMA 2.1. *Assume that for any* $\phi \in \Lambda_h$,

$$\mathcal{Q}_{h,i}\phi = 0, \ 1 \le i \le n, \ \textit{implies that } \phi = 0. \tag{2.14}$$

Then there exists a unique solution of (2.10)–(2.12).

REMARK 2.2. Condition (2.14) says that the mortar space cannot be too rich compared to the normal traces of the subdomain velocity spaces. A richer Λ_h gives a better local mass conservation across Γ, by (2.12); however, if the space is too rich, i.e., too much local mass conservation across Γ is demanded, unique solvability is lost. (See also Remark 3.1 below.)

Proof. Since (2.10)–(2.12) is a square system, it is enough to show uniqueness. Let $f = 0$, $g = 0$. Setting $\mathbf{v} = \mathbf{u}_h$, $w = p_h$, and $\mu = -\lambda_h$, adding (2.10)–(2.12) together, and summing over $1 \le i \le n$, implies that $\mathbf{u}_h = 0$. Denote, for $1 \le i \le n$,

$$\overline{p_{h,i}} = \frac{1}{|\Omega_i|}\int_{\Omega_i} p_h\,dx, \qquad \overline{\mathcal{Q}_{h,i}\lambda_h} = \frac{1}{|\partial\Omega_i|}\int_{\partial\Omega_i} \mathcal{Q}_{h,i}\lambda_h\,ds,$$

and consider the auxiliary problem

$$\begin{aligned} -\nabla\cdot\nabla\varphi_i &= p_h - \overline{p_{h,i}} && \text{in } \Omega_i,\\ -\nabla\varphi_i\cdot\nu &= -\left(\mathcal{Q}_{h,i}\lambda_h - \overline{\mathcal{Q}_{h,i}\lambda_h}\right) && \text{on } \partial\Omega_i, \end{aligned}$$

where $\lambda_h = 0$ on $\partial\Omega \cap \partial\Omega_i$. Note that the problem is well posed and regular with φ_i determined up to a constant. Setting $\mathbf{v} = -\Pi_i\nabla\varphi_i$ in (2.10), we have

$$(p_h, p_h - \overline{p_{h,i}})_{\Omega_i} + \left\langle \mathcal{Q}_{h,i}\lambda_h, \mathcal{Q}_{h,i}\lambda_h - \overline{\mathcal{Q}_{h,i}\lambda_h}\right\rangle_{\partial\Omega_i} = 0,$$

implying

$$p_h|_{\Omega_i} = \overline{p_{h,i}}, \quad \mathcal{Q}_{h,i}\lambda_h = \overline{\mathcal{Q}_{h,i}\lambda_h}.$$

Since now (2.10) is

$$p_h|_{\Omega_i}(1, \nabla\cdot\mathbf{v})_{\Omega_i} - \mathcal{Q}_{h,i}\lambda_h\langle 1, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i} = 0,$$

the divergence theorem implies $p_h|_{\Omega_i} = \mathcal{Q}_{h,i}\lambda_h$.

Since $\lambda_h = 0$ on $\partial\Omega$, $p_h|_{\Omega_i} = \mathcal{Q}_{h,i}\lambda_h = 0$ for those domains i with $\partial\Omega_i \cap \partial\Omega \ne \emptyset$. For any j such that $\partial\Omega_i \cap \partial\Omega_j = \Gamma_{i,j} \ne \emptyset$, (2.13) implies that

$$0 = \mathcal{Q}_{h,i}\lambda_h|_{\Gamma_{i,j}} = \frac{1}{|\Gamma_{i,j}|}\int_{\Gamma_{i,j}} \lambda_h\,ds = \mathcal{Q}_{h,j}\lambda_h|_{\Gamma_{i,j}}.$$

We conclude that $\mathcal{Q}_{h,i}\lambda_h = 0$ for all $1 \le i \le n$; hence, $p_h = 0$ and $\lambda_h = 0$ by the hypothesis of the lemma. □

REMARK 2.3. This proof could be simplified by using the Π_0 projection operator defined in the next section.

3. The space of weakly continuous velocities. We first introduce some projection operators needed in the analysis. Let $\mathcal{P}_h$ be the $L^2(\Gamma)$ projection onto Λ_h satisfying for any $\psi \in L^2(\Gamma)$,

$$\langle \psi - \mathcal{P}_h\psi, \mu\rangle_\Gamma = 0, \quad \mu \in \Lambda_h.$$

For any $\varphi \in L^2(\Omega)$, let $\hat{\varphi} \in W_h$ be its $L^2(\Omega)$ projection satisfying

$$(\varphi - \hat{\varphi}, w) = 0, \quad w \in W_h.$$

These projections and Π and $\mathcal{Q}_{h,i}$ defined earlier have the following approximation properties, wherein l is associated with the degree of the polynomials in W_h:

$$\|\psi - \mathcal{P}_h\psi\|_{-s,\Gamma_{i,j}} \le C\|\psi\|_{r,\Gamma_{i,j}} h^{r+s}, \qquad 0 \le r \le k+2,\ 0 \le s \le k+2, \tag{3.1}$$

$$\|\varphi - \hat{\varphi}\|_0 \le C\|\varphi\|_r h^r, \qquad 0 \le r \le l+1, \tag{3.2}$$

$$\|\mathbf{q} - \Pi_i\mathbf{q}\|_{0,\Omega_i} \le C\|\mathbf{q}\|_{r,\Omega_i} h^r, \qquad 1 \le r \le k+1, \tag{3.3}$$

$$\|\nabla\cdot(\mathbf{q} - \Pi_i\mathbf{q})\|_{0,\Omega_i} \le C\|\nabla\cdot\mathbf{q}\|_{r,\Omega_i} h^r, \qquad 0 \le r \le l+1, \tag{3.4}$$

$$\|\psi - \mathcal{Q}_{h,i}\psi\|_{-s,\Gamma_{i,j}} \le C\|\psi\|_{r,\Gamma_{i,j}} h^{r+s}, \qquad 0 \le r \le k+1,\ 0 \le s \le k+1, \tag{3.5}$$

$$\|(\mathbf{q} - \Pi_i\mathbf{q})\cdot\nu_i\|_{-s,\Gamma_{i,j}} \le C\|\mathbf{q}\|_{r,\Gamma_{i,j}} h^{r+s}, \qquad 0 \le r \le k+1,\ 0 \le s \le k+1, \tag{3.6}$$

where $\|\cdot\|_r$ is the H^r-norm and $\|\cdot\|_{-s}$ is the norm of H^{-s}, the dual of H^s (*not* H_0^s). Moreover, from (2.5)–(2.6),

$$\nabla\cdot\Pi_i\mathbf{q} = \widehat{\nabla\cdot\mathbf{q}}, \tag{3.7}$$

$$(\Pi_i\mathbf{q})\cdot\nu_i = \mathcal{Q}_{h,i}(\mathbf{q}\cdot\nu_i). \tag{3.8}$$

Bounds (3.1),(3.2), and (3.4)–(3.6) are standard L^2-projection approximation results [11]; bound (3.3) can be found in [9, 27]. We use the nonstandard trace theorem

$$\|q\|_{r,\Gamma_{i,j}} \le C\|q\|_{r+1/2,\Omega_i}$$

in this paper; it can be found in [21, Theorem 1.5.2.1].

Let

$$\mathbf{V}_{h,0} = \left\{\mathbf{v} \in \mathbf{V}_h : \sum_{i=1}^n \langle \mathbf{v}|_{\Omega_i}\cdot\nu_i, \mu\rangle_{\Gamma_i} = 0 \ \ \forall\ \ \mu \in \Lambda_h\right\}$$

be the space of weakly continuous velocities, with respect to the mortar space. Then the mixed method (2.10)–(2.12) can be rewritten in the following way. Find $\mathbf{u}_h \in \mathbf{V}_{h,0}$, $p_h \in W_h$ such that

$$(K^{-1}\mathbf{u}_h, \mathbf{v}) = \sum_{i=1}^n (p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle g, \mathbf{v}\cdot\nu\rangle_{\partial\Omega}, \quad \mathbf{v} \in \mathbf{V}_{h,0}, \tag{3.9}$$

$$\sum_{i=1}^n (\nabla\cdot\mathbf{u}_h, w)_{\Omega_i} = (f, w), \qquad w \in W_h. \tag{3.10}$$

The above form of the scheme will be used for analysis only. It is not necessarily suitable for computing, since it is difficult to construct a basis for the weakly continuous velocity space $\mathbf{V}_{h,0}$.

Our goal for the rest of this section is to construct a projection operator Π_0 onto $\mathbf{V}_{h,0}$ with optimal approximation properties such that, for any $\mathbf{q} \in (H^1(\Omega))^d$,

$$(\nabla\cdot(\Pi_0\mathbf{q} - \mathbf{q}), w)_\Omega = 0, \quad w \in W_h. \tag{3.11}$$

By an abuse of notation, define

$$\begin{aligned}\mathbf{V}_h \cdot \nu = \{(\phi_L, \phi_R) \in L^2(\Gamma) \times L^2(\Gamma) : \phi_L|_{\Gamma_{i,j}} \in \mathbf{V}_{h,i} \cdot \nu_i \text{ and} \\ \phi_R|_{\Gamma_{i,j}} \in \mathbf{V}_{h,j} \cdot \nu_j \ \ \forall\ \ 1 \le i < j \le n\}\end{aligned}$$

and

$$\begin{aligned}\mathbf{V}_{h,0} \cdot \nu = \{(\phi_L, \phi_R) \in L^2(\Gamma) \times L^2(\Gamma) : \exists \mathbf{v} \in \mathbf{V}_{h,0} \text{ such that} \\ \phi_L|_{\Gamma_{i,j}} = \mathbf{v}|_{\Omega_i} \cdot \nu_i \text{ and } \phi_R|_{\Gamma_{i,j}} = \mathbf{v}|_{\Omega_j} \cdot \nu_j \ \ \forall\ \ 1 \le i < j \le n\}.\end{aligned}$$

Henceforth, for any $\phi = (\phi_L, \phi_R) \in (L^2(\Gamma))^2$, we write $\phi|_{\Gamma_{i,j}} = (\phi_i, \phi_j)$, $1 \le i < j \le n$. Define a projection $\mathcal{Q}_{h,0} : (L^2(\Gamma))^2 \to \mathbf{V}_{h,0} \cdot \nu$ such that, for any $\phi \in (L^2(\Gamma))^2$,

$$\sum_{i=1}^{n} \langle \phi_i - (\mathcal{Q}_{h,0}\phi)_i, \xi_i \rangle_{\Gamma_i} = 0, \quad \xi \in \mathbf{V}_{h,0} \cdot \nu. \tag{3.12}$$

LEMMA 3.1. *Assume that* (2.14) *holds. For any* $\phi \in (L^2(\Gamma))^2$, *there exists* $\lambda_h \in \Lambda_h$ *such that on* $\Gamma_{i,j}$, $1 \le i < j \le n$,

$$\mathcal{Q}_{h,i}\lambda_h = \mathcal{Q}_{h,i}\phi_i - (\mathcal{Q}_{h,0}\phi)_i, \tag{3.13}$$

$$\mathcal{Q}_{h,j}\lambda_h = \mathcal{Q}_{h,j}\phi_j - (\mathcal{Q}_{h,0}\phi)_j, \tag{3.14}$$

$$\langle \lambda_h, 1 \rangle_{\Gamma_{i,j}} = \tfrac{1}{2}\langle \phi_i + \phi_j, 1 \rangle_{\Gamma_{i,j}}. \tag{3.15}$$

Proof. Consider the following auxiliary problem. Given $\phi \in (L^2(\Gamma))^2$, find $\psi_h \in \mathbf{V}_h \cdot \nu$ and $\lambda_h \in \Lambda_h$ such that

$$\sum_{i=1}^{n} \langle \phi_i - \psi_{h,i} - \lambda_h, \xi_i \rangle_{\Gamma_i} = 0, \qquad \xi \in \mathbf{V}_h \cdot \nu, \tag{3.16}$$

$$\sum_{i=1}^{n} \langle \psi_{h,i}, \mu \rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h. \tag{3.17}$$

To show existence and uniqueness of a solution of (3.16)–(3.17), take $\phi = 0$, $\xi = \psi_h$, and $\mu = \lambda_h$ to conclude that $\psi_h = 0$. Now (3.16) and (2.14) imply that $\lambda_h = 0$.

With $\xi \in \mathbf{V}_{h,0} \cdot \nu$ in (3.16) we have

$$\sum_{i=1}^{n} \langle \phi_i - \psi_{h,i}, \xi_i \rangle_{\Gamma_i} = 0.$$

Also, from (3.17), $\psi_h \in \mathbf{V}_{h,0} \cdot \nu$. Therefore $\psi_h = \mathcal{Q}_{h,0}\phi$. Equation (3.16) now implies (3.13) and (3.14). Since any constant function is in $\mathbf{V}_{h,i} \cdot \nu_i$, $\mathbf{V}_{h,j} \cdot \nu_j$, and $\Lambda_{h,i,j}$, we have

$$\begin{aligned}2\langle \lambda_h, 1 \rangle_{\Gamma_{i,j}} &= \langle \mathcal{Q}_{h,i}\lambda_h, 1 \rangle_{\Gamma_{i,j}} + \langle \mathcal{Q}_{h,j}\lambda_h, 1 \rangle_{\Gamma_{i,j}} \\ &= \langle \mathcal{Q}_{h,i}\phi_i - (\mathcal{Q}_{h,0}\phi)_i, 1 \rangle_{\Gamma_{i,j}} + \langle \mathcal{Q}_{h,j}\phi_j - (\mathcal{Q}_{h,0}\phi)_j, 1 \rangle_{\Gamma_{i,j}} \\ &= \langle \mathcal{Q}_{h,i}\phi_i, 1 \rangle_{\Gamma_{i,j}} + \langle \mathcal{Q}_{h,j}\phi_j, 1 \rangle_{\Gamma_{i,j}} \\ &= \langle \phi_i + \phi_j, 1 \rangle_{\Gamma_{i,j}},\end{aligned}$$

and (3.15) follows. ∎

The next lemma shows that, under a relatively mild assumption on the mortar space Λ_h, $\mathcal{Q}_{h,0}$ has optimal approximation properties for normal traces:

$$\phi = (\mathbf{u}\cdot\nu_i, \mathbf{u}\cdot\nu_j) = (\mathbf{u}\cdot\nu_i, -\mathbf{u}\cdot\nu_i).$$

LEMMA 3.2. *Assume that there exists a constant C, independent of h, such that*

$$\|\mu\|_{0,\Gamma_{i,j}} \le C(\|\mathcal{Q}_{h,i}\mu\|_{0,\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}\mu\|_{0,\Gamma_{i,j}}), \quad \forall \mu\in\Lambda_h,\ \ 1\le i<j\le n. \tag{3.18}$$

Then, for any ϕ such that $\phi|_{\Gamma_{i,j}} = (\phi_i, -\phi_i)$, there exists a constant C, independent of h, such that

$$\left(\sum_{1\le i<j\le n} \|\mathcal{Q}_{h,i}\phi_i - (\mathcal{Q}_{h,0}\phi)_i\|^2_{-s,\Gamma_{i,j}}\right)^{1/2} \le C \sum_{1\le i<j\le n} \|\phi_i\|_{r,\Gamma_{i,j}} h^{r+s},$$
$$0\le r\le k+1,\ 0\le s\le k+1. \tag{3.19}$$

REMARK 3.1. Condition (3.18) implies the solvability condition (2.14), which is simply (3.18) wherein we allow C to vary with h. So (3.18) strengthens (2.14) so that it holds uniformly as h tends to zero. This is not a very restrictive condition, and it is easily satisfied in practice. It can be shown [30] that (3.18) holds for both continuous and discontinuous mortar spaces, if the the mortar grid on each interface is a coarsening by two in each direction of the trace of either one of the subdomain grids. This choice is reminiscent of the one in the case of standard or spectral finite element subdomain discretizations [5].

Proof. By Lemma 3.1, there is a $\lambda_h\in\Lambda_h$ such that

$$\mathcal{Q}_{h,i}\lambda_h = \mathcal{Q}_{h,i}\phi_i - (\mathcal{Q}_{h,0}\phi)_i. \tag{3.20}$$

Since $\sum_{i=1}^n \langle(\mathcal{Q}_{h,0}\phi)_i, \lambda_h\rangle_{\Gamma_i} = \sum_{i=1}^n\langle\phi_i,\lambda_h\rangle_{\Gamma_i} = 0$,

$$\begin{aligned}
\sum_{i=1}^n \|\mathcal{Q}_{h,i}\lambda_h\|^2_{0,\Gamma_i} &= \sum_{i=1}^n \langle \mathcal{Q}_{h,i}\lambda_h, \lambda_h\rangle_{\Gamma_i}\\
&= \sum_{i=1}^n \langle \mathcal{Q}_{h,i}\phi_i - \phi_i, \lambda_h\rangle_{\Gamma_i}\\
&\le \left(\sum_{i=1}^n \|\mathcal{Q}_{h,i}\phi_i - \phi_i\|^2_{0,\Gamma_i}\right)^{1/2}\left(\sum_{i=1}^n \|\lambda_h\|^2_{0,\Gamma_i}\right)^{1/2}\\
&\le C\left(\sum_{i=1}^n \|\mathcal{Q}_{h,i}\phi_i - \phi_i\|^2_{0,\Gamma_i}\right)^{1/2}\left(\sum_{i=1}^n \|\mathcal{Q}_{h,i}\lambda_h\|^2_{0,\Gamma_i}\right)^{1/2},
\end{aligned}$$

by (3.18), and (3.19) with $s=0$ follows from (3.20) and (3.5).

On any interface $\Gamma_{i,j}$ take any $\rho\in H^s(\Gamma_{i,j})$, $0\le s\le k+1$, and write

$$\begin{aligned}
\langle \mathcal{Q}_{h,i}\lambda_h, \rho\rangle_{\Gamma_{i,j}} &= \langle\lambda_h, \mathcal{Q}_{h,i}\rho - \rho\rangle_{\Gamma_{i,j}} + \langle\lambda_h,\rho\rangle_{\Gamma_{i,j}}\\
&\le C\|\lambda_h\|_{0,\Gamma_{i,j}} h^s \|\rho\|_{s,\Gamma_{i,j}} + \langle\lambda_h,\rho\rangle_{\Gamma_{i,j}}.
\end{aligned} \tag{3.21}$$

The last term is

$$\begin{aligned}\langle \lambda_h, \rho\rangle_{\Gamma_{i,j}} &= \langle \lambda_h, \rho - \tfrac{1}{2}(\mathcal{Q}_{h,i}\rho + \mathcal{Q}_{h,j}\rho)\rangle_{\Gamma_{i,j}} \\ &\quad + \tfrac{1}{2}\langle \lambda_h, \mathcal{Q}_{h,i}\rho + \mathcal{Q}_{h,j}\rho\rangle_{\Gamma_{i,j}} \\ &\le C\|\lambda_h\|_{0,\Gamma_{i,j}} h^s \|\rho\|_{s,\Gamma_{i,j}} + \tfrac{1}{2}\langle \lambda_h, \mathcal{Q}_{h,i}\rho + \mathcal{Q}_{h,j}\rho\rangle_{\Gamma_{i,j}}.\end{aligned} \tag{3.22}$$

Using Lemma 3.1, for the last term in (3.22) we have

$$\begin{aligned}&\langle \lambda_h, \mathcal{Q}_{h,i}\rho + \mathcal{Q}_{h,j}\rho\rangle_{\Gamma_{i,j}} \\ &\quad = \langle \mathcal{Q}_{h,i}\lambda_h, \mathcal{Q}_{h,i}\rho\rangle_{\Gamma_{i,j}} + \langle \mathcal{Q}_{h,j}\lambda_h, \mathcal{Q}_{h,j}\rho\rangle_{\Gamma_{i,j}} \\ &\quad = \langle \phi_i - (\mathcal{Q}_{h,0}\phi)_i, \mathcal{Q}_{h,i}\rho\rangle_{\Gamma_{i,j}} + \langle \phi_j - (\mathcal{Q}_{h,0}\phi)_j, \mathcal{Q}_{h,j}\rho\rangle_{\Gamma_{i,j}} \\ &\quad = \langle \phi_i - (\mathcal{Q}_{h,0}\phi)_i, \mathcal{Q}_{h,i}\rho - \mathcal{P}_h\rho\rangle_{\Gamma_{i,j}} + \langle \phi_j - (\mathcal{Q}_{h,0}\phi)_j, \mathcal{Q}_{h,j}\rho - \mathcal{P}_h\rho\rangle_{\Gamma_{i,j}} \\ &\quad \le Ch^r\|\phi_i\|_{r,\Gamma_{i,j}} h^s\|\rho\|_{s,\Gamma_{i,j}}, \quad 0 \le r \le k+1.\end{aligned} \tag{3.23}$$

Combining (3.21)–(3.23) with (3.18), we obtain (3.19). ☐

We are now ready to construct our projection. For any $\mathbf{q} \in (H^{1/2+\varepsilon}(\Omega_i))^d \cap \mathbf{V}_i$ define

$$\Pi_0\mathbf{q}|_{\Omega_i} = \Pi_i(\mathbf{q} + \delta\mathbf{q}_i),$$

where $\delta\mathbf{q}_i$ solves

$$\delta\mathbf{q}_i = -\nabla\pi_i \qquad \text{in } \Omega_i, \tag{3.24}$$

$$\nabla\cdot\delta\mathbf{q}_i = 0 \qquad \text{in } \Omega_i, \tag{3.25}$$

$$\delta\mathbf{q}_i\cdot\nu_i = -\mathcal{Q}_{h,i}\mathbf{q}\cdot\nu_i + (\mathcal{Q}_{h,0}\mathbf{q}\cdot\nu)_i \qquad \text{on } \Gamma_i, \tag{3.26}$$

$$\delta\mathbf{q}_i\cdot\nu_i = 0 \qquad \text{on } \partial\Omega_i\cap\partial\Omega, \tag{3.27}$$

wherein, on any $\Gamma_{i,j}$, $\mathbf{q}\cdot\nu|_{\Gamma_{i,j}} = (\mathbf{q}\cdot\nu_i, \mathbf{q}\cdot\nu_j)$. Note that the Neumann problems (3.24)–(3.27) are well posed, since (3.15) and (3.13) imply that

$$\langle \mathcal{Q}_{h,i}\mathbf{q}\cdot\nu_i - (\mathcal{Q}_{h,0}\mathbf{q}\cdot\nu)_i, 1\rangle_{\Gamma_i} = 0.$$

Also, note that the piece-wise constant Neumann data is in $H^{1/2-\varepsilon}(\partial\Omega_i)$, so $\delta\mathbf{q}_i \in (H^r(\Omega_i))^d$, where $r = 1-\varepsilon$ if we have enough regularity but at least $r \ge 1/2+\varepsilon$; thus, Π_i can be applied to $\delta\mathbf{q}_i$.

We first notice that by (3.8),

$$\sum_{i=1}^n \langle (\Pi_0\mathbf{q})\cdot\nu_i, \mu\rangle_{\Gamma_i} = \sum_{i=1}^n \langle (\mathcal{Q}_{h,0}\mathbf{q}\cdot\nu)_i, \mu\rangle_{\Gamma_i} = 0, \quad \forall\mu\in\Lambda_h;$$

therefore $\Pi_0\mathbf{q}\in\mathbf{V}_{h,0}$. Also, by (3.7),

$$(\nabla\cdot\Pi_0\mathbf{q}, w)_{\Omega_i} = (\nabla\cdot\Pi_i\mathbf{q}, w)_{\Omega_i} + (\nabla\cdot\Pi_i\delta\mathbf{q}, w)_{\Omega_i} = (\nabla\cdot\mathbf{q}, w)_{\Omega_i}, \quad \forall w\in W_{h,i}.$$

It remains to estimate the approximability of Π_0. Since on Ω_i

$$\Pi_0\mathbf{q} - \mathbf{q} = \Pi_i\mathbf{q} - \mathbf{q} + \Pi_i\delta\mathbf{q}_i,$$

with (3.3) we need only bound the correction $\Pi_i\delta\mathbf{q}_i$. By elliptic regularity [21, 22], for any $0\le s\le 1/2$

$$\|\delta\mathbf{q}_i\|_{1/2-s,\Omega_i} \le C\sum_j \|\mathcal{Q}_{h,i}\mathbf{q}\cdot\nu_i - (\mathcal{Q}_{h,0}\mathbf{q}\cdot\nu)_i\|_{-s,\Gamma_{i,j}}. \tag{3.28}$$

We now have

$$\begin{aligned}\|\Pi_i \delta\mathbf{q}_i\|_{0,\Omega_i} &\le \|\Pi_i \delta\mathbf{q}_i - \delta\mathbf{q}_i\|_{0,\Omega_i} + \|\delta\mathbf{q}_i\|_{0,\Omega_i} \\ &\le Ch^{1/2}\|\delta\mathbf{q}_i\|_{1/2,\Omega_i} + \|\delta\mathbf{q}_i\|_{0,\Omega_i} \\ &\le C\sum_j \{\|\mathcal{Q}_{h,i}\mathbf{q}\cdot\nu_i - (\mathcal{Q}_{h,0}\mathbf{q}\cdot\nu)_i\|_{0,\Gamma_{i,j}} h^{1/2} \\ &\qquad + \|\mathcal{Q}_{h,i}\mathbf{q}_i\cdot\nu_i - (\mathcal{Q}_{h,0}\mathbf{q}\cdot\nu)_i\|_{-1/2,\Gamma_{i,j}}\}, \end{aligned} \tag{3.29}$$

using an estimate by Mathew [23] for any divergence free vector ψ

$$\|\Pi_i\psi - \psi\|_{0,\Omega_i} \le Ch^{\varepsilon}\|\psi\|_{\varepsilon,\Omega_i}, \quad 0 < \varepsilon < 1.$$

Note that the result in [23] is for Raviart-Thomas spaces, but can be trivially extended to any of the mixed spaces under consideration. Together with Lemma 3.2, (3.29) gives

$$\|\Pi_0\mathbf{q} - \Pi\mathbf{q}\|_0 \le C\sum_{i=1}^n \|\mathbf{q}\|_{r+1/2,\Omega_i} h^{r+1/2}, \quad 0 \le r \le k+1, \tag{3.30}$$

and, with (3.3),

$$\|\Pi_0\mathbf{q} - \mathbf{q}\|_0 \le C\sum_{i=1}^n \|\mathbf{q}\|_{r,\Omega_i} h^{r}, \quad 1 \le r \le k+1. \tag{3.31}$$

4. Error estimates for the velocity. We start this section with a lemma needed later in the analysis.

LEMMA 4.1. *For any function* $\mathbf{v} \in \mathbf{V}_{h,i}$,

$$\|\mathbf{v}\cdot\nu\|_{0,\partial\Omega_i} \le Ch^{-1/2}\|\mathbf{v}\|_{0,\Omega_i}.$$

Proof. All spaces under consideration admit nodal bases that include the degrees of freedom of the normal traces on the element boundaries. Since for any element E and any of its faces (or edges) e, $|e| \le Ch^{-1}|E|$, the lemma follows. ☐

4.1. Optimal convergence. Subtracting (3.9)–(3.10) from (2.3)–(2.4) gives the error equations

$$(K^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{v}) = \sum_{i=1}^n ((p - p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle p, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i}) \quad \mathbf{v} \in \mathbf{V}_{h,0}, \tag{4.1}$$

$$\sum_{i=1}^n (\nabla\cdot(\mathbf{u} - \mathbf{u}_h), w)_{\Omega_i} = 0, \qquad w \in W_h. \tag{4.2}$$

We first notice that (4.2) implies that

$$\nabla\cdot(\Pi_0\mathbf{u} - \mathbf{u}_h) = \nabla\cdot(\Pi\mathbf{u} - \mathbf{u}_h) = 0. \tag{4.3}$$

We now take $\mathbf{v} = \Pi_0\mathbf{u} - \mathbf{u}_h$ to get

$$(K^{-1}(\Pi_0\mathbf{u} - \mathbf{u}_h), \Pi_0\mathbf{u} - \mathbf{u}_h)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \langle \mathcal{P}_h p - p, (\Pi_0 \mathbf{u} - \mathbf{u}_h)\cdot \nu_i\rangle_{\Gamma_i} + (K^{-1}(\Pi_0\mathbf{u} - \mathbf{u}), \Pi_0\mathbf{u} - \mathbf{u}_h)\\
&\le \sum_{i=1}^{n} \|\mathcal{P}_h p - p\|_{0,\Gamma_i} \|(\Pi_0 \mathbf{u} - \mathbf{u}_h)\cdot \nu_i\|_{0,\Gamma_i} + (K^{-1}(\Pi_0\mathbf{u} - \mathbf{u}), \Pi_0\mathbf{u} - \mathbf{u}_h)\\
&\le C\left(\sum_{i=1}^{n} \|p\|_{r+1,\Omega_i} h^{r+1/2} \|\Pi_0\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i} h^{-1/2} + \sum_{i=1}^{n} \|\mathbf{u}\|_{r,\Omega_i} h^r \|\Pi_0\mathbf{u} - \mathbf{u}_h\|_0\right),\\
&\qquad\qquad 1 \le r \le k+1, \qquad (4.4)
\end{aligned}
$$

where we used (3.1), Lemma 4.1, and (3.31) for the last inequality. With (4.3)–(4.4), (3.4), and (3.31) we have shown the following theorem.

THEOREM 4.2. *For the velocity* $\mathbf{u}_h$ *of the mixed method* (2.10)–(2.12), *if* (2.14) *holds, then there exists a positive constant* C *independent of* h *such that*

$$
\|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|_0 \le C \sum_{i=1}^{n} \|\nabla\cdot\mathbf{u}\|_{r,\Omega_i} h^r, \quad 1 \le r \le l+1.
$$

Moreover, if (3.18) *holds,*

$$
\|\mathbf{u} - \mathbf{u}_h\|_0 \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i}) h^r, \quad 1 \le r \le k+1.
$$

4.2. Superconvergence. In this subsection we restrict to the case of diagonal tensor K and RTN spaces on rectangular type grids. In this case superconvergence of the velocity is attained at certain discrete points. To show this we modify the last inequality in (4.4). In particular, (3.1) gives, for $1 \le r \le k+1$,

$$
\sum_{i=1}^{n} \|\mathcal{P}_h p - p\|_{0,\Gamma_i} \|(\Pi_0 \mathbf{u} - \mathbf{u}_h)\cdot \nu_i\|_{0,\Gamma_i} \le C \sum_{i=1}^{n} \|p\|_{r+3/2,\Omega_i} h^{r+1} \|\Pi_0\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i} h^{-1/2},
$$

and (3.30) gives, for $1 \le r \le k+1$,

$$
(K^{-1}(\Pi_0\mathbf{u} - \Pi\mathbf{u}), \Pi_0\mathbf{u} - \mathbf{u}_h) \le C \sum_{i=1}^{n} \|\mathbf{u}\|_{r+1/2,\Omega_i} h^{r+1/2} \|\Pi_0\mathbf{u} - \mathbf{u}_h\|_0,
$$

which, combined with the estimate (see [24] and [15, Theorem 3.1])

$$
(K^{-1}(\Pi_i\mathbf{u} - \mathbf{u}), \Pi_0\mathbf{u} - \mathbf{u}_h)_{\Omega_i} \le C\|\mathbf{u}\|_{r+1,\Omega_i} h^{r+1} \|\Pi_0\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i}, \quad 0 \le r \le k+1,
$$

implies

$$
\|\Pi_0\mathbf{u} - \mathbf{u}_h\|_0 \le C \sum_{i=1}^{n} (\|p\|_{r+3/2,\Omega_i} + \|\mathbf{u}\|_{r+1/2,\Omega_i}) h^{r+1/2}, \quad 1 \le r \le k+1. \qquad (4.5)
$$

This estimate implies superconvergence along the Gaussian lines. Consider (for $d = 3$) an element $E = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$. Denote by $g_1^i, ..., g_{k+1}^i$, $i = 1, 2, 3$, the Gaussian points on $[a_i, b_i]$, i.e., the roots of the Legendre polynomials of degree $k+1$ on $[a_i, b_i]$. As in [17, 15], for a vector $\mathbf{q} = (q_1, q_2, q_3)$ define

$$
|||q_1|||_{1,E}^2 = \sum_{j_2=1}^{k+1} A_{j_2}(b_2 - a_2) \sum_{j_3=1}^{k+1} A_{j_3}(b_3 - a_3) \int_{a_1}^{b_1} |q_1(x_1, g_{j_2}^2, g_{j_3}^3)|^2\, dx_1,
$$

$$|||q_2|||_{2,E}^2 = \sum_{j_1=1}^{k+1} A_{j_1}(b_1 - a_1) \sum_{j_3=1}^{k+1} A_{j_3}(b_3 - a_3) \int_{a_2}^{b_2} |q_2(g_{j_1}^1, x_2, g_{j_3}^3)|^2 \, dx_2,$$

$$|||q_3|||_{3,E}^2 = \sum_{j_1=1}^{k+1} A_{j_1}(b_1 - a_1) \sum_{j_2=1}^{k+1} A_{j_2}(b_2 - a_2) \int_{a_3}^{b_3} |q_3(g_{j_1}^1, g_{j_2}^2, x_3)|^2 \, dx_3,$$

where A_{j_i}, $j_i = 1, ..., k+1$ are the coefficients of Gaussian quadrature in $[-1, 1]$. Define

$$|||\mathbf{q}|||^2 = \sum_{i=1}^{3} \sum_{E \in \mathcal{T}_h} |||q_i|||_{i,E}^2.$$

Note that, for $\mathbf{q} \in \mathbf{V}_h$, $|||\mathbf{q}|||$ is equal to the L^2-norm of $\mathbf{q}$.

THEOREM 4.3. *Assume that the tensor K is diagonal and the mixed finite element spaces are* RTN *on rectangular type grids. For the velocity $\mathbf{u}_h$ of the mixed method* (2.10)–(2.12), *if* (3.18) *holds, then there exists a positive constant C independent of h such that*

$$|||\mathbf{u} - \mathbf{u}_h||| \le C \sum_{i=1}^{n} (\|p\|_{r+3/2,\Omega_i} + \|\mathbf{u}\|_{r+1/2,\Omega_i}) h^{r+1/2}, \quad 1 \le r \le k+1.$$

Proof. By the triangle inequality,

$$|||\mathbf{u} - \mathbf{u}_h||| \le |||\mathbf{u} - \Pi \mathbf{u}||| + |||\Pi \mathbf{u} - \Pi_0 \mathbf{u}||| + |||\Pi_0 \mathbf{u} - \mathbf{u}_h|||.$$

The theorem follows from (3.30), (4.5), and the bound (see [15])

$$|||\mathbf{u} - \Pi \mathbf{u}|||_{\Omega_i} \le C \|\mathbf{u}\|_{r+1,\Omega_i} h^{r+1}, \quad 1 \le r \le k+1.$$

□

5. Error estimates for the pressure. In this section we use a duality argument to derive a superconvergence estimate for $\hat{p} - p_h$. We will assume full H^2-regularity of the problem on Ω for simplicity (reduced superconvergence is obtained for reduced regularity, as can be seen in the argument below). Let φ be the solution of

$$\begin{aligned} -\nabla \cdot K \nabla \varphi &= -(\hat{p} - p_h) \quad && \text{in } \Omega, \\ \varphi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By elliptic regularity,

$$\|\varphi\|_2 \le C \|\hat{p} - p_h\|_0. \tag{5.1}$$

Take $\mathbf{v} = \Pi_0 K \nabla \varphi$ in (4.1) to get

$$\begin{aligned} \|\hat{p} - p_h\|_0^2 &= \sum_{i=1}^{n} (\hat{p} - p_h, \nabla \cdot \Pi_0 K \nabla \varphi)_{\Omega_i} \\ &= \sum_{i=1}^{n} ((K^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi_0 K \nabla \varphi)_{\Omega_i} + \langle p - \mathcal{P}_h p, \Pi_0 K \nabla \varphi \cdot \nu_i \rangle_{\Gamma_i}). \end{aligned} \tag{5.2}$$

The first term on the right can be manipulated as

$$\begin{aligned}
&\sum_{i=1}^{n}(K^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi_0 K\nabla\varphi)_{\Omega_i}\\
&\quad=\sum_{i=1}^{n}\left((K^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi_0 K\nabla\varphi-K\nabla\varphi)_{\Omega_i}+(\mathbf{u}-\mathbf{u}_h,\nabla\varphi)_{\Omega_i}\right)\\
&\quad=\sum_{i=1}^{n}\Big((K^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi_0 K\nabla\varphi-K\nabla\varphi)_{\Omega_i}\\
&\qquad\qquad+(\nabla\cdot(\mathbf{u}-\mathbf{u}_h),\varphi-\hat{\varphi})_{\Omega_i}-\langle(\mathbf{u}-\mathbf{u}_h)\cdot\nu_i,\varphi-\mathcal{P}_h\varphi\rangle_{\Gamma_i}\Big) \qquad (5.3)\\
&\quad\le C\sum_{i=1}^{n}\left(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_i}h+\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega_i}h+\|(\mathbf{u}-\mathbf{u}_h)\cdot\nu_i\|_{0,\Gamma_i}h^{3/2}\right)\|\varphi\|_{2,\Omega_i}
\end{aligned}$$

using (3.31), (3.2), and (3.1) for the last inequality with $C = C(\max_i \|K\|_{1,\infty,\Omega_i})$. The last term on the right is

$$\begin{aligned}
&\|(\mathbf{u}-\mathbf{u}_h)\cdot\nu_i\|_{0,\Gamma_i}h^{3/2}\\
&\qquad\le(\|(\mathbf{u}-\Pi_i\mathbf{u})\cdot\nu_i\|_{0,\Gamma_i}+\|(\Pi_i\mathbf{u}-\mathbf{u}_h)\cdot\nu_i\|_{0,\Gamma_i})h^{3/2}\\
&\qquad\le C(h^r\sum_j\|\mathbf{u}\|_{r,\Gamma_{i,j}}+h^{-1/2}\|\Pi_i\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_i})h^{3/2}\\
&\qquad=C(\sum_j\|\mathbf{u}\|_{r,\Gamma_{i,j}}h^{r+3/2}+\|\Pi_i\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_i}h),\quad 0\le r\le k+1, \qquad (5.4)
\end{aligned}$$

using (3.6) and Lemma 4.1.

For the second term on the right in (5.2) we have

$$\begin{aligned}
&\langle p-\mathcal{P}_h p,\Pi_0 K\nabla\varphi\cdot\nu_i\rangle_{\Gamma_i}\\
&\quad=\langle p-\mathcal{P}_h p,(\Pi_0 K\nabla\varphi-\Pi_i K\nabla\varphi)\cdot\nu_i+(\Pi_i K\nabla\varphi-K\nabla\varphi)\cdot\nu_i+K\nabla\varphi\cdot\nu_i\rangle_{\Gamma_i}\\
&\quad\le\sum_j\|p-\mathcal{P}_h p\|_{0,\Gamma_{i,j}}(\|\delta(K\nabla\varphi)_i\cdot\nu_i\|_{0,\Gamma_{i,j}}+\|(\Pi_i K\nabla\varphi-K\nabla\varphi)\cdot\nu_i\|_{0,\Gamma_{i,j}})\\
&\qquad\qquad+\sum_j\|p-\mathcal{P}_h p\|_{-1/2,\Gamma_{i,j}}\|K\nabla\varphi\cdot\nu_i\|_{1/2,\Gamma_{i,j}}.
\end{aligned}$$

With (3.1), (3.26), (3.19), and (3.6) we have

$$\begin{aligned}
&\|p-\mathcal{P}_h p\|_{0,\Gamma_{i,j}}\le C\|p\|_{r+1/2,\Gamma_{i,j}}h^{r+1/2},\quad 0\le r\le k+1,\\
&\|p-\mathcal{P}_h p\|_{-1/2,\Gamma_{i,j}}\le C\|p\|_{r+1/2,\Gamma_{i,j}}h^{r+1},\quad 0\le r\le k+1,\\
&\|\delta(K\nabla\varphi)_i\cdot\nu_i\|_{0,\Gamma_i}\le C\|\varphi\|_{2,\Omega_i}h^{1/2},\\
&\|(\Pi_i K\nabla\varphi-K\nabla\varphi)\cdot\nu_i\|_{0,\Gamma_i}\le C\|\varphi\|_{2,\Omega_i}h^{1/2};
\end{aligned}$$

therefore,

$$\langle p-\mathcal{P}_h p,\Pi_0 K\nabla\varphi\cdot\nu_i\rangle_{\Gamma_i}\le Ch^{r+1}\|p\|_{r+1,\Omega_i}\|\varphi\|_{2,\Omega_i},\quad 0\le r\le k+1. \qquad (5.5)$$

A combination of (5.1)–(5.5), Theorem 4.2, and (3.3) gives the following theorem.

THEOREM 5.1. *For the pressure p_h of the mixed method* (2.10)–(2.12), *if* (3.18) *holds, then there exists a positive constant C, independent of h, such that*

$$\|\hat{p} - p_h\|_0 \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i} + \|\nabla\cdot\mathbf{u}\|_{r,\Omega_i}) h^{r+1},$$
$$\|p - p_h\|_0 \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i} + \|\nabla\cdot\mathbf{u}\|_{r,\Omega_i}) h^{r},$$

where $1 \le r \le \min(k+1, l+1)$.

6. An interface operator. In this section we introduce a reduced problem involving only the mortar pressure. This reduced problem arose naturally in the work of Glowinski and Wheeler [20] on substructuring domain decomposition methods for mixed finite elements and is closely related to the inter-element multiplier formulation of Arnold and Brezzi [4]. The reason to consider the interface operator is twofold. First, we use it to derive a bound on the error in the mortar space. Second, it is the basis for our parallel domain decomposition implementation.

6.1. The reduced problem. Define a bilinear form $d_h : L^2(\Gamma) \times L^2(\Gamma) \to \mathbf{R}$ for $\lambda, \mu \in L^2(\Gamma)$ by

$$d_h(\lambda,\mu) = \sum_{i=1}^{n} d_{h,i}(\lambda,\mu) = -\sum_{i=1}^{n} \langle \mathbf{u}_h^*(\lambda)\cdot\nu_i, \mu\rangle_{\Gamma_i},$$

where $\mathbf{u}_h^*(\lambda)$ is a component of the solution $(\mathbf{u}_h^*(\lambda), p_h^*(\lambda)) \in \mathbf{V}_h \times W_h$ of, for $1 \le i \le n$,

$$(K^{-1}\mathbf{u}_h^*(\lambda), \mathbf{v})_{\Omega_i} = (p_h^*(\lambda), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{6.1}$$
$$(\nabla\cdot\mathbf{u}_h^*(\lambda), w)_{\Omega_i} = 0, \quad w \in W_{h,i}. \tag{6.2}$$

Define a linear functional $g_h : L^2(\Gamma) \to \mathbf{R}$ by

$$g_h(\mu) = \sum_{i=1}^{n} g_{h,i}(\mu) = \sum_{i=1}^{n} \langle \bar{\mathbf{u}}_h\cdot\nu_i, \mu\rangle_{\Gamma_i},$$

where $(\bar{\mathbf{u}}_h, \bar{p}_h) \in \mathbf{V}_h \times W_h$ solve, for $1 \le i \le n$,

$$(K^{-1}\bar{\mathbf{u}}_h, \mathbf{v})_{\Omega_i} = (\bar{p}_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle g, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\setminus\Gamma}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{6.3}$$
$$(\nabla\cdot\bar{\mathbf{u}}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_{h,i}. \tag{6.4}$$

It is straightforward to show (see [20]) that the solution $(\mathbf{u}_h, p_h, \lambda_h)$ of (2.10)–(2.12) satisfies

$$d_h(\lambda_h, \mu) = g_h(\mu), \quad \mu \in \Lambda_h, \tag{6.5}$$

with

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \qquad p_h = p_h^*(\lambda_h) + \bar{p}_h. \tag{6.6}$$

LEMMA 6.1. *The interface bilinear form $d_h(\cdot,\cdot)$ is symmetric and positive semi-definite on $L^2(\Gamma)$. If* (2.14) *holds, then $d_h(\cdot,\cdot)$ is positive definite on Λ_h.*

Proof. With $\mathbf{v} = \mathbf{u}_h^*(\mu)$ in (6.1) for some $\mu \in L^2(\Gamma)$, we have

$$d_{h,i}(\mu,\lambda) = -\langle \lambda, \mathbf{u}_h^*(\mu)\cdot\nu_i\rangle_{\Gamma_i} = (K^{-1}\mathbf{u}_h^*(\lambda), \mathbf{u}_h^*(\mu))_{\Omega_i} = d_{h,i}(\lambda,\mu), \tag{6.7}$$

which shows that $d_h(\cdot,\cdot)$ is symmetric and

$$d_{h,i}(\mu,\mu) = (K^{-1}\mathbf{u}_h^*(\mu), \mathbf{u}_h^*(\mu))_{\Omega_i} \geq 0. \tag{6.8}$$

For $\mu \in \Lambda_h$, if (2.14) holds, the argument from Lemma 2.1 shows that $d_h(\mu,\mu) = 0$ implies $\mu = 0$. ▯

6.2. Error estimates for the mortar pressure. Denote by $\|\cdot\|_{d_h}$ the semi-norm induced by $d_h(\cdot,\cdot)$ on $L^2(\Gamma)$, i.e.,

$$\|\mu\|_{d_h} = d_h(\mu,\mu)^{1/2}, \quad \mu \in L^2(\Gamma).$$

THEOREM 6.2. *For the mortar pressure λ_h of the mixed method* (2.10)–(2.12)*, if* (3.18) *holds, then there exists a positive constant C, independent of h, such that*

$$\|p-\lambda_h\|_{d_h} \leq C\sum_{i=1}^n (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i})h^r, \quad 1\leq r\leq k+1, \tag{6.9}$$

$$\|\mathcal{P}_h p-\lambda_h\|_{d_h} \leq C\sum_{i=1}^n (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i})h^r, \quad 1\leq r\leq k+1. \tag{6.10}$$

In the case of diagonal tensor K and RTN *spaces on rectangular type grids,*

$$\|p-\lambda_h\|_{d_h} \leq C\sum_{i=1}^n (\|p\|_{r+3/2,\Omega_i} + \|\mathbf{u}\|_{r+1/2,\Omega_i})h^{r+1/2}, \quad 1\leq r\leq k+1, \tag{6.11}$$

$$\|\mathcal{P}_h p-\lambda_h\|_{d_h} \leq C\sum_{i=1}^n (\|p\|_{r+3/2,\Omega_i} + \|\mathbf{u}\|_{r+1/2,\Omega_i})h^{r+1/2}, \quad 1\leq r\leq k+1. \tag{6.12}$$

Proof. With (6.8) we have

$$\|p-\lambda_h\|_{d_h} \leq C\|\mathbf{u}_h^*(p) - \mathbf{u}_h^*(\lambda_h)\|_0, \tag{6.13}$$

using that $\mathbf{u}_h^*(\cdot)$ depends linearly on its argument. Define, for $\mu \in L^2(\Gamma)$,

$$\mathbf{u}_h(\mu) = \mathbf{u}_h^*(\mu) + \bar{\mathbf{u}}_h, \quad p_h(\mu) = p_h^*(\mu) + \bar{p}_h,$$

and note that $(\mathbf{u}_h(\mu), p_h(\mu)) \in \mathbf{V}_h \times W_h$ satisfy, for $1 \leq i \leq n$,

$$\begin{aligned} (K^{-1}\mathbf{u}_h(\mu), \mathbf{v})_{\Omega_i} &= (p_h(\mu), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \mu, \mathbf{v}\cdot\nu\rangle_{\Gamma_i} \\ &\quad - \langle g, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\setminus\Gamma}, \qquad \mathbf{v}\in\mathbf{V}_{h,i}, \end{aligned} \tag{6.14}$$

$$(\nabla\cdot\mathbf{u}_h(\mu), w)_{\Omega_i} = (f,w)_{\Omega_i}, \qquad w \in W_{h,i}. \tag{6.15}$$

We now have

$$\begin{aligned} \|\mathbf{u}_h^*(p) - \mathbf{u}_h^*(\lambda_h)\|_0 &= \|\mathbf{u}_h(p) - \mathbf{u}_h(\lambda_h)\|_0 \\ &= \|\mathbf{u}_h(p) - \mathbf{u}_h\|_0 \\ &\leq \|\mathbf{u}_h(p) - \mathbf{u}\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 \end{aligned} \tag{6.16}$$

Bound (6.9) now follows from (6.13), (6.16), Theorem 4.2, and the standard mixed method estimate for (2.3)–(2.4) and (6.14)–(6.15) [28, 26, 14]

$$\|\mathbf{u}_h(p) - \mathbf{u}\|_{0,\Omega_i} \le C(\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i})h^r, \quad 1 \le r \le k+1.$$

To show (6.11), we modify (6.16) as

$$\|\mathbf{u}_h(p) - \mathbf{u}_h\|_0 \le \|\mathbf{u}_h(p) - \Pi\mathbf{u}\|_0 + \|\Pi\mathbf{u} - \Pi_0\mathbf{u}\|_0 + \|\Pi_0\mathbf{u} - \mathbf{u}_h\|_0. \tag{6.17}$$

Bound (6.11) now follows from (6.13), (6.17), (3.30), (4.5), and a superconvergence estimate for the standard mixed method [15] (see also [24, 17])

$$\|\mathbf{u}_h(p) - \Pi_i\mathbf{u}\|_{0,\Omega_i} \le C\|\mathbf{u}\|_{r+1,\Omega_i}h^{r+1}, \quad 1 \le r \le k+1.$$

To prove (6.10) and (6.12), note that, by (6.1),

$$\begin{aligned}
(K^{-1}\mathbf{u}_h^*(\mathcal{P}_h p - p), \mathbf{u}_h^*(\mathcal{P}_h p - p))_{\Omega_i} &= -\langle \mathcal{P}_h p - p, \mathbf{u}_h^*(\mathcal{P}_h p - p)\cdot\nu\rangle_{\Gamma_i} \\
&\le \sum_j \|\mathcal{P}_h p - p\|_{0,\Gamma_{i,j}} \|\mathbf{u}_h^*(\mathcal{P}_h p - p)\cdot\nu\|_{0,\Gamma_{i,j}} \\
&\le C\sum_j \|p\|_{r,\Gamma_{i,j}} h^r \|\mathbf{u}_h^*(\mathcal{P}_h p - p)\|_{0,\Omega_i} h^{-1/2}, \quad 0 \le r \le k+2,
\end{aligned}$$

using (3.1) and Lemma 4.1 for the last inequality. Therefore, with (6.8),

$$\|\mathcal{P}_h p - p\|_{d_h} \le C\sum_{i=1}^n \|p\|_{r+3/2,\Omega_i} h^{r+1/2}, \quad 0 \le r \le k+1. \tag{6.18}$$

Bounds (6.10) and (6.12) follow from (6.9) and (6.11), respectively, using the triangle inequality and (6.18). □

REMARK 6.1. In the case of the lowest order RTN spaces, it is proven in [12] that, for any $\phi \in \Lambda_h$, $d_{h,i}(\phi,\phi)$ is equivalent to $|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\phi|^2_{1/2,\partial\Omega_i}$, where $\mathcal{I}^{\partial\Omega_i}$ is an interpolation operator onto the space of continuous piece-wise linears on $\partial\Omega_i$. Therefore $\|\cdot\|_{d_h}$ can be characterized as a certain discrete $H^{1/2}$-semi-norm on Γ (see [30]). This is also in accordance with the numerically observed $O(h^2)$ convergence for the mortars in a discrete L^2-norm (see Section 8).

7. A substructuring domain decomposition algorithm. In this section we discuss implementation of a parallel domain decomposition algorithm for solving the resulting linear system. We apply a substructuring algorithm by Glowinski and Wheeler [20] to the lowest order RTN discretization on non-matching multiblock rectangular type grids. In our case we solve an interface problem in the space of mortar pressures. We use the conjugate gradient method to solve the interface problem (6.5). Note that Lemma 6.1 guarantees convergence of the iterative procedure in Λ_h.

Every iteration of the conjugate gradient requires an evaluation of the bilinear form $d_h(\cdot,\cdot)$, and therefore, solving subdomain problems (6.1)–(6.2) with a given Dirichlet data in the mortar space Λ_h. Because of the property

$$d_{h,i}(\lambda,\mu) = d_{h,i}(\mathcal{Q}_{h,i}\lambda, \mathcal{Q}_{h,i}\mu),$$

the subdomain solves only use projections of the mortar data onto the local spaces. Therefore, no change in the local solvers is needed for the implementation. Moreover, the conjugate gradient is performed in the space

$$\{(\phi_L,\phi_R) \in (L^2(\Gamma))^2 : \phi_L|_{\Gamma_{i,j}} \in \mathcal{Q}_{h,i}\Lambda_h \text{ and } \phi_R|_{\Gamma_{i,j}} \in \mathcal{Q}_{h,j}\Lambda_h, \ 1 \le i < j \le n\}.$$

The conjugate gradient residual is the jump in the fluxes across subdomain boundaries. The jump is computed after projecting the local boundary fluxes onto the mortar space, as indicated by

$$d_h(\lambda,\mu) = -\sum_{i=1}^{n} \langle \mathbf{u}_h^*(\lambda)\cdot\nu_i,\mu\rangle_{\Gamma_i} = -\sum_{i=1}^{n} \langle \mathcal{P}_h\mathbf{u}_h^*(\lambda)\cdot\nu_i,\mu\rangle_{\Gamma_i}, \quad \lambda,\mu\in\Lambda_h.$$

Therefore the only additional computational cost compared to the case of matching grids is computing the projections $\mathcal{Q}_{h,i}:\Lambda_h\to\mathbf{V}_{h,i}\cdot\nu_i$ and $\mathcal{P}_h:\mathbf{V}_{h,i}\cdot\nu_i\to\Lambda_h$.

8. Numerical results. In this section we present three numerical tests confirming the theoretical convergence rates. All examples are on the unit square, use only the lowest order RTN spaces on rectangles (so $k=l=0$), and use a diagonal K.

In the first example we solve a problem with known analytic solution

$$p(x,y) = x^3y^4 + x^2 + \sin(xy)\cos(y)$$

and tensor coefficient

$$K = \begin{pmatrix} (x+1)^2+y^2 & 0 \\ 0 & (x+1)^2 \end{pmatrix}.$$

The boundary conditions are Dirichlet on the left and right edge and Neumann on the rest of the boundary. The domain is divided into four subdomains with interfaces along the $x=1/2$ and $y=1/2$ lines. The initial non-matching grids are shown in Figure 8.1. We test both continuous and discontinuous mortars. The initial mortar grids on all interfaces have 4 elements with 5 degrees of freedom in the continuous case and 2 elements with 4 degrees of freedom in the discontinuous case, therefore satisfying the solvability condition (2.14).

Convergence rates for this test case are given in Table 8.1. The rates were established by running the test case and 4 levels of grid refinement (we halve the element diameters for each refinement) and computing a least squares fit to the error. We observe numerically convergence rates corresponding to those predicted by the theory. The pressure error, $|||p-p_h|||$, is the discrete L^2-norm induced by the midpoint rule on $\mathcal{T}_h$. It is $O(h^2)$-close to $\|\hat{p}-p_h\|_0$, which itself is superconvergent of $O(h^2)$ from Theorem 5.1. The discrete velocity error $|||\mathbf{u}-\mathbf{u}_h|||$ is superconvergent of $O(h^{3/2})$ by Theorem 4.3. Finally, the discrete interface pressure error $|||p-\lambda_h|||$ is actually computed by summing over blocks i the discrete L^2-norm of $p-\mathcal{Q}_{h,i}\lambda_h$ induced by the midpoint rule on the traces of $\mathcal{T}_{h,i}$ on $\partial\Omega_i\cap\Gamma$. This is essentially the L^2-norm, and thus we might expect it to be $1/2$ power of h better than $\|p-\lambda_h\|_{d_h}$, since the latter is essentially an $H^{1/2}$-semi-norm by Remark 6.1. Since Theorem 6.2 implies that $\|p-\lambda_h\|_{d_h}$ is superconvergent of $O(h^{3/2})$, we might expect to see $|||p-\lambda_h|||$ converging as $O(h^2)$; indeed we do.

The computed pressure and velocity with continuous and discontinuous mortars on the first level of refinement are shown in Figure 8.2. Although both solutions look the same, Table 8.1 indicates that they differ. This can also be seen in Figure 8.3, where the magnified numerical error is shown. The error in the continuous mortar case is concentrated at the cross-points, where the only discontinuities in the mortar space occur. The error in the discontinuous mortar case is distributed along the interfaces and is somewhat larger. We should point out, however, that the discontinuous mortars provide flux continuity in a more local sense, as indicated by the flux matching condition (2.12).

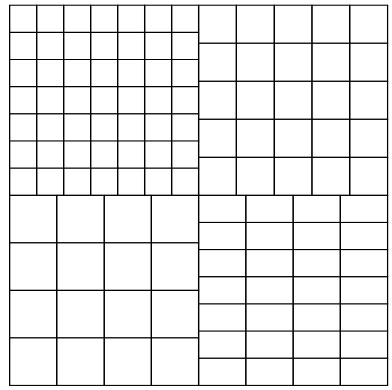


FIG. 8.1. *Initial non-matching grids for Example 1.*

	Continuous mortars			Discontinuous mortars		
$1/h$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - \lambda_h\|\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - \lambda_h\|\|\|$
8	9.62E-03	2.95E-02	1.31E-02	9.52E-03	4.12E-02	1.36E-02
16	2.41E-03	8.54E-03	3.31E-03	2.40E-03	1.36E-02	3.45E-03
32	6.04E-04	2.42E-03	8.30E-04	6.03E-04	4.55E-03	8.68E-04
64	1.51E-04	6.66E-04	2.08E-04	1.51E-04	1.54E-03	2.19E-04
128	3.91E-05	1.88E-04	5.39E-05	3.75E-05	5.29E-04	5.35E-05
rate	$O(h^{1.99})$	$O(h^{1.83})$	$O(h^{1.99})$	$O(h^{2.00})$	$O(h^{1.57})$	$O(h^{2.00})$

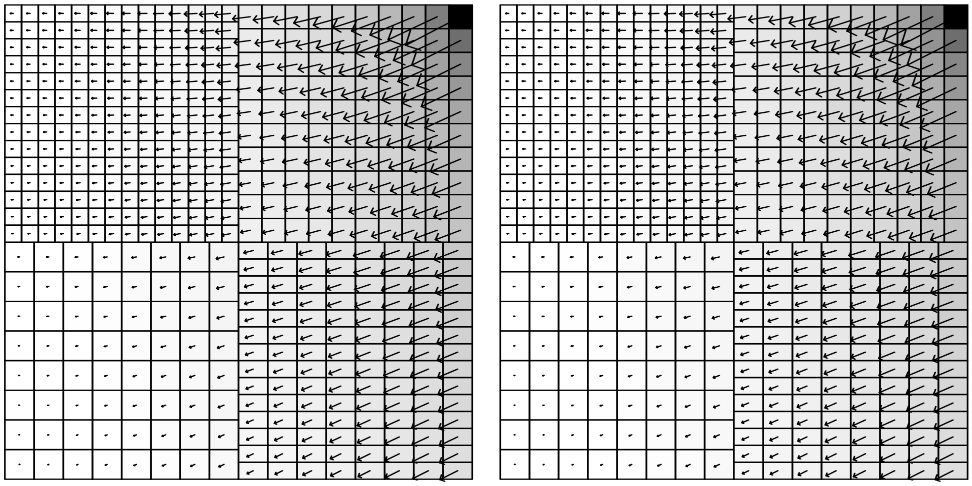
TABLE 8.1
Discrete norm errors and convergence rates for Example 1.

In the second example we test a problem with a discontinuous coefficient. We choose $K = I$ for $0 \le x < 1/2$ and $K = 10 * I$ for $1/2 < x \le 1$. The solution

$$p(x, y) = \begin{cases} x^2 y^3 + \cos(xy), & 0 \le x \le 1/2, \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos\left(\frac{2x+9}{20} y\right), & 1/2 \le x \le 1, \end{cases}$$

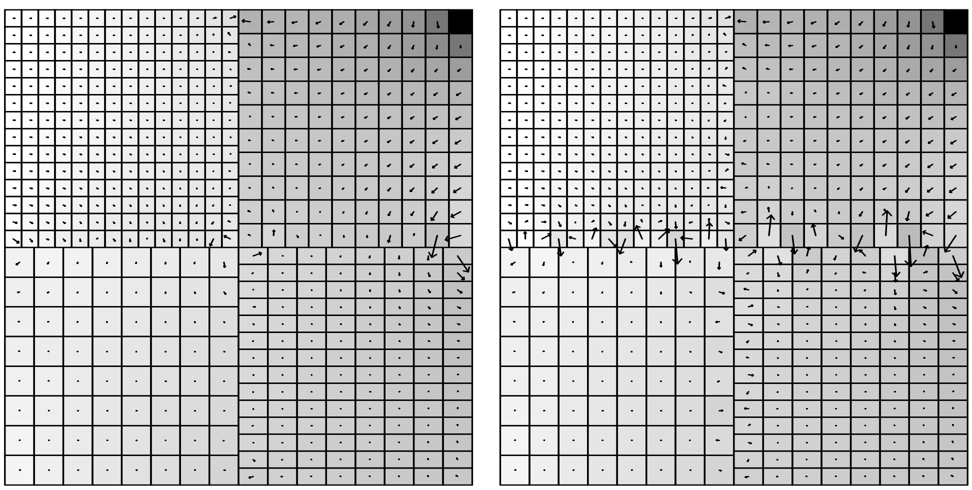
is chosen to be continuous and have continuous normal flux at $x = 1/2$. The domain has two subdomains with an interface along $x = 1/2$. The initial grids are 4×8 on the left and 4×11 on the right. Continuous mortars on a grid of 7 elements with 8 degrees of freedom or discontinuous mortars on a grid of 4 elements with 8 degrees of freedom are introduced on the interface. Convergence rates for the test case are given in Table 8.2; again they agree with the theory, even though K is mildly discontinuous.

In the third and last example we compare the mortar element mixed method on locally refined grids to the "slave" or "worker" nodes local refinement technique [16, 18]. In the latter, the fine grid interface fluxes within a coarse cell are forced to be equal to the coarse grid flux. We note that this scheme can be recovered as a special case of the mortar element method with discontinuous mortars, if the trace of the fine grid is a refinement by two of the interface grid. Indeed, in this case the flux matching condition (2.12) becomes a local condition over two (four if $d = 3$) fine grid boundary elements and forces all fine grid fluxes to be equal to the coarse grid flux. Our theory also recovers the convergence and superconvergence results derived by Ewing and Wang [18]. In the mortar method, however, the flux continuity condition



A. *Continuous mortars* B. *Discontinuous mortars*

FIG. 8.2. *Computed pressure (shade) and velocity (arrows) for Example 1.*



A. *Continuous mortars* B. *Discontinuous mortars*

FIG. 8.3. *Pressure and velocity error for Example 1.*

	Continuous mortars			Discontinuous mortars		
$1/h$	$\|\|\|p-p_h\|\|\|$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$\|\|\|p-\lambda_h\|\|\|$	$\|\|\|p-p_h\|\|\|$	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$\|\|\|p-\lambda_h\|\|\|$
8	3.20E-04	1.60E-02	3.32E-04	3.38E-04	3.27E-02	1.22E-03
16	8.38E-05	4.27E-03	8.18E-05	8.55E-05	1.13E-02	3.17E-04
32	2.12E-05	1.18E-03	2.01E-05	2.14E-05	3.93E-03	8.01E-05
64	5.35E-06	3.41E-04	4.89E-06	5.34E-06	1.37E-03	2.01E-05
128	1.38E-06	1.05E-04	1.15E-06	1.35E-06	4.82E-04	5.02E-06
rate	$O(h^{1.97})$	$O(h^{1.81})$	$O(h^{2.04})$	$O(h^{1.99})$	$O(h^{1.52})$	$O(h^{1.98})$

TABLE 8.2

Discrete norm errors and convergence rates for Example 2 (discontinuous K).

can be relaxed by choosing a coarser mortar space. In this case the fine grid fluxes are not forced to be equal and approximate the solution better. Our numerical experience shows that this may reduce the flux error on the interface by up to a factor of two.

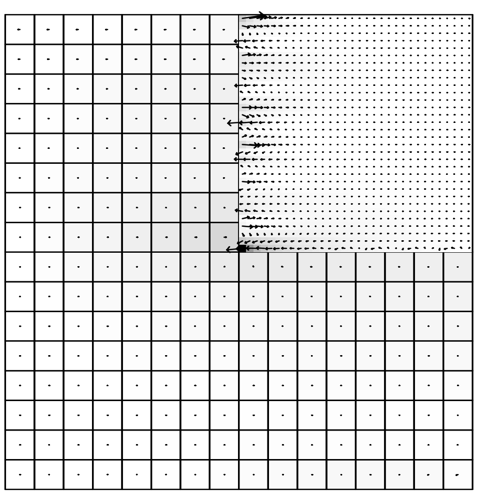
We solve a problem on locally refined grids with solution and coefficient

$$p(x,y) = x^3y^2 + \sin(xy) \quad \text{and} \quad K = \begin{pmatrix} 10 + 5\cos(xy) & 0 \\ 0 & 1 \end{pmatrix}.$$

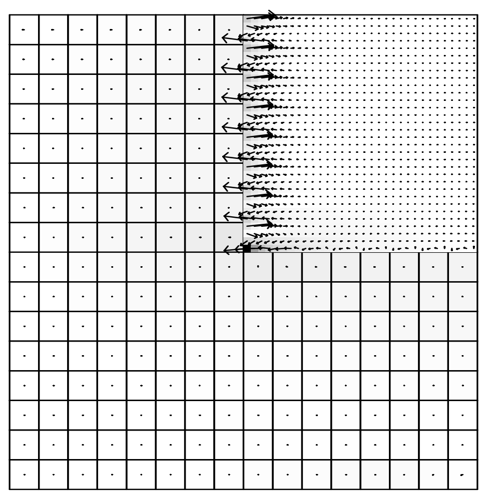
The domain is divided into four subdomains with interfaces along the $x = 1/2$ and $y = 1/2$ lines. The domains are numbered starting from the lower left corner and first increasing x. The initial grids are 4×4 on Ω_1–Ω_3 and 16×16 on Ω_4. We use discontinuous piece-wise linear mortars on the non-matching interface. We report the numerical error on the grid and three levels of refinement for two cases. If the coarse grid is $n \times n$, we take a mortar grid with $n - 1$ elements in the first case and $2n$ elements in the second case, which is equivalent to the "slave" nodes method. The results are summarized in Table 8.3. The pressure and velocity error on the first level of refinement are shown in Figure 8.4.

	Discontinuous mortars			"Slave" nodes		
$1/h$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - \lambda_h\|\|\|$	$\|\|\|p - p_h\|\|\|$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\|\|p - \lambda_h\|\|\|$
8	1.12E-3	6.70E-2	3.80E-3	1.30E-3	1.45E-1	5.74E-3
16	2.67E-4	2.48E-2	1.03E-3	2.90E-4	5.00E-2	1.39E-3
32	6.57E-5	9.77E-3	2.72E-4	6.86E-5	1.74E-2	3.41E-4
64	1.64E-5	3.62E-3	6.93E-5	1.66E-5	6.09E-3	8.42E-5
rate	$O(h^{2.03})$	$O(h^{1.40})$	$O(h^{1.93})$	$O(h^{2.09})$	$O(h^{1.52})$	$O(h^{2.03})$

TABLE 8.3
Discrete norm errors and convergence rates for Example 3 (locally refined grids).



A. *Discontinuous mortars.*



B. *"Slave" nodes.*

FIG. 8.4. *Pressure and velocity error for Example 3 (locally refined grids).*

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REFERENCES

[1] T. Arbogast, C. N. Dawson, P. T. Keenan, M. F. Wheeler, and I. Yotov, *Enhanced cell-centered finite differences for elliptic equations on general geometry*, SIAM J. Sci. Comp., 19 (1998), pp. 404–425.

[2] T. Arbogast, M. F. Wheeler, and I. Yotov, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997), pp. 828–852.

[3] T. Arbogast and I. Yotov, *A non-mortar mixed finite element method for elliptic problems on non-matching multiblock grids*, Comp. Meth. in Appl. Mech. and Engng., 149 (1997), pp. 225–265.

[4] D. N. Arnold and F. Brezzi, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, RAIRO Modèl. Math. Anal. Numèr., 19 (1985), pp. 7–32.

[5] C. Bernardi, Y. Maday, and A. T. Patera, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear partial differential equations and their applications, H. Brezis and J. L. Lions, eds., Longman Scientific & Technical, UK, 1994.

[6] F. Brezzi, J. Douglas, Jr., R. Duràn, and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.

[7] F. Brezzi, J. Douglas, Jr., M. Fortin, and L. D. Marini, *Efficient rectangular mixed finite elements in two and three space variables*, RAIRO Modèl. Math. Anal. Numèr., 21 (1987), pp. 581–604.

[8] F. Brezzi, J. Douglas, Jr., and L. D. Marini, *Two families of mixed elements for second order elliptic problems*, Numer. Math., 88 (1985), pp. 217–235.

[9] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, 1991.

[10] Z. Chen and J. Douglas, Jr., *Prismatic mixed finite elements for second order elliptic problems*, Calcolo, 26 (1989), pp. 135–148.

[11] Ph. Ciarlet, *The Finite Element Method for Elliptic Problems*, North Holland, 1976.

[12] L. C. Cowsar, J. Mandel, and M. F. Wheeler, *Balancing domain decomposition for mixed finite elements*, Math. Comp., 64 (1995), pp. 989–1015.

[13] L. C. Cowsar and M. F. Wheeler, *Parallel domain decomposition method for mixed finite elements for elliptic partial differential equations*, in Fourth Int'l Symposium on Domain Decomposition Methods for Partial Diff. Eqns., R. Glowinski et al., eds., SIAM, 1991.

[14] J. Douglas, Jr. and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp., 44 (1985), pp. 39–52.

[15] R. Durán, *Superconvergence for rectangular mixed finite elements*, Numer. Math., 58 (1990), pp. 287–298.

[16] R. E. Ewing, R. D. Lazarov, T. F. Russell, and P. S. Vassilevski, *Analysis of the mixed finite element method for rectangular Raviart-Thomas elements with local refinement*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan et al., eds., SIAM, 1990, pp. 98–114.

[17] R. E. Ewing, R. D. Lazarov, and J. Wang, *Superconvergence of the velocity along the Gauss lines in mixed finite element methods*, SIAM J. Numer. Anal., 28 (1991), pp. 1015–1029.

[18] R. E. Ewing and J. Wang, *Analysis of mixed finite element methods on locally refined grids*, Numer. Math., 63 (1992), pp. 183–194.

[19] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[20] R. Glowinski and M. F. Wheeler, *Domain decomposition and mixed finite element methods for elliptic problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski et al., eds., SIAM, 1988, pp. 144–172.

[21] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston, 1985.

[22] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, vol. 1, Springer-Verlag, 1972.

[23] T. P. Mathew, *Domain decomposition and iterative refinement methods for mixed finite element discretizations of elliptic problems*, PhD Thesis, Courant Institute of Mathematical Sciences, New York University, 1989. Tech. Rep. 463.

[24] M. Nakata, A. Weiser, and M. F. Wheeler, *Some superconvergence results for mixed finite element methods for elliptic problems on rectangular domains*, in The Mathematics of Finite Elements and Applic. V, J. R. Whiteman, ed., Academic Press, 1985, pp. 367–389.

[25] J. C. Nedelec, *Mixed finite elements in* $\mathbf{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[26] R. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic*

problems, in Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics, vol. 606, Springer-Verlag, New York, 1977, pp. 292–315.
[27] J. E. Robertrs and J.-M. Thomas, *Mixed and hybrid methods*, in Handbook of Numerical Analysis, Vol. II, P. G. Ciarlet and J. Lions, eds., Elsevier, 1991, pp. 523–639.
[28] J. M. Thomas, *These de Doctorat d'etat*, 'a l'Universite Pierre et Marie Curie, 1977.
[29] A. Weiser and M. F. Wheeler, *On convergence of block-centered finite-differences for elliptic problems*, SIAM J. Numer. Anal., 25 (1988), pp. 351–375.
[30] I. Yotov, *Mixed finite element methods for flow in porous media*, PhD Thesis, Rice University, Houston, Texas, 1996.